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by

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THE INVERSE PROBLEM FOR THE PHASE SHIFT AND RELATED MOMENT PROBLEMS

Mihai Putinar

Introduction

The paper deals with a Hilbert space interpretation of some classical results of Ahiezer and Krein [2] concerning the L-problem of moments on the real axis.This geometric viewpoint allows an easy generalization from scalar to operator valued measures,contained in the second part of the paper.

The interest for the moments of a bounded measurable function on the axis goes back to A.A.Markov,at the end of the last century.It was M.G. Krein and his school who took again this problem in the early thirties, with the new methods at that time of modern function theory and functional analysis,see [2] and [8].Since then up to nowadays their ideas have influenced and continue to influence many branches of pure and applied mathematics.

The original approach of Ahiezer and Krein to L-problems of moments was based on the elementary theory of orthogonal polynomials and on two integral representation formulas (additive and multiplicative) for a class of analytic functions in the unit disk.Later,more or less the same tools have been used by M.G.Krein [9] in the perturbation theory of self-adjoint operators.

On the other hand,a well founded tradition at present consists in treating moment problems by operator and Hilbert space methods,cf. [11], [14], [15], [16].So,it is not surprinsing at all to explain the former results concerning the L-problems of moments by the theory of perturbation of self-adjoint operators.Although this interpretation is not finally very far from the original approach,it has some certain advantages.Quite specifically,it offers an easy and natural way of generalizing the classical results to operator valued measures or to the multidimensional setting. The present paper contains some indication on the first possible generalization.The second way of multidimensional moment problems was initiated in [13].

Except the usual notions and results of perturbation theory,briefly recalled in Section 1 below, the quite involved concept of operator phase shift due to R.W.Carey [5] will be a main technical tool in our study.

I would like to thank my colleague Tiberiu Constantinescu for some fruitful discussions on moment problems and related topics.

1. Preliminaries

A function which contains all relevant information on the modification of self-adjoint spectra by trace-class perturbations was introduced by M.G.Krein in [9]. This object, called the phase shift, answered some concrete desires of theoretical phisicists. Later, an enormous number of studies were devoted to ramifications of its theory, cf. [10], [4], [6].

To briefly recall the definition and the properties of the phase shift we shall confine ourselves to treat, though not always necessary, only the case of bounded operators acting on a separable complex Hilbert space H.

Let A and B be bounded self-adjoint operators on H, with the property that $\mathrm{Tr}|A-B| < \infty$. In that case a well-known theorem of Kato and Rosenblum asserts that their absolutely continuous spectra coincide: $\sigma_{ac}(A) = \sigma_{ac}(B)$, see for instance [14]. However, the singular spectra may differ. In order to control the spectral displacement from A to B is very useful after M.G. Krein to study the next <u>perturbation determinant</u>:

$$\Delta_{B/A}(z) = \det[(B-z)(A-z)^{-1}] \quad , \qquad z \in \mathbb{C} \setminus \sigma(A).$$

The infinite determinant exists because

$$(B-z)(A-z)^{-1} = (A-z+B-A)(A-z)^{-1} = I+(B-A)(A-z)^{-1} \in I+C_1,$$

where C_1 stands for the ideal of trace-class operators acting on H. One of the main results of the landmarking paper [9] consists in representing multiplicatively the analytic function $\Delta_{B/A}$ as follows:

$$\Delta_{B/A}(z) = \exp\left(\int_{\mathbb{R}} \varphi(t)(t-z)^{-1}dt\right) , \qquad z \in \mathbb{C} \setminus \mathbb{R} ,$$

where $\varphi \in L^1_{comp}(\mathbb{R},dt)$.

The function φ is called the phase shift of the perturbation $A \longrightarrow B$ and it has a series of remarkable invariance properties reflecting the spectral behaviour of the pair of self-adjoint operators (A,B), see [9], [10] and [4].

