REDUCTION IDEALS FOR MAXIMAL

BUCHSBAUM MODULES

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## 1. INTRODUCTION

The representation theory for maximal Cohen-Macaulay (shortly MCM) modules has achieved a remarkable progress in the recent years. The general idea is to suitably extend the techniques which have been developed in representation theory of Artin algebras. This strategy works very well indeed (see e.g. [Yo], [Di], [Po], [PR]). Motivated by these papers, we were led to study some objects closely related to Cohen-Macaulay modules - namely, the Buchsbaum modules. In some respects, they share many pleasant features with modules having Cohen-Macaulay property. It is the aim of this paper to show a similar behaviour from the representation theory point of view.

Of course, there is no deeper reason why one should restrict one's attention to maximal Buchsbaum (shortly MB) modules. The great technical advantage of this class of modules is that we dispose of Goto's Structure Theorem ([Go]; the precise statement is recalled in (2.2)).

Let $(R,\underline{m},k)$ be a local Cohen-Macaulay ring. Suppose that R has closed singular locus defined by the ideal $I \subsetneq \underline{m}$. Denote by A the completion of R with respect to I and let IMB(R) (resp. IMB(A)) be the isomorphism classes of indecomposable MB modules over R (resp. over A). In this setting the main result of this paper is the following:

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**THEOREM A** (see (4.9)). Suppose that R is a reduced excellent henselian Cohen-Macaulay ring and k is perfect. If R contains a field, then the base change functor $- \otimes_R A$ induces a bijection $IMB(R) \longrightarrow IMB(A)$.

This statement is obtained in section 4 by showing that a certain ideal $\underline{b}$ is a reduction ideal in the sense that the functor $- \otimes_R R/\underline{b}$ reflects isomorphisms, preserves indecomposability and separates isomorphism classes of MB R-modules. This line of proof parallels to Maranda's approach for lattices over orders. We shall pursue the analogy with some works done by Yoshino, Dieterich, Roczen and the second author.

To answer the question of existence of a reduction ideal we choose the method from [Po], [PR]. Thus, in section 3 we shall discuss the bound properties for MB modules which provide a main tool in the subsequent part. Though the proof pattern is as given by [PR], we shall need some new results.

In the second section are given several definitions and properties from the theory of Buchsbaum modules. The main results are stated in (2.6), (2.8) and these represent the technical core of the paper.

We explicitely mention that some of the hypotheses from Theorem A are superfluous. They are carried away because we know the existence of reduction ideals only in rather restrictive conditions. By making use of the hard theory of Artin approximation property, it is possible to get a stronger result (see (4.10)).

We would like to thank Professor W. Vogel who suggested to us the possibility to extend [PR, (2.8)] in the frame of MB modules using Goto's Structure Theorem.

## 2. MAXIMAL BUCHSBAUM MODULES

**(2.1)** Throughout this paper, the rings are understood to be commutative, unitary, Noetherian and all modules are supposed to be finitely generated.

Let $(A,\underline{m})$ be a local ring. An A-module M is said to be <u>Buchsbaum</u> (resp. <u>generalized Cohen-Macaulay</u>) if there exists an integer $I_A(M)$ such that for all parameter ideals $\underline{q}$ of M one has $I_A(M) = \text{length}_A(M/\underline{q}M) - e(\underline{q},M)$ (resp.

$I_A(M) = \sup_{\underline{q}} \{\mathrm{length}_A(M/_{\underline{q}M}) - e(\underline{q},M)\}$, where $\underline{q}$ runs over parameter ideals of M). Here $e(\underline{q},M)$ denotes the multiplicity of $\underline{q}$ with respect to M. A Buchsbaum module having the same dimension as the ring itself is called <u>maximal Buchsbaum</u> (shortly MB module).

Let $H^i(\underline{m},M)$ denote the i-th cohomology module for the Koszul complex generated by a minimal system of generators for $\underline{m}$ with respect to M. Then there exist canonical homomorphisms $\lambda^i(M) : H^i(\underline{m},M) \longrightarrow H^i_{\underline{m}}(M)$, where $H^i_{\underline{m}}(M)$ denotes the i-th local cohomology module of M relative to $\underline{m}$ (see e.g. [SV, Ch. 0, § 1]). Recall that M is a Buchsbaum A-module iff $\lambda^i(M)$ are surjective for all $i = 0,1,\ldots,\dim M - 1$ [SV, Ch. I, (2.15)].

Now assume that A is a module-finite extension of a regular local ring $(R,\underline{n})$ such that $R/\underline{n} \cong A/\underline{m}$ and $\dim_A M = \dim R = d$. Let $E_i$ denote the i-th syzygy module of the residue field $R/\underline{n}$ of R. For a given R-module E and an integer $h \geq 0$, let hE denotes the direct sum of h copies of E. With the above notations, the part we need from Goto's Structure Theorem is stated as follows:

**(2.2) THEOREM** ([Go, (1.1)]). Let $\underline{x} = x_1,\ldots,x_d$ be a regular system of parameters for R and put $\underline{q} = (x_1,\ldots,x_d)A$. Then the following conditions are equivalent:

**(i)** M is a generalized Cohen-Macaulay A-module and $I_A(M) = \mathrm{length}_A(M/\underline{q}M) - e(\underline{q},M)$;

**(ii)** M is a Buchsbaum R-module;

**(iii)** $M \cong \bigoplus_{i=0}^{d} h_i E_i$ as R-modules for some non-negative integers $h_i$.

In particular, the properties **(ii)** and **(iii)** are true if M is a MB A-module.

