MULTI-ANALYTIC OPERATORS AND SOME

FACTORIZATION THEOREM. II

by

G. POPESCU*)

March, 1990

*) Department of Mathematics, INCREST, Bd. Pacii 220, 79622 Bucharest, Romania.

MULTI-ANALYTIC OPERATORS AND SOME

FACTORIZATION THEOREM. II

by

Gelu Popescu

This paper is a continuation of [5] and deals with $\mathcal{S}$-Toeplitz operators. (For the terminology see Section 1).

Section 2 is devoted to an extension of the Fejér-Riesz theorem [6] to our setting and to a concrete realization for the Fock space [1].

In the last section we extend the abstract Szegö infimum theorem [2], to $\mathcal{S}$-Toeplitz operators.

1. PRELIMINARIES

Throughout this paper Λ stands for the set $\{1,2,\dots,k\}$ $(k \in \mathbb{N}^*)$ or the set $\mathbb{N}^* = \{1,2,\dots\}$. For every $n \in \mathbb{N}^*$ let $F(n,\Lambda)$ be the set of all functions from the set $\{1,2,\dots,n\}$ to Λ and

$$\mathcal{F} = \bigcup_{n=0}^{\infty} F(n,\Lambda), \qquad \text{where } F(0,\Lambda) \text{ stands for the set } \{0\}.$$

A sequence $\mathcal{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ of unilateral shifts on a Hilbert space $\mathcal{H}$ with orthogonal final spaces is called a Λ-orthogonal shift if the operator matrix $[S_1,S_2,\dots]$ is nonunitary, i.e., $\mathcal{L} := \mathcal{H} \ominus (\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{H}) \neq \{0\}$. This definition is essentially the same as that from [3,4]. The dimension of $\mathcal{L}$ is called the multiplicity of the Λ-orthogonal shift. One can show that a Λ-orthogonal shift is determined up to unitary equivalence by its multiplicity.

Let us recall from [5] some definitions. An operator $T \in B(\mathcal{H})$ is called

(i) $\mathcal{S}$-Toeplitz if $S_\lambda^* T S_\lambda = T$ for any $\lambda \in \Lambda$ and $S_\lambda^* R S_\mu = 0$ for $\lambda \neq \mu$; $\lambda, \mu \in \Lambda$;

(ii) $\mathcal{S}$-analytic (or multi-analytic) if $TS_\lambda = S_\lambda T$ for any $\lambda \in \Lambda$;

(iii) $\mathcal{S}$-inner if T is $\mathcal{S}$-analytic and partially isometric;

(iv) $\mathcal{S}$-outer if T is $\mathcal{S}$-analytic and $\overline{T\mathcal{H}}$ reduces each S_λ $(\lambda \in \Lambda)$.

In the next two section we shall use some of the results from [5].

2. FEJÉR-RIESZ THEOREM ON THE FOCK SPACE

Let us consider $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ a Λ-orthogonal shift on $\mathcal{H}$ and $\mathcal{L} = \mathcal{H} \ominus (\bigoplus_{\lambda \in \Lambda} S_\lambda \mathcal{H})$. The following is an abstract extension of the Fejer-Riesz theorem [7, p. 118], [6] to our setting.

THEOREM 2.1. Let $T \in B(\mathcal{H})$ be a nonnegative $\mathcal{S}$-Toeplitz operator and let $\mathcal{L}'$ be a dense subset of $\mathcal{L}$ such that for all $l' \in \mathcal{L}'$ there is $n_{l'} \in N^*$ such that

$$S_f^* T l' = 0, \qquad \text{for any } f \in F(n_{l'}, \Lambda),$$

where for any $f \in F(n, \Lambda)$, S_f stands for the product $S_{f(1)} S_{f(2)} \dots S_{f(n)}$.

