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COCYCLIC COHOMOLOGY OF CROSSED PRODUCTS BY ALGEBRAIC GROUPS

by

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CYCLIC COHOMOLOGY OF CROSSED PRODUCTS BY ALGEBRAIC GROUPS

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Introduction

The computation of K-theory groups of crossed product C^*-algebras in one of the central problems in operator algebras. Important steps have been made [3, 13, 19, 20] but the solution is not complete.

In [4] Connes has defined the cyclic cohomology groups $HC^*(A)$ of an algebra A over the complex numbers and a pairing $K_*(A) \otimes HC^*(A) \to \mathbb{C}$. This pairing was succesfully used by Connes and Moscovici to obtain important geometric applications.

The computation of cyclic cohomology groups of suitable "smooth" crossed product algebras has shown that they behave very much like the K-theory groups of the corresponding C^*-algebras. This is true for Z [17,19], R [3, 7], groups having a special manifold as classifying space [13, 18].

It is the purpose of this paper to study the cyclic cohomology of crossed products by Lie groups, trying to recover results known to be true for their K-theory. An interesting feature is that there seems to be no γ-obstruction [13].

Let G be a Lie group acting smoothly on a locally convex algebra A. Then $C_c^\infty(G, A)$ becomes a locally convex algebra for the convolution product, denoted here $A \rtimes G$. This will be the crossed product we shall be concerned with. Note that as in [18] we have choosen the most restrictive behaviour at infinity, i.e. the vanishing in a neighborhood of infinity. This choice is justified by technical reasons : the computation of the cohomology of G with values in such modules is easier.

By a real algebraic group we shall mean the real form of a complex algebraic group. $C_{inv}^\infty(G)$ denotes the ring of smooth class functions on a Lie group G. Here is our main result.

Theorem Let G be a real algebraic group, K a maximal compact subgroup and $q = \dim G/K$. The periodic cyclic homology groups $PHC_*(A \rtimes G)$ and $PHC_*(A \rtimes K)$ are modules over $C^\infty_{inv}(G)$ such that

$$PHC_*(A \rtimes G)_m \simeq PHC_{*+q}(A \rtimes K)_m$$

for any maximal ideal m of $C^\infty_{inv}(G)$.

The proof is based on the study of L(G, G), a global form of the construction we have used in [18]. There we have shown that for G discrete the Connes' complex decomposes as a direct sum of subcomplexes indexed by the conjugacy classes of G. For Lie groups such a result is no longer true and is replaced by a $C^\infty_{inv}(G)$-module structure on the Connes' complex. Instead of looking at a given conjugacy class we use localization at the corresponding maximal ideal. That is why localisation enters the statement of the theorem. Actually it also follows that $PHC_*(A \rtimes G)$ and $PHC_{*+q}(A \rtimes K)$ are isomorphic as vector spaces, however we do not obtain this cannonically but simply by counting dimensions. Typically the situation is that of the modules of sections in two vector bundles over a compact space having the same dimension at each point.

L(G, G) is a quasicyclic object, a class introduced in the first section . Another feature of it is that it has a smooth G-action such that $L(G,G) \otimes_G \mathbb{C} \simeq (A \rtimes G)^{\natural}$. Then we use a dual form of the van Est's complex to obtain a resolution of $A \rtimes G$ by quasicyclic objects. PHC_*-groups are available only for reduced quasicyclic objects (satisfying $T^{n+1}_{n+1} = 1$) which L(G,G) is not. The Main Lemma contains the technique necessary to bridge this difficulty. It uses a sort of "dual Dirac element", different to the usual one [13] but G-invariant , not only K-invariant, however it can be defined only locally. The output of the Main Lemma is an exact sequence of reduced quasicyclic objects the extreme terms of which are the ones we are interested in (localisations of $(A\rtimes G)^{\natural}$ and $(A\rtimes K)^{\natural}$) and the middle terms have vanishing PHC_*-groups.

Here is briefly the contents of the sections. In the first section we introduce the notion of a quasicyclic object and we discuss their properties. It also contains the Main Lemma, the key point in proving the isomorphism of PHC_*-groups. The second and the third section contain the construction of the data needed for the use of the Main Lemma. The second section also contains a theorem for the cyclic cohomology of coverings:"If $G_1 \to G$ is a finite covering then

$$HC^*(A\rtimes G) \simeq HC^*(A\rtimes G_1)^H$$

where H is the covering group". The fourth section contains a Weyl theorem in cyclic cohomology :"$PHC^*(A\rtimes G) \simeq PHC^*(A\rtimes T)^W$ if G is a compact connected Lie group, $T\subset G$ a maximal torus and W the Weyl group". Needless to say, these two last theorems are generalisations of well-known results in the theory of compact group representation.

The third section also contains some applications to nonalgebraic groups. If G is the universal covering group of $SL_2(R)$ then

$$PHC_*(A\rtimes G) \simeq PHC_{*+3}(A)$$

The Appendix collects some results on Lie groups. The reader should keep in mind the algebraic case for which these results are mainly intended.

I would like to express my gratitude to Professor A. Buium for usefull discussions about algebraic groups.

1. Quasicyclic objects

In this section we introduce the class of quasicyclic objects, a class with properties close to those of cyclic objects and whose definition is inspired by [18]. We carefully study its properties and show how to exploit certain exact sequences of quasicyclic objects.

1.1. Let us recall the definition of a cyclic object in an abelian category M [5]. It is a simplicial object $(X_n)_{n\geq 0}$ in M with an extra structure given by an action of Z_{n+1} on X_n. The face and degeneracy operators

$d_i : X_n \to X_{n-1}$, $s_i : X_n \to X_{n+1}$ $i = 0,\ldots, n$ satisfy the usual simplicial identities [15] :

$$(S_1) \qquad d_i d_j = d_{j-1} d_i \qquad i < j$$

$$(S_2) \qquad s_i s_j = s_{j+1} s_i \qquad i \leq j$$

$$(S_3) \qquad d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ 1 & i = j \text{ and } i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

And if $\{ t_{n+1} \in Z_{n+1} , n \geq 0 \}$ is the standard system of generators we also have :

$$(C_1) \qquad d_i t_{n+1} = \begin{cases} t_n d_{i-1} & i = 1,\ldots,n \\ d_n & i = 0 \end{cases}$$

$$(C_2) \qquad s_i t_{n+1} = \begin{cases} t_{n+2}\, s_{i-1} & i = 1,\ldots, n \\ t_{n+2}^2\, s_n & i = 0 \end{cases}$$

and

$$(C_3) \qquad t_{n+1}^{n+1} = 1 \quad n \geq 0.$$

The main example is $\mathfrak{A}^{\natural} = (\mathfrak{A}^{\otimes n+1})_{n\geq 0}$ where $\mathfrak{A}$ is a unital algebra over C and

$$d_i(a_0 \otimes \dots \otimes a_n) = \begin{cases} a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n & i = 0,\dots, n-1 \\ a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} & i = n \end{cases}$$

$$s_i(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_n \quad i = 0,\dots, n$$

$$t_{n+1}(a_0 \otimes \dots \otimes a_n) = a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$$

If $\mathfrak{A}$ is a locally convex algebra we understand to replace $\otimes$ by $\hat{\otimes}$ = the projective tensor product [10] as in [4].

1.2. We now define another class of objects containing objects like $\mathfrak{A}^{\natural}$ even if $\mathfrak{A}$ is nonunital, as well as the objects $\tilde{L}(A, G, x)$ defined in [18] (see also 1.3 iii).

Definition. A quasicyclic object in an abelian category M is a graded object $(X_n)_{n\geq 0}$, $X_n \in Ob(M)$ together with morphisms $d_i : X_n \to X_{n-1}$, $i = 0,\dots,n$ and $T_{n+1} : X_n \to X_n$ satisfying (S_1), (C_1) and

(Q) $\qquad d_j T_{n+1}^{n+1} = T_n^n d_j$

1.3. Examples of quasicyclic objects.

i) Cyclic objects.

ii) $\mathfrak{A}^{\natural}$ for a not necessary il unital algebra. Actualy we will be interested in the following situation. Let A be a complete locally convex algebra, $1 \in A$, G a Lie group acting smothly on A, in the sense that a morphism $\alpha : G \to Aut(A)$ is given such that α_g is continuous and unital for each $g \in G$ and the map $G \ni g \to \alpha_g(a) \in A$ is smoth for each $a \in A$. Then $A \rtimes G$, the smoth crossed product of A with G is defined as $C_c^\infty(G, A) = \{\varphi \in C^\infty(G, A)\ ,\ \operatorname{supp}\varphi \text{ compact}\}$ with the convolution product

$$\varphi * \psi(g) = \int_G \varphi(h)\alpha_h(\psi(h^{-1}g))\, dh$$

for $\varphi, \psi \in A \rtimes G$ and dh a fixed left Haar measure on G.

The explicit formulae in this case are $\mathcal{U}^\natural_n = C_c^\infty(G^{n+1}, A^{\otimes n+1})$, $\mathcal{U} = A \rtimes G$,

$$(d_j\varphi)(g_0,\dots,g_{n-1}) = \int_G d_j\circ(1\otimes\dots\otimes 1\otimes\alpha_h\otimes 1\otimes\dots\otimes 1)(\varphi(g_0,\dots,g_{j-1}, h, h^{-1} g_j,\dots,g_{n-1})\, dh$$

for $j = 0, \dots, n-1$

$$(d_n\varphi)(g_0,\dots,g_{n-1}) = \int_G d_n\circ(\alpha_h\otimes 1\otimes\dots\otimes 1)(\varphi(h^{-1}g_0, g_1,\dots,g_{n-1},h))\, dh$$

and

$$(T_{n+1}\varphi)(g_0,\dots,g_n) = t_{n+1}(\varphi(g_1,\dots,g_n, g_0)),\quad \varphi\in (A\rtimes G)^\natural_n.$$

In these formulae the left hand d_j is for $(A \rtimes G)^\natural$ while the right hand d_j is for $A^\natural$; h appears on j^{th} position in the first formula.

iii) Let A and G be as in ii) and suppose that there are given $\rho : G \to G_1$ a morphism of Lie groups and $U \subset G$ an open set.