Next we consider only the simplest non-trivial perturbation $B=A+\xi\otimes\xi$, denoting by $\xi\otimes\xi$ the rank one operator: $(\xi\otimes\xi)(\eta)=\langle\eta,\xi\rangle\xi$, $\eta\in H$. In that case the determinant $\Delta_{B/A}$ can be easily computed:

$$\Delta_{B/A}(z) = 1+\langle(A-z)^{-1}\xi,\xi\rangle \quad , \quad z\in\mathbb{C}\setminus\sigma(A).$$

This explicit form yields a second integral representation formula for $\Delta_{B/A}$:

$$\Delta_{B/A}(z) = 1+\int_{\mathbb{R}}(t-z)^{-1}d\mu(t), \qquad z\in\mathbb{C}\setminus\sigma(A),$$

where μ denotes the spectral measure of A, localized at the vector $\xi\in H$.

Under the preceding assumptions one easily computes the corresponding phase shift as a boundary value:

$$(1) \qquad \varphi(t) = \lim_{\varepsilon\searrow 0} \pi^{-1}\arg\Delta_{B/A}(t+i\varepsilon), \qquad t\in\mathbb{R}.$$

But a direct computation shows that $\operatorname{Im}z.\operatorname{Im}\Delta_{B/A}(z)>0$ for $z\in\mathbb{C}\setminus\mathbb{R}$. Accordingly, one finds that the phase shift φ must satisfy the inequalities

$$0\leqslant\varphi\leqslant 1.$$

The key point of the above two integral representations consists in their natural parametrization with measures μ or functions φ. More precisely the parameters μ and φ are completely free, as it comes out from the next result, implicitly contained in the work of M.G.Krein, see [10].

Theorem 1. (M.G.Krein) *Let H be a separable complex Hilbert space. There exists a bijection between the following classes:*

$\{(A, \xi)$; $A=A^* \in L(H)$ with the cyclic vector $\xi \in H\}$/ unitary eq. $\cong$

$\{\mu ; \mu$ -Borel measure on $\mathbb{R}$, positive and with compact support$\}$ $\cong$

$\{\varphi ; \varphi \in L^1_{comp}(\mathbb{R}),\ 0 \leq \varphi \leq 1\}$.

The bijection is established by the formulae:

$$(2) \qquad 1+\langle (A-z)^{-1}\xi, \xi\rangle = 1+\int_{\mathbb{R}} (t-z)^{-1} d\mu(t) = \exp\Big(\int_{\mathbb{R}} \varphi(t)(t-z)^{-1}dt\Big),\ z \in \mathbb{C}\setminus\mathbb{R}.$$

The cyclicity hypothesis on A is natural because the perturbation determinant $\Delta_{B/A}$ does not distinguish between possible different orthocomplements of the A-cyclic subspace generated by ξ.

It can be shown by purely function theoretic arguments that the function φ above takes only the values 0 and 1 on an interval (a,b) if and only if the measure $\mu|(a,b)$ is singular with respect to the linear Lebesgue measure, see [10] or [12].

The phase shift of a general trace-class perturbation is far from being a complete unitary invariant of the perturbation data, as turns out to be in the particular situation contained in Theorem 1. However, there exists an operator valued substitute of the phase shift which fills this gap, but is less easier to be handled. Its existence and properties were established by R.W.Carey [5] by means of some sharp methods of operator theory. Summing up a part of Carey's results, one can state the next.

Theorem 2. (R.W.Carey) Let H be a separable infinite dimensional Hilbert space. There exists a bijection between the following classes:

$\{(A,K);\ K, A \in L(H), K \geq 0, A=A^*$ and $H=\bigvee_{n=0}^{\infty} A^n \mathrm{Ran} K\}$ / unitary eq. $\cong$

$\{\sigma ; \sigma$ -positive L(H)-valued measure compactly supported on $\mathbb{R}\}$ /unit.eq. $\cong$

$\{B ;\ B \in L^1_{comp}(\mathbb{R}, L(H)), 0 \leq B \leq I\}$ / unitary eq.

