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STIINTIFICA SI TEHNICA

ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No. 12/1990

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COMPRESSIONS IN H-CONES

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N. BOBOC*) and Gh. BUCUR**)

March, 1990

*) *Faculty of Mathematics, University of Bucharest, Str. Academiei No. 14, 70109 Bucharest, Romania.*

**) *Department of Mathematics, INCREST, Bd. Pacii 220, 79622 Bucharest, Romania.*

SUBORDINATIONS IN EXCESIVE STRUCTURES. COMPRESSIONS IN H-CONES

N. BOBOC and G.BUCUR

We give an analytic proof of the following result from the theory of Markov processes: If Y, Y' are two standard processe such that the hitting distributions of Y are dominated by the hitting distributions of Y' then Y' is obtained from a random time change in a subprocess of Y (see [14]).

If $\mathcal{V}=(V_\alpha)_{\alpha>0}$ is a proper resolvent on a measurable space $(X,\mathcal{B})$ $\mathcal{E}_{\mathcal{V}}$ is the set of all $\mathcal{V}$-excessive functions on X which are finite $\mathcal{V}$-a.s. and if P is a proper kernel on $(X,\mathcal{B})$ such that $Pf\in\mathcal{E}_{\mathcal{V}}$ for any positive measurable function f with $Pf<\infty$ then (see [5]) the convex cone

$$\{ s\in\mathcal{E}_{\mathcal{V}} \mid Ps \preceq_{\mathcal{E}_{\mathcal{V}}} s \}$$

coincides with the set of all excesive functions with respect to a new resolvent $\mathcal{V}^P$ whose initial kernel V_0^P is given by

$$V_0^P=\sum_{n=0}^{\infty} P^n V_0$$

The same result may be obtained if P is only a map

$$P: \mathcal{E}_{\mathcal{V}} \longrightarrow \tilde{\mathcal{E}}_{\mathcal{V}}$$

which is additive, $Ps\in\mathcal{E}_{\mathcal{V}}$ if $Ps<\infty$ $\mathcal{V}$-a.s.

$$s\leq t \Rightarrow Ps \preceq_{\mathcal{E}_{\mathcal{V}}} Pt \quad \text{if } Pt<\infty \quad \mathcal{V}\text{- a.s.}$$

$$s_n \uparrow s \Rightarrow Ps_n \uparrow Ps$$

where $\tilde{\mathcal{E}}_\vartheta$ is the set of all ϑ-excesive functions on X.

In the first part of the present paper we develop a similar construction starting with an abstract H-cone $\mathcal{S}$ instead of $\mathcal{E}_\vartheta$ and with a map (called compression operator on S)

$$P : D(P) \longrightarrow S$$

where D(P) is a solid convex subcone of S and such that P is additive,

$$s \leq t \Rightarrow Ps \underset{\mathcal{S}}{\precsim} Pt$$

and

$$s_i \uparrow s \Rightarrow Ps_i \uparrow Ps.$$

Then we prove that the convex cone

$$S' := \{ s \in D(P) \mid Ps \precsim_S s \}$$

is also an H-cone such that

$$F \subset S' \Rightarrow \bigwedge_{\mathcal{S}} F \in S'$$

$$F \subset S' \Rightarrow \curlywedge_{\mathcal{S}} F \in S'$$

and for any upper directed and dominated family $F \subset S'$ we have

$$\bigvee_{\mathcal{S}} F \in S'$$

Also if, B is a balayage on S then the map $B':S'\to S'$ defined by

$$B's = \bigwedge\{t\in S' \mid t\geq Bs\}$$

is a balayage on S' and we have

$$Bf \leq B'f \quad (\forall) f\in(S'-S')_+.$$

A sub H-cone S' of S for which this last relation is fullfiled is termed a compression of S. In the second part of this paper we give some supplimentary conditions such that any compression S' of S may be obtained with the aid of a compression operator P on S as above.

More precisely we start with two standard H-cones of functions S and S' on a set X such that; a) $S'\subset S$; b) the natural topology on X associated with S' is greater than the natural topology on X associated with S c) there exists a strictly positive potential on X with respect to S'. d) Any absorbant point of X with respect to S is also absorbent with respect to S'. We denote for any balayage B on S, by B' the corresponding balayage on S' (i.e. $B's=\bigwedge\{t\in S' \mid t\geq Bs\}$ for any $s\in S'$).

In this frame we prove that there exists a compression operator P on S such that

$$S'=\{s\in D(P) \mid Ps \preceq_S s\}$$

if for any balayage B on S and any $f\in(S'-S')_+$ we have

$$Bf\leq B'f$$

It is known that in the previous conditions we may construct two strong Ray processes Y and Y' on X such that S (resp. S') becomes the set (resp. ') of all excessive functions on X with respect to Y (resp. Y') (see [7]

The condition

$$Bf \leq B'f \quad (\forall) \quad f \in (S'-S)_+$$

is equivalent with the fact that hitting distributions with respect to Y are dominated by the hitting distributions with respect to Y'. The relation

$$S' = \{ s \in D(P) \mid Ps \precsim_S s \}$$

is equivalent with the fact that Y is obtained from a subprocess of Y' using a random time change .

In this way we give an analytic treatment of a well known problem considered by Šur([15], [16]), Shih ([14]) and others ([2], [3], [14], [17]).

We mention also a particular analytic approach ([9], [10], [1]) considered for the case when S and S' satisfy the suplimentary condition of natural sheaf property (or equivalent when the processes Y and Y' are continuous).

Throughout this paper S will be a general H-cone ([4]). We use $\leq$ for the natural order in S and $\wedge$, $\vee$ for the lattice operations with respect to this order relation, Also we denote by $\precsim$ (or more precisely $\precsim_S$) the specific order relation in S and by $\curlywedge$, $\curlyvee$ the lattice operations with respect to this order relation. The order relations $\leq$, $\precsim$ are extended naturally in the liniar vector space S-S generated by S.

If $f \in S-S$ we remember that $Rf = R^S f$ means the reduite of f with respect to S i.e.

$$Rf = \bigwedge\{s \in S \mid s \geq f\},$$

and if $f = s - t$, $s, t \in S$ then we have

$$Rf \preceq s.$$

1. Compression operators on an H-cone

Definition. A map P defined on a convex subcone D(P) of S, which is solid with respect to the natural order on S is called a compression operator in S if we have

a) $P(s_1+s_2) = Ps_1 + Ps_2 \quad (\forall) s_1, s_2 \in D(P)$

b) $s_1 \leq s_2 \Rightarrow Ps_1 \preceq Ps_2 \quad (\forall) \ s_1, s_2 \in D(P)$

c) $(Ps_i)_i \uparrow Ps$ for any $s \in D(P)$ and and increasing sequence $(s_i)_i$ in D(P) such that $(s_i)_i \uparrow s$.

Definition. Let $P: D(P) \to S$ be a compression operator in S. An element $s \in D(P)$ is termed P-excessive if $Ps \preceq s$. The set of all P-excessive elements of D(P) will be denoted by S(P).

Proposition 1.1. For any compression operator P in S the set S(P) is a convex subcone of S such that

a) $\bigwedge_S F \in S(P) \quad (\forall) \quad F \subset S(P)$

b) $\bigvee_S F \in S(P)$ for any upper directed and dominated family F in S(P).

<u>Proof.</u> a) If $F \subset S(P)$ then $F \subset D(P)$ and therefore $\bigwedge_S F \in D(P)$. We have

$$P(\bigwedge_S F) \precsim Ps \precsim s \quad (\forall)\ s \in F$$

and therefore

$$P(\bigwedge_S F) \precsim \bigwedge_S F, \quad \bigwedge_S F \in S(P)$$

b) Suppose that $F \subset S(P)$ is upper directed and dominated by $s_o \in S(P)$. We have

$$\bigvee_S F \leq s_o, \quad \bigvee F \in D(P),$$

$$P(\bigvee_S F) = \bigvee_{s \in F} Ps \precsim \bigvee_{s \in F} s,$$

$$\bigvee_S F \in S(P)$$

<u>Corollary 1.2.</u> Let P be a compression operator in S and let $\leq'$ be an order relation on S such that $(S, \leq')$ is an H-cone and such that

$$s \precsim t \Rightarrow s \leq' t, \quad s \leq' t \Rightarrow s \leq t,$$

for any $s, t \in S$. Then we have

a) $\bigwedge_{(S, \leq')} F \in S(P)$ $(\forall)$ $F \subset S(P)$

b) $\bigvee_{(S, \leq')} F \in S(P)$ for any upper directed and dominated family F in $(S(P), \leq')$.