Let $L_n(U, G_1) = C_c^\infty(U \times G_1^{n+1}, A^{\otimes n+1})$ and define

$$(d_0\varphi)(\gamma,\hat{g}_0,\dots,g_n) = \int_G d_0\circ(1\otimes\alpha_\gamma\otimes 1\otimes\dots\otimes 1)(\varphi(\gamma, g_0,\dots,g_n))\, dg_0$$

$$(T_{n+1}\varphi)(\gamma,g_0,\dots,g_n) = (1\otimes\alpha_\gamma^{-1}\otimes 1\otimes\dots\otimes 1)\circ t_{n+1}(\varphi(\gamma,g_1,\dots,g_n,\rho(\gamma)g_0)$$

for $\gamma \in U$, $g_0,\dots,g_n \in G_1$, $\varphi \in L_n(U, G_1)$.

Also let $d_j = T_n^j\, d_0\, T_{n+1}^{-j}$ for $j = 1,\dots, n$. Explicitely

$$(d_j\varphi)(\gamma, g_0,\dots,\hat{g}_j,\dots,g_n) = \int_G d_j(\varphi(\gamma, g_0,\dots,g_n))\, dg_j.$$

Then $(L_n(U, G_1))_{n\geq 0}$ is a quasicyclic object.

We stress that examples ii) and iii) will be effectively used in computation.

1.4. <u>Definition</u> . A quasicyclic object $(X_n,\ n \geq 0,\ d_j,\ T_{n+1})$ will be called reduced if $T_{n+1}^{n+1} = 1$.

If $(X_n)_{n \geq 0}$ is an arbitrary quasicyclic object then $(X_n / (1-T_{n+1}^{n+1})X_n)_{n\geq 0}$ and $(\ker(T_{n+1}^{n+1} : X_n \to X_n))_{n\geq 0}$ are examples of reduced quasicyclic objects.

Given a reduced quasicyclic object $X = (X_n,\ n\geq 0,\ d_j,\ T_{n+1})$ we can form the Connes' complex $C(X)$ as in the cyclic case [5, 14] . Let us recall its definition.

Let $\partial, \partial' : X_n \to X_{n-1}$ be defined by $\partial' = \sum_{j=0}^{n-1} (-1)^j d_j$, $\partial = \partial' + (-1)^n d_n$. Also let $\varepsilon = 1-(-1)^n T_{n+1}$, $N = \sum_{j=0}^{n} (-1)^{nj} T_{n+1}^{j}$. The Connes' complex associated with X is the total complex associated with the periodic bicomplex

$$
\begin{tikzcd}
\cdots & \cdots & \cdots & \\
X_{n+1} \arrow[d, "\partial"'] & X_{n+1} \arrow[l, "\varepsilon"'] \arrow[d, "-\partial'"] & X_{n+1} \arrow[l, "N"'] \arrow[d, "\partial"] & \cdots \arrow[l, "\varepsilon"'] \\
X_n \arrow[d, "\partial"'] & X_n \arrow[l, "\varepsilon"'] \arrow[d, "-\partial'"] & X_n \arrow[l, "N"'] \arrow[d, "\partial"] & \cdots \arrow[l, "\varepsilon"'] \\
\cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "-\partial'"] & \cdots \arrow[d, "\partial"] & \\
X_0 & X_0 \arrow[l, "\varepsilon"'] & X_0 \arrow[l, "N"'] & \cdots \arrow[l, "\varepsilon"']
\end{tikzcd}
$$

Then $HC_*(X)$, the cyclic homology of X, is by definition , the homology of $C(X)$. The homology of the infinite twosided periodic version of this bicomplex defines $PHC_*(X)$, the periodic cyclic homology of X.

If $X = \mathfrak{A}$ we shall write $HC_*(\mathfrak{A})$ and $PHC_*(\mathfrak{A})$ instead of $HC_*(\mathfrak{A}^{\natural})$ and $PHC_*(\mathfrak{A}^{\natural})$.

The above definitions are the same as in the cyclic case [5, 14] .

1.5. We now briefly investigate the extension of Connes' exact sequence [4].

Let $X = (X_n, n \geq 0, d_j, T_{n+1})$ be a quasicyclic object. Denote by $\mathcal{E}(X)$ the bicomplex

$$\begin{tikzcd}
\vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "-\partial'"] \\
X_{n+1} \arrow[d, "\partial"'] & X_{n+1} \arrow[l, "\varepsilon"'] \arrow[d, "-\partial'"] \\
X_n \arrow[d, "\partial"'] & X_n \arrow[l, "\varepsilon"'] \arrow[d, "-\partial'"] \\
\vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "-\partial'"] \\
X_0 & X_0 \arrow[l, "\varepsilon"']
\end{tikzcd}$$

and by $HH_*(X)$ its homology . Here ∂ and ∂' are as in 1.4. and $\varepsilon = 1-(-1)^n T_{n+1}$.

If X is a reduced quasicyclic object then, as for cyclic objects [4], there exists an exact sequence

$$\ldots \xrightarrow{I} HC_n(X) \xrightarrow{S} HC_{n-2}(X) \xrightarrow{B} HH_{n-1}(X) \xrightarrow{I} HC_{n-1}(X) \xrightarrow{S} \ldots$$

$HC_*(X)$, $PHC_*(X)$ and S are related by the following exact sequence

$$0 \to \varprojlim{}^1 (HC_{n+2k+1}(X), S) \to PHC_n(X) \to \varprojlim (HC_{n+2k}(X), S) \to 0$$

It shows that $PHC_*(X) = 0$ whenever there exists a natural number m such that $S^m : HC_{n+2m}(X) \to HC_n(X)$ is the 0 morphism for any $n \geq 0$.

1.6. The following vanishing criterion for S is basically proved in [18].

<u>Proposition.</u> Let $X = (X_n, n \geq 0, d_j, T_{n+1})$ be a quasicyclic object such that $1-T_{n+1}^{n+1}$ is injective. Then $S = 0$ on $HC_*(X_n/(1-T_{n+1}^{n+1})X_n)$.

<u>Proof.</u> Consider the composition of

$$\mathcal{E}(X) \to \mathcal{E}(X_n/(1-T_{n+1}^{n+1})X_n)_{n\geq 0} \to C(X_n/(1-T_{n+1}^{n+1})X_n)_{n\geq 0}$$

The line filtration of the first and third bicomplex shows that the above map induces an isomorphism on homology. This shows that I is onto and hence S vanishes in the Connes' exact sequence of the reduced quasicyclic object $(X_n/(1-T_{n+1}^{n+1})X_n)_{n\geq 0}$.

1.7. We are going now to introduce a sort of "dual Dirac element", an important device in establishing isomorphisms of PHC_*-groups.

Suppose the following data is given:

a) Two exact sequences of quasicyclic complete locally convex spaces

$$(E_1) \qquad 0 \longrightarrow \mathcal{X}^{(q)} \xrightarrow{\delta} \mathcal{X}^{(q-1)} \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{X}^{(0)} \longrightarrow Y \rightarrow 0$$

$$(E_2) \qquad 0 \longrightarrow \mathcal{X}^{(0)} \xrightarrow{\sigma} \mathcal{X}^{(1)} \xrightarrow{\sigma} \dots \xrightarrow{\sigma} \mathcal{X}^{(q)} \longrightarrow \mathcal{Z} \rightarrow 0$$

where Y is reduced.

b) A C^∞-action of R, $\eta : R \rightarrow GL(\mathcal{X}^{(j)})$ for any $j = 0,\dots, q$ such that $\eta_1 = T^{n+1}_{n+1}$ and if ∇ denotes the derivative of η_t at $t = 0$ then $\nabla = \delta\sigma + \sigma\delta$ (we understand that $\delta(\mathcal{X}^{(0)}) = \sigma(\mathcal{X}^{(q)}) = 0$).

c) $1 - T^{n+1}_{n+1}$ is injective on $\mathcal{X}^{(j)}_n$ for any $j = 0,\dots, q$ and any $n \geq 0$.

Let $\tilde{\mathcal{X}}^{(0)} = \mathcal{X}^{(0)} / \nabla \mathcal{X}^{(0)}$, $\tilde{\mathcal{X}}^{(j)} = \mathcal{X}^{(j)} / (\nabla \mathcal{X}^{(j)} + \sigma \mathcal{X}^{(j-1)})$ for $j = 1,\dots, q-1$.

<u>Main lemma.</u> i) $\tilde{\mathcal{X}}^{(j)}$ is a reduced quasicyclic object for any $j = 0,\dots,q-1$.

ii) The complex

$$0 \rightarrow \mathcal{Z} \longrightarrow \tilde{\mathcal{X}}^{(q-1)} \xrightarrow{\delta} \dots \xrightarrow{\delta} \tilde{\mathcal{X}}^{(1)} \xrightarrow{\delta} \tilde{\mathcal{X}}^{(0)} \longrightarrow \mathcal{Y} \rightarrow 0$$

is acyclic.

iii) $PHC_*(\tilde{\mathcal{X}}^{(j)}) = 0$ for any $j = 0,\dots, q-1$.

iv) $PHC_*(Y) \simeq PHC_{*+q}(\mathcal{Z})$.

<u>Proof.</u> Since a short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ of reduced quasicyclic objects gives rise to the six term exact sequence

$$\begin{array}{ccccc} PHC_0(X') & \longrightarrow & PHC_0(X) & \longrightarrow & PHC_0(X'') \\ \uparrow & & & & \downarrow \\ PHC_1(X'') & \longleftarrow & PHC_1(X) & \longleftarrow & PHC_1(X') \end{array}$$

we immediately see that iv) is a consequence of i), ii) and iii). (PHC_* is a $\mathbb{Z}_2$-graded theory).

The relation $1 - T^{n+1}_{n+1} = \int_0^1 \nabla \eta_t \, dt = \nabla(\int_0^1 \eta_t \, dt)$ shows that $\operatorname{Ran}(1 - T^{n+1}_{n+1}) \subset \operatorname{Ran} \nabla$ and hence $\tilde{\mathcal{X}}^{(j)}$ is reduced. We also obtain that ∇ is injective.