The equivalence is established by the relations :

$$(3) \qquad I+K(A-z)^{-1}K = I+\int_{\mathbb{R}} (t-z)^{-1} d\sigma(t) = \exp\Big(\int_{\mathbb{R}} B(t)(t-z)^{-1}dt\Big), \qquad z \in \mathbb{C}\setminus\mathbb{R}.$$

<u>Moreover, in that case</u> $\mathrm{supp}(\sigma)\subset\mathrm{supp}(B)$ <u>and these sets have the same lower bound.</u>

We have denoted as usual by $L(H)$ the algebra of bounded operators on H, by RanK the range of the oprtaor K and by $\bigvee E_i$ the closed linear span of the sets $E_i\subset H$.

Actually the above equivalences are obtained in [5] under the additional assumption : $\mathrm{Tr}(K^2)<\infty$. However, this restriction is not necessary in Theorem 2, as it is proved for instance in [12] Chap. IX.

With the notations in Theorem 2, the measure $\sigma(.)$ turns out to be the compression $KE(.)K$ of the spectral measure E of the self-adjoint operator A. In particular one gets:

$$(4)\qquad K=\sigma(\mathbb{R})^{1/2}=\left(\int_{\mathbb{R}} B(t)dt\right)^{1/2}.$$

It is worth remarking that, for a fixed closed subspace $H_0\subset H$, finite or infinite dimensional, $\mathrm{Ran}B(t)\subset H_0$ for all $t\in\mathbb{R}$ if and only if $\mathrm{Ran}\,\sigma(\mathbb{R})\subset H_0$. Thus the second isomorphism in Theorem 2 remains valid on finite dimensional spaces.

Most of the properties of the phase shift function are preserved in the latter operator valued context, cf. [5] and [12]. We shall use a part of these results without mentioning them separately in this section.

2. <u>Some operator L-problems of moments</u>

The resemblance between Theorems 1 and 2 above will be reflected below by a parallel between their applications to some moment problems. The common treatment of scalar and operator valued moment problems will not be technically more difficult than the classical approach of Ahiezer and Krein [2].

Besides the vector valued generalization, this section contains some novel proofs of the main existence and uniqueness results of Ahiezer and Krein. The only additional difficulty in the operatorial case is related to the possibility of translating the truncated Hamburger moment problem into positivity conditions. Fortunately this question is completely independent on the principal result below, and it was treated by Ando [3].

Fix a positive real number L and a separable complex Hilbert space H, finite or infinite dimensional. We are seeking solvability conditions of the L-<u>problem of moments</u>:

$$(5) \quad A_n = \int_{\mathbb{R}} t^n F(t)dt, \qquad n \geq 0,$$

where $F \in L^1(\mathbb{R};L(H))$ and $0 \leq F(t) \leq L$ a.e..

In other terms, given the sequence of self-adjoint operators $(A_n)_{n=0}^{\infty}$, one asks when there is a solution F of (5), possibly with the support contained in a prescribed bounded or unbounded interval.

A related question is to decide whether the function F with the given moments is unique or not, and to parametrize in the latter case all solutions.

In order to resolve these problems, two formal transformations of the moments sequence are needed. We shall work in the formal series ring $L(H)[[X]]$, with X as indeterminate. To a sequence of operators $(A_n)_{n=0}^{\infty}$ one associates the exponential transforms:

$$(6) \quad \sum_{n=0}^{\infty} B_n X^{n+1} = 1-\exp(-L^{-1} \sum_{n=0}^{\infty} A_n X^{n+1}),$$

and

$$(6)' \quad \sum_{n=0}^{\infty} B'_n X^{n+1} = -1+\exp(L^{-1} \sum_{n=0}^{\infty} A_n X^{n+1}).$$

Both right hand terms converge in the (X)-adic topology. We call (6) and (6)' the L-<u>exponential transform</u>, respectively the -L-<u>exponential transform</u> of the sequence (A_n).

It is worth remarking the dependence of the coefficients (B_n) of (A_n). Each B_n is a non-commutative polynomial in $A_0, A_1, \ldots, A_n$ and L^{-1}, and conversely, each A_n is a polynomial in $B_0, B_1, \ldots, B_n$ and L, for $n \geq 0$. Similarly depends (B'_n) on (A_n).

The exponential transforms linearize in certain sense the problem of moments (5). Quite specifically, the next result holds.