For every ring A one denotes by X(A) the set of all non-maximal prime ideals of A. A homomorphism of rings $u : B \longrightarrow A$ is said to be <u>flat on the punctured spectrum</u> if for all $\underline{q} \in X(A)$ the induced map $u_{\underline{q}} : B_{\underline{q} \cap B} \longrightarrow A_{\underline{q}}$ is flat. Similarly, an A-module N is called <u>free on the punctured spectrum</u> if $N_{\underline{q}}$ is a free $A_{\underline{q}}$-module for all $\underline{q} \in X(A)$. This terminology is useful in stating the next results. Bellow we shall denote by $H_{A/B}$ the non-smooth locus of the B-algebra A, i.e. $H_{A/B}$ is the intersection of all prime ideals $\underline{q} \subset A$ such that $B \longrightarrow A_{\underline{q}}$ is not smooth.

**(2.3) LEMMA.** Let $u : B \longrightarrow A$ be a finite ring homomorphism, $Q \subset A$ a primary ideal, $\underline{q} = \sqrt{Q}$, $\underline{p} = u^{-1}(\underline{q})$, $P = u^{-1}(Q)$ and x an element from $A \setminus \underline{q}$. Suppose that $B_{\underline{p}} \longrightarrow A_{\underline{q}}$ is smooth. Then for every A-module N it holds

$$(QN : x)_N \subseteq N \cap PN_{\underline{q}} = N \cap PN_{\underline{p}}$$

**Proof.** Note that the map $B_{\underline{p}} \longrightarrow A_{\underline{q}}$ is étale because it is smooth and essentially finite. Thus we get $\underline{p}A_{\underline{q}} = \underline{q}A_{\underline{q}}$. Let N be an A-module. If $z \in N \cap PN_{\underline{q}}$ then there exists $y \in A \setminus \underline{q}$ such that $yz \in PN$. Since the induced extension $B/\underline{p} \longrightarrow A/\underline{q}$ is finite we get $u^{-1}(yA) \not\subset \underline{p}$. Thus changing y by one of its multiple we may suppose $y \in B \setminus \underline{p}$. Hence $z \in PN_{\underline{p}}$, i.e. the equality holds. The above inclusion will be established in several steps.

**Step 1.** Case when the residue field extension of $B_{\underline{p}} \longrightarrow A_{\underline{q}}$ is trivial.

Then the extension $B_{\underline{p}} \longrightarrow A_{\underline{q}}$ is dense and so $PA_{\underline{q}} = QA_{\underline{q}}$. Let $w \in N$ be such that $xw \in QN$. Thus $w \in QN_{\underline{q}} = PN_{\underline{q}}$ and so the inclusion holds.

**Step 2.** Case when there exist a finite B-algebra C and a prime ideal $\underline{q}' \subset D := C \otimes_B A$ such that $\underline{q}'$ lies on $\underline{q}$, the residue field extension of $C_{\underline{p}'} \longrightarrow D_{\underline{q}'}$, $\underline{p}' := (C \otimes u)^{-1}\underline{q}'$ is trivial and the map $B \longrightarrow C_{\underline{p}'}$ is smooth.

As above the morphisms $B_{\underline{p}} \longrightarrow C_{\underline{p}'}$, $A_{\underline{q}} \longrightarrow D_{\underline{q}'}$ are étale and so $\underline{p}C_{\underline{p}'} = \underline{p}'C_{\underline{p}'}$, $\underline{q}D_{\underline{q}'} = \underline{q}'D_{\underline{q}'}$. Clearly $QD_{\underline{q}'}$, $PC_{\underline{p}'}$ are primary ideals and so $Q' = D \cap QD_{\underline{q}'}$, $P' = C \cap PC_{\underline{p}'}$ are primary too.

Let $x' := 1 \otimes x$, $u' := C \otimes u$, $E := (QN : x)_N$ and $N' := D \otimes_A N$. By Step 1 we get

(1) $$E' := (Q'N' : x')_{N'} \subseteq N' \cap P'N'_{\underline{p}'} = N' \cap PN'_{\underline{p}'}$$

Obviously E is the kernel of the composed morphism $N \xrightarrow{x} N \longrightarrow N/QN$. Since $C_{\underline{p}'}$ is flat over B we get the following exact sequence of $C_{\underline{p}'}$-modules

(2) $$0 \longrightarrow C_{\underline{p}'} \otimes_B E \longrightarrow C_{\underline{p}'} \otimes_B N \longrightarrow C_{\underline{p}'} \otimes_B (N/QN)$$

where the last map is in fact the composite homomorphism

(3) $$C_{\underline{p}'} \otimes_B N \cong N'_{\underline{p}'} \xrightarrow{x'} N'_{\underline{p}'} \longrightarrow N'_{\underline{p}'}/Q'N'_{\underline{p}'} \cong C_{\underline{p}'} \otimes_B (N/QN)$$

For the last isomorphism note that $Q'N'_{\underline{q}'} = QN'_{\underline{q}'}$ by construction.

Thus if $z \in Q'N'_{\underline{q}'}$ then $yz \in QN'$ for a certain $y \in D \setminus \underline{q}'$. Since the map $C/\underline{p}' \longrightarrow D/\underline{q}'$ is finite we get $u'^{-1}(yA) \not\subset \underline{p}'$. Thus changing y by one of its multiple we may suppose that $y \in C \setminus \underline{p}'$, i.e. $Q'N'_{\underline{q}'} = QN'_{\underline{p}'}$.