Then there is an $\mathcal{S}$-outer operator $A \in B(\mathcal{H})$ such that

$$T = A^* A, \quad A_o := P_o A P_o \geq 0,$$

where P_o is the orthogonal projection of $\mathcal{H}$ on $\mathcal{L}$, and

$$S_f^* A l' = 0, \qquad \text{for any } f \in F(n_{l'}, \Lambda).$$

PROOF. By hypothesis, for any $l' \in \mathcal{L}'$, $h \in \mathcal{H}$ and $n \geq n_{l'}$, we have

$$\langle S_f^* T l', h \rangle = 0, \qquad \text{for any } f \in F(n, \Lambda).$$

From this we get

$$\lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \sup \left\{ |\langle T l', S_f h \rangle|^2 ;\ h \in \mathcal{H},\ \|T^{\frac{1}{2}} h\| = 1 \right\} = 0.$$

Applying Theorem 4.1 in [5], we find an $\mathcal{S}$-outer operator $A \in B(\mathcal{H})$ such that

$$T = A^*A, \quad A_o = P_o A P_o \geq 0.$$

Let us show that $S_f^* Al' = 0$, for any $l' \in \mathcal{L}'$, $f \in F(n_{l'}, \Lambda)$. Since A is an $\mathcal{S}$-outer operator on $\mathcal{H}$, the subspace $\mathcal{M} := \overline{A\mathcal{H}}$ of $\mathcal{H}$ reduces each S_λ $(\lambda \in \Lambda)$. Therefore, there is an $\mathcal{S}$-inner operator B on $\mathcal{H}$ such that $\mathcal{M} = B^*\mathcal{H}$, namely $B = P_{\mathcal{M}}$, where $P_{\mathcal{M}}$ stands for the orthogonal projection of $\mathcal{H}$ on $\mathcal{M}$. Since each S_λ $(\lambda \in \Lambda)$ commutes with B it follows that

$$S_f^* \mathcal{M} = S_f^* B^* \mathcal{H} = B^* S_f^* \mathcal{H} \subset B^* \mathcal{H} = \mathcal{M}$$

whence

(2.1) $$S_f^* Al' \in \mathcal{M}, \qquad \text{for any } l' \in \mathcal{L}'.$$

On the other hand

$$A^*(S_f^* Al') = S_f^* Tl' = 0, \qquad \text{for any } f \in F(n_{l'}, \Lambda)$$

and hence

(2.2) $$S_f^* Al' \in \operatorname{Ker} A^* = \mathcal{M}^\perp.$$

From (2.1) and (2.2) we deduce $S_f^* Al' = 0$, for any $f \in F(n_{l'}, \Lambda)$. The proof is complete.

Now, let us give a concrete form of the above theorem on the full Fock space [1]

$$\mathcal{F}(H_n) = \mathbb{C}1 \oplus \bigoplus_{m \geq 1} H_n^{\otimes m},$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \dots, e_n$.

For this, let us define the isometries S_λ, $\lambda \in \Lambda = \{1, 2, \dots, n\}$ by

$$S_\lambda h = e_\lambda \otimes h, \qquad h \in \mathcal{F}(H_n)$$

It is easy to see that $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ is a Λ-orthogonal shift on $\mathcal{F}(H_n)$ with the multiplicity one, i.e.,

$$\dim(I - S_1S_1^* - S_2S_2^* - \dots - S_nS_n^*)\mathcal{F}(H_n) = 1 .$$

THEOREM 2.2. If $T \in B(\mathcal{F}(H_n))$ is a nonnegative $\mathcal{S}$-Toeplitz operator and there is $k \in N$ such that

$$S_f^* T(1) = 0 \qquad \text{for any } f \in F(k,\Lambda),$$

then T has a factorization $T = A^*A$ and A is an $\mathcal{S}$-analytic operator on $\mathcal{F}(H_n)$ defined by

$$Ah = h \otimes \varphi, \qquad h \in \mathcal{F}(H_n),$$

where $\varphi = \varphi_0 \oplus \varphi_1 \oplus \dots \oplus \varphi_k \oplus 0 \oplus 0 \oplus \dots$ and $\varphi_0 \in \mathbb{C}$, $\varphi_m \in H_n^{\otimes m}$ $(m \in \{1,2,\dots,k\})$.

PROOF. By Theorem 2.1, in the particular case when $\mathcal{H} = \mathcal{F}(H_n)$, $\Lambda = \{1,2,\dots,n\}$, it follows that $T = A^*A$ for some $\mathcal{S}$-analytic operator $A \in B(\mathcal{F}(H_n))$ and $S_f^* A(1) = 0$, for any $f \in F(k,\Lambda)$.