Theorem 3. The sequence of self-adjoint operators $(A_n)_{n=0}^{\infty}$ represents the moments of a function $F \in L^1(\mathbb{R};L(H))$, $0 \leq F \leq L$, if and only if the coefficients $(B_n)_{n=0}^{\infty}$ of the L-exponential transform of $(A_n)_{n=0}^{\infty}$ are the moments of a positive operator valued measure on $\mathbb{R}$.

A similar statement holds for the $-L$-exponential transform.

Proof. Assume for the moment that $(A_n)_{n=0}^{\infty}$ are the moments of a function $F \in L^1_{comp}(\mathbb{R};L(H))$, satisfying $0 \leq F \leq L$. According to Theorem 2, F/L is the phase operator of a measure σ with values in $L(H)_+$. In view of relations (3) and (6) one has, for $|z|$ sufficiently large:

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n z^{-n-1} &= 1-\exp(-L^{-1}\sum_{n=0}^{\infty} A_n z^{-n-1}) \\
&= 1-\exp(-\sum_{n=0}^{\infty} \int_{\mathbb{R}} t^n z^{-n-1} F(t) L^{-1} dt) \\
&= 1-\exp(\int_{\mathbb{R}} F(t) L^{-1} (t-z)^{-1} dt) \\
&= -\int_{\mathbb{R}} (t-z)^{-1} d\sigma(t) \\
&= \sum_{n=0}^{\infty} (\int_{\mathbb{R}} t^n d\sigma(t)) z^{-n-1}.
\end{aligned}$$

Notice that all of the above series converge absolutely for $|z| \gg 0$.

Conversely, the same sequence of equalities shows that, if $(B_n)_{n=0}^{\infty}$ are the moments of a $L(H)_+$-valued measure σ with compact support in $\mathbb{R}$, then there is a unique function $F \in L^1_{comp}(\mathbb{R};L(H))$, satisfying the conditions of the statement.

Let us remark that, for a fixed real number a, $\mathrm{supp}(F) \subset [a,\infty)$ if and only if $\mathrm{supp}(\sigma) \subset [a,\infty)$, see Theorem 2.

A similar argument yields the conclusion of Theorem 3 in the case of compact supports and for the $-L$-exponential transform (6)'. Specifically, Carey's equivalences can also be derived from relations like:

$$I-K'(A'-z)^{-1}K' = I-\int_{\mathbb{R}}(t-z)^{-1}d\,\sigma'(t) = \exp(-\int_{\mathbb{R}}B(t)(t-z)^{-1}dt),$$

see for instance [5]. However, the objects (A,K) and σ associated to B by relations (3) are in general different of (A',K') and σ' above.

This second parametrization has the property that, for a fixed $b\in\mathbb{R}$, $\mathrm{supp}(B)\subset(-\infty,b]$ if and only if $\mathrm{supp}(\sigma')\subset(-\infty,b]$.

More interesting is the case of unbounded supports. Assume that the function $F\in L^1(\mathbb{R};L(H))$ has the moments $(A_n)_{n=0}^{\infty}$ and satisfies $0\leqslant F\leqslant L$, a.e.. Let χ_r denote the characteristic function of the interval $[-r,r]\subset\mathbb{R}$, and put $F_r=\chi_r F$, with the corresponding moments $(A_n(r))_{n=0}^{\infty}$. In view of the preceding considerations there is a positive operator valued measure σ_r, compactly supported by $\mathbb{R}$ and with the property:

$$(7)\qquad I+\int_{\mathbb{R}}(t-z)^{-1}d\,\sigma_r(t) = \exp(L^{-1}\int_{\mathbb{R}}(t-z)^{-1}F_r(t)),\qquad z\in\mathbb{C}\setminus\mathbb{R}.$$