Let us observe that $\delta\nabla = \nabla\delta$. Since $\mathcal{Y}$ is reduced and $\delta\sigma = \nabla - \sigma\delta$ we obtain that the complex in ii) is well defined. We now prove its exacteness.
At $\mathcal{Y}$ and $\tilde{\mathcal{X}}^{(0)}$: this follows immediately from the exacteness of (E_1).
At $\mathcal{X}^{(j)}$, $j = 1,\dots, q-1$, and $\mathcal{Z}$: observe first that $\delta\mathcal{X}^{(j+1)} \cap \sigma\mathcal{X}^{(j-1)} = 0$. Indeed if $x \in \delta\mathcal{X}^{(j+1)} \cap \sigma\mathcal{X}^{(j-1)}$ then $\delta x = \sigma x = 0$ and hence $\nabla x = 0$. Since ∇ is injective we obtain $x = 0$. Let $v_j \in \mathcal{X}^{(j)}$ be such that $\delta v_j = \nabla v_{j-1} + \sigma v_{j-2}$ for some $v_{j-1} \in \mathcal{X}^{(j-1)}$ and $v_{j-2} \in \mathcal{X}^{(j-2)}$.

We obtain $\delta(v_j - \sigma v_{j-1}) = 0$ as follows from the above discussion. Then $v_j = \sigma v_{j-1} + \delta v_{j+1}$ for some $v_{j+1} \in \mathcal{X}^{(j+1)}$. (The undefined terms are to be replaced by 0).So we have proved i) and ii).

For iii) let us note that $\mathcal{X}^{(j)}/\sigma\mathcal{X}^{(j-1)}$ is a quasicyclic object with R acting on it by quasicyclic endomorphisms. This follows since ∇ commutes with σ and with the structural morphisms of $\mathcal{X}^{(j)}$ (use $\nabla = \delta\sigma + \sigma\delta$). We show that $1 - T^{n+1}_{n+1}$ is injective on $\mathcal{X}^{(j)}/\sigma\mathcal{X}^{(j-1)}$ for $j = 0,\dots, q-1$.

Suppose that $(1 - T^{n+1}_{n+1})v_j = \sigma v_{j-1}$ for some $v_j \in \mathcal{X}^{(j)}$, $v_{j-1} \in \mathcal{X}^{(j-1)}$.

Then $\sigma(1 - T^{n+1}_{n+1})v_j = 0$ and hence $\sigma v_j = 0$ since $1 - T^{n+1}_{n+1}$ is assumed to be injective. The exactness of (E_2) shows that there exists $\vartheta'_{j-1} \in \mathcal{X}^{(j-1)}$ such that $\vartheta_j = \sigma \vartheta'_{j-1}$. The action of R on $\mathcal{X}^{(j)}/\sigma\mathcal{X}^{(j-1)}$ factors to an action of the compact group $\mathbb{T} = R/Z$ on $(\mathcal{X}_n^{(j)}/\sigma\mathcal{X}_n^{(j-1)})/$ $/(1 - T_{n+1}^{n+1})(\mathcal{X}_n^{(j)}/\sigma\mathcal{X}_n^{(j-1)})$ whose set of fixed vectors is just $\tilde{\mathcal{X}}^{(j)}$. The proof is completed using proposition 1.6. and the fact that $F \longrightarrow F^{\mathbb{T}}$ is an exact functor.

2. Preliminary computations

In this section we shall be concerned with specific properties of the quasicyclic objects $L(U, G_1)$ defined in 1.3. iii). We shall also establish the technical details needed for the proof of the exacteness of (E_1) appearing in the Main Lemma .

2.1 Let A, G, G_1 , $\alpha : G \to \mathrm{Aut}(A)$ and $\rho : G \to G_1$ be as in 1.3. iii) . Also let $U \subset G$ be an Ad_G-invariant open set. Define $\beta : G \to GL(L(U, G_1))$ by

$$(\beta_{\gamma}\varphi)(\gamma_1, g_0, \dots, g_n) = \alpha_{\gamma}^{\otimes n+1}(\varphi(\gamma^{-1}\gamma_1\gamma, \rho(\gamma)^{-1}g_0, \dots, \rho(\gamma)^{-1}g_n)$$

for any $\gamma, \gamma_1 \in G$, $g_0, \dots, g_n \in G_1$ and $\varphi \in L_n(U, G_1)$. β defines a smooth action of G.

In case $G = G_1$, $\rho = \mathrm{id}$ and $U = G$ we define $p: L(G,G) \to (A \rtimes G)^{\natural}$ by

$$(p\varphi)(h_0, \dots, h_n) = \int_G \tilde{\Psi}((\beta_{\gamma}\varphi)(g_n, g_0, g_1, \dots, g_n))\, d\gamma$$

if $g_0 = h_0$, $g_1 = h_0h_1, \dots,$ $g_n = h_0h_1 \dots h_n$, $\tilde{\Psi} = (\alpha_{g_n} \otimes \alpha_{g_0} \otimes \dots \otimes \alpha_{g_{n-1}})^{-1}$ and $\varphi \in L_n$.

Let C_{Δ} be the G-module having C as underlying space and the G-module structure given by $g \cdot \lambda = \Delta(g)\lambda$ for any $g \in G$ and any $\lambda \in C$. Here Δ is the modular function of G. Let $L = L(G, G)$.

<u>Proposition.</u> i) β_{γ} and p are morphisms of quasicyclic objects.

ii) $p^{*} : (A \rtimes G)^{\natural *} \to L^{*}$ establishes an isomorphism of $(A \rtimes G)^{\natural *}$ onto $\mathrm{Hom}_G(C_{\Delta}, L^{*})$. (The action of G on L is $\langle gf, \varphi \rangle = \langle f, g^{-1}\varphi \rangle$.)

<u>Proof.</u> i) is a tedious but straight forward computation which we shall omitt.

Let $\Phi : L_n \to C_c^{\infty}(G, C_c^{\infty}(G^{n+1}, A^{\otimes n+1})) = C_c^{\infty}(G, (A \rtimes G)_n)$ be given by

$$\Phi(\varphi)(g^{-1}, h_0, \dots, h_n) = \tilde{\Psi}((\beta_g\varphi)(g_n, g_0, g_1, \dots, g_n))$$

(here Ψ , $g_0,\dots,g_n$ have the same meaning as above). Φ is obviously an isomorphism which is also G-equivariant since

$$\Phi(\beta_\gamma\varphi)(g^{-1},h_0,\dots,h_n) = \Psi(\beta_{g\gamma}\varphi(g_n,g_0,\dots,g_n))) = \Phi(\varphi)(\gamma^{-1}g^{-1},h_0,\dots,h_n)$$

The proof of the proposition will be accomplished once the following lemma is proved.

2.2. Let G be as above, X a smooth principal G-bundle and $B = G\backslash X$. Also let E be a complete locally convex space. Then $C_c^\infty(X, E)$ is a complete smooth G-module. Let $F = C_\Delta \otimes C_c^\infty(G, E)$.

Lemma. i) $\mathrm{Hom}_G(C_\Delta, C_c^\infty(G, E)^*) \simeq H^0(G, F^*) \simeq C_c^\infty(B, E)^*$,

ii) $H^j(G, F^*) = 0$ for $j > 0$.

Here $H^j(G, F_0)$, $j \geq 0$ denote the cohomology groups of G with values in the continuous G-module F_0. As in the case G discrete they represent the derived functors of $F_0 \to F_0^G (= H^0(G, F_0))$ [9].

Proof. It is proved in [9, Observation III .1.3.] that $H^0(G, C^\infty(G, E^*)) = E^*$ and $H^j(G, C^\infty(G, E^*)) = 0$ for $j > 0$. The formulae for the homotopies used in the proof extend by continuity to prove that $H^0(G, C_c^\infty(G, E)^*) = E^*$ and $H^j(G, C_c^\infty(G, E)^*) = 0$ for $j > 0$. The isomorphism $C_c^\infty(G, E) \simeq C_\Delta \otimes C_c^\infty(G, E)$ as G-modules proves the theorem for $X = G$. Since $C_c^\infty(G\times B, E) \simeq C_c^\infty(G, C_c^\infty(B, E))$ the case X trivial reduces to the case $X = G$. The general case follows by a partition of unity argument and the $C^\infty(B)$-linearity of the homotopies for $X = G \times B$.

The isomorphism $\mathrm{Hom}_G(C_\Delta, C_c^\infty(G, E)^*) \simeq C_c^\infty(B, E)^*$ is the dual of

$p_0 : C_c^\infty(X, E) \to C_c^\infty(B, E)$

$$(p_0 f)(x) = \int_G f(g^{-1} y)\, dg$$

where y is an arbitrary lifting of x.

2.3. Let $U \subset G$ be an Ad_G-invariant open set. Define $J: L(U, G_1) \to L(U, \{e\})$ by

$$J\varphi(\gamma) = \int_{G_1^{n+1}} \varphi(\gamma', g_0, \dots, g_n)\, dg_0 \dots dg_n$$

<u>Lemma.</u> J is a morphism of quasicyclic objects and $HH_n(J)$ is an isomorphism.

<u>Proof.</u> The first part is obvious.

Let $F = \ker J$. It is enough to show that $HH_n(F) = 0$. Define $F^{(-1)} = 0$,

$$F^{(j)} = \{\varphi \in F, \int_{G_1^{j+1}} \varphi(\gamma', g_0, \dots, g_n)\, dg_0, \dots, dg_j = 0 \text{ if } \varphi \in L_n(U, G_1)$$

Then $F^{(j)} \subset F^{(j+1)}$, $\bigcup F^{(j)} = F$.

Let us observe that $F^{(j)}$ is invariant for both ∂ and ∂' :
$d_i F_n^{(j)} \subset F_{n-1}^{(j)}$ for $i = 0, \dots, n$; $d_i F_n^{(j)} \subset F_{n-1}^{(j-1)}$ for $i = 0, \dots, j$.

Choose $\chi \in C_c^\infty(G_1)$, $\int_{G_1} \chi(g)\, dg = 1$. We define

$$(s\varphi)(\gamma, g_0, \dots, g_{n+1}) = s_j(\varphi(\gamma, g_0, \dots, g_j, g_{j+2}, \dots, g_{n+1}))\chi(g_{j+1})$$

s_j being the degeneracy of $A^\natural$. Using $\partial = (-1)^{j+1}(d_{j+1} - d_{j+2} + \dots + (-1)^{n-j+1} d_n)$

on $F^{(j)} / F^{(j-1)}$ we obtain $\partial s + s\partial = (-1)^{j+1}$ on $F^{(j)} / F^{(j-1)}$ and similarily

$\partial' s + s\partial' = (-1)^{j+1}$ on $F^{(j)} / F^{(j-1)}$. This proves the lemma.