By (2) and (3) we get $C_{\underline{p}'} \otimes_B E \cong E'_{\underline{p}'}$. According to (1) the inclusion $PN_{\underline{p}} \subseteq E_{\underline{p}} + PN_{\underline{p}}$ becomes equality after tensorization with $C_{\underline{p}'}$. By faithfully flatness it follows $PN_{\underline{p}} = E_{\underline{p}} + PN_{\underline{p}}$ i.e. $E_{\underline{p}} \subseteq PN_{\underline{p}}$. Thus $E \subseteq N \cap PN_{\underline{p}}$.

**Step 3.** General case-reduction to Step 2.

We need the following Lemma - a particular case of [Po, (3.4)].

**(2.3.1.) LEMMA.** Let $S \subset R$ be a finite ring homomorphism, $\underline{p} \subset S$ a prime ideal and

$$d_{R,\underline{p}} = \min_{\substack{\underline{q} \in \operatorname{Spec} R \\ \underline{q} \cap S = \underline{p}}} ([k(\underline{q}) : k(\underline{p})] - 1)$$

where $k(\underline{p})$ denotes the residue field of $S_{\underline{p}}$. Suppose that $d_{R,\underline{p}} > 0$. Then there exists a prime ideal $\underline{p}' \subset R$ lying on $\underline{p}$ such that $d_{R \otimes_S R, \underline{p}'} < d_{R,\underline{p}}$.

By the above Lemma a finite B-algebra C of the form $A \otimes_B A \otimes_B \cdots \otimes_B A$ satisfies all the hypothesis required for Step 2.

**(2.4) LEMMA.** Let $u : B \longrightarrow A$ be a finite ring homomorphism, $\underline{a} \subset A$ an ideal, $\underline{b} = u^{-1}(\underline{a})$ and $x \in H_{A/B}$ an element such that for all $\underline{q} \in X(A)$, x is not a zero-divisor on $(A/\underline{a})_{\underline{q}}$. Suppose that u is flat on the punctured spectrum. Then for every A-module N which is free on the punctured spectrum of B it holds

$$(\underline{a}N : x)_N \subseteq N \cap \underline{b}N_{\underline{p}}$$

for every $\underline{p} \in X(B)$.

**Proof.** Let $\underline{a} = \bigcap_{i=1}^{e} Q_i$ be an irredundant primary decomposition of $\underline{a}$, $P_i := u^{-1}(Q_i)$, $\underline{p}_i = \sqrt{P_i}$, $\underline{q}_i := \sqrt{Q_i}$. If $\underline{q}_i \in X(A)$ then $x \notin q_i$ because x is not a zero-divisor on $(A/\underline{a})_{\underline{q}_i}$ be hypothesis. Thus the map $B \longrightarrow A_{q_i}$ is smooth ($x \in H_{A/B}$!). Let N be as in our Lemma. Applying Lemma (2.3) we get

$$(\underline{a}N : x)_N \subseteq (Q_iN : x)_N \subseteq N \cap P_iN_{\underline{p}_i}$$

It follows

$$(\underline{a}N : x)_N \subseteq \bigcap_{\substack{i=1 \\ \underline{p}_i \in X(B)}}^{e} (N \cap P_iN_{\underline{p}_i})$$

because $\underline{q}_i \in X(A)$ iff $\underline{p}_i \in X(B)$, u being finite. Let $\underline{p} \in X(B)$. Since $N_{\underline{p}}$ is free over $B_{\underline{p}}$ we get

$$\underline{b}N_{\underline{p}} = \bigcap_{\substack{i=1 \\ \underline{p}_i \subseteq \underline{p}}}^{e} (N_{\underline{p}} \cap P_iN_{\underline{p}_i}) \supseteq (\underline{a}N : x)_N \ .$$

**(2.5) LEMMA.** Let $u : (B,\underline{n}) \longrightarrow (A,\underline{m})$ be a finite homomorphism of local rings, $\underline{a} \subset A$ an ideal, $x \in H_{A/B}$ and $\mathcal{F}$ a finite set of B-modules. Suppose that u is flat on the punctured spectrum. Then there exists a positive integer r such that for every A-module N having the properties

a) $N$ is free on the punctured spectrum of $B$;

b) $N \cong \bigoplus_{F \in \mathcal{F}} t_F F$ as a $B$-module, for certain non-negative integers $t_F$, it holds

$$(\underline{a}N : x^r)_N = (\underline{a}N : x^{r+1})_N .$$

**Proof.** Clearly the equality holds for every $r \in \mathbb{N}$ if $x \notin \underline{m}$ and so we may assume $x \in \underline{m}$. By noetherianity, the increasing chain of ideals $\{(\underline{a} : x^s)\}_s$ stops beyond a certain non-negative integer $t$. Let us say $\underline{a}' := (\underline{a} : x^t)$. As is readily seen, if $r'$ satisfies the conclusion of Lemma for $x$ and $\underline{a}'$, then $r := t + r'$ will satisfy the conclusion for $x$ and $\underline{a}$. Thus changing $\underline{a}$ by $\underline{a}'$ one may suppose $x$ is not a zero-divisor modulo $\underline{a}$.