In particular $\|\sigma_r(\mathbb{R})\|\leqslant\|F\|_1L^{-1}$, by (4). Moreover, the universal polynomials which relate the moments of σ_r to those of F_r show recurrently that:

$$(8)\qquad \sup_{r>0}\left\|\int_{\mathbb{R}}t^n d\,\sigma_r(t)\right\|<\infty,$$

for any $n\geqslant 0$. Therefore $(\sigma_r)_{r>0}$ is a family of functionals on the separable locally convex space P of continuous functions on $\mathbb{R}$, with polynomial growth and with values in the trace class ideal $C_1\subset L(H)$. Since this family is uniformly bounded with respect to a fundamental system of seminorms of P, Alaoglu's Theorem provides the existence of a weak limit σ of $(\sigma_r)_{r>0}$. In view of (8) one gets:

$$\text{so-}\lim_{r\to\infty}\int_{\mathbb{R}}t^n d\,\sigma_r(t) = \int_{\mathbb{R}}t^n d\,\sigma(t).$$

This suffices to conclude that the coefficients $(B_n)_{n=0}^{\infty}$ are precisely the moments of the measure σ.

Conversely, let $\sigma: \mathcal{B}(\mathbb{R}) \longrightarrow L(H)_+$ be a measure whose moments $(B_n)_{n=0}^{\infty}$ exist. By performing a similar truncation device, one associates to every measure $\sigma_r = \chi_r \sigma$ a phase shift operator $F_r . L^{-1}$ with compact support.

By taking into account relation (7), one finds by the same recurrent dependence between (B_n) and (A_n) that the moments of F_r are uniformly bounded with respect to $r > 0$. Whence arguing as before there exists a weak limit ν of the family $(F_r)_{r>0}$, in the space of operator valued measures on $\mathbb{R}$. Since

$$\left|\left\langle \left(\int_a^b F_r(t)dt\right)\xi, \eta\right\rangle\right| \leq L(b-a)\|\xi\|\|\eta\|$$

for any vectors $\xi, \eta \in H$ and any interval $(a,b) \subset \mathbb{R}$, we infer that ν is an absolutely continuous measure with respect to the linear Lebesgue measure, and its weight $F = d\nu/dt$ satisfies $0 \leq F(t) \leq L$ a.e..

In conclusion the moments of F_r converge to those of F, and therefore they are necessarily equal to $(A_n)_{n=0}^{\infty}$.

This completes the proof of Theorem 3.

At this point we have transformed the L-problem of moments (5) into a Hamburger problem. This has certain advantages in translating the conclusion of Theorem 3 into positivity conditions, even with control on the support of the indeterminate function F. More precisely one has the next result.

Corollary 4. <u>Let</u> (a,b) <u>be an interval of the real axis. With the assumptions and notation of Theorem</u> 3, <u>the following assertions are equivalent:</u>

1) <u>The L-problem of moments with data</u> $(A_n)_{n=0}^{\infty}$ <u>is solvable;</u>
2) <u>The blok-operatorial matrix</u> $(B_{n+m})_{n,m=0}^{\infty}$ <u>is non-negative definite;</u>
3) <u>The blok-operatorial matrix</u> $(B'_{n+m})_{n,m=0}^{\infty}$ <u>is non-negative definite.</u>

<u>In that case the solution</u> $F \in L^1(\mathbb{R}, L(H))$ <u>satisfies</u> $\mathrm{supp}(F) \subset [a,b]$ <u>if and only if the matrices</u> $(B_{n+m+1} - aB_{n+m})_{n,m=0}^{\infty}$ <u>and</u> $(bB'_{n+m} - B'_{n+m+1})_{n,m=0}^{\infty}$ <u>are non-negative definite.</u>

<u>Proof</u>. Recall that the sequence of self-adjoint operators $(B_n)_{n=0}^{\infty}$ repre-

sents the moments of an operator valued measure σ on the axis, which is positive, if and only if the Hankel matrix $(B_{n+m})_{n,m=0}^{\infty}$ is non-negative definite. This is the operatorial counterpart of the classical theorem of Hamburger. The proof (in the scalar case) of this result contained in [14] Chap.X extends with minor modifications to the operator valued setting, see also [15] and [7].

In conclusion, Theorem 3 proves that assertions 1),2) and 3) are equivalent.