2.4. We now give a consequence of the lemma above.

<u>Proposition.</u> Let $K \subset G$ be a compact subgroup, then

$$HC_*(L(K, G)^K) \simeq HC_*(A \rtimes K)$$

Proof. The above lemma shows that J induces an isomorphism $HH_*(L(K, G)^K) \simeq HH_*(L(K, \{e\})^K)$ since K is compact. The Connes' exact sequence shows that it also induces an isomorphism $HC_*(L(K, G)^K) \simeq HC_*(L(K, \{e\})^K)$. Then we use the isomorphism $(A \rtimes K)^{h} \simeq L(K, K)^K$ (see proposition 2.1. ii).

2.5. Let K be a compact Lie group. Consider $C^\infty(K)$ with the convolution product.

Corollary. [16] $HC_*(C^\infty(K)) \simeq C^\infty(K)^K \otimes HC_*(C)$.

Proof. Use the above proposition and observe that $L(K,\{e\}) = C^\infty(K)$. Hence $L(K,\{e\})^K$ is the space of smooth class-functions on K.

2.5. Let E and E' be two locally convex spaces wich are also continuous G-modules in the sense that there are given morphisms $G \to GL(E)$ and $G \to GL(E')$ such that $G \ni g \to g\xi \in E$ is continuous for each $\xi \in E$ (respectively $\xi \in E'$). Then $E \otimes_G E'$ will denote the quotient of $E \hat{\otimes} E'$ by the closed subspace generated by $g^{-1}\xi \otimes \xi' - \xi \otimes g\xi'$.

If E is a complete smooth G-module and G is compact then $E \otimes_G C$ is isomorphic to E^G. For any G $(E \otimes_G C)^* \simeq E^{*G}$.

2.6. We now give a complex whose homology might be called the G-homology of a given smooth G-module F. It is closely connected to van Est's theorem [8, 9].

Let $K \subset G$ be a maximal compact subgroup of G, t= Lie $K \subset g$ = Lie G.

Define

$$\delta : (\Lambda^j(g/t) \otimes F) \otimes_K C \to (\Lambda^{j-1}(g/t) \otimes F) \otimes_K C$$

by

$$\delta(\mathring{X}_1 \wedge \ldots \wedge \mathring{X}_j \otimes \xi) = \sum_{i=1}^{j} (-1)^{i+1} \mathring{X}_1 \wedge \ldots \hat{\mathring{X}}_i \ldots \wedge \mathring{X}_j \otimes X_i(\xi) -$$

$$- \sum_{i<k} (-1)^{i+k} \overset{\circ}{\overbrace{[X_i, X_k]}} \otimes \mathring{X}_1 \wedge \ldots \hat{\mathring{X}}_i \ldots \hat{\mathring{X}}_k \ldots \wedge \mathring{X}_j \otimes \xi .$$

for $X_1, \ldots, X_j \in g$, $\xi \in F$ and $\dot{X}$ denoting the class of $X \in g$ in g/t.

We shall be interested only in the case $F = C_\Delta \hat{\otimes} C_c^\infty(G, E)$ for E an arbitrary locally convex space.

Proposition. Let G be a finite component Lie group, K, E and F be as above. Let $q = \dim(G/K)$. Then the complex

$$0 \to (\Lambda^q(g/t) \hat{\otimes} F) \hat{\otimes}_K C \xrightarrow{\delta} \ldots \xrightarrow{\delta} (\Lambda^0(g/t) \hat{\otimes} F) \hat{\otimes}_K C \to F \hat{\otimes}_G C \to 0$$

is well defined and acyclic .

$$(C_\Delta \hat{\otimes} L(G, G)) \hat{\otimes}_G C \simeq (A \rtimes G)^h .$$

Proof. It is easy to verify that it is well defined (this follows also from 3.7.). If we discard the augmentation $(\Lambda^0(g/t) \hat{\otimes} F) \hat{\otimes}_K C = F \hat{\otimes}_K C \to F \hat{\otimes}_G C$ and pass to the dual complex we obtain the complex computing $H^j(G, F^*)$ [9]. Since $H^j(G, F^*) = 0$ for $j > 0$ and $H^0(G, F^*) = F^{*G} = (F \hat{\otimes}_G C)^*$ by lemma 2.2 we obtain the result. The last statement follows from proposition 2.1.

2.7. Let $H \subset G$ be a central compact subgroup, H acting trivially on A, $\rho : G \to G_1 = G/H$ the quotient morphism. There exists an action of H on $L(G, G)$ and $L(G, G_1)$ given by

$$(h\varphi)(\gamma, g_0, \ldots, g_n) = \varphi(h\gamma, g_0, \ldots, g_n)$$

commuting with the action of G defined in 2.1.

Let $J_0 : L(G, G) \to L(G, G_1)$ be given by

$$J_0\varphi(\gamma, \dot{g}_0, \ldots, \dot{g}_n) = \int_{H^{n+1}} \varphi(\gamma, h_0 g_0, \ldots, h_n g_n)\, dh_0 \ldots dh_n$$

($\dot{g}_i$ is the class of g_i in G_1).

Then J_0 is a morphism of quasicyclic objects commuting with the actions of G and H. Lemma 2.3. shows $HH_*(J_0)$ is an isomorphism. Since H is compact and central it is contained in all maximaly compact subgroups of G.

This shows that the complex of lemma 2.6. for $L(G, G_1)$ is exact. A spectral sequence argument shows that

$$HH_*(J_0 \otimes_G C) : HH_*((C_\Delta \otimes L(G, G)) \otimes_G C) \to HH_*((C_\Delta L(G, G_1)) \otimes_G C)$$

is an isomorphism and hence also

$$HC_*(J_0 \otimes_G C) : HC_*((C_\Delta \otimes L(G, G)) \otimes_G C) \to HC_*((C_\Delta \otimes L(G, G_1)) \otimes_G C)$$

is an isomorphism. Since $((C_\Delta \otimes L(G, G)) \otimes_G C \simeq (A \rtimes G)$ by 2.1. and 2.6. and $L(G, G_1)^H \simeq L(G_1, G_1)$ we obtain

Theorem. Let $H \subset G$ be a compact central subgroup, $G_1 = G/H$, then

$$HC_*(A \rtimes G_1) \simeq HC_*(A \rtimes G)^H \quad \text{and}$$
$$HC^*(A \rtimes G_1) \simeq HC^*(A \rtimes G)^H .$$

This theorem may be viewed as a generalisation of a well known theorem in the theory of compact group representation :"if G is compact then $C \otimes R(G_1) \simeq (C \otimes R(G))^H$" [1].

3. Reduction to the maximal compact subgroup

In this section we shall show that $PHC_*(A \rtimes G)$ and $PHC_{*+q}(A \rtimes K)$ are isomorphic when localised at suitable maximal ideals of $C^{\infty}_{inv}(G) = \{ f \in C^{\infty}(G),\ f(\gamma g \gamma^{-1}) = f(g)$ for any $\gamma, g \in G\}$. Here K is a maximal compact subgroup of G and q = dim G/K. We shall accomplish this by using the Main Lemma, so most of this section will be concerned with obtaining the data needed to put us in position to use the Main Lemma.

3.1. Let A, G, G_1, $\rho : G \to G_1$ and $\alpha : G \to \mathrm{Aut}(A)$ be as in 1.3. Also let $V \subset G$ be an Ad_G-invariant open set. Define a $C^{\infty}_{inv}(G)$- module structure on $L(V, G_1)$ by

$$(\varphi\psi)(\gamma, g_0, \ldots, g_n) = \varphi(\gamma)\psi(\gamma, g_0, \ldots, g_n)$$

for any $\varphi \in C^{\infty}_{inv}(G)$, $\psi \in L_n(V, G_1)$. It is easy to see that d_j, T_{n+1} and β_γ ($\gamma \in G$) are $C^{\infty}_{inv}(G)$ - module endomorphisms.

We shall need the following

<u>Lemma.</u> Let $m_x = \{ \varphi \in C^{\infty}_{inv}(G), \varphi(x) = 0\}$. Then m_x, $x \in G$ exhaust the set of all closed maximal ideals of $C^{\infty}_{inv}(G)$. If $I \subset C^{\infty}_{inv}(G)$ is an ideal not contained in any m_x and $K_0 \subset G$ is an arbitrary compact subset then there exists $\varphi \in I$ such that $\varphi = 1$ on K_0.

<u>Proof.</u> Let $I \subset C^{\infty}_{inv}(G)$ be an ideal not contained in any m_x. Then for any $x \in G$ there exists $\varphi_x \in I$ such that $\varphi_x(x) \neq 0$. Replacing φ_x by $|\varphi_x|^2$ if necessary we may suppose that $\varphi_x \geq 0$. Let $K_0 \subset G$ be a compact subset. An easy argument shows that there exists $\varphi \in I$ such that $\varphi \geq 1$ on K_0. Let $g \in C^{\infty}(R)$, such that $g(t) = t^{-1}$ for $t > 1$. Then $\psi = (g \circ \varphi)\varphi$ belongs to I

and $\psi = 1$ on K_0.

If I is also closed let $G = \bigcup K_n$, $n \geq 1$ with K_n compact for any $n \geq 1$. Choose $\varphi_n \in I$ such that $\varphi_n > 0$ on K_n. There exists $\lambda_n > 0$ such that $\sum \lambda_n \varphi_n$ is convergent in $C^\infty(G)$. Then $\sum \lambda_n \varphi_n$ belongs to I and is invertible in $C^\infty_{inv}(G)$.