Using a primary decomposition over $B$, for every $F \in \mathcal{F}$ we can write $\underline{b}F = F' \cap F_o$ where $F_o$ is a $\underline{n}$-primary sub-$B$-module of $F$ and $F' \subset F$ is a sub-$B$-module such that $\mathrm{Ass}_B(F/F') \subset X(B)$. Let $s \in \mathbb{N}$ be a non-negative integer such that $\underline{n}^s F \subseteq F_o$ for all $F \in \mathcal{F}$. Since $u$ is finite we get $\underline{m}^r \subseteq \underline{n}^s A$ for a certain $r \in \mathbb{N}$. We claim that this $r$ is the wanted one.

Indeed, let $N$ be as in our Lemma. By b) we get $N = \bigoplus_{F \in \mathcal{F}} t_F F$ for some $t_F \in \mathbb{N}$. Then $\underline{b}N = \bigoplus_{F \in \mathcal{F}} t_F(\underline{b}F)$. Denote $N_o = \bigoplus_{F \in \mathcal{F}} t_F F_o$ and $N' = \bigoplus_{F \in \mathcal{F}} t_F F'$. Clearly we have $\underline{b}N = N_o \cap N'$ and $\underline{m}^r N \subseteq \underline{n}^s N = \bigoplus_{F \in \mathcal{F}} t_F(\underline{n}^s F) \subseteq \bigoplus_{F \in \mathcal{F}} t_F F_o = N_o$.

By construction $\mathrm{Ass}_B(N/N') \subset X(B)$ and so $N'$ is uniquely determined by

$$N' = \bigcap_{\underline{p} \in X(N)} (N \cap \underline{b}N_{\underline{p}}) , \quad X(N) := X(B) \cap \mathrm{Supp}\, N$$

Now let $z \in N$ be such that $x^{r+1} z \in \underline{a}N$. By (2.4) we get $z \in \underline{b}N_{\underline{p}}$ for all $\underline{p} \in X(N)$ and so $z \in N'$. Also $x^r z \in N_o$ because $x \in \underline{m}$. Consequently $x^r z \in N_o \cap N' = \underline{b}N \subseteq \underline{a}N$, i.e. $(\underline{a}N : x^{r+1})_N = (\underline{a}N : x^r)_N$.

**(2.6.) PROPOSITION.** Let $B \subset A$ be a finite extension of local rings, $\underline{a} \subset A$ an ideal and $x \in H_{A/B}$. Suppose $B$ is regular, $A$ is a generalized Cohen-Macaulay ring and the residue field extension is trivial. Then there exists a natural number r such that for every MB A-module N it holds

$$(\underline{a}N : x^r)_N = (\underline{a}N : x^{r+1})_N$$

**Proof.** According to [Go, (4.1)], for every $\underline{q} \in X(A)$ $A_{\underline{q}}$ is a Cohen-Macaulay ring of dimension $d := \dim A - \dim A/\underline{q}$. If $\underline{p} := \underline{q} \cap B$, then $B_{\underline{p}}$ is a regular ring of dimension $\dim B - \dim B/\underline{p} = \dim A - \dim A/\underline{q} = d$. Hence it follows $A_{\underline{q}}$ is free over $B_{\underline{p}}$ , i.e. the structural homomorphism $B \to A$ is flat on the punctured spectrum.

Let N be a MB A-module and $E_i$ , $i = 0,\dots,s := \dim A$ denote the i-th syzygy module of the residue field of B. By Goto's theorem (2.2) we know that N fulfils the condition b) from (2.5) with $\mathcal{F} = \{E_o, E_1, \dots, E_s\}$ . Applying [Go,(4.1)] to N we note that $N_{\underline{p}}$ is a MCM over $B_{\underline{p}}$ for $\underline{p} \in X(B) \cap \mathrm{Supp}_B N$. Thus N is free on the punctured spectrum of B and the assertion follows from (2.5).

Now we recall that for an arbitrary R-algebra S, the Noether different is defined by

$$\mathcal{N}^S_R := \mu((O : I)_{S \otimes_R S}),$$

where I denotes the kernel of the multiplication map $\mu : S \otimes_R S \longrightarrow S$, $\mu(a \otimes b) := ab$.

**(2.7) LEMMA.** Let $(B,\underline{n}) \subset (A,\underline{m})$ be a finite extension of local rings and let $J := \mathcal{N}^A_B$ be the Noether different. Suppose $\mathcal{F}$ is a finite set of B-modules. Then there exists a positive integer r such that for every A-module M having the properties:

a) M is free on the punctured spectrum of B;

b) $M \cong \bigoplus_{F \in \mathcal{F}} t_F F$ as B-modules for certain non-negative integers $t_F$,

it holds

$$\underline{n}^r \cdot J \cdot \mathrm{Ext}^1_A(M,N) = 0 \qquad \text{for every A-module N.}$$

**Proof.** If M and N are as above, consider

(+) $$0 \longrightarrow K \xrightarrow{w} L \xrightarrow{v} M \longrightarrow 0$$

a short exact sequence of A-modules with L finitely generated and free. Clearly one gets the exact sequences over B

(*) $$0 \longrightarrow E' := \mathrm{Hom}_B(M,N) \xrightarrow{\tilde{v}} E := \mathrm{Hom}_B(L,N) \xrightarrow{h} D \longrightarrow 0$$

(**) $$0 \longrightarrow D \xrightarrow{g} E'' := \mathrm{Hom}_B(K,N)$$

where D denotes the image of $\tilde{w} := \mathrm{Hom}_B(w,N)$ and h, g are obtained from $\tilde{w}$ in an obvious way.