The statement concerning the support of the function F follows from the observation that $\mathrm{supp}(\sigma)\subset[a,\infty)$ if and only if $a(B_{n+m})_{n,m=0}^{\infty}\leq (B_{n+m+1})_{n,m=0}^{\infty}$. This in turn is equivalent to $\mathrm{supp}(F)\subset[a,\infty)$, as we have mentioned in the proof of Theorem 3. Correspondingly, the matrix $(B'_{n+m})_{n,m=0}^{\infty}$ controlls the upper bound of supp(F). Thus the proof of Corollary 4 is complete.

Notice that the interval (a,b) may by unbounded in Corollary 4.

Some proofs of Theorem 3 and Corollary 4 are given in [2] and [11], in the scalar case (dimH=1). These authors use some slightly different exponential transforms depending on a and b, in the case with prescribed support into [a,b].

When one imposes conditions on supp(F) like: $\mathrm{supp}(F)\subset\bigcup_{j=1}^{k}[a_j,b_j]$, the conditions found by M.G.Krein and Nudelman [11] seem to be optimal. More exactly, Corollary 4 can be completed as follows.

Corollary 5. <u>Keeping the notation of Theorem 3, let</u> $J=\bigcup_{j=1}^{k}[a_j,b_j]$ <u>be a finite disjoint union of intervals, and let</u> $F\in L^1([a_1,b_k];L(H)), 0\leq F\leq L$.

<u>Then</u> $\mathrm{supp}(F)\subset J$ <u>if and only if the Hankel matrix</u> $(C_{n+m})_{n,m=0}^{\infty}$ <u>is non-negative definite, where the operators</u> C_n <u>are defined by the formal identity:</u>

$$I-\sum_{n=0}^{\infty}C_nX^{n+1}=\frac{(a_2-X)\dots(a_k-X)}{(b_1-X)\dots(b_{k-1}-X)}\quad(I-\sum_{n=0}^{\infty}B_nX^{n+1}).$$

<u>Proof</u>. Let [a,b] be a compact interval of the real axis. The starting point is the obvious identity:

$$\exp\left(\int_a^b (t-z)^{-1}dt\right)=(b-z)(a-z)^{-1},\qquad z\in\mathbb{C}\setminus\mathbb{R}.$$

The L-exponential transform (B_n) of the moment sequence (A_n) of the given function F defines the analytic function:

$$\phi(z) = I - \sum_{n=0} B_n z^{-n-1} = \exp(L^{-1}\int_{\mathbb{R}}(t-z)^{-1}F(t)dt), \qquad \mathrm{Im}z > 0.$$

Assume that $\mathrm{supp}(F) \subset J$. Then the analytic function defined in the upper half plane by the expression:

$$\psi(z) = \frac{(a_2-z)\ldots(a_k-z)}{(b_1-z)\ldots(b_{k-1}-z)}\phi(z) = \exp(\int_{\mathbb{R}}(F(t)L^{-1} + \chi_{[b_1,a_2]}(t) + \ldots +$$

$$+\chi_{[b_{k-1},a_k]}(t))(t-z)^{-1}dt)$$

is of the form

$$\psi(z) = I + \int_{\mathbb{R}}(t-z)^{-1}d\tau(t), \qquad \mathrm{Im}z > 0,$$

with a positive operator valued measure τ.

Indeed, it suffices to remark that $0 \leq F.L^{-1} + \chi_{[b_1,a_2]} + \ldots + \chi_{[b_{k-1},a_k]} \leq I$ if and only if $\mathrm{supp}(F) \subset J$. Then it remains to apply Theorem 2.

The same observation finishes the proof of the converse implication, by using this time Corollary 4.

Notice that the cases $a_1 = -\infty$ or/and $b_k = \infty$ are not excluded in Corollary 5.

Henceforth we keep the notation of Theorem 3 but we impose restrictions on dimH. First, the case dimH=1 is analized. The next result is classical [2], but the proof below exploits the theory of the phase shift.