3.2. Fix $x \in G$. In order to study the localisations at m_x we make some assumptions on x.

Let $G_x = \{\gamma \in G, \gamma x = x\gamma\}$, $g_x = \text{Lie } G_x \subset g = \text{Lie } G$. If U is an Ad_{G_x} - invariant neighborhood of 0 in g_x we let $G \times_{G_x} U = (G \times U)/ G_x$ for the following action of G_x on $G \times U$: $(\gamma_1, X)\gamma = (\gamma_1\gamma, Ad^{-1}_\gamma(X))$, $\gamma_1 \in G$, $X \in U$ and $\gamma \in G_x$. We define $c: G \times_{G_x} U \to G$ by $c(\gamma, X) = \gamma x \exp(X)\gamma^{-1}$. c is obviously well defined.

We shall assume for $x \in G$ fixed that there exists $U \subset g_x$ satisfying:

(A1) c is a diffeomorphism onto an open set $V \subset G$.

Let $R_0 = \{\psi \in C^\infty_{inv}(G), \text{supp}\,\psi \subset V\}$. $W = x\exp(U)$ is an open set in G_x provided (A1) is true. Let $R_1 = \{\varphi \in C^\infty_{inv}(G_x), \text{supp}\,\varphi \subset W\}$. Define $\Phi: R_0 \to R_1, \Phi(\psi)(g) = 0$ if $g \notin W, \Phi(\psi)(g) = \psi(g)$ if $g \in W$. We shall also assume

(A2) Q is an isomorphism, and

(A3) There exist $\varphi, \varphi' \in R_1$ satisfying $\varphi(x) = 1$ and $\varphi\varphi' = \varphi$.

Conditions for x, U and G to satisfy (A1)-(A3) will be given in the Appendix.

3.3. <u>Conventions</u>. Let H be a Lie group and M be a $C^\infty_{inv}(H)$-module. We shall denote by M_y the localization of M at m_y for $y \in H$.

If not otherwise stated we shall assume (A1)-(A3) for fixed $x \in G$ and $U \subset g_x$. V and W will always have the meaning of 3.2.

3.4. An immediate consequence of (A2) and (A3) is that $L(V, G)_x \to L(G, G)_x$ and $L(W, G_x)_x \to L(G_x, G_x)_x$ are isomorphisms for the $C^{\infty}_{inv}(G)$ (respectively $C^{\infty}_{inv}(G_x)$) -module structure. This justifies the study of $L(V, G)$ and $L(W, G_x)$.

Let us observe that $V \times G^{n+1}$ and $W \times G^{n+1}$ are G (respectively G_x) principal bundles for the action $\gamma(\gamma_1, g_0, \ldots, g_n) = (\gamma\gamma_1\gamma^{-1}, \gamma g_0, \ldots, \gamma g_n)$ where $g_0, \ldots, g_n \in G$, $\gamma \in G$, $\gamma_1 \in V$ (respectively $\gamma \in G_x$, $\gamma_1 \in W$). Moreover due to (A1) the inclusion $W \times G^{n+1} \to V \times G^{n+1}$ factors to give a diffeomorphism $G_x \backslash W \times G^{n+1} \to G \backslash V \times G^{n+1}$. Lemma 2.2. then gives $(C_{\Delta} \otimes L(V, G)) \otimes_G C \simeq (C_{\Delta'} \otimes L(W, G)) \otimes_{G_x} C$, Δ' being the modular function of G_x. We obtain

$$(A \rtimes G)^{h}_x \simeq ((C_{\Delta} \otimes L(G,G)) \otimes_G C)_x \simeq ((C_{\Delta} \otimes L(V,G)) \otimes_G C)_x \simeq ((C_{\Delta'} \otimes L(W,G)) \otimes_{G_x} C)_x$$

3.5. Let $\sigma : G_x \backslash G \to G$ be a locally bounded borelian section for $G \to G_x \backslash G$ [21 , 5.1.1].

Lemma. i) There exists a measure μ on $G_x \backslash G$ such that

$$\int_G h(g)\, dg = \int_{G_x \, G_x \, G} h(\gamma\sigma(t))\, d \quad d \ (t).$$

ii) If we let $E\varphi(\gamma) = \int_{G_x \backslash G} \varphi(\gamma\sigma(t))\, d\mu(t)$ then we obtain a continuous linear map $E : C^{\infty}_c(G) \to C^{\infty}_c(G_x)$ satisfying

$$\int_G \varphi(g)\, dg = \int_{G_x} E\varphi(\gamma)\, d\gamma$$

Proof. Let $\psi \in C_c(G_x \backslash G)$. For any $\varphi \in C_c(G_x)$ let $I_{\psi}(\varphi) = \int_G (\varphi \times \psi) \circ f^{-1}(g)\, dg$ where $f(\gamma, t) = \gamma\sigma(t)$ defines a borelian isomorphism $G_x \times G_x \backslash G \to G$. I_{ψ} defines a continuous linear functional $C_c(G_x) \to C$ which is G_x invariant. Hence there exists $\mu : C_c(G_x\backslash G) \to C$ such that $I_\psi(\varphi) = \mu(\psi)\int_{G_x}\varphi(\gamma)\,d\gamma$ for $\psi \in C_c(G_x\backslash G)$ and $\varphi \in C_c(G_x)$. μ is a continuous linear functional and hence there exists a measure $d\mu$ on $G_x\backslash G$ such that $\mu(\psi) = \int_{G_x\backslash G}\psi(t)\,d\mu(t)$. This proves i). ii) follows from the definition of $d\mu$ and the locally boundedness of σ.

3.6. Let $(A \rtimes G)_V = (C \underset{\Delta}{\otimes} L(V, G)) \otimes_G C$.

Define $E_0 : L(W, G) \to L(W, G_x)$ by

$$E_0\varphi(\gamma, g_0, \ldots, g_n) = \int_{(G_x\backslash G)^{n+1}} \varphi(\gamma, g_0\sigma(x_0), \ldots, g_n\sigma(x_n))\,d\mu(x_0)\ldots d\mu(x_n)$$

Proposition. i) E_0 is a morphism of quasicyclic objects commuting with the action of G_x.

ii) If $x \in G$ satisfies (A1) then $HC_*((A \rtimes G)_V) \simeq HC_*((A \rtimes G_x)_W)$.

iii) If also (A2) and (A3) are satisfied then

$HC_*(A \rtimes G)_x \simeq HC_*(A \rtimes G_x)_x$.

Proof. i) follows from the above lemma.

The above lemma and lemma 2.3. show that $HH_*(E_0)$ is an isomorphism. Then a standard reasoning (see the proof of theorem 2.7.) shows that

$HC_*((C \underset{\Delta}{\otimes} E_0) \otimes_{G_x} C) : HC_*((C \underset{\Delta}{\otimes} L(W,G)) \otimes_{G_x} C) \to HC_*((C \underset{\Delta}{\otimes} L(W,G_x)) \otimes_{G_x} C)$

is also an isomorphism. Since $(C \underset{\Delta}{\otimes} L(V,G)) \otimes_G C \simeq (C \underset{\Delta}{\otimes} L(W,G)) \otimes_{G_x} C$ we obtain ii). iii) follows from ii) and 3.4.

3.7. We are now continuing to define the data needed for the use of the Main Lemma.

A element y in a topological group is called a topologically torsion element if (y), the closed group generated by y, is compact. If (y) is not compact then we shall say that y is topologically torsion free. Let K_x be a maximal compact subgroup of G_x. Then x is topologically torsion precisely when $x \in K_x$.

Let $F = C_{\Delta'} \otimes L(W, G_x)$ if $x \in K_x$,

$F = C_{\Delta'} \otimes L(W, G_x) \;/\; (1-x)(C_{\Delta'} \otimes L(W, G_x))$ if $x \notin K_x$.

Also let $L = K_x$ if $x \in K_x$, otherwise let L be a maximal compact subgroup of $G_x/(x)$. Define t = Lie L and $\mathcal{X}^{(j)} = (\Lambda^j(g/t) \otimes F) \otimes_L C$.

The definitions of δ and σ as well as their properties are closely connected with the complex computing the g_x-homology of F.

Let $C_j = \Lambda^j g_x \otimes F$ and $\delta_0 : C_j \to C_{j-1}$ be defined by a formula similar to that of δ in 2.6. :

$$\delta_0(X_1 \wedge \ldots \wedge X_j \otimes \xi) = \sum_{i=1}^{j} (-1)^{i+1} X_1 \wedge \ldots \hat{X}_i \ldots \wedge X_j \otimes X_i(\xi) -$$

$$- \sum_{i<k} (-1)^{i+k} [X_i, X_k] \wedge X_1 \wedge \ldots \hat{X}_i \ldots \hat{X}_k \ldots \wedge X_j \otimes \xi$$

As it is well known (C_j, δ_0) computes $H_*(g_x, F)$ = the homology of the g_x-module F.

$\mathcal{X}^{(j)}$ is isomorphic to C_j/C_j', C_j' being the closed submodule of C_j generated by $t \wedge C_{j-1}$ and $(\gamma-1)C_j$ for $\gamma \in K_x \cup (x)$. Observe that $t(C_j) \subset C_j'$. Recall that $\delta_0(X \wedge \omega) = X(\omega) - X \wedge \delta_0(\omega)$ for $X \in g$ and $\omega \in C_{j-1}$, this equation and the G_x-invariance of δ_0 show that $\delta_0(C_j') \subset C_j'$. δ ,already defined in 2.6, coincides with the quotient of δ_0. (This also completes the proof of 2.6.)

Let $d_j : C_k \to C_k$, $d_j(X_1 \wedge \ldots \wedge X_k \otimes \lambda \otimes \xi) = X_1 \wedge \ldots \wedge X_k \otimes \lambda \otimes d_j \xi$

if $X_1, \ldots, X_k \in g$, $\lambda \in C_\Delta$, $\xi \in F$. C_k becomes a quasicyclic object if we similarily define T_{n+1} to act only on the factor F. C_k' is invariant for the structural morphisms and we shall endow $\mathcal{X}^{(j)} = C_j/C_j'$ with the quotient quasicyclic structure.

3.8. Let $Z \in C^\infty(W, g_x)$, $Z(x \exp(X)) = X$. We define $\sigma_0 : C_k \to C_{k+1}$ by $\sigma_0(\omega) = Z \wedge \omega$.

Lemma. i) $\sigma_0(Y \wedge \omega) = -Y \wedge \sigma_0(\omega)$,

ii) $Y(\sigma_0(\omega)) = \sigma_0(Y(\omega))$ for any $Y \in g_x$, $\omega \in C_j$.

iii) $\sigma_0(C_j') \subset C_j'$.

