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by

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1. INTRODUCTION

Let H be a separable, infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H. A planar compact set K is said to be a spectral set for $T \in L(H)$ if $K \supset \sigma(T)$ and $\|f(T)\| \leq \sup \{ |f(z)| ; z \in K \}$ for every rational function f with poles off K. If $\sigma(T)$ itself is a spectral set for T then T is called a von Neumann operator. One easily sees that every subnormal operator (i.e. restriction of a normal operator to an invariant subspace) is a von Neumann operator. In [14], R. Olin and J. Thompson proved, using the Scott Brown's technique, a structure theorem for the predual of the dual algebra generated by a subnormal operator. Moreover, they showed that every such operator is reflexive. For basic facts about subnormal operators see [9].

The aim of this paper is to extend the above results to the class of von Neumann operators. Our proof is based on recent results on the structure of contractions with isometric functional calculus (see [2], [6] and [7]).

2. PRELIMINARIES

We assume the reader is familiar with the basic definitions and results in the theory of dual algebras (see [3]). However, recall that a dual algebra $A \subset L(H)$ (i.e. a weak* closed subalgebra containing 1_H) is said to have property $(A_1(r))$ for some $r > 0$ if for each element φ in the predual Q_A of A, and each $s > r$, there are vectors x and y in

H such that $\varphi(T) = (Tx,y)$, $T \in A$ and $\|x\| \|y\| < s \|\varphi\|$. Let $H^{\infty}(G)$ denote the algebra of all bounded analytic functions on the planar open set G. Recently, H. Bercovici [2] and B. Chevreau [7], proved independently that every dual algebra $A \subset L(H)$ which is isometrically isomorphic and weak* homeomorphic with $H^{\infty}(D)$, where D denote the unit disc, has property $(A_1(r))$ (Bercovici gets $r = 1$). Moreover, S. Brown and B. Chevreau proved (see [6]) that every such algebra is reflexive, i.e. $A = \mathrm{AlgLat}\, A$, where $\mathrm{AlgLat}\, A = \{T \in L(H); \mathrm{Lat}\, A \subset \mathrm{Lat}\, T\}$.

These results can be used to prove that the dual algebra generated by a von Neumann operator has both the above properties.

Before doing this, we need a decomposition theorem related to spectral sets. For any compact set $K \subset \mathbf{C}$, $R(K)$ denotes the uniform closure in $C(K)$ of rational functions with pales off K. $R(K)$ is said to be Dirichlet if $\mathrm{Re}R(K)_{|\partial K}$ is uniformly dense in $C_R(\partial K)$. The following theorem is a particular case of stronger result due to Lautzenheiser [11] and Mlak [12].

THEOREM 2.1. Let $T \in L(H)$ and assume that K is a spectral set for T such that $R(K)$ is Dirichlet and $U = \mathrm{Int}\, K$ is nonvoid. Let $\{U_i\}_{i \geq 1}$ denote the components of U. Then $T = \bigoplus_{i \geq 0} T_i$ corresponding to a decomposition $H = \bigoplus_{i \geq 0} H_i$ where:

- T_0 is a normal operator with $\sigma(T_0) \subset \partial K$;

- for each $i \geq 1$, there are weak* continuous contractive representations

$$\Phi_i : H^{\infty}(U_i) \longrightarrow L(H_i)$$

such that $\Phi_i(1) = I_{H_i}$ and $\Phi_i(z) = T_i$.

If moreover, $\sigma(T) \cap U_i \neq \emptyset$, then T_i is non trivial.

The above result was also used by J. Agler [1], where he showed that every von Neumann operator has a nontrivial invariant subspace. Recall that a subset $S \subset G$, where $G \subset \mathbf{C}$ is an open set, is said to be dominating in G if $\sup_{z \in S} |f(z)| = \|f\|_{\infty}$ for every

$f \in H^\infty(G)$. For the proof of the next result see [5, Theorem 3].

LEMMA 2.2. Let $L \subset \mathbb{C}$ be a compact set such that $C(L) \neq R(L)$. Then there exists a compact set $K \supset L$ with $\operatorname{Int} K \neq \emptyset$ such that:

- $R(K)$ is Dirichlet

and

- $L \cap \operatorname{Int} K$ is dominating in $\operatorname{Int} K$.

3. A STRUCTURE THEOREM

The main result of the paper is the following.

THEOREM 3.1. If $T \in L(H)$ is a von Neumann operator, then the dual algebra A_T generated by T has property $(A_1(1))$ and is reflexive.

Proof

If $R(\sigma(T)) = C(\sigma(T))$ then by a result of von Neumann [13], T is a normal operator. For such operators it is easy to see that A_T has property $(A_1(1))$ and is reflexive (cf. [15]).

Thus, we may assume $R(\sigma(T)) \neq C(\sigma(T))$. By Lemma 2.2, there exists a compact set $K \supset \sigma(T)$ such that $R(K)$ is Dirichlet and $\sigma(T) \cap \operatorname{Int} K$ is dominating in $\operatorname{Int} K$. Let $\{U_i\}_{i \geq 1}$ denote the components of $\operatorname{Int} K$ and let $T = \bigoplus_{i \geq 0} T_i$ be the decomposition of T with respect to the spectral set K, given by Theorem 2.1. Let also $\Phi_i : H^\infty(U_i) \to L(H_i)$ denote the weak* continuous representations satisfying $\Phi_i(z) = T_i$, $i \geq 1$. Since $\sigma(T_i) \subset \bar{G}_i$, $i \geq 1$, $i \geq 0$ and $\sigma(T_0) \subset \partial K$, one easily sees that $\sigma(T) \cap U_i = \sigma(T_i) \cap U_i$ for each $i \geq 1$. By assumption, $\sigma(T) \cap \operatorname{Int} K$ is dominating in $\operatorname{Int} K$, hence $\sigma(T_i) \cap U_i$ is dominating in U_i, for $i \geq 1$. Since $f(\lambda) \in \sigma(\Phi_i(f))$ for each $\lambda \in \sigma(T_i) \cap U_i$, it follows that $\Phi_i : H^\infty(U_i) \to L(H_i)$ is an isometry. By a standard application of the Krein-Smulian theorem one gets that Φ_i is a weak* homeomorphism from $H^{\infty}(U_i)$ onto its image $A_i = \phi_i(H^{\infty}(U_i))$. Since each U_i is imply connected (cf. [9, Theorem VI 7.2]), $H^{\infty}(U_i)$, hence A_i , are isometrically isomorphic and weak* homeomorphic with $H^{\infty}(\mathbf{D})$. By the already mentioned results from [2], [6] and [7], it follows that every dual algebra A_i has property $(A_1(1))$ and is reflexive. Because T_o is normal $A_o = A_{T_o}$ has both the above properties. Therefore, by [10, Proposition 2.5], the direct sum $A = \bigoplus_{i \geq 0} A_i$ has $(A_1(1))$ and obviously A is also reflexive. By [10, Proposition 2.5], every dual subalgebra of A, in particular A_T has $(A_1(1))$ and is reflexive. The proof is complete.

An operator $T \in L(H)$ is said to be reflexive if the weakly closed subalgebra W_T generated by T in L(H) is reflexive. An immediate corollary of Theorem 3.1 is the following extension of [14, Theorem 3].

COROLLARY 3.2. Every von Neumann operator is reflexive.

Proof

One knows ([3]) that if a dual algebra $A \subset L(H)$ has property $(A_1(r))$, for some $r > 0$, then A is also weakly closed. In particular, this is true for the algebra A_T , generated by a von Neumann operator T on L(H). Apply now Theorem 3.1.

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