Proposition 6. <u>Let</u> $L > 0$ <u>and</u> $N \in \mathbb{N}$ <u>be fixed. Consider a sequence</u> $(a_n)_{n=0}^{2N}$ <u>of real numbers and its L-exponential transform</u> $(b_n)_{n=0}^{2N}$ <u>. Then:</u>

a) <u>There exists a function</u> $f \in L^1(\mathbb{R}), 0 \leq f \leq L$ <u>with the first</u> 2N+1 <u>moments</u>

$a_0, a_1, \ldots, a_{2N}$ *if and only if the matrix* $(b_{n+m})_{n,m=0}^{N}$ *is non-negative definite.*

b) *Problem* a) *has a solution* $f \in L^1(\mathbb{R})$ *satisfying* $0 \leq f \leq L$ *if and only if the matrix* $(b_{n+m})_{n,m=0}^{N}$ *is positive definite.*

c) *Assume that the matrix* $(b_{n+m})_{n,m=0}^{N}$ *is non-negative definite and* $\det(b_{n+m})_{n,m=0}^{N}=0$. *Then and only then there is a unique solution* f *of problem* a). *Moreover, in that case* f *is piecewise constant and it assumes only the values* 0 *and* L.

Proof. A part of the assertions of Proposition 6 are simple applications of Corollary 4. Next we deal only with the non-trivial implications.

First of all it should be recalled that any non-negative definite Hankel matrix $(b_{n+m})_{n,m=0}^{N}$ can be extended to an infinite non-negative definite Hankel matrix. This can be shown by a variety of methods, for instance by constructing a finite rank self-adjoint operator A, with cyclic vector ξ and with the property:

$$(9) \qquad b_n = \langle A^n \xi, \xi \rangle, \quad 0 \quad n \leq 2N.$$

The Hilbert space K on which A acts efectively is finite dimensional, but its dimension exceeds N.

Then the phase shift φ of the perturbation $A \longrightarrow A + \xi \otimes \xi$ has some moments whose 1-exponential transform are, in virtue of Theorem 1, $(b_n)_{n=0}^{\infty}$. Since the operator A has only point spectrum, the phase shift φ is piecewise constant and takes only the values 0 and 1. Thus the function $f = L\varphi$ is a solution of problem a).

If $\det(b_{n+m})_{n,m=0}^{N} > 0$, then the L'-problem a) has a solution f', provided $L' < L$ is sufficiently close to L. Indeed, it suffice to recall that $b_0, \ldots, b_{2N}$ are polynomial functions in $L^{-1}, a_0, \ldots, a_{2N}$. Whence $\det(b_{n+m}(L'^{-1}, a_0, \ldots, a_{2N}))$ is still positive for L-L' small enough. Let us remark also that in this case there is an infinity of solutions of problem a), obtained for instance by varying the constant L'.

If $\det(b_{n+m})_{n,m=0}=0$, then relation (9) shows that the vectors $\xi, A\xi, \ldots, A^N\xi$ are linearly dependent (in K). Since ξ was supposed to be a cyclic vector for $A \in L(K)$, it follows that $\dim K \leq N$. Hence the pair (A, ξ), and a fortiori the function $f=L\varphi$,are uniquely determined by the finite sequence $b_0,\dots,b_{2N}$.Thus the only solution of problem a) is forced in that case to be $f=L\varphi$ .As we have already remarked,the phase shift φ is piecewise constant and it assumes only the values 0 and L, Q.E.D.

The adaptation of Proposition 6 to the L-problem of moments with prescribed supports is now a simple routine.

Actually some implications of Proposition 5 remain valid in the operator valued case (i.e. when $\dim H>1$),as direct consequences of Theorem 3 and Corollary 4.Next a less trivial generalization of Proposition 6.b) is discussed.

Proposition 7. Let H be a finite dimensional Hilbert space and let $L>0, N\in\mathbb{N}$ be fixed.Consider a sequence of 2N+1 self-adjoint operators $A_0,A_1,\dots,A_{2N}\in L(H)$,and its exponential transform $B_0,B_1,\dots,B_{2N}$.

If the block-matrix $(B_{n+m})_{n,m=0}^{N}$ is positive definite,then there is an infinity of functions $F\in L^1(\mathbb{R};L(H))$ with the first moments $A_0,\dots,A_{2N}$ and satisfying $0\leq F(t)<L$ a.e..