Proof. i) follows from the definition. Z is Ad_{G_x}-invariant and hence $Y(Z) = 0$. Then $Y(\sigma_0(\omega)) = Y(Z \wedge \omega) = Y(Z) \wedge \omega + Z \wedge Y(\omega) = \sigma_0(Y(\omega))$. This proves ii). The last part follows from i), ii) and the G_x-invariance of σ_0.

We define $\sigma : \mathcal{X}^{(j)} \to \mathcal{X}^{(j+1)}$ as the quotient of σ_0 according to iii) of the lemma.

3.9. We now come to the definition of η. Let $\eta' : R \to GL(L(W, G_x))$.

$$(\eta_t' \varphi)(x \exp(X), g_0, \ldots, g_n) = (\beta_{\exp(tX)} \varphi)(x \exp(X), g_0, \ldots, g_n)$$

for any $\varphi \in L_n(W, G_x)$, $t \in R$. Also let R act on C_j by

$$t \cdot (X_1 \wedge \ldots \wedge X_k \otimes \lambda \otimes \xi) = X_1 \wedge \ldots \wedge X_k \otimes \lambda \otimes \eta_t'(\xi)$$

for $X_1, \ldots, X_k \in g$, $\lambda \in C_\Delta$ and $\varphi \in L(W, G_x)$. Let ∇_0 be the derivative of this action at 0.

Lemma. $\nabla_0 = \delta_0\sigma_0 + \sigma_0\delta_0$.

Proof. Let $\omega = \lambda\otimes\varphi \in C_0 = C_{\Delta}\otimes L(W, G_x)$. Choose a basis $Y_1,\dots,Y_m$ of g_x and let $y^1,\dots, y^m$ be the dual basis. Then $Z = \sum_{j=1}^m f^j\, Y_j$ if

$f^j(x\exp(X)) = y^j(X)$.

Let $[Y_i, Y_j] = \sum_{k=1}^m C_{ij}^k\, Y_k$, then $Y_i(f^j) = -\sum_{k=1}^m C_{ik}^j\, f^k$. Using the relation $d\Delta = -\operatorname{tr}\circ\operatorname{ad}$ we obtain for ω as above $\delta_0(\sigma_0(\omega)) =$

$= \delta_0(\sum_{i=1}^m Y_i\otimes\lambda\otimes f^i) = -\sum_{i=1}^m \operatorname{tr}(\operatorname{ad}_{Y_i})\lambda\otimes f^i\varphi + \sum_{i=1}^m \lambda\otimes Y_i(f^i)\varphi +$

$+ \sum_{i=1}^m \lambda\otimes f^i\, Y_i(\varphi)$.

Now $\sum_{i=1}^m \operatorname{tr}(\operatorname{ad}_{Y_i})\lambda\otimes f^i\varphi = \lambda\otimes\operatorname{tr}(\operatorname{ad}_Z)\varphi$ and $\sum_{i=1}^m Y_i(f^i) =$

$= -\sum_{i,j=1}^m C_{ij}^i\, f^j = \operatorname{tr}(\operatorname{ad}_Z)$. Since $\nabla_0\varphi = \sum_{i=1}^m f^i\, Y_i(\varphi)$ we obtain the statement of the lemma on C_0 .

Next we proceed by induction. Using lemma 3.8. we get :

$(\delta_0\sigma_0 + \sigma_0\delta_0)(X\wedge\omega) = -\delta_0(X\wedge\sigma_0(\omega)) + \sigma_0(X(\omega) - X\wedge\delta_0(\omega)) =$

$= -X(\sigma_0(\omega)) + X\wedge\delta_0\sigma_0(\omega) + \sigma_0(X(\omega)) + X\wedge\sigma_0(\delta_0(\omega)) = X\wedge\nabla_0(\omega) = \nabla_0(X\wedge\omega)$.

From this lemma or directly from the definition we obtain that C_j' is invariant for this action of R and hence, passing to quotients we obtain the desired action of R on $\mathcal{X}^{(j)}$, $\eta : R \to GL(\mathcal{X}^{(j)})$. It satisfies $\eta_1 = T_{n+1}^{n+1}$ since x acts trivially on F and $\nabla = \sigma\delta + \delta\sigma$ if ∇ is the derivative of η_t at 0.

3.10. Let $\mathcal{Y} = (C_{\Delta}\otimes L(W, G_x))\otimes_{G_x} C_{\cdot} = (A\ G_x)_W$. Also let $\mathcal{Z} = L(W\cap K_x, G_x\otimes_{K_x} C$, if $x\in K_x$, $Z = L(W\cap M, G_x)\otimes_M C$ if $x\notin K_x$ and M is the inverse image of L in G_x.

3.11. We have defined all objects needed for the use of the Main Lemma. We are now concentrating to prove that the conditions of the Main Lemma are satisfied. Let $q = \dim(G/L)$.

If G_x is a finite component Lie group the exactness of (E_1) was treated in section 2 (2.1., 2.2. and mainly 2.6.). For $x \notin K_x$ we also use the Serre-Hochschild spectral sequence for the normal subgroup $(x) \subset G_x$ [9].

Let $Y_1, \dots, Y_m$ be a basis of g such that $Y_{q+1}, \dots, Y_m$ is a basis of t = Lie L. Also let $y^1, \dots, y^m$ be the dual basis and $f^j(x\,\exp(X)) = y^j(X)$. Then $(\Lambda^j(g_x/t) \otimes F, \sigma_0)$ is the Koszul complex associated with the regular sequence $(f^1, \dots, f^q)$ and the $C^\infty(W)$-module F. The ideal I generated by $(f^1, \dots, f^q)$ in C (W) is the ideal of $K_x \cap W$ if $x \in K_x$, respectively the ideal of $M \cap W$ if $x \notin K_x$.

$$\mathrm{Tor}_j^{C^\infty(W)}(I, F) = 0 \text{ for } j > 0 \text{ and } \mathrm{Tor}_0^{C^\infty(W)}(I, F) = I \otimes_{C^\infty(W)} F =$$

$= L(W \cap K_x, G_x)$ if $x \in K_x$, respectively $L(W \cap M, G_x)/(1-x)L(W \cap M, G_x)$ if $x \notin K_x$. (Note that we have omitted $C_\Delta' \otimes$ since it does not affect the underlying locally convex space, it only changes the action of G_x. Since in the above formulae such an action does not exist or is not important, our procedure is justified.) The exactness of (E_2) follows from the exactness of $\otimes_L C$ (L is compact).

3.12 We are going now to study conditions for the injectivity of $1-T_{n+1}^{n+1}$; it may fail to be injective and this happens for example if G_x is compact or $G = SL_2(R)$ and $x = e$. However it is injective for $G \times R$ and any x.

<u>Lemma.</u> i) Suppose $x \in K_x$. If $(1-T_{n+1}^{n+1})\varphi = 0$ for a certain $\varphi \in L_n(W, G_x)$ then $\varphi(\gamma, g_0, \dots, g_n) = 0$ for any topologically torsion free γ .

ii) If $G \simeq H \times R$ then $1 - T^{n+1}_{n+1}$ is injective for any x.

Proof. The assumption of i) shows that $\varphi(\gamma, g_0, \dots, g_n) =$

$= (\alpha_\gamma^{\otimes n+1})^{-m}(\varphi(\overbrace{\gamma^m}^{\gamma,} g_0, \dots, \gamma^m g_n))$ for any $m \in Z$. Since φ has compact support this may happen for topologically torsion free γ only if $\varphi(\gamma, g_0, \dots, g_n) = 0$. This proves i). ii) is proved similarily.

3.13. We are ready to draw some conclusions from the above discussion and the Main Lemma.

Let G be a finite component Lie group, $K \subset G$ a maximal compact subgroup. Also let $x \in G$ and $U \subset g_x$ satisfying $(A_1) - (A_3)$. V, W, K_x, L and M will have the same meaning as before. Let $q = \dim G/K$, $q' = \dim G_x/K_x$, K being chosen such that $K_x \subset K$.

Proposition. i) $PHC_*(A \rtimes R) \simeq PHC_{*+1}(A)$.

ii) $PHC_*(A \rtimes R)_0 \simeq PHC_*(A \rtimes R)$,

$PHC(A \rtimes R)_t = 0$ for any $t \in R$, $t \neq 0$.

iii) If $x \in K_x$, G_x is a finite component Lie group and $U' = c(K \times_{K_x} (t \cap U))$, then $PHC_*((A \rtimes G)_V) \simeq PHC_{*+q}((A \rtimes K)_{U'})$.

iv) If $x \notin K_x$, $W \cap K_x = \emptyset$ and $G_x \simeq H \times Z^n$ where H is a finite component Lie group then

$$PHC_*((A \rtimes G)_V) = 0$$

(Here K_x is a maximal compact subgroup of H.)

Proof. $$PHC_*((A \rtimes G)_V) \simeq PHC_*((A \rtimes G_x)_W) \tag{1}$$

by proposition 3.6. . Suppose that $1 - T^{n+1}_{n+1}$ is injective on each $\mathcal{X}^{(j)}$.

Then the Main Lemma shows that

$$PHC_*((A \rtimes G_x)_W) \simeq PHC_{*+q'}(Z) \tag{2}$$

If $x \in K_x$ lemma 2.3. shows that

$$PHC_*(Z) \simeq PHC_*((A \rtimes K_x)_{W \cap K_x}) \tag{3}$$

Using once again proposition 3.6. we obtain

$$PHC_*((A \rtimes K_x)_{W \cap K_x}) \simeq PHC_*((A \rtimes K)_{U'})$$

For $G = R$ and $x = e$ we may take $U = R$ and we obtain i) and ii). The proof of i) for $(A \rtimes G) \rtimes R$ shows that $PHC_*(A \rtimes (G \times R)_{U \times R}) \simeq PHC_{*+1}((A \rtimes G)_U)$. This shows that we may replace G by $G \times R$. Since for $G \times R$ $1 - T_{n+1}^{n+1}$ is injective the above discussion proves iii) since q and q' have the same parity.

iv) It is enough to prove that $PHC_*((A \rtimes G_x)_W) = 0$. Suppose first that $W \cap H = \emptyset$. Then the statement follows from the vanishing of the periodic cyclic homology of the inhomogeneous components of crossed products by Z^n [18]. If $W \subset H$ then we show that $PHC_*((A \rtimes G_x)_{WZ^n}) = 0$.