<u>Corollary 1.3.</u> Let P be a compression operator in S. Then we have

a) $\bigwedge F \in S(P) \quad (\forall) \quad F \subset S(P)$

b) $\bigvee F \in S(P)$ for any upper directed and dominated family F in $(S(P), \preccurlyeq)$.

<u>Proposition 1.4.</u> If for any $f, g \in S-S$ we put

$$f \wedge g =: g - R(g-f), \quad f \vee g = f + R(g-f)$$

then we have

a) $f \wedge g \leq f, \ f \wedge g \preccurlyeq g,$

$$f \wedge g = \bigvee \{ h \in S-S / h \leq f, \ h \preccurlyeq g \}$$

$$(f+h) \wedge (g+h) = (f \wedge g) + h \quad (\forall) \quad h \in S-S$$

$$u \in S-S, \ u \preccurlyeq f, \ u \preccurlyeq g \Rightarrow u \preccurlyeq f \wedge g$$

b) $f \vee g \succcurlyeq f, \ f \vee g \succcurlyeq g,$

$$f \vee g = \bigwedge \{ h \in S-S \mid h \succcurlyeq f, \ h \succcurlyeq g \}$$

$$(f+h) \vee (g+h) = (f \vee g) + h$$

$$u \in S-S, \ u \succcurlyeq f, \ u \succcurlyeq g \Rightarrow u \succcurlyeq f \vee g.$$

c) $f \wedge g + f \vee g = f + g$

d) $f, g \in S \Rightarrow f \wedge g$ and $f \vee g \in S$.

<u>Proof.</u> The assertion c) is obvious. The assertion d) follows from the deffinitions of $f \wedge g$ and $f \vee g$ using the fact that

$$f,g\in S \Rightarrow R(g-f)\precsim g$$

a) Since $R(g-f)\in S$ we have $f\wedge g\precsim g$. Since $g-f\leq R(g-f)$ it follows that $f\wedge g\leq f$.

Let now $h\in S-S$ be such that $h\leq f$, $h\precsim g$.

We have $g-h\in S$, $g-h\geq g-f$ and therefore

$$g-h\geq R(g-f),$$

$$f\wedge g = g-R(g-f)\geq h.$$

The relation

$$(f+h)\wedge(g+h)=(f\wedge g)+h$$

follows just form the definition of the element $(f+h)\wedge(g+h)$.

Let now $u\in S-S$ be such that $u\precsim f, u\precsim g$ and let $s,t\in S$ be such that

$$f=u+s,\ g=u+t$$

We have

$$f\wedge g=(u+t)-R(t-s)\succsim u.$$

b) The relations

$$f\vee g\succsim g,\ f\vee g\succsim f$$

follows from the relations

$$R(g-f)\geq g-f,\ R(g-f)\in S.$$

Let $h \in S-S$ be such that $h \geq g$, $h \geq f$. We have

$$h-f \in S, \quad h-f \geq g-f$$

and therefore

$$h-f \geq R(g-f)$$

$$h \geq f+R(g-f)=f \gamma g$$

The relation

$$(h+f) \gamma (h+g)=h+(f \gamma g)$$

follows just from the definition of $(h+f) \gamma (h+g)$.

Let now $u \in S-S$ be such that

$$u \geq f, u \geq g$$

and let $s,t \in S$ be such that

$$u=f+s, \quad u=g+t.$$

We have $g-f=s-t$,

$$R(g-f)=R(s-t) \leq s$$

and therefore

$$f \gamma g=f+R(g-f)=(u-s)+R(s-t) \leq u.$$

Lemma 1.5. If P is a compression operator in S then any $s \in S(P)$, $t \in S$ such that

$$Ps \precsim t \leq s$$

we have $t \in S(P)$.

Proof. We have

$$Pt \precsim Ps \precsim t, \qquad t \in S(P).$$

Lemma 1.6. If P is a compression operator in S then for any $s, t \in S(P)$ we have

$$s \wedge (Ps + t - Pt) \in S(P).$$

Proof. Since $Pt \precsim t$ we deduce, using Proposition 1.4, d), that the element

$$u := s \wedge (Ps + t - Pt)$$

belong to S. On the other hand since $Ps \precsim s$ we get, using Proposition 1.4, a)

$$Ps \precsim u \leq s$$

The assertion follows now form the previous lemma

Theorem 1.7. If P is a compression operator in S then for any $s, t \in S(P)$ we have

$$\mathcal{R}^{S(P)}(s-t)=\bigvee s_n$$

where $(s_n)_n$ is the sequence from D(P) inductively defined by

$$s_1=(Ps-Pt)\curlyvee(s-t), \quad s_{n+1}=Ps_n\curlyvee s_n$$

Moreover, we have

$$s-R^{S(P)}(s-t)\in S(P).$$

Proof. Obviously we may suppose that $t\leq s$. With the above notations we show inductively that we have, for $n\geq 1$

$$s_n\in D(P),\ s_n\precsim s,$$
$$Ps+s_n\precsim s+Ps_n,$$
$$s_n\in S.$$

Indeed, if we put

$$s_o=s-t$$

we have

$$s_o+t=s,\ s_o\precsim s,$$
$$s_o+Pt\precsim s_o+t=s.$$

Since $Ps_o:=Ps-Pt\in S$ we deduce that

$$s_1:=Ps_o+R(s_o-Ps_o)\in S.$$

On the other hand we have, using Proposition 4,

$$Ps+s_1=Ps+Ps_o\curlyvee s_o=(Ps+Ps_o)\curlyvee(Ps+s_o),$$
$$Ps+Ps_o\precsim s+Ps_1, \quad Ps+s_o=Ps_o+Pt+s_o\precsim Ps_1+s$$

Hence from Proposition 1.4, b) we get

$$Ps+s_1\precsim Ps_1+s, \quad s_1\precsim s.$$

We suppose now that for $n\in N$, $n\geq 1$ we have

$$s_n\precsim s, \quad Ps+s_n\precsim Ps_n+s$$

Obviously the element $s_{n+1}:=Ps_n\curlyvee s_n$ belongs to S, $s_n\leq s_{n+1}$ and therefore

$$Ps+s_{n+1}=(Ps+Ps_n)\curlyvee(Ps+s_n)\precsim s+Ps_{n+1}$$

On the other hand since $s_n\precsim s$ we get $Ps_n\precsim s$ and therefore, using again Proposition 1.4,b), we get

$$s_{n+1}\precsim s$$

Since the sequence $(s_n)_n$ from D(P) is increasing and dominated by $s\in D(P)$ we deduce that the element $r:\bigvee_n s_n$ belongs to D(P) and we have

$$Pr=\bigvee_n Ps_n\precsim\bigvee_n s_{n+1}, \quad r\in S(P), \quad r\precsim s.$$

Now from the relations

$$Ps + s_n \lesssim s + Ps_n \quad (\forall) n \in N$$

we get

$$Ps+r \lesssim s + Pr$$

i.e.

$$P(s-r) \lesssim s-r, \ s-r \in S(P).$$

Obviously $r \geq s_o$ and therefore

$$r \geq R^{S(P)}(s-t)$$

Let now $v \in S(P)$ be such that $v \geq s_o = s-t$. By induction one can verify that

$$v \geq s_n \quad (\forall) \ n \in N$$

and therefore $v \geq r$. Hence

$$r = R^{S(P)}(s-t).$$

Theorem 1.8. Let $P:D(P) \longrightarrow S$ be a compression operator in S. Then S(P) is an H-cone with respect to the order relation on S(P) induced by the natural order of S.

Proof. The assertion follows from Proposition 1.1 and Theorem 1.7.

<u>Corollary 1.9.</u> Let $P:D(P)\to S$ be a compression operator in S. Then S(P) is an H-cone with respect to any order relation on S(P) induced by an order relation $\leq'$ on S for which

$$s \lesssim t \Rightarrow s \leq' t \ , \quad s \leq' t \Rightarrow s \leq t \quad (\forall) s,t \in S$$

and such that $(S, \leq')$ is an H-cone.

Particularly S(P) is an H-cone with respect to the order relation on S(P) induced by the specific order on S.