Proof. It suffices to remark that,under the assumptions of the statement, the Hankel matrix $(B_{n+m})_{n,m=0}^{N}$ can be extended to an infinite Hankel matrix,preserving the positivity.The extension can be obtained recurrently as follows.

Choose for B_{2N+1} an arbitrary self-adjoint matrix and take $B_{2N+2}=\mathrm{diag}(x_1,\dots,x_d)$ where $x_1,\dots,x_d$ are real numbers and $d=\dim H$.Then the numbers $x_1,\dots,x_d$ can be fixed succesively in order to get positive principal minors in the extended Hankel matrix:

$$\begin{pmatrix} B_0 & B_1 & \dots & B_N & B_{N+1} \\ B_1 & B_2 & \dots & B_{N+1} & B_{N+2} \\ & \vdots & & \vdots & \\ B_N & B_{N+1} & \dots & B_{2N} & B_{2N+1} \\ B_{N+1} & B_{N+2} & \dots & B_{2N+1} & \mathrm{diag}(x_1,\dots,x_d) \end{pmatrix}.$$

Whenever such an extension is possible,Corollary 4 insures the existence of a solution $F \in L^1(\mathbb{R};L(H))$ with the prescribed moments $A_0,\dots,A_{2N}$ and satisfying $0 \leq F \leq L$.

By repeating the argument given in the proof of Proposition 6, the condition $\det(B_{n+m})_{n,m=0}^{N} > 0$ can be exploited in order to find solutions F with the property $0 < F(t) < L$.

The non-uniqueness may be obtained for instance by small additive perturbations, as follows. Let $\xi(t)$ be a measurable family of unit eigenvectors corresponding to the maximal eigenvalue of F(t). Take a measurable set $\delta \subset \mathbb{R}$ on which $\varepsilon < \langle F(t)\xi(t), \xi(t)\rangle < L-\varepsilon$ for a suitable small constant $\varepsilon > 0$ and $t \in \delta$. To any scalar function $g \in L^{\infty}(\delta)$, $g \neq 0$, orthogonal in $L^2(\delta)$ to the finite system $1, t, \dots, t^{2N}$, there corresponds the family of operator valued functions:

$$F_s(t) = F(t) + sg(t)\xi(t) \otimes \xi(t), \qquad s \in (0, \varepsilon/\|g\|_{\infty}), t \in \mathbb{R}.$$

Finally one remarks that $0 \leq F_s < L$ and

$$\int_{\mathbb{R}} t^n F_s(t)dt = \int_{\mathbb{R}} t^n F(t)dt = A_n,$$

for every $n \leq 2N$ and s as above.

This concludes the proof of Proposition 7.

3. Final remarks

1) Unlike in the scalar case, the reciprocal to Proposition 7 fails to be true. This can be seen for instance on an example arising from a trivial direct sum: $F \oplus 0 \in L^1(R;L(H \oplus H))$, where $0 \leq F \leq L$. The moments of the function $F \oplus 0$ are $(A_n \oplus 0)_{n=0}^{2N}$ and their L-exponential transform is of course of the form $(B_n \oplus 0)_{n=0}^{2N}$.

2) Proposition 6.a) still has an analogue in the operator valued case. However, in that case the positivity assumption $(B_{n+m})_{n,m=0}^{N} \geq 0$ is not sufficient. To that condition it must be added a quite involved additional

property discovered by Ando [3] Theorem 1.

3) It is important to notice that the proof of Theorem 3 above shows that there exists a bijection between the set of solution of the L-problem of moments (5) and the Hamburger moment problem with data the L-exponential transform of the initial moments.

Some abstract parametrizations of the solutions of the operator valued Hamburger moment problem exist. The reader can consult [7].

4) All the operator valued moment problems discussed in the present paper on the real axis have a natural counterpart on the one-dimensional torus. The corresponding solutions on the torus exist and they are in perfect analogy with the main results above. For L-problems of moments on the torus, in the scalar case, see [2] and [9].

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