Let us observe that $PHC_*((A \rtimes G_x)_{WZ^n}) \simeq PHC_*(((A \rtimes Z^n) \rtimes H)_W)$. Denote by L a maximal compact subgroup of $H/(x)$ and by M its preimage in H. L has the same homotopy type as $H/(x)$ and hence has a finite number of connected components. Let $\mathfrak{z}$ be the center of $\mathfrak{t}$ = Lie M. There exists a suitable power of x such that $x^n = \exp(X)$ for some $X \in \mathfrak{z}$. Then $R \ni s \to \exp(sX) \in M$ identifies R with a closed central subgroup of M such that M/R is compact. Then $M \simeq M/R \ltimes R$ and this isomorphism is unique if we require $M/R \to M/R \times R \simeq M \to M/R$ to be identity (use $H^1(M/R, R) = H^2(M/R, R) = 0$). We identify M/R to the corresponding subgroup of M. We still have (1) and (2) for $q' = \dim H/M$.

However (3) has to be replaced with

$$PHC_*(Z) \simeq PHC((A \rtimes M)_{M \cap W})$$

Since $M \cap W \subset M/R \times R$ we obtain iv) from ii).

Corollary [7] Let G be a connected nilpotent Lie group, $K \subset G$ a maximal compact subgroup then $PHC_*(A \rtimes G) \simeq PHC_{*+q}(A \rtimes K)$.

Proof. K is a central subgroup . Then use i) and induction.

3.14. We are going now to prove the theorem in the introduction.

Proof of theorem

Let $m \subset C^{\infty}_{inv}(G)$ be a maximal ideal. If m is not closed then $L(G, G)_m = 0$ by lemma 3.1. . Suppose now $m = m_y$ for some $y \in G$. Let $y = xx_u$ be the Jordan decomposition of y :x is semisimple, x_u unipotent and $xx_u = x_u x$ [2]. Chose $\rho : G \to GL(V)$ an injective morphism of algebraic groups [2]. Then $\rho(x)$ and Ad_x are semisimple and hence x and $U = U_\varepsilon(\rho)$ for small $\varepsilon > 0$ satisfy (A_1)-(A_3) (see the Appendix). Here $U_\varepsilon(\rho) = \{ X \in g_x , (d\rho(X)) \subset B(0,\varepsilon)\}$. Moreover the function φ appearing in (A_3) satisfies $\varphi(y) = 1$ (5.3).

Let us observe that G_x, being an algebraic group, has a finite number of components.

If $x \in K_x$ the statement of the theorem follows localising 3.13. iii) at m_y.

If $x \notin K_x$ then $\sigma(\rho(x))$ is not contained in $\mathbb{T}$ and the same will be true of $c(\gamma, X)$ for any $\gamma \in G$, $X \in U_\varepsilon(\rho)$ for small ε . Then since $PHC_*(A \rtimes G)_y \simeq PHC_*((A \rtimes G)_V)_y = 0$ and $PHC_*(A \rtimes K)_y = 0$ the proof is completed.

3.15. We are going now to give some more examples. Their purpose is it to show how to use the techniques we have developed also for some nonalgebraic groups.

i) $G = \widetilde{SL_2(R)}$ - the simply connected covering group of $SL_2(R)$. Let x be a generator of the center, $U = \{ X \in sl_2(R), \det(X) < \pi^2\}$.

The proposition 3.13. iii) shows that $PHC_*((A \rtimes G)_V) \simeq PHC_{*+3}(A)$ if $V = \exp(U)$, and proposition 3.13. iv) show that $PHC((A \rtimes G)_V) = 0$ if $V = x^n \exp(U)$ and $n \neq 0$. Since every orbit is contained in a set $x^n \exp(U)$ we obtain

$$PHC_*(A \rtimes G) \simeq PHC_{*+3}(A).$$

ii) Let $G = \widetilde{SL_2(R)} \times R/N$ where $N = \{(x^n, n), n \in Z\}$, x being as above. Let $X \in sl_2(R)$ be such that $\exp(X) = x$ then $K = \{\exp(s(X, -1)), s \in R/Z\}$ is a maximal compact subgroup of G. Let U be as above and let $V = \exp(U \times (-1/2, 1/2))$, $V' = \exp(U \times (0, 1))$. Using the same reasoning as before we obtain that $PHC_*((A \rtimes G)_{x^nV}) = 0$ (respectively $PHC_*((A \rtimes G)_{x^nV'}) = 0$ for $n \neq 0$ and $PHC_*((A \rtimes G)_V) \simeq PHC_{*+3}((A \rtimes K)_I)$ where $I = \{\exp s(X,-1), s \in (-\frac{1}{2}, \frac{1}{2})\}$ and $PHC_*((A \rtimes G)_{V'}) \simeq PHC_{*+3}((A \rtimes K)_{I'})$ where $I' = \{\exp s(X,-1), s \in (-1,0)\}$.

This shows the local isomorphism of $PHC_*(A \rtimes G)$ and $PHC_{*+3}(A \rtimes K)$.

4. More on the case G compact

In this section G will denote a connected compact Lie, group, T a maximal torus of G and W the Weyl group of the pair (G, T) [1].

Let $L(G) = L(G, \{e\})^G$.

4.1. Restriction defines a map $L(G, \{e\}) \to L(T, \{e\})$ and hence also a map $r : L(G) \to L(T)$. Since $HC^*(L(G)) \simeq HC^*(A \rtimes G)$ (by lemma 2.3.) we obtain a morphism

$$r^*: PHC^*(A \rtimes T) \to PHC^G(A \rtimes G)$$

Our aim is to prove the following .

<u>Theorem.</u> r^* gives an isomorphism

$$PHC^*(A \rtimes G) \simeq PHC^*(A \rtimes T)^W$$

The proof will be splitt in steps.

4.2. <u>Lemma.</u> Suppose $\rho : G_1 \to G$ is a finite covering of connected groups and the theorem is true for G_1, then it is true also for G.

<u>Proof.</u> Let $H = \ker \rho$, $T_1 = \rho^{-1}(T)$, then $H \subset T_1$ and T_1 is a maximal torus of G_1. Moreover the Weyl group of (G_1, T_1) naturaly identifies with W. Recall (theorem 2.7.) that there exists a morphism $I : L(G_1, \{e\}) \to L(G, \{e\})$ defined by "integration along the fibers of $G_1 \to G$":

$$I\varphi(\dot{\gamma}) = \sum_{h \in H} \varphi(\gamma h)$$

which defines an isomorphism $PHC^*(A \rtimes G_1)^H \simeq PHC^*(A \rtimes G)$. The diagram

$$\begin{array}{ccc} L(G_1) & \xrightarrow{r} & L(T_1) \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\ L(G) & \xrightarrow{r} & L(T) \end{array}$$

is commutative. Since the actions of W and H on $L(T_1)$ commute we obtain

$$PHC^*(A \rtimes G) \simeq PHC^*(A \rtimes G_1)^H \simeq (PHC^*(A \rtimes T_1)^W)^H \simeq (PHC^*(A \rtimes T_1)^H)^W \simeq PHC^*(A \rtimes T)^W$$

This lemma shows that we may suppose that $G \simeq H \times Z$ with H simply-connected simisimple and Z abelian. Since $A \rtimes G \simeq (A \rtimes Z) \rtimes H$ we may suppose that G itself is simply-connected and semisimple.

4.3. We define now a filtration invariant for r on L(G) and L(T).

Let $X_j \subset T$, $X_j = \{x \in T, \dim Z_G(x) \geq j\}$ (Here $Z_G(x)$ is the centralizer of x in G).

Let $L(G)^{(j)} = \{\varphi \in L(G)$, all derivatives of φ vanish on $X_j\}$ and define $L(T)^{(j)}$ similarily. It is immediate from the definition that $r(L(G)^{(j)}) \subset L(T)^{(j)W}$.

4.3. <u>Lemma</u>. $L(G)^{(j+1)} / L(G)^{(j)} \longrightarrow (L(T)^{(j+1)} / L(T)^{(j)})^W$ defines an isomorphism for the PHC^*-groups.

<u>Proof</u>. Let T' be a torus contained in X_j but not in X_{j+1}. Consider the cyclic modules $A = A' / A''$, $B = B' / B''$ where

$A' = \{\varphi \in L(T)$, all derivatives of φ vanish on $T' \cap X_{j+1}\}$,
$A'' = \{\varphi \in L(T)$, all derivatives of φ vanish on $T'\}$
$B' = r^{-1}A'$ and $B'' = r^{-1}A''$.

Then A and B are direct sumands of the modules in the statement of the lemma. It is enough to show that

$$PHC^*(A)^W \longrightarrow PHC^*(B)$$

is an isomorphism. Let $A^{(n)} = \{\varphi \in A$, the derivatives of φ of order $< n$ vanish on $T'\}$, $B^{(n)} = r^{-1}(A^{(n)})$.

Let W_1 be the Weyl group of $H = Z_G(T')$. W_1 is the stabilizer of T' in W. It coincides with the stabilizer in W of any $x \in T' \setminus X_{j+1}$ [1]. Let $h = \mathrm{Lie}\, H$ and $h' = [hh]$. h' is the fiber of the normal bundle of $\mathrm{Ad}_G(T' \setminus X_{j+1})$ at any point of $T' \setminus X_j$. (with respect to the metric defined by the Killing form). Then

$$A^{(n)} / A^{(n+1)} \simeq C_0^{\infty}(T' \setminus X_{j+1})^{W_2} \hat{\otimes} \mathrm{Hom}(t'^{\otimes n+1}, A^{\natural})$$

$$B^{(n)} / B^{(n+1)} \simeq C_0^{\infty}(T' \setminus X_{j+1})^{W_2} \hat{\otimes} \mathrm{Hom}_{H'}(h'^{\otimes n+1}, A^{\natural})$$

Here $C_0^{\infty}(T' \setminus X_{j+1})$ is the space of smooth functions on T' all of whose derivatives on X_{j+1} vanish. W_2 is the normalizer of T' in W. H' is the commutant of H, of course $h' = \mathrm{Lie}\, H'$. $t' = t \cap h'$ is the Lie algebra of a maximal torus of H', it is <u>not</u> equal to $\mathrm{Lie}\, T'$!