<u>Theorem 1.10.</u> Let

$$P:D(P)\longrightarrow S$$

$$Q:D(Q)\longrightarrow S$$

be two compression operators in S. We denote by Q_p the map

$$Q_p:D(Q_p)\longrightarrow S(P)$$

defined by

$$D(Q_p):=\{s\in S(P)\cap D(Q) \mid Qs\in \bigcap_n D(P^n), \sum_{n=0}^{\infty} P^n Qs \in D(P)\}$$

$$Q_p s=\sum_{n=0}^{\infty} P^n Qs.$$

and by P+Q the map

$$P+Q : D(P) \cap D(Q) \longrightarrow S$$

defined by

$$(P + Q)s = Ps + Qs$$

Then P+Q is a compression operator in S, Q_P is a compression operator in S(P) and we have

$$S(P)(Q_p) = S(P+Q)$$

Proof. For any $s \in D(P) \cap D(Q)$ we have

$$(P+Q)s \precsim s \implies Ps \precsim s, \quad Qs \precsim s$$

and therefore

$$S(P+Q) \subset S(P) \cap S(Q).$$

Obviously P+Q is a compression operator in S.

If $s \in D(Q_p)$ then $Q_p s \in D(P)$,

$$Qs + P(Q_p s) = Q_p s,$$
$$P(Q_p s) \precsim Q_p s$$

and therefore $Q_p s \in S(P)$. On the other hand if $s_1, s_2 \in D(Q_p)$, $s_1 \leq s_2$ then we have

$$Q_p s_2 - Q_p s_1 = Qs_2 - Qs_1 + P(Q_p s_2 - Q_p s_1)$$

and since $Qs_2 - Qs_1 \in S$ we get

$$Q_p s_2 - Q_p s_1 \in S$$

and

$$P(Q_p s_2 - Q_p s_1) \lesssim Q_p s_2 - Q_p s_1$$
$$Q_p s_2 - Q_p s_1 \in S(P).$$

Hence Q_p is a compression operator in $S(P)$.

Let $s \in S(P+Q)$. We have

$$s \in D(P) \cap D(Q), \quad (P+Q)s \lesssim s,$$

and therefore

$$Ps \lesssim s \quad , \quad Qs \lesssim s$$

We deduce inductively

$$\sum_{k=0}^{n} P^k Qs \lesssim s$$

and therefore

$$Q_p s := \sum_{n=0}^{\infty} P^n Q s \quad \lesssim s$$
$$Q_p s \in D(P), \quad s \in D(Q_p),$$

We have

$$(1-P)Q_p s = Qs$$

and therefore

$$(1-P)(1-Q_p)s=\big(1-(P+Q)\big)s \gtrsim 0.$$

Hence

$$s-Q_p s \in S(P),$$
$$s\in S(P)(Q_p).$$

Conversely, if $s\in S(P)(Q_p)$ then

$$s\in S(P), \quad s\in D(Q),$$

and from

$$0 \lesssim (1-P)(1-Q_p)s=\big(1-(P+Q)\big)s$$

we get

$$s\in S(P+Q).$$

<u>Lemma 1.11.</u> Let P be a compression operator in S and let $\mathcal{A}$ be a subset of S-S such that there exists $s\in S(P)$ with the property $f\leq s-Ps$ for any $f\in\mathcal{A}$. Then the set

$$\mathcal{M}(\mathcal{A}):=\{t\in S(P) \mid f\leq t-Pt \ (\forall)\ f\in\mathcal{A}\}$$

has a smallest element $t_{\mathcal{A}}$ and moreover we have

$$\bigwedge_{S-S}\{t-Pt \mid t\in\mathcal{M}(\mathcal{A})\} = t_{\mathcal{A}} - Pt_{\mathcal{A}}$$

Proof. With the above notations we put

$$t_{\mathcal{A}} := \bigwedge_{S}\{t \mid t\in\mathcal{M}(\mathcal{A})\}$$

Since for any $f\in\mathcal{A}$ and any $t\in\mathcal{M}(\mathcal{A})$ we have

$$f\leq t-Pt\leq t-Pt_{\mathcal{A}},$$

we deduce that

$$f\leq t_{\mathcal{A}}-Pt_{\mathcal{A}} \qquad (\forall) f\in\mathcal{A}$$

and therefore

$$t_{\mathcal{A}}\in\mathcal{M}(\mathcal{A}).$$

If $t\in\mathcal{M}(\mathcal{A})$ we put

$$v:=(t-Pt)\wedge(t_{\mathcal{A}}-Pt_{\mathcal{A}})+Pt_{\mathcal{A}}$$

Obviously $v\in S$ and moreover

$$Pt_{\mathcal{A}}\lneqq v\leq t_{\mathcal{A}}.$$

From Lemma 1.5. we deduce $v\in S(P)$. On the other hand for any $f\in\mathcal{A}$ we have

$$f\leq v-Pt_{\mathcal{A}}\leq v-Pv.$$

Hence $v \in \mathcal{M}(\mathcal{A})$ and therefore $v = t_{\mathcal{A}}$ i.e.

$$t_{\mathcal{A}} - Pt_{\mathcal{A}} = v - Pt_{\mathcal{A}} \leq t - Pt \qquad (\forall)\ t \in \mathcal{M}(\mathcal{A})$$

$$t_{\mathcal{A}} - Pt_{\mathcal{A}} = \bigwedge_{S-S} \{t - Pt \mid t \in \mathcal{M}(\mathcal{A})\}$$

<u>Theorem 1.12.</u> Let $P:D(P) \longrightarrow S$ be a compression operator in S. Then for any $u \in D(P)$ and any $s \in S(P)$ there exists $t \in S(P)$ such that

$$u \wedge (s - Ps) = t - Pt$$

<u>Proof</u>. We put

$$B := \{t \in S(P) \mid u \wedge (s - Ps) \leq t - Pt\}$$

From the previous lemma the element

$$t_o := \bigwedge B$$

belongs to B. We want to show that

$$u \wedge (s - Ps) = t_o - Pt_o$$

Since $s \in B$ we have

$$t_o \leq s,$$
$$u \wedge (s - Ps) \leq t_o - Pt_o.$$

On the other hand we have

$$s-Ps+Pt_o\in D(P)$$

$$Pt_o\lesssim(u+Pt_o)\wedge(s-Ps+Pt_o)\wedge t_o\le t_o$$

and therefore, using Lemma 1.5, we get

$$v:=(u+Pt_o)\wedge(s-Ps+Pt_o)\wedge t_o\in S(P).$$

From this relation we deduce

$$v\le t_o,$$
$$u\wedge(s-Ps)\wedge(t_o-Pt_o)=v-Pt_o\le v-Pv$$
$$u\wedge(s-Ps)=u\wedge(s-Ps)\wedge(t_o-Pt_o)\le v-Pv,$$
$$t_o\le v,\ t_o=v.$$

Hence

$$(u+Pt_o)\wedge(s-Ps+Pt_o)\geq t_o,$$
$$u\wedge(s-Ps)\geq t_o-Pt_o,$$
$$u\wedge(s-Ps)=t_o-Pt_o$$

Corollary 1.13. For any compression operator P in S the set

$$\{\ s-Ps\,|\,s\in S(P)\ \}$$

is a solid convex subcone of S. Particularly, it is an H-cone with respecto to the natural order induced by S.

Proposition 1.14. Let P be a compression operator in S. Then for any pseudo balayage $\mathcal{B}$ on S the map

$$B':S(P)\longrightarrow S(P)$$

defined by

$$B's=\bigwedge\{t\in S(P) \mid t\geq Bs\}$$

is a pseudobalayage on S(P) and verifies the relations

$$BB's=Bs \quad (\forall)\ s\in S(P)$$
$$B's-Bs=P(B's)-BP(B's) \quad (\forall)\ s\in S(P)$$

Moreover if B is a balayage on S then B' is also a balayage on S(P).

<u>Proof.</u> From the definition we have

$$Bs\leq B's\leq s$$
$$s\leq t \Rightarrow B's\leq B't,$$
$$B'(s+t)\leq B's + B't,$$

for any $s,t\in S(P)$. Since B is a pseudo balayage on S then we get, using the preceding relations,

$$BB's=Bs \qquad B'B's=B's \ (\forall)\ s\in S(P).$$

For any $s,t\in S(P)$ let $s',t'\in S(P)$ be such that

$$B'(s+t)=s'+t', \ s'\leq B's, \ t'\leq B't.$$

Since we have

$$B(s+t)=B(B'(s+t))=Bs'+Bt'\leq BB's+BB't\leq Bs+Bt$$

we get $Bs'=Bs$, $Bt'=Bt$ and therefore

$$s' \geq B's, \quad t' \geq B't ,$$

$$s'=B's, \quad t'=B't, \qquad B'(s+t)=B's+B't.$$

Hence B' is a pseudobalayage on S(P).