In the following diagram

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "\tilde{v}"] \arrow[dd, "\delta'"] & E \arrow[r, "h"] \arrow[dd, "\delta"] & D \arrow[r] \arrow[d, "g"] & 0 \\
 & & & E'' \arrow[d, "\delta''"] & \\
0 \arrow[r] & \mathrm{Hom}_B(A,E') \arrow[r, "v^*"] & \mathrm{Hom}_B(A,E) \arrow[r, "w^*"] & \mathrm{Hom}_B(A,E'') &
\end{tikzcd}$$

the homomorphism $\delta$ is defined by

$$(\delta f)(x) := xf - fx\,, \qquad f \in E, \ x \in A$$

and similarly $\delta'$ and $\delta''$. Clearly the rows are exact and since $\tilde{w} = gh$ the diagram is commutative. As $\ker \delta' = \mathrm{Hom}_A(M,N)$ (see e.g. [Pi, (11.2) + (10.4)]) we get by Snake Lemma the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(L,N) \xrightarrow{\alpha} D \cap \mathrm{Ker}\, \delta'' \longrightarrow \mathrm{Coker}\, \delta'$$

In fact, the last homomorphism has the image in the Hochschild cohomology module $H^1_B(A,E') \subset \mathrm{Coker}\, \delta'$. It is well known that the Noether different J annihilates $H^1_B(A,P)$ for every A-bimodule P (see e.g. [Yo, (2.2)]). Therefore

(1) $$J.D_0 \subseteq \mathrm{Im}\, \alpha, \qquad \text{where } D_0 := D \cap \mathrm{Ker}\, \delta''.$$

Now we need the following Lemma

**(2.7.1.) LEMMA.** Let E be a finitely generated module over a local ring $(R,\underline{n})$ such that it is free on the punctured spectrum of R. Then there exists a positive integer $s \in \mathbf{N}$ such that

$$\underline{n}^s \mathrm{Ext}_R^1(E,P) = 0$$

for every R-module P.

By the above Lemma there exists $r \in \mathbf{N}$ such that $\underline{n}^r \mathrm{Ext}_B^1(F,P) = 0$ for every $F \in \mathcal{F}$ and every B-module P. In particular we have

$$\underline{n}^r \mathrm{Ext}_B^1(M,N) = 0$$

for all M,N as in our Lemma. It follows

(2) $$\underline{n}^r \mathrm{Hom}_B(K,N) \subseteq D$$

using (+). Since $H_B^o(A,-)$ is a functor it preserves the multiplications and so we get

(3) $$\underline{n}^r \mathrm{Hom}_A(K,N) = \underline{n}^r H_B^o(A, \mathrm{Hom}_B(K,N)) \subseteq D \cap \mathrm{Ker}\, \delta'' = D_o .$$

Combining (1) and (3) it follows

$$J.\underline{n}^r.\mathrm{Hom}_A(K,N) \subseteq \mathrm{Im}\, \alpha$$

Since $\alpha$ has the same image as $\mathrm{Hom}_A(w,N)$ we get $J\underline{n}^r \mathrm{Ext}_A^1(M,N) = 0$ for all M,N as in our Lemma.

**Proof of Lemma (2.7.1).** Choose a finitely generated free R-module L and a surjective R-linear map $p : L \longrightarrow E$. Let $x \in \underline{n}$. Then $E_x$ is projective by assumption and so $R_x \otimes p$ has a section q. Since $(\mathrm{Hom}_R(E,L))_x \cong \mathrm{Hom}_{R_x}(E_x, L_x)$, there exists a homomorphism $\varphi : E \longrightarrow L$ such that $R_x \otimes \varphi = q$. Thus $\mathrm{id}\, E_x = (R_x \otimes p) \circ \circ (R_x \otimes \varphi) = R_x \otimes (p \circ \varphi)$ and therefore we may find a positive integer t such that $x^t$ annihilates $p \circ \varphi - \mathrm{id}\, E$. If $\mu : E \longrightarrow E$ denotes the multiplication by $x^t$, then it results $\mu$ factorises through L. As $\mathrm{Ext}_R^1(L,P) = 0$, it follows

(4) $$x^t \mathrm{Ext}_R^1(E,P) = 0$$

for every R-module P.

Choose a system of generators $x_1,\dots,x_e$ of $\underline{n}$ and a positive integer t for which (4) holds for all $x = x_i$. Then taking $s = te$ we get

$$\underline{n}^s \cdot \mathrm{Ext}^1_R(E,P) \subseteq (x_1^t,\dots,x_e^t)\mathrm{Ext}^1_R(E,P) = 0$$

**(2.8) PROPOSITION.** Let $(B,\underline{n}) \subset (A,\underline{m})$ be a finite extension of local rings and let J be the Noether different. Suppose that B is regular and the residue field extension is trivial. Then there exists a natural number r such that

$$\underline{m}^r J\mathrm{Ext}^1_A(M,N) = 0$$

for every MB A-module M and for every A-module N.

The proof follows by the preceding Lemma using the same argumentation as in the proof of (2.6). Moreover, we note that $\underline{n}A$ is a $\underline{m}$-primary ideal and so $\underline{m}^t \subseteq \underline{n}A$ for a certain $t \in \mathbb{N}$.

We record here some simple facts concerning the transfer properties for the Buchsbaum property.

**(2.9) LEMMA.** Let $f : R \longrightarrow S$ be a flat local homomorphism of local rings. Suppose M is a Buchsbaum S-module of positive dimension $d = \dim_S M$. Then M is a Buchsbaum module over R.