The rest of the proof is contained in the following three lemmata.

4.5. <u>Lemma.</u> Let K be a connected compact Lie group acting on A. Suppose that this action commutes with the action of G then

$$PHC^{*}(L(G)) \simeq PHC^{*}(L(G)^{K_1})$$

is an isomorphism for any closed subgroup K_1 of K.

<u>Proof.</u> The statement follows from the homotopy invariance of the periodic cyclic cohomology [4].

4.6. <u>Lemma.</u> $PHC^{*}(A^{(n)} / A^{(n+1)})^{W} \to PHC^{*}(B^{(n)} / B^{(n+1)})$ is an isomorphism.

<u>Proof.</u> The above lemma shows that

$$C_0^{\infty}(T' \setminus X_{j+1})^{W_2} \otimes (t'^{\otimes n+1})^{*W_1} \hat{\otimes} A^{\natural W_1} \to (A^{(n)} / A^{(n+1)})^{W_1}$$

and

$$C_0(T' \setminus X_{j+1})^{W_2} \otimes (h'^{\otimes n+1})^{*H'} \hat{\otimes} A^{\natural H'} \to (B^{(n)} / B^{(n+1)})^{H'}$$

induce isomorphisms for the PHC^*-groups. Since $C[h']^{H'} \simeq C[t']^{W_1}$ [12] the lemma is proved.

4.7. Lemma. $A^* \simeq \varinjlim (A / A^{(n)})^*$ and $B^* \simeq \varinjlim (B / B^{(n)})^*$.

Proof. Any distribution on a compact manifold has finite order.

5. Appendix

In this section we give some more manageable conditions on $x \in G$ and U ensuring $(A_1) - (A_3)$ to hold true.

Fix $x \in G$.

5.1. If $\rho : G \to GL(V)$ is a linear representation let $U_\varepsilon(\rho) = \{X \in g_x, \sigma(d\rho(x)) \subset B(0,\varepsilon)\}$. Here $B(0,\varepsilon) = \{|z| < \varepsilon\}$ and σ referes to the spectrum in $V \otimes C$.

5.2 Suppose Ad_x is semisimple and $U \subset U_\varepsilon(Ad)$ is Ad_{G_x}-invariant. Then for small $\varepsilon > 0$ $c : G \times_{G_x} U \to G$ is a local diffeomorphism.

Proof. We first identify $T_\gamma G$ with $g = T_e G$ by means of left translations and $T_X U$ with g_x for $X \in U$.

If $(\gamma, X) \in G \times_{G_x} U$ then $T_{(\gamma, X)}(G \times_{G_x} U) = g \; g_x / \{(Y, [XY]), Y \in g_x\}$.

We shall show that $dc_{(\gamma, X)}$ is injective, since $\dim T_{(\gamma,X)}(G \times_{G_x} U) = \dim T_{c(\gamma,X)} G$ we shall obtain that c is a local diffeomorphism.

Now c is G-equivariant so we may suppose that $\gamma = e$. We compute

$$dc_{(e,X)}(Y, X_0) = Ad^{-1}_{x \exp(X)}(Y) - Y + f(ad_X)(X_0)$$

where $f(t) = (1 - e^{-t})\, t^{-1}$, $f(ad_X)$ is defined by analytic functional calculus and is the differential of exp at X [11].

If $dc_{(e,X)}(Y, X_0) = 0$ then, for small ε , it follows that $Y \in g_x$ and

$$f(ad_X)(X_0) = (1 - Ad^{-1}_{x\ \exp(X)})(Y) = f(ad_X)(ad_X(Y))$$

and hence $X_0 = [XY]$ again for ε small enough.

5.3. We are looking now for conditions ensuring the other conditions on x.

<u>Lemma.</u> Suppose there exists $\rho : G \to GL(V)$ a complex representation with $d\rho$ injective, $\rho(x)$ semisimple and $U \subseteq U_\varepsilon(\rho)$. If U is Ad_{G_x}-invariant and $\varepsilon > 0$ is small enough then c is injective and (A_2) is satisfied. If moreover $U = U_\varepsilon(g)$ then also (A_3) is satisfied.

<u>Proof.</u> Let $\Omega \subset C$ be a disjoint union of open balls each centered at a point of $\sigma(\rho(x))$. We choose $\varepsilon > 0$, $\varepsilon < \pi$ so small that $\sigma(\rho(x \exp(X))) \subset \Omega$ for any $X \in U_\varepsilon(\rho)$. We split the proof into several steps.

<u>Step 1.</u> <u>c is injective.</u> Let $c(\gamma, X) = c(e, X_0)$, then

(1) $\rho(\gamma x \gamma^{-1}) \exp(\rho(\gamma) d\rho(X) \rho(\gamma^{-1})) = \rho(x) \exp(d\rho(X_0))$.

Let $\chi_0 : \Omega \to C$ be the unique locally constant function such that $\chi_0 | \sigma(\rho(x)) = \mathrm{id}_{\sigma(\rho(x))}$. χ_0 being an analytic function $\chi_0(\rho(x)\exp(d\rho(X_0)))$ may be defined by analytic functional calculus and is equal to $\rho(x)$. Hence $\rho(\gamma x \gamma^{-1}) = \rho(x)$ and $\exp(\rho(\gamma)d\rho(X)\rho(\gamma^{-1})) = \exp(d\rho(X_0))$. We obtain that $d\rho(Ad_\gamma(X)) = d\rho(X_0)$ and hence $Ad_\gamma(X) = X_0$ since $d\rho$ is injective. This shows that $\gamma x \gamma^{-1} = x$ and hence $(\gamma, X) = (e, X_0)$ in $G \times_{G_x} U$.

<u>Step 2.</u> <u>$\mathcal{O}_{G_0}(x_0) = \{\gamma x_0 \gamma^{-1} , \gamma \in G_0\}$ is closed in G_0.</u> Here we have denoted $G_0 = \rho(G)$, $x_0 = \rho(x)$.

$\mathcal{O}_{G_0}(x_0)$ is a submanifold of $GL(V)$. (We do not assume it to be closed).

$\mathcal{O}_{GL(V)}(x_0)$ is closed since it is equal to $\{\gamma \in GL(V)\ ,\ P(\gamma) = 0\}$ if P is the minimal polinomial of x_0. Let $y \in \mathcal{O}_{GL(V)}(x_0) \cap G_0$. Then

$T_y \mathcal{O}_{GL(V)}(x_0) = (Ad_y^{-1} - 1)(gl(V))$, $T_y \mathcal{O}_{GL(V)}(x_0) \cap T_y G_0 = (Ad_y^{-1} - 1)(g) =$

$= T_y \mathcal{O}_{G_0}(y)$ since we may choose an Ad_y-invariant complement of g in $gl(V)$.

This shows that $\mathcal{O}_{G_0}(y)$ is open in $\mathcal{O}_{GL(V)}(x_0) \cap G_0$. The latter being a union of such orbits it follows that $\mathcal{O}_{G_0}(x_0)$ is closed in $\mathcal{O}_{GL(V)}(x_0) \cap G_0$ and hence also in G_0.

Step 3. $\mathcal{O}_{\underline{G}}(x)$ is closed. ρ restricts to an obvious local diffeomorphism $\mathcal{O}_G(x) \to \mathcal{O}_{G_0}(x_0)$. Since $\ker \rho$ is discrete $\mathcal{O}_G(x)$ is closed iff $\mathcal{O}_{G_0}(x_0)$ is closed.

Step 4. Suppose $L \subset U$ is a closed $Ad_{\underline{G}_x}$-invariant subset of $\underline{g}_x$. Then $c(G \times_{\underline{G}_x} L)$ is closed in G. We assume first that ρ is injective and identify G with G_0. Let $c(\gamma_n, X_n) \to y \in G$. Then $\mathcal{X}_0(c(\gamma_n, X_n)) = \gamma_n x \gamma_n^{-1} \to \mathcal{X}_0(y) \in$

$\in \mathcal{O}_G(x)$. Since $\mathcal{O}_G(x)$ is closed we may write $\gamma_n = \gamma'_n \gamma''_n$ with $\gamma''_n \in G_x$, γ'_n convergent to say γ. Then $\exp(\gamma''_n X_n \gamma''^{-1}_n)$ converges to $x^{-1}\gamma^{-1} y \gamma$ and hence also $\gamma''_n X_n \gamma''^{-1}_n$ is convergent in g_x. Let $X = \lim \gamma''_n X_n \gamma''^{-1}_n$, then $X \in L$. This shows that $y = \lim c(\gamma_n, X_n) =$

$= \lim c(\gamma'_n, \gamma''_n X_n \gamma''^{-1}_n) = c(\gamma, X) \in c(G \times_{G_x} L)$.

For $\ker(\rho)$ discrete $\neq \{e\}$ observe that $c(G \times_{G_x} L)$ is closed in $\rho^{-1}(\rho(c(G \times_{G_x} L)))$.

Step 5. (A_2) is satisfied.

Observe that $\bar{\Phi}$ is well defined and injective if (A_1) holds true(see 3.2). If $\psi \in R_1$ let $\varphi(\gamma x \exp(X) \gamma^{-1}) = \psi(x \exp(X))$. φ is smooth on V, step 4 implies that φ is smooth also on G\V. Since $\bar{\Phi}(\varphi) = \psi$, $\bar{\Phi}$ is also onto.

Step 6. If $U = U_\varepsilon(\rho)$ then also (A_3) is satisfied.

Let $\varphi_0, \varphi_0' \in C_c^\infty(C^1)$ where $1 = \dim V$ be such that $\varphi_0(0) = 1$, $\varphi_0' \varphi_0 = \varphi_0$. Let $\varphi(x \exp(X)) = \varphi_0(\operatorname{tr}\rho(X), \operatorname{tr}\Lambda^2\rho(X), \ldots, \operatorname{tr}\Lambda^1\rho(X))$ and 0 else define φ' similarily. Then φ and φ' have the required properties provided φ_0' has small support.

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