On the other hand, we have, for any $s \in S(P)$,

$$B(s-Ps) \leq s-Ps$$

$$Ps+B(s-Ps) \in S,$$

$$Ps \leq Ps+B(s-Ps) \leq s$$

and therefore, from Lemma 1.4,

$$Ls:=Ps+B(s-Ps) \in S(P).$$

Since, for any $s \in S(P)$ we have $Ls \geq Bs$ we deduce, from the definition of B',

$$B's \leq Ls \leq s$$

and therefore

$$LB's = B's,$$

or equivalently

$$B's-Bs=B's-BB's=PB's-BPB's.$$

Suppose now that B is a balayage on S and let $(s_i)_{i \in I}$ be an increasing family from S(P) with

$$\bigvee_{i\in I} s_i = s.$$

We have

$$\bigvee_{i\in I} B's_i \leq B's,$$

$$\bigvee_{i\in I} B's_i \geq \bigvee_{i\in I} Bs_i = Bs$$

and therefore, using the definition of B'

$$\bigvee_{i\in I} B's_i \geq B's,$$

$$\bigvee_{i\in I} B's_i = B's.$$

Hence B' is also a balayage on S(P).

Corollary 1.15. Let P be a compression operator in $\mathcal{S}$, B be a pseudobalayage on S and let B' be the pseudo-balayage on S(P) associated with B as in the preceding proposition. Then we have

$$f\in(S(P)-S(P))_{+} \Rightarrow B'f-Bf\in S_B.$$

Proof. We have

$$B'f\in(S(P)-S(P))_{+},\ P(B'f)\in S$$

and therefore

$$B'f-Bf=PB'f-BPB'f\in S_B.$$

2. Compression of an H-cone

Definition. Let S be an H-cone. A convex subcone S' of S is called an H-subcone if S' is an H-cone with respect to the order relation induced by the natural order of S and moreover

a) $\bigwedge_{S'} F = \bigwedge_{S} F$ for any non-empty subset F of S'

b) $\bigvee_{S'} F = \bigvee_{S} F$ for any non empty upper directed and dominated subset F of S'.

If the sequel we mark by $\curlywedge, \curlyvee$ the lattice operations with respect to the specific order $\preccurlyeq$ on S-S given by S.

Theorem 2.1. Let S be an H-cone and let S' be an H-subcone of S. Then for any psuedo-balayage B on S the map

$$B' : S' \longrightarrow S'$$

defined by

$$B's = \bigwedge \{ t \in S' \mid t \geq Bs \}$$

is a pseudobalayage on S'. In fact B' is the smallest pseudobalayage on S' such that

$$B's \geq Bs \quad (\forall) s \in S'$$

or equivalently

$$BB's = Bs \quad (\forall) \; s \in S'$$

Moreover if B is a balayage on S then B' is also a balayage on S'.

<u>Proof.</u> From the definition we have

$$s \geq B's \geq Bs, \quad BB's = Bs \quad (\forall) s \in S'$$
$$B'(B's) = B's \quad (\forall) s \in S'$$
$$B's_1 \leq B's_2 \quad (\forall) s_1, s_2 \in S', \ s_1 \leq s_2$$
$$B'(s_1 + s_2) \leq B's_1 + B's_2 \ (\forall) s_1, s_2 \in S'.$$

Since S' is an H-cone it follows that for any $s_1, s_2 \in S'$ there exists $u_1, u_2 \in S'$ with

$$u_1 + u_2 = B'(s_1 + s_2), \ u_1 \leq B's_1, \ u_2 \leq B's_2$$

From these relation we get

$$Bu_1 + Bu_2 = B(u_1 + u_2) = BB'(s_1 + s_2) = B(s_1 + s_2) = Bs_1 + Bs_2$$
$$Bu_1 \leq BB's_1 = Bs_1, \qquad Bu_2 \leq BB's_2 = BS_2$$

and therefore

$$u_i \geq Bu_i = Bs_i, \quad i = 1, 2,$$
$$u_i \geq B's_i, \qquad i = 1, 2.$$

Hence

$$u_i = B's_i, \quad i = 1, 2,$$
$$B'(s_1 + s_2) = B's_1 + B's_2.$$

Let now T be a pseudobalayage on S' such that

$$Ts \geq Bs \qquad (\forall) s \in S'$$

Then, from the definition of B', we get

$$Ts \geq B's, \quad (\forall)\ s \in S'.$$

Suppose now that B is a balayage on S and let $(s_i)_{i \in I}$ be an increasing and dominated family of S' and

$$s = \bigvee_{i \in I} s_i.$$

We have obviously

$$B's \geq \bigvee_{i \in I} B's_i.$$

Since

$$B's_i \geq Bs_i \quad (\forall)\ i \in I$$

we get

$$\bigvee_{i \in I} B's_i \geq \bigvee_{i \in I} Bs_i = Bs$$

and therefore

$$\bigvee_{i \in I} B's_i \geq B's.$$

Hence B' is a balayage on S'.

Remark. 1. Whenever we will use the pseudo-balayage B' we call it the pseudo-balayage on S' associated to the pseudo-balayage B.

2. If S is represented as an H-cone of functions on a set X and A is a subbasic subset of X with respect to S then A is also a subbasic subset of X with respect to S' and the balayage on S'

associated with B^A (i.e. $(B^A)'$) coincides with the balayage on S' given by

$$(B')^A s = \bigwedge \{ t \in S' \mid t \geq s \text{ on } A \}$$

3. Generally, if $f \in (S'-S')_+$ and B_f is the balayage on S given by ([4])

$$s \in S \Rightarrow B_f s := \bigwedge \{ t \in S \mid t \geq s \wedge nf \ (\forall) \ n \in N \}$$

then the balayage on S' associated with B_f (ie. B'_f) coincides with the balayage on S' given by

$$s \in S' \Rightarrow B'_f s := \bigwedge \{ t \in S' \mid t \geq s \wedge (nf) \ (\forall) \ n \in N \}$$

<u>Definition</u>. An H-subcone S' of S is called a <u>compression of S</u> if for any balayage B on S the balayage B' on S' associated with B is such that

$$B'f \geq Bf \qquad (\forall) \ f \in (S'-S')_+$$

<u>Remark</u>. If P is a compression operator on S then the H-cone S(P) is a compression of S. Moreover for any pseudo-balayage B on S we have

$$B's - Bs = PB's - BPB's \ (\forall) \ s \in S'$$

where B' is the pseudo-balayage on S' associated with B (see Proposition 1.14)

Proposition 2.2. Let S' be a compression of S. Then for any balayage B on S and any $s \in S'$ there exists $t \in S$ such that

$$s - B's = t - Bt$$

where B' is the balayage on S' associated with B.

Proof. Let S_B be the H-cone ([4]) of all elements of the form

$$t - Bt, \quad t \in S.$$

We remarks that

$$s - B's = (s - Bs) - (B's - BB's)$$

i.e. $s - B's$ is of the form

$$s - B's = u - v$$

where $u, v \in S_B$. To prove that $u - v \in S_B$ is equivalently to show that

$$T(u - v) \leq u - v$$

for any balayage T on S_B. Let now T be a balayage on S_B. One can see that the map

$$s \rightarrow T(s - Bs) + Bs$$

is a balayage B_1 on S, $B_1 s \geq Bs$ $(\forall) s \in S$. We have

$$Tf = B_1 f \quad (\forall) f \in S_B$$

and therefore

$$T(s-B's)=Tu-Tv=B_1u-B_1v=B_1(u-v)=$$
$$=B_1(s-B's) \leq B_1'(s-B's)=$$
$$=B_1's-B's \leq s-B's.$$

Theorem 2.3. Let S' be a sub H-cone of S. Then the following assertions are equivalent:

For any $f \in (S'-S')_+$ and any balayage B on S we have $B'f \geq Bf$.

2. For any $f \in (S'-S')_+$ and any balayage B on S we have $B'f-Bf \in S_B$.

3. For $f \in (S'-S')_+$, $t \in S'$, $f \leq t$ and for any balayage B on S we have

$$B'f-Bf \precsim t.$$

4. For any finite family $(f)_{i\in I}$ in $(S'-S')_+$ any family $(B_i)_{i\in I}$ of balayages on S and for any $t \in S'$ with $\sum_{i\in I} B_i' f_i \leq t$ we have

$$\sum_{i\in I} (B_i' f_i - B_i f_i) \precsim t.$$

Proof. The assertions $2) \Rightarrow 1)$, $4) \Rightarrow 3)$ are obvious.