But this condition holds if and only if

$$A(1) = \varphi_0 \oplus \varphi_1 \oplus \dots \oplus \varphi_k \oplus 0 \oplus 0 \oplus \dots ,$$

where $\varphi_0 \in \mathbb{C}$, $\varphi_m \in H_n^{\otimes m}$ $(m \in \{1,2,\dots,k\})$.

Let us point out that $\varphi := A(1)$ can be viewed as a polynomial in "n" noncommuting indeterminates.

On the other hand, since A is $\mathcal{S}$-analytic, it is uniquely determined by φ. This follows since for any $f \in \mathcal{F} = \bigcup_{k=0}^{\infty} F(k,\Lambda)$ we have $AS_f(1) = S_f\varphi$ and $\bigvee_{f \in \mathcal{F}} S_f(\mathbb{C}) = \mathcal{F}(H_n)$. Now it is easy to see that

$$Ah = h \otimes \varphi, \qquad \text{for any } h \in \mathcal{F}(H_n). \tag{2.3}$$

The proof is complete.

REMARK 2.3. a) If we replace H_n by an infinite dimensional Hilbert space with orthogonal basis $\{e_i\}_{i=1}^{\infty}$, a similar result holds true.

b) As in the proof of Theorem 2.2 we can see that an $\mathcal{S}$-analytic operator A on $\mathcal{F}(H_n)$ is given by

(5.3) $$Ah = h \otimes \varphi \ , \qquad h \in \mathcal{F}(H_n),$$

where φ stands for A(1).

c) A characterization (as in the classical case) of those "φ" for which (5.3) provides a bounded operator on $\mathcal{F}(H_n)$ would be interesting.

3. ABSTRACT SZEGÖ INFIMUM FOR $\mathcal{S}$-TOEPLITZ OPERATORS

In this Section we extend the Moore's treatment [2] of Szegö infimum problem, to $\mathcal{S}$-Toeplitz operators.

Let us consider $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ a Λ-orthogonal shift on $\mathcal{H}$, $\mathcal{L} = \mathcal{H} \ominus (\bigoplus_{\lambda \in \Lambda} S_\lambda \mathcal{H})$ and let $T \in B(\mathcal{H})$ be a nonnegative $\mathcal{S}$-Toeplitz operator. For each $\lambda \in \Lambda$ we define the Lowdenslager's isometry $S_{T,\lambda}$ on $\mathcal{H}_T := T^{\frac{1}{2}}\mathcal{H}$ by setting $S_{T,\lambda}(T^{\frac{1}{2}}h) = T^{\frac{1}{2}}S_\lambda h$, $(h \in \mathcal{H})$. It is easy to see that $\mathcal{S}_T := \{S_{T,\lambda}\}_{\lambda \in \Lambda}$ is a sequence of isometries with orthogonal final spaces.

After these preliminaries we can prove the following theorem.

THEOREM 3.1. Let $T \in B(\mathcal{H})$ be a nonnegative $\mathcal{S}$-Toeplitz operator. If $1 \in \mathcal{L}$, then

$$\inf\{\langle T(1 - \sum_{\lambda \in \Lambda} S_\lambda h_\lambda), 1 - \sum_{\lambda \in \Lambda} S_\lambda h_\lambda\rangle ;\ h_\lambda \in \mathcal{H},\ \sum_{\lambda \in \Lambda} \|h_\lambda\|^2 < \infty\} > 0$$

if and only if 1 has a nonzero projection on $(\mathcal{L} \cap T^{\frac{1}{2}}\mathcal{H})^-$.