**Proof.** By [SV, Ch. I, (1.10)] M is a Buchsbaum S-module iff for every system of parameters $y_1,\dots,y_d \in S$ of M we have for all $i = 0,\dots,d-1$

$$((y_1,\dots,y_i)M : y_{i+1})_M = ((y_1,\dots,y_i)M : y_{i+1}^2)_M$$

Let us consider $x_1,\dots,x_t \in R$ a system of parameters for M as a R-module. Since f is flat, it results $t = \dim_R M \leq \dim_S M$ and $x_1,\dots,x_t$ is a part of a system of parameters for the S-module M. The assertion follows from the above characterization.

**(2.10) LEMMA.** Let $f : (R,\underline{m}) \longrightarrow (S,\underline{n})$ be a flat local homomorphism of rings with the same dimension and let $M$ be a finitely generated $R$-module of positive dimension $d$. If $N := M \otimes_R S$ is a Buchsbaum $S$-module, then $M$ is a Buchsbaum $R$-module. The converse implication holds if $\underline{m}S = \underline{n}$.

**Proof.** Since $\underline{m}S \subseteq \underline{n}$ and $f$ is flat, we get for each $i = 0,1,\dots,d-1$ a commutative diagram with canonical maps

$$\begin{array}{ccccc}
H^i(\underline{n},N) & \xrightarrow{u} & H^i(\underline{m}S,N) & \xrightarrow{\sim} & H^i(\underline{m},M) \otimes_R S \\
\downarrow{\scriptstyle \lambda^i(N)} & & \downarrow & & \downarrow{\scriptstyle \lambda^i(M)\otimes S} \\
H^i_{\underline{n}}(N) & \xrightarrow{v} & H^i_{\underline{m}S}(N) & \xrightarrow{\sim} & H^i_{\underline{m}}(M) \otimes_R S
\end{array}$$

As $\underline{m}S$ is a $\underline{n}$-primary ideal, actually $v$ is an isomorphism. Suppose that $N$ is a Buchsbaum $S$-module. Then the left vertical homomorphism $\lambda^i(N)$ is onto. By faithfully flatness it follows $\lambda^i(M)$ is surjection too.

Conversely, the additional assumption $\underline{m}S = \underline{n}$ implies $u$ is onto and thus the assertion is obvious.

## 3. RINGS WITH BOUND PROPERTIES ON MAXIMAL BUCHSBAUM MODULES

**(3.1)** In this section $(R,\underline{m},k)$ will be a ring having unique maximal ideal $\underline{m}$ and residue field $k$. Suppose that the set

$$\operatorname{Reg} R := \{\underline{p} \in \operatorname{Spec} R : R_{\underline{p}} \text{ regular}\}$$

is open (this is the case if $R$ is complete or quasi-excellent). Then the singular locus is a closed set defined by a radical ideal denoted $I_s(R)$. Since the situation R regular is covered by (2.2), we shall furthermore assume $I_s(R) \subsetneq \underline{m}$.

The next definition is inspired by [PR].

**(3.2)** We say that R <u>has bound properties on MB-modules</u> if the following conditions are fulfilled:

**(B1)** there exists a positive integer r such that $I_s(R)^r \mathrm{Ext}^1_R(M,N) = 0$ for every MB module M and every R-module N;

**(B2)** for every ideal $\underline{a}$ in R and for every element y of $I_s(R)$ there exists a positive integer e such that

$$(\underline{a}M : y^e)_M = (\underline{a}M : y^{e+1})_M$$

for every MB module M.

**(3.3) REMARK.** A moment of thought reveals that it is sufficient to consider in **(B1)** and **(B2)** only indecomposable MB modules M in order to establish if R satisfies the definition (3.2).

The next result roughly says the descent of bound properties on MB-modules.

**(3.4) LEMMA.** Let $f : (R,\underline{m}) \longrightarrow (A,\underline{n})$ be a flat homomorphism of local rings such that $I_s(R)A \subseteq I_s(A) \subsetneq \underline{n}$. Suppose $\underline{m}A = \underline{n}$ and A has bound properties on MB-modules. Then R has bound properties on MB-modules.

**Proof.** Let r be the natural number given for A by **(B1)**. We shall show that R satisfies the condition **(B1)** for the same value r.

Let M be a MB R-module and N a R-module (finitely generated, as always). By (2.10) we note that $M \otimes_R A$ is a MB module over A and by flatness $A \otimes_R \mathrm{Ext}^1_R(M,N) \cong$ $\cong \mathrm{Ext}^1_A(M \otimes_R A, N \otimes_R A)$. According to the assumption $I_s(R)A \subseteq I_s(A)$ and from **(B1)** one gets

$$I_s(R)^r(A \otimes_R \mathrm{Ext}^1_R(M,N)) \subseteq I_s(A)^r \mathrm{Ext}^1_A(M \otimes_R A, N \otimes_R A) = 0$$

whence $I_s(R)^r \mathrm{Ext}^1_R(M,N) = 0$ since f is faithfully flat.

Next one examines the condition (B2). If $\underline{a}$ is an ideal of R and $y \in I_s(R)$, then $f(y) \in I_s(A)$ and by hypothesis there exists a natural number e such that $(\underline{a}N : f(y)^e)_N = (\underline{a}N : f(y)^{e+1})_N$ for every MB A-module N.

Let M be a MB R-module. As above $N := M \otimes_R A$ is a MB A-module and one has

$$(\underline{a}N : f(y)^s)_N \cong A \otimes_R (\underline{a}M : y^s)_M$$

for every non-negative integer s. Thus the desired conclusion follows again by the faithfully flatness of the morphism f.