$1) \Rightarrow 2)$ Let f $(S'-S')_+$ and let B be a balayage on S. For any balayage T on S_B there exists a balayage B_1 on S, $B_1 \geq B$ such that

$$T(u-Bu)=B_1u-Bu \quad (\forall) u \in S.$$

We have

$$T(B'f-Bf)=T(B'f-BB'f)=B_1B'f-BB'f \leq$$
$$\leq B_1'B'f-Bf = B'f-Bf$$

and therefore, T being arbitrary,

$$B'f-Bf \in S_B.$$

2)$\Rightarrow$3) From Proposition 2.2 it follows that there exists $u \in S$ such that

$$t-B't=u-Bu.$$

On the other hand from 2) there exists $v \in S$ such that

$$B'(t-f)-B(t-f)=v-Bv.$$

We have

$$t-(B'f-Bf)=(t-B't)+(B'(t-f)-B(t-f))+Bt=$$
$$=(u+v)-B(u+v)+Bt$$

Since

$$(u+v)-B(u+v)\leq t-Bt$$

it follows that

$$u+v-B(u+v)+Bt \in S,$$
$$t-(B'f-Bf) \in S, \quad B'f-Bf \lesseqgtr t$$

3)$\Rightarrow$1) Let $f\in(S'-S')_+$, $f=t-s$ where $s,t\in S'$ and

$$s\leq t.$$

We have

$$B's\leq B't$$

and therefore, from 3), we get

$$B's-Bs=B'(B's)-B(B's)\lesssim B't$$

If u is the element of S such that

$$B's-Bs+u=B't$$

we deduce that

$$Bu=BB't=Bt$$

and therefore

$$B't-Bt=B's-Bs+u-Bu\gtrsim B's-Bs,$$
$$B'f\gtrsim Bf$$

3)$\Rightarrow$4) The assertion from 4) will be proved inductively following the cardinal of I. If card I=1 the assertion from 4) is just the assertion 3). Suppose that the assertion from 4) is true for any I with card I=n and let I be such that card I=n+1.

For any $i\in I$ we denote

$$u_i := t + \sum_{j \in I \setminus \{i\}} (B_j f_j - B_j' f_j)$$

and we put

$$f := t + \sum_{i \in I} (B_i f_i - B_i' f_i)$$

$$u := \bigwedge_{i \in I} u_i .$$

Obviously we have

$$f \leq u.$$

We want to show that $f = R^S f$ or equivalently

$$\alpha R^S f \leq f \quad (\forall) \quad \alpha \in (0,1).$$

Let $\alpha \in (0,1)$ and let us denote

$$g := f - f \wedge (\alpha R^S f) \neq 0.$$

We consider the balayage B on S defined by (see[4])

$$Bs = \bigvee_{n \in N} R^S (s \wedge ng) .$$

We have (see [8])

$$B(R^S f) = R^S f$$

On the other hand we have

$$B \geq B_i \quad (\forall) \quad i \in I$$

and therefore, since $\sum_{i\in I} B'_i f_i \leq t$ we get

$$Bf=B(\sum_{i\in I} B_i f_i)+B(t-\sum_{i\in I} B'_i f_i)=$$

$$=\sum_{i\in I} B_i f_i+B(t-\sum_{i\in I} B'_i f) \leq$$

$$\leq \sum_{i\in I} B_i f_i+B'(t-\sum_{i\in I} B'_i f_i)=\sum_{i\in I} B_i f_i+B't-\sum_{i\in I} B'_i f_i \leq f$$

$$Bf \leq f.$$

From the preceding considerations we get

$$R^S f=B(R^S f)=\frac{1}{\alpha} B(\alpha R^S f)\leq \frac{1}{\alpha} Bf \leq \frac{1}{\alpha} f$$

$$\alpha R^S f \leq f \quad (\forall) \quad \alpha \in (0,1).$$

Definition. Let S' be a compression of S. For any balayage B on S we denote by P_B the map

$$P_B : S' \longrightarrow S$$

defined by

$$P_B s=B's-(B's)\underset{S}{\wedge} Bs := B's - B's \wedge Bs$$

where B' is the balayage on S' associated with B.

Proposition 2.4. The map P_B satisfies the following properties:

1) $P_B(s+t)=P_B s+P_B t \quad (\forall)\ s,t \in S'$
2) $P_B s \precsim P_B t \precsim t \quad (\forall)\ s,t\in S',\ s\leq t$
3) $P_B(B's)=P_B s \quad (\forall)\ s\in S'$
4) $\bigvee_{i\in I} P_B s_i=P_B(\bigvee_{i\in I} s_i)$

for any increasing and dominated family $(s_i)_{i\in I}$ in S'.

Moreover for any compression operator P of S such that S(P)= =S' we have

$$P_B f \lesssim P(B'f) \ (\forall)\ f\in(S'-S')_+$$

Proof. Let s,t∈S'. We have

$$B's-Bs=P_B s-BP_B s$$

$$B't-Bt=P_B t-BP_B t$$

$$B'(s+t)-B(s+t)=P_B(s+t)-BP_B(s+t)$$

and therefore

$$(P_B s+P_B t)-B(P_B s+P_B t)=P_B(s+t)-BP_B(s+t).$$

Since

$$P_B s \curlywedge Bu=P_B t \curlywedge Bu=P_B(s+t)\curlywedge Bu=0$$

for any u∈S we deduce

$$P_B s + P_B t = P_B(s+t).$$

The relation

$$s\in S' \Rightarrow P_B(B's)=P_B s$$

follows immediately from the definition of P_B.

Let now $s,t\in S'$ be such that $s\leq t$. We have

$$P_B s - BP_B s = B's - Bs$$
$$P_B t - BP_B t = B't - Bt$$
$$(P_B t - P_B s) - B(P_B t - P_B s) = B'(t-s) - B(t-s).$$

Since S' is a compression of S then

$$B'(t-s) - B(t-s) \in S_B$$

and therefore there exists $u\in S$ such that

$$u \underset{S}{\wedge} Bu = 0$$

and

$$B'(t-s) - B(t-s) = u - Bu.$$

Hence

$$P_B t - BP_B t = P_B s - BP_B s + u - Bu,$$

Since

$$P_B t \wedge Bv = P_B s \wedge Bv = u \wedge Bv = 0$$

for any $v\in S$ we deduce

$$P_B t = P_B s + u,$$
$$P_B s \leqslant P_B t.$$

On the other hand we have

$$t-P_B t=t-B't+B't \wedge Bt=$$
$$=(t+B't) \wedge (t+Bt)-B't.$$

To prove the relation

$$P_B t \lesssim t$$

is equivalently. to show that

$$B't \lesssim t+Bt.$$

This relation follows from the fact that S' is a compression of S using Theorem 2.3.

Let $(s_i)_{i\in I}$ be an increasing and dominated family of S'. We have

$$P_B s_i - BP_B s_i = B's_i - Bs_i$$

Since

$$i \leq j \Rightarrow P_B s_i \lesssim P_B s_j$$

and since

$$P_B s_i + Bs_i = B's_i + BP_B s_i$$

we get

$$\bigvee_{i\in I} P_B s_i + B(\bigvee_{i\in I} s_i) = B'(\bigvee_{i\in I} s_i) + B(\bigvee_{i\in I} P_B s_i)$$

$$B'(\bigvee_{i\in I} s_i)-B(\bigvee_{i\in I} s_i)=\bigvee_{i\in I} P_B s_i - B(\bigvee_{i\in I} P_B s_i).$$

On the other hand we have

$$(\bigvee_{i\in I} P_B s_i) \underset{S}{\curlywedge} Bv = 0 \quad (\forall)\ v\in S$$

and therefore

$$P_B(\bigvee_{i\in I} s_i)=\bigvee_{i\in I} P_B s_i .$$

Let now P be a compression operator of S such that $S(P)=S'$ and let $f\in(S'-S')_+$. We have, using Proposition 1.15

$$P_B f-BP_B f=B'f-Bf=P(B'f)-BP(B'f).$$

Since

$$P_B f \underset{S}{\curlywedge} Bv = 0 \ (\forall)\ v\in S$$

we get

$$P_B f \preceq P(B'f).$$

In the sequel we suppose that S' is a standard H-cone, S and S' are H-cones of functions on a set X, S' is a compression of S and X is semisaturated with respect to S'. Moreover we suppose that any element of S is lower semicontinuous with respect to the fine topology generated by S'.