PROOF. Let $\mathcal{H}_T$ and $\mathcal{S}_T = \{S_{T,\lambda}\}_{\lambda \in \Lambda}$ be defined as above. If we set

(3.1) $$\mathcal{L}_T := \mathcal{H}_T \ominus (\bigoplus_{\lambda \in \Lambda} S_{T,\lambda} \mathcal{H}_T)$$

then

(3.2) $$T^{\frac{1}{2}}\mathcal{L}_T = \mathcal{L} \cap T^{\frac{1}{2}}\mathcal{H} .$$

Indeed, if $l_o \in \mathcal{L}_T$, then for any $h \in \mathcal{H}$ and $\lambda \in \Lambda$ we have

$$\langle S_\lambda^* T^{\frac{1}{2}} l_o, h\rangle = \langle l_o, T^{\frac{1}{2}} S_\lambda h\rangle = \langle l_o, S_{T,\lambda} T^{\frac{1}{2}} h\rangle = 0 .$$

Hence, taking into account (3.1) and that $T^{\frac{1}{2}}\mathcal{H}$ is dense in $\mathcal{H}_T$ we deduce $T^{\frac{1}{2}} l_o \in \bigcap_{\lambda\in\Lambda} \operatorname{Ker} S_\lambda^* = \mathcal{L}$, so

$$T^{\frac{1}{2}}\mathcal{L}_T \subset \mathcal{L} \cap T^{\frac{1}{2}}\mathcal{H} \tag{3.3}$$

Conversely, if $h \in \mathcal{H}$ such that $Th \in \mathcal{L}$, then for any $k \in \mathcal{H}$ and $\lambda \in \Lambda$ we get

$$0 = \langle Th, S_\lambda k\rangle = \langle T^{\frac{1}{2}} h, T^{\frac{1}{2}} S_\lambda k\rangle = \langle T^{\frac{1}{2}} h, S_{T,\lambda} T^{\frac{1}{2}} k\rangle .$$

Hence, it follows that $T^{\frac{1}{2}} h \in \mathcal{L}_T$, whence $Th \in T^{\frac{1}{2}}\mathcal{L}_T$. Therefore $\mathcal{L} \cap T^{\frac{1}{2}}\mathcal{H} \subset T^{\frac{1}{2}}\mathcal{L}_T$, which together with (3.3) proves (3.2).

Now, we have

$$\langle T(1 - \sum_{\lambda\in\Lambda} S_\lambda h_\lambda), 1 - \sum_{\lambda\in\Lambda} S_\lambda h_\lambda\rangle = \| T^{\frac{1}{2}} 1 - \sum_{\lambda\in\Lambda} T^{\frac{1}{2}} S_\lambda h_\lambda \|^2 =$$

$$= \| T^{\frac{1}{2}} 1 - \sum_{\lambda\in\Lambda} S_{T,\lambda} T^{\frac{1}{2}} h_\lambda \|^2$$

for any $h_\lambda \in \mathcal{H}$, $\sum_{\lambda\in\Lambda} \|h_\lambda\|^2 < \infty$.

Since $T^{\frac{1}{2}}\mathcal{H}$ is dense in $\mathcal{H}_T$ and (3.1), (3.2) hold, the infimum in the theorem is 0 if and only if $T^{\frac{1}{2}} 1 \perp \mathcal{L}_T$, that is, $1 \perp T^{\frac{1}{2}}\mathcal{L}_T$ or $1 \perp \mathcal{L} \cap T^{\frac{1}{2}}\mathcal{H}$. The result follows.

REFERENCES

1. D.E. Evans, On $\mathcal{O}_n$, Publ. Res. Inst. Math. Sci., 16 (1980), 915-927.
2. B. Moore, The Szegö infimum, Proc. Amer. Math. Soc. 29 (1971), 55-62, Erratum, ibid. 31 (1972), 638.
3. G. Popescu, Isometric dilations for infinite sequences of noncommuting operators, Trans. Amer. Math. Soc., to appear.
4. G. Popescu, Characteristic functions for infinite sequences of noncommuting operators, J. Operator Theory, 22 (1989), 51-71.
5. G. Popescu, Multi-analytic operators and some factorization theorems, Indiana Univ. Math. J., vol. 38, no. 3 (1989), 693-710.
6. M. Rosenblum, Vectorial Toeplitz operators and the Fejer-Riesz theorem, J. Math. Anal. Appl. 23 (1968), 139-147.
7. M. Rosenblum; J. Rovnyak, "Hardy Classes and Operator Theory", New York: Oxford University Press, 1985.

Department of Mathematics
INCREST, Bd. Pacii 220
79622 Bucharest, Romania.