Now we provide an answer to the question of existence of rings having bound properties on MB-modules.

**(3.5) PROPOSITION.** Let R be a reduced complete ring with perfect residue field. If R is Cohen-Macaulay and contains a field, then R has bound properties on MB modules.

**Proof.** Let $x = x_1,\ldots,x_n$ be a system of parameters for R. By Cohen's Structure Theorem, R is a finite extension of the regular local ring $S(x) := k[[x_1,\ldots,x_n]]$. Moreover, k is the residue field of S(x). Let us denote by J(x) the Noether different of the S(x) - algebra R. Since R is Cohen-Macaulay and k is perfect, by [Yo,(2.5)] we have

$$I_s(R) = \mathrm{Rad}(J)$$

with $J := \sum J(x)$, where the sum is taken over all system of parameters x for R. Note that R is flat over S(x).

Choose some systems of parameters $x^{(i)}$ such that $J = \sum_{i=1}^{t} J(x^{(i)})$. Applying (2.8) we find for every $i = 1,2,\ldots,t$ an integer $s_i$ such that $\underline{m}^{s_i}.\, J(x^{(i)}).\mathrm{Ext}^1_R(M,N) = 0$ for every MB R-module M and every N. Therefore, the ideal $I := \prod_{i=1}^{t} \underline{m}^{s_i}$ satisfies $I.J.\mathrm{Ext}^1_R(M,N) = 0$ for M,N as above. Since $I_s(R) = \mathrm{Rad}(I.J)$ we conclude that **(B1)** holds.

Now let $\underline{a} \subset R$ be an ideal and $y \in I_s(R)$. If there exists x as above such that $y \in J(x) \subset H_{R/S(x)}$ (see e.g. [Po, (2.10)]) we obtain the equality asserted in **(B2)** by (2.6) applied to the flat extension $S(x) \longrightarrow R$. In the general case choose in $I_s(R)$ a system of elements $r_1,\dots,r_t$ such that $I_s(R) = \mathrm{Rad}(\sum r_i R)$ and for every $i = 1,\dots,t$ there exists a system of parameters $x^{(i)}$ of R such that $r_i \in J(x^{(i)})$.

Then for every i one has $yr_i \in J(x^{(i)})$ and as above there exists a natural number $e_i$ such that $(\underline{a}M : (yr_i)^{e_i})_M = (\underline{a}M : (yr_i)^{e_i+1})_M$ for every MB R-module M.

We claim that the condition **(B2)** is fulfilled for $e := v + \max_{1 \le i \le t} e_i$ , where v is a positive integer for which it holds $I_s(R)^v \subseteq \sum r_i^{e_i} R$.

Indeed, let M be a MB R-module. If $y^s z \in \underline{a}M$ for a certain $z \in M$ and $s \in N$, then $(yr_i)^s z \in \underline{a}M$ and so $(yr_i)^{e_i} z \in \underline{a}M$ for every $i = 1,\dots,t$. Thus one has

$$y^e z \in y^{e-v}(\sum r_i^{e_i} R)z \subseteq \sum (yr_i)^{e_i} Rz \subseteq \underline{a}M$$

This finishes the proof of (3.5).

By combining the preceding facts one gets the main result of this section, which ressembles to [PR, (1.5)].

**(3.6) THEOREM.** Let R be an excellent Cohen-Macaulay reduced ring with perfect residue field. If R contains a field, then R has bound properties on MB-modules.

**Proof.** Let A be the completion of R. As R is excellent, the canonical homomorphism $f : R \longrightarrow A$ is regular. Hence it follows by [Ma, (33.B)]

$$\mathrm{Reg}\,A = \{ \underline{q} \in \mathrm{Spec}\,A : f^{-1}(\underline{q}) \in \mathrm{Reg}\,R \}$$

and therefore $I_s(A) = \mathrm{Rad}\,(I_s(R)A)$.

Also, A is reduced Cohen-Macaulay. Thus, by (3.5) it results A has bound properties on MB-modules and so the assertion is obtained from (3.4).

## 4. RINGS WITH MB-REDUCTION IDEALS

**(4.1)** Throughout this section $(R,\underline{m},k)$ will denote a Cohen-Macaulay local ring with non-empty closed singular locus.

Let $\underline{a} \subset R$ be an ideal. The couple $(R,\underline{a})$ is said to be <u>MB-approximation</u> if there exists a function $\nu : N \longrightarrow N$, $\nu \geq id_N$ such that for every $t \in N$, every MB R-modules M, N and every $f \in Hom_R(M, N/\underline{a}^{\nu(t)}N)$, there exists a R-linear map $g : M \longrightarrow N$ such that

$$R/\underline{a}^t \otimes_R f \cong R/\underline{a}^t \otimes_R g$$

**(4.2) LEMMA.** Suppose that R has bound properties on MB-modules. Then for every ideal $\underline{a} \subset R$ and $y \in I_s(R)$ there exists a function $\nu : N \longrightarrow N$, $\nu \geq id_N$ such that for every $t \in N$, every MB modules M, N and every $f \in Hom_R(M, N/(\underline{a},x^{\nu(t)})N)$ there exists $g \in Hom_R(M, N/\underline{a},N)$ for which the following diagram is commutative:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, dashed, "g"'] & N/(\underline{a},x^{\nu(t)})N \arrow[d] \\
N/\underline{a}N \arrow[r] & N/(\underline{a},x^{t})N
\end{tikzcd}$$

**Proof.** Define $\nu(t) := r(1 + \max(e,t))$, where the positive integers r and e are given by **(B1)**, **(B2)** respectively. Then proceed as in [PR, Proof of (2.2)].