The specific order relation in S is denoted by $\preceq$ and we put $\curlyvee$, $\curlywedge$ for the lattice operations with respect to this order relation.

<u>Definition.</u> If B is a balayage on S and P_B is the map

$$P_B : S' \to S$$

defined by

$$P_B s = B's - B's \wedge Bs$$

we denote for any $x \in X$, by $P_{B,x}$ the H-measure (with respect to S') on X given by the relation

$$P_{B,x}(s) = P_B s(x), \quad s \in S'$$

For any positive Borel function f on X we denote by $P_B f$ the function on X defined by

$$P_B f(x) = P_{B,x}(f).$$

<u>Proposition 2.5.</u> If B is a balayage on S and f is a positive Borel function on X dominated by an element of S' we have $P_B f \in S$. Moreover if $(s_i)_{i \in I}$ is a finite family of S' and if $s \in S'$ are such that

$$\sum_{i \in I} s_i \leq s$$

then

$$\sum_{i \in I} P_{B_i} s_i \precsim s$$

for any family, $(B_i)_{i \in I}$ of balayages on S

<u>Proof.</u> For any positive bounded lower semicontinuous function g on X there exists an increasing sequence $(g_n)_n$ from $(S'-S')_+$ such that

$$g = \sup_n g_n$$

Using Proposition 2.4 we get

$$P_B g_n \in S$$

and therefore

$$P_B g = \sup_n P_B g_n \in S.$$

On the other hand if g_1, g_2 are lower semicontinuous positive bounded functions on X such that $g_1 \leq g_2$ we have

$$P_B g_1 \precsim P_B g_2$$

Indeed, let $(g_1^{(n)})_n$, $(g_2^{(n)})_n$ be two increasing sequences in $(S'-S')_+$ with

$$\sup_n g_1^{(n)} = g_1 \ , \ \sup g_2^{(n)} = g_2$$

Obviously we may suppose that

$$g_1^{(n)} \leq g_2^{(n)} \quad (\forall) \ n \in N.$$

Using Proposition 2.4 it follows that

$$P_B g_1 = \curlyvee_n P_B g_1^{(n)} \precsim \curlyvee_n P_B g_2^n = P_B g_2.$$

Let now f be a positive bounded Borel function on X. From the above considerations the family

$$(P_B g)_{g \in \mathcal{G}}$$

where $\mathcal{G}$ is the set of all bounded lower semicontinuous functions g on X such that $g \geq f$, is a family in S, which is specifically lower directed and

$$\inf_{g \in \mathcal{G}} P_B g = P_B f$$

Hence

$$P_B f = \inf_{g \in \mathcal{G}} P_B g = \bigwedge_{g \in \mathcal{G}} P_B g \in S$$

Let f be a positive Borel function on X dominated by an element of S'. We have

$$P_B f = \bigvee_n P_B(\inf(f,n)) \in S.$$

Let now $(s_i)_{i \in I}$ be a finite family from S', and let $s \in S'$ be such that

$$\sum_{i \in I} s_i \leq s.$$

If we consider a family $(B_i)_{i \in I}$ of balayages on S we deduce, from Theorem 2.3,

$$\sum_{i \in I} (B_i' s_i - B_i s_i) \precsim s.$$

From the definition of P_{B_i} we deduce that the relation

$$\sum_{i \in I} P_{B} s_i \precsim s$$

is equivalent with the following one

$$\sum_{i\in I} B'_i s_i \precsim s + \sum_{i\in I} B'_i s_i \curlywedge B_i s_i .$$

On the other hand we have immediately

$$s+\sum_{i\in I} B'_i s_i \curlywedge B_i s_i = \bigwedge_{J\in\mathcal{P}(I)} \Big(s+\sum_{j\in J} B'_j s_j + \sum_{j\in I\setminus J} B_j s_j\Big)$$

From the preceding considerations we have for any $J\in\mathcal{P}(I)$,

$$\sum_{j\in I\setminus J} B'_j s_j \precsim s + \sum_{j\in I\setminus J} B_j s_j$$

or equivalently

$$\sum_{i\in I} B'_i s_i \precsim s + \sum_{j\in J} B'_j s_j + \sum_{j\in I\setminus J} B_j s_j$$

and therefore

$$\sum_{i\in I} B'_i s_i \precsim \bigwedge_{J\in\mathcal{P}(I)} \Big(s+\sum_{j\in J} B'_j s_j + \sum_{j\in I\setminus J} B_j s_j\Big) =$$
$$= s+\sum_{i\in I} B'_i s_i \curlywedge B_i s_i .$$

<u>Theorem 2.6.</u> Let f be a positive fine lower semicontinuous (with resp. to S') function on X which is dominated by an element of S'. If for any balayage B on S and any $g\in(S'-S')_+$ such that $g\leq f$ we have

$$P_B g \precsim f$$

then $f\in S'$.

<u>Proof</u>. Let $s,t\in S'$ be such that, s, t bounded and $0\leq s-t\leq f$. Obviously since $P_B f \precsim f$ then $f\in S$.

We want to show that

$$B'(s-t)\leq f.$$

Indeed we have

$$P_B(s-t)\leq P_B(f) \leq f$$

and therefore there exists $u\in S$ such that

$$P_B(s-t)+u=f,$$

or equivalently

$$B'(s-t)+u=f+(B's\wedge Bs-B't\wedge Bt).$$

Since

$$BB'(s-t)=B(s-t)$$

$$B(B's\wedge Bs)=B's\wedge Bs,$$

$$B(B't\wedge Bs)=B't\wedge Bt$$

we get

$$B's\wedge Bs-B't\wedge Bt + Bf = B(s-t)+Bu$$

and therefore

$$f+B(s-t)+Bu=Bf+u+B'(s-t).$$

From this relation and from the obvious relations

$$Bu\leq u, \qquad B(s-t)\leq Bf$$

we get

$$B'(s-t) \leq f.$$

We show now that

$$R^{S'}(s-t) \leq f$$

Indeed, for any $\alpha \in (0,1)$, we have (see [8])

$$R(s-t) = B'_g(R^{S'}(s-t)) \leq \frac{1}{\alpha} B'_g(s-t)$$

where

$$g = ((s-t) - \alpha R^{S'}(s-t))_+$$

and B_g is the balayage on S given by

$$B_g s = \bigvee_{n \in N} R^S(s \wedge ng).$$

Since

$$B'_g(s-t) \leq f$$

it follows that

$$R^{S'}(s-t) \leq \frac{1}{\alpha} f \quad (\forall) \ \alpha \in (0,1),$$
$$R^{S'}(s-t) \leq f.$$

The fact that $f \in S'$ follows now from the preceding considerations using the fact that there exists an increasing family $(g)_{i \in I}$

in $(S_b'-S_b')_+$ such that

$$\sup_{i\in I} g_i = f.$$

<u>Definition.</u> Let S and S' be as above. We denote by P the map from S' into S given by

$$Ps = \bigvee\left\{\sum_{i\in I} P_{B_i} s_i ;\ I \text{ finite}, (s_i)_{i\in I} \subset S', \sum_{i\in I} s_i \leq s, (B_i)_i \subset B(S)\right\}$$

where B(S) is the set of all balayages on S.

<u>Proposition 2.7.</u> Let $s\in S'$ and u be a fine lower semicontinuous positive function on X such that

$$Ps \precsim u \leq s.$$

Then $u\in S'$

<u>Proof.</u> Let $g\in(S'-S')_+$ be such that

$$g \leq u$$

We have $g\leq s$,

$$P_B g \precsim P_B s \precsim Ps \precsim u$$

and therefore, using Theorem 2.6, it follows that $u\in S'$.

<u>Theorem 2.8.</u> We have

1) $Ps\in S'$ $(\forall)$ $s\in S'$

2) $s,t\in S'$, $s\leq t \Rightarrow Ps \precsim Pt \precsim t$

3) $s,t\in S' \Rightarrow P(s+t)=Ps+Pt$

4) $\bigvee_i Ps_i = P(\bigvee_i s_i)$ for any increasing and dominated family $(s_i)_{i\in I}$ from S'.

5) for any $s\in S'$ and any balayage B on S we have

$$B's-Bs=PB's-BPB's$$

6) $Ps(a)=0$ for any $s\in S'$ and any absorbent point a with respect to S'.

Proof. The assertion 1) follows from Proposition 2.7. The assertion 2) follows immediately from the definition and from Proposition 2.4

3) If $s,t\in S'$ then, from the definition, we have

$$Ps + Pt \lesssim P(s+t)$$

Let now $(s_i)_{i\in I}$ be a finite family from S' such that

$$\sum_i s_i \leq s + t$$

Using Riesz decomposition property there exist two families $(s_i')_{i\in I}$, $(s_i'')_{i\in I}$ on S' such that

$$s_i = s_i' + s_i'' \; (\forall)\; i\in I,$$
$$\sum_{i\in I} s_i' \leq s, \; \sum_{i\in I} s_i'' \leq t$$

If $(B_i)_{i\in I}$ is a family of balayages on S we have

$$\sum_{i\in I} P_{B_i} s_i = \sum_{i\in I} P_{B_i} s_i' + \sum_{i\in I} P_{B_i} s_i'' \lesssim Ps + Pt,$$

$$P(S+t) \quad Ps+Pt$$

4) Let $(s_i)_{i\in I}$ be an increasing and dominated family from S'.

Obviously we have

$$\bigvee_{i\in I} Ps_i \leq P(\bigvee_{i\in I} s_i)$$

Let now $(t_j)_{j\in J}$ be a finite family from S' such that

$$\sum_{j\in J} t_j \leq s := \bigvee_{i\in I} s_i$$

From ([4], Proposition 2.2.3) for any $j\in J$ there exists an increasing family $(s_{ij})_{i\in I}$ in S' such that

$$\sum_{j\in J} s_{i,j} \leq s_i \qquad \bigvee_i s_{i,j} = t_j \ (\forall)\ j\in J$$

If $(B_j)_{j\in J}$ is a family of balayages on S we have, using Proposition 2.4

$$\sum_j P_{B_j} t_j = \sum_{j\in J} (\bigvee_{i\in I} P_{B_j} s_{ij}) = \bigvee_{i\in I} \sum_{j\in J} P_{B_j} s_{ij} \leq \bigvee_{i\in I} Ps_i ,$$

and therefore

$$Ps \leq \bigvee_{i\in I} Ps_i$$

5) Let $s\in S'$. We have

$$P_B s = P_B B's \leq PB's$$

and therefore

$$B's - Bs = P_B B's - BP_B B's \leq PB's - BPB's$$

On the other hand we have

$$PB's \leq B's$$

and therefore

$$PB's - BPB's \leq B's - Bs$$

6) Let $a \in X$ be an absorbent point with respect to S' and let B be a balayage on S and B' be the balayage on S' associated with B. Certainly a is also an absorbent point with respect to S. Obviously to prove that $Ps(a)=0$ $(\forall)$ $s \in S'$ it will be sufficient to show that $P_B s(a)=0$ $(\forall)$ $s \in S'$. Let now B be a balayage on S. If a does not belong to the base of B' then we have $B's(a)=0$ and therefore $P_B s(a)=$ $=0$. If a belongs to the base of B' then a belongs also to the base of B. If we denote by T_a (resp. T'_a) the balayage on $\{a\}$ in S (resp. S') then we have for any $s \in S'$

$$s - T'_a s \in S' \qquad s - T_a s \in S$$

and therefore

$$B's = B'(s - T'_a s) + T'_a s$$

$$Bs = B(s - T_a s) + T_a s$$

$$B's \geq B's \wedge Bs \geq T'_a s \wedge T_a s = T_a s$$

$$P_B s(a) = B's(a) - (B's \wedge Bs)(a) = 0.$$

<u>Definition</u>. Suppose that S and S' are as above. For any $x \in X$ we denote by P_x the H-measure (with respect to S') on X given by the relation

$$P_x s = Ps(x) \quad (\forall) \quad s \in S'$$

For any positive Borel function f on X we denote by Pf the function on X defined by

$$Pf(x) = P_x(f).$$

Proposition 2.9. The map

$$f \to Pf, \qquad f \in \mathcal{F}$$

is a bounded kernel on X such that for any $f \in \mathcal{F}$ for which Pf is dominated by an element of S we have

$$Pf \in S$$

Proof. If g is a positive bounded lower semicontinuous function on X then there exists an increasing sequence $(g_n)_n$ from $(S'-S')_+$ such that

$$g = \sup_n g_n$$

From Theorem 2.7 we have $P(S') \subset S'$, $Pg_n \in S \cap (S'-S')_+$ and therefore

$$Pg = \sup_n Pg_n \in S \cap \mathcal{F}$$

On the other hand if g_1, g_2 are lower semicontinuous positive bounded functions on X such that $g_1 \leq g_2$ we have

$$Pg_1 \precsim Pg_2$$

Indeed, there exists twe increasing sequence $(g_1^{(n)})_n$, $(g_2^{(n)})_n$ from $(S'-S')_+$ such that

$$g_1^{(n)} \leq g_2^{(n)} \quad (\forall)\ n\in N, \ \sup_n g_1^{(n)} = g_1, \ \sup_n g_2^{(n)} = g_2$$

We have, using Theorem 2.8

$$Pg_1 = \bigvee_n Pg_1^{(n)} \leq \bigvee_n Pg_2^{(n)} = Pg_2$$

Let now f be a positive bounded Borel function on X. From the above considerations the family

$$(Pg)_{g\in \mathcal{G}}$$

where $\mathcal{G}$ is the set of all bounded lower semicontinuous function g on X, $g\geq f$, is a family in S which is specifically lower directed and

$$Pf = \inf_{g\in\mathcal{G}} Pg = \quad_g \quad Pg$$

Since S is a standard H-cone we have

$$\bigwedge_{g\in\mathcal{G}} Pg = \bigwedge_n Pg_n = \inf_n Pg_n$$

for a suitable decreasing sequence $(g_n)_n$ and therefore

$$Pf \in S \cap \mathcal{F}$$

Hence P is a bounded kernel on X such that

$$Pf \in S$$

for any bounded function $f \in \mathcal{F}$. If $f \in \mathcal{F}$ is such that there exists $s \in S$, with $Pf \leq s$ then we have

$$Pf = \sup_n P(f \wedge n), \quad P(f \wedge n) \leq s$$

and therefore $Pf \in S$.

Theorem 2.10. The set

$$\{ s - Ps \mid s \in S' \}$$

is a solid part (with respect to the natural order) in S.

Proof. Let $u \in S$ and $s \in S'$. We consider the element

$$v := u \wedge (s - Ps).$$

We denote:

$$s' := \bigwedge \{ t \in S' \mid t - Pt \geq v \}$$

Obviously we have $s' \leq s$ and

$$s' - Ps' \geq v.$$

We have

$$s' \wedge (Ps' + v) = Ps' + v,$$

$$Ps' \leq s' \wedge (Ps' + v) \leq s'$$

and therefore, using Proposition 2.7 we get

$$w := P s' + v = s' \wedge (P s' + v) \in S', \qquad w \leq s'.$$

On the other hand we have

$$w - Pw \geq w - Ps' = v$$

and therefore

$$w \geq s', \qquad w = s'.$$

Hence

$$v = s' - Ps'.$$

Definition. We denote by S'_f the set of all finite elements of S' and for any $s \in S'_f$ we put

$$Hs = \inf_{n \in N} P^n s$$

Proposition 2.11. We have

1) $H(s+t) = Hs + Ht \qquad (\forall) s, t \in S'_f$
2) $s \leq t \quad Hs \preceq Ht \preceq Pt \preceq t \qquad (\forall) \ s, t \in S'_f$
3) $H^2 s = Hs \ (\forall) s \in S'_f$
4) $PHs = Hs \quad (\forall) s \in S'_f$

Proof. The assertions 1), 4) are obvious. The assertion 3) follows directly from 4). If $s, t \in S'_f$, $s \leq t$ we have

$$P^n s \preceq P^n t \preceq t \qquad (\forall) \ n \in N$$

$$\bigwedge_n P^n s = \bigwedge_n P^n s \lneq \bigwedge_n P^n t = \bigwedge_n P^n t \lneq t.$$

From now on we suppose that S and S' are standard H-cones of functions on X, S' is a compression of S, the natural topology on X with respect to S is smaller than the natural topology on X with respect to S' and that there exists a strictly positive pontential on X with respect to S'.

Remark. If S and S' are as above then any element of S is lower semicontinuous with respect to the natural topology generated by S' and moreover X is semisaturated with respect to S' and S. Therefore we have fullfiled the previous hypotheses which allowed the construction of kernel P.