**(4.3) LEMMA.** Suppose that R has bound properties on MB-modules. Then for every ideal $\underline{a} \subseteq I_s(R)$ the couple $(R,\underline{a})$ is a MB-approximation.

The proof uses (4.2) and goes exactly as in [PR, (2.4)].

(4.4) The ideal $\underline{b} \subset R$ is said to be a MB-<u>reduction ideal</u> if the following statements hold:

1) a MB R-module M is indecomposable iff $M/\underline{b}M$ is indecomposable over $R/\underline{b}$;

2) two indecomposable MB R-modules M, N are isomorphic iff $M/\underline{b}M$ and $N/\underline{b}N$ are isomorphic as $R/\underline{b}$-modules.

By [Po, (4.5) + (4.6)] we get immediately:

**(4.5) LEMMA.** Let R be a henselian ring and $\underline{a} \subset R$ an ideal such that the couple $(R,\underline{a})$ is a MB-approximation with associated function $\nu$. Then $\underline{a}^r$ is a MB-reduction ideal, where $r = \nu(1)$.

The next proposition sums up our results on MB-reduction ideals. By combining it with (3.6) we obtain a specific class of rings for which we know a positive answer to the problem of existence of MB-reduction ideals.

**(4.6) PROPOSITION.** Let R be a henselian local ring having bound properties on MB-modules. Then for every ideal $\underline{a} \subseteq I_s(R)$ there exists a natural number r such that $\underline{a}^r$ is a MB-reduction ideal.

**(4.7) THEOREM.** Let $(R,\underline{m})$ be a reduced excellent henselian Cohen-Macaulay ring with perfect residue field. If R contains a field, then $I_s(R)^r$ is a MB-reduction ideal for a certain positive integer r.

Let IBM(R) be the set of isomorphism classes of indecomposable MB R-modules. Then

**(4.8) PROPOSITION.** Let R be an excellent henselian ring having bound properties on MB-modules. Let A be the completion of R with respect to $I_s(R)$. Then the base change functor $- \otimes_R A$ induces a bijection $\mathrm{IMB}(R) \longrightarrow \mathrm{IMB}(A)$.

**Proof.** Since R is excellent, the canonical map $R \longrightarrow A$ is regular. Therefore A is a Cohen-Macaulay ring and $I_s(A) = \mathrm{Rad}(I_s(R)A)$ by [Ma, (33.B)].

Take M from IMB(R). Then $M \otimes_R A$ is a MB A-module by (2.10). According to (4.6) $\underline{b} := I_s(R)^r$ is a MB-reduction ideal for a certain $r \in \mathbb{N}$ and so $M \otimes_R R/\underline{b}$ is indecomposable as a $R/\underline{b} \cong A/\underline{b}A$-module. This means $M \otimes_R A/\underline{b}(M \otimes_R A)$ is indecomposable, whence $M \otimes_R A$ is indecomposable by Nakayama's Lemma. Thus the base change functor $- \otimes_R A$ defines a function

$$\varphi: \mathrm{IMB}(R) \longrightarrow \mathrm{IMB}(A)$$

Now we show that $\varphi$ is one-to-one. If $M, N \in \mathrm{IMB}(R)$ are such that $M \otimes_R A \cong N \otimes_R A$, then

$$M/\underline{b}M \cong (M \otimes_R A)/(M \otimes_R \underline{b}A) \cong (N \otimes_R A)/(N \otimes_R \underline{b}A) \cong N/\underline{b}N$$

and so $M \cong N$, because $\underline{b}$ is a MB-reduction ideal.

Lastly, for every MB A-module N and for every $\underline{p} \in \mathrm{Supp}_A N \cap X(A)$ we have by [Go, (4.1)] that $N_{\underline{p}}$ is a Cohen-Macaulay $A_{\underline{p}}$-module of $\dim_{A_{\underline{p}}} N_{\underline{p}} = \dim_A N - \dim A/\underline{p} =$ $= \dim A - \dim A/\underline{p} = \dim A_{\underline{p}}$. Thus N is locally free on Reg A and so $\varphi$ must be onto by [El, Th. 3].

**(4.9) THEOREM.** Suppose that R is an excellent henselian reduced Cohen-Macaulay ring with perfect residue field. Let A be the completion of R with respect to $I_s(R)$. If R contains a field, then the base change functor $- \otimes_R A$ induces a bijection $\mathrm{IMB}(R) \longrightarrow \mathrm{IMB}(A)$.

**(4.10) REMARK.** We discuss, in broad outline, how the above result may be improved.

Firstly, one must impose conditions on the ring R such that a well-defined map $\varphi: IMB(R) \longrightarrow IMB(A)$ is induced by the base change functor $- \otimes_R A$. To this end one may use the Artin approximation property as in [PR, (3.4) - (3.8)]. If R is an excellent henselian ring, then from [PR, (3.8)] it results $\varphi$ is one-to-one.

Secondly, Elkik's theorem implies $\varphi$ is a surjection, provided one could show that every MB A-module is locally free on the regular locus. As above, this is the case if A is a generalized Cohen-Macaulay ring. Since the various local cohomology modules of A are isomorphic to the extensions of the corresponding local cohomology modules of R, it is sufficient that R is a generalized Cohen-Macaulay ring.

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