Proposition 2.12. There exists a balayage T on S' such that

$$Hs = Ts \quad (\forall) \quad s \in S'_o.$$

Moreover we have

$$s \leq t \Longrightarrow Ts \lneq Tt \lneq t.$$

S'_o is the set of all universally continuous elements of S')

Proof. Since for any $s \in S'_o$ we have

$$Hs \lneq s$$

and using the fact that the natural topology on X associated with S is smaller than the natural topology on X associated with S we deduce that Hs is continuous on X with respect to the natural topology associated with S'. From ([6], Theorem 2.1) it follows that Hs is nearly continuous.

For any $x \in X$ the map

$$S'_o \ni s \to T_x s = Hs(x) \in R_+$$

id additive and increasing. Therefore T'_x is the restriction at S'_o of an H-measure on S'. For any $s \in S'$ we put

$$Ts(x) = T_x s.$$

Since

$$s \in S'_o \Rightarrow Ts = Hs \leq s$$

It follows that

$$s, t \in S'_o, \ s \leq t \Rightarrow Ts \leq Tt \leq t$$

and therefore

$$s, t \in S', s \leq t \Rightarrow Ts \leq Tt \leq t$$

We show that

$$THs = Hs \ (\forall) \ s \in S'_o.$$

Indeed, from the above considerations there exists a sequences $(s_n)_n$ in S'_o such that

$$Hs = \sum_n s_n \ .$$

From $H^2 s = Hs$ we deduce

$$H\delta_n = \delta_n \quad (\forall)\ n \in N,$$

and therefore

$$THs = \sum_n T\delta_n = \sum_n H\delta_n = \sum_n \delta_n = Hs.$$

We show now that $T^2 = T$. Indeed we have

$$s \in S'_o \Rightarrow T(Ts) = T(Hs) = Hs = Ts.$$

Proposition 2.13. If $T \neq 0$ then there exists an absorbent set $A \neq \emptyset$ with respect to S such that any $a \in A$ is an absorbent point with respect to S and such that

$$Ts = \bigwedge \{ t \in S' \mid t \geq s \text{ on } A \}$$

Proof. Let $(p_n)_n$ be a sequence in S'_o such that for any $s \in S'$ we have

$$s = \bigvee \{ p_n \mid p_n \leq s \}$$

We put

$$p = \sum \alpha_n p_n$$

where $\alpha_n > 0$, $p_n \leq \frac{1}{2^n \alpha_n}$ $(\forall)$ $n \in N$ and we denote

$$A := [Tp = p]$$

and for any $x \in X$ let T_x be the measure on X defined by

$$T_x s = Ts(x) \quad (\forall) s' \in S'.$$

Since $T \neq 0$ and $T^2 = T$ it follows that T_x is a measure carried by A and therefore $A \neq \emptyset$. Since $p - Tp \in S$ we deduce that A is absorbent with respect to S. On the other hand for any $a, b \in A$, $a \neq b$ there exists $s, t \in S'$ such that $s \leq t$, $(t-s)(a) = 0$, $(t-s)(b) > 0$. Since $T(t-s) \in S$ and $B'(t-s) = t-s$ on A it follows that any point $a \in A$ is absorbent with respect to S. Obviously we have

$$Ts = \bigwedge \{ t \in S' \mid t \geq s \text{ on } A \}.$$

Theorem 2.14. (Shih). Suppose that any absorbent point of X with respect to S is an absorbent point of X with respect to S'. Then the map

$$P : D(P) \longrightarrow S$$

where

$$D(P) = \{ f \in S \mid (\exists) s \in S', \ f \leq s \}$$

is a compression operator on S and we have

$$S' = \{ s \in D(P) \mid Ps \leq s \}$$

Proof. Using the hypothesis and Proposition 2.13 we have for any $s \in S'$

$$Ts = \bigwedge \{ t \in S' \mid t \geq s \text{ on } A \}$$

where A is an absorbent set with respect to S such that any point of A is an absorbent point with respect to S'. On the other hand we have

$$Ts = Hs = \bigwedge_n P^n s \quad (\forall)\ s \in S'_o.$$

From Theorem 2.8 we have

$$Ps = 0 \quad \text{on } A$$

and therefore

$$Ts = 0 \quad \text{on } A\ (\forall)\ s \in S'_o,$$

$$Ts = 0 \quad (\forall)\ s \in S'_o$$

$$\bigwedge_n \mathcal{P}^n s = 0 \quad (\forall)\ s \in S'_o$$

Hence

$$\bigwedge_n \mathcal{P}^n f = 0$$

for any positive Borel function f on X dominated by an element $s \in S'_o$. We remark also that for any $s \in S'_o$ the sequence $(\mathcal{P}^n s)_n$ is specifically decreasing in S and therefore we have

$$\bigwedge_n \mathcal{P}^n s = \curlywedge \mathcal{P}^n s = \inf_n \mathcal{P}^n s.$$

Hence for any positive Borel function f on X dominated by an element $s \in S'_o$ we have

$$\inf_n \mathcal{P}^n f \leq \inf_n \mathcal{P}^n s = 0$$

From the previous considerations we deduce that for any $s \in S'_o$ we have

$$s = \sum_n (s_n - \mathcal{P} s_n)$$

where $s_o = s$, $s_{n+1} = \mathcal{P} s_n$ $(\forall)$ $n \in N$. From this fact and from Theorem 2.10 it follows that the set

$$\{ s - \mathcal{P}s \mid s \in S' \}$$

is a solide and increasingly dense subcone of S.

Let now $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ be a submarkovian resolvent on X such that S' is a solid and increasingly dense subset of the cone $\mathcal{E}_{\mathcal{V}}$ of all $\mathcal{V}$-excessive functions on X which are finite $\mathcal{V}$- a.s. We may suppose also that the initial kennel V of $\mathcal{V}$ is such that Vf is a bounded continuous function on X for any positive bounded Borel function f on X. We put

$$Wf = Vf - PVf \quad (\forall)\ f \in \mathcal{F}_b$$

From Theorem 1.8 we get

$$Wf \in \mathcal{S} \quad (\forall)\ f \in \mathcal{F}_b$$

and W is a kernel on X. Let now $s, t \in S'$ and $f \in \mathcal{F}_b$. We have

$$\mathcal{P}s \lesssim s \wedge (\mathcal{P}s + t - Pt + Pf) \leq s$$

and therefore from Proposition 2.7 we get

$$s \wedge (Ps + t - Pt + Pf) \in S'$$

Now using a well know result of Mokobodzky (see [13]) we deduce that there exists a submarkovian resolvent $\mathcal{W} = (W_\alpha)_{\alpha > 0}$ on X such that W is the initial kernel of $\mathcal{W}$.

From the above considerations it follows that $\mathcal{S}$ is a solid and increasingly dense subcone of the set $\mathcal{E}_{\mathcal{W}}$ of all $\mathcal{W}$-excessive functions.

Let now $\mathcal{F}_o$ be the set of all positive bounded Borel function f on X such that $Vf \in S'_o$. From the first part of the proof we get

$$Vf = \sum_{n=0}^{\infty} P^n Wf$$

Since V and W are kernels and since $\mathcal{F}_o$ is sufficiently large subset of $\mathcal{F}_b$ we get

$$Vf = \sum_{n=0}^{\infty} P^n Wf \quad (\forall) \ f \in \mathcal{F}_b$$

Hence for any $f \in \mathcal{F}_b$ we have

$$\sum_{n=0}^{\infty} P^n Wf \in S'$$

and therefore, using Hunt theorem, we deduce that we have

$$\sum_{n=0}^{\infty} P^n u \in S'$$

for any $u \in S$ for which

$$\sum_{n=0}^{\infty} P^n u$$

is dominated by an element of S'.

Let now $g \in D(P)$ (be) such that

$$Pg \precsim g.$$

If $s \in S'_o$ we put

$$g_s = g \wedge s.$$

Obviously $g_s \in D(P)$ and

$$Pg_s \preccurlyeq g_s$$

Since $g_s \leq s$ we have $\inf_n \mathcal{P}^n g_s = 0$ and therefore if we put

$$u_s = g_s - \mathcal{P} g_s$$

we get, $u_s \in S$ and

$$g_s = \sum_{n=0}^{\infty} \mathcal{P}^n u_s .$$

From the previous remark it follows

$$g_s \in S', \quad g = \bigvee_{s \in S'_0} g_s \in S'.$$

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