INTERNAL APPROXIMATION OF QUASI-VARIATIONAL INEQUALITIES

by

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April 1990

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Internal Approximation of Quasi-Variational Inequalities

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Summary. In this paper the internal approximation of a quasi-variational inequality is considered. An algorithm of Bensoussan-Lions type is proposed for which the convergence is proved. These results are applied to Signorini problem with friction for which two error estimates and numerical examples are also given.

Subject Classifications: AMS(MOS): 65N30, 65N15; 49A29

1. Introduction

An important class of variational inequalities consists of the so-called quasi-variational inequalities which were first introduced by Bensoussan and Lions [2], for the study of stochastic impulse control problems. There after, the quasi-variational inequalities have been used by many authors to study nonlinear problems in continuum mechanics including a wide variety of free-boundary problems (flow of fluids through porous media, unilateral contact problems with friction, etc.).

The present work is concerned with the numerical analysis of the quasi-variational inequalities of the second kind. In Sect.2 we consider an abstract quasi-variational inequality for which we prove the existence and uniqueness of the solution (Theorem 2.1). From the existence proof an algorithm of Bensoussan-Lions type for the numerical approximation of the problem follows. The convergence of this algorithm is proved in Sect.3.

In Sect.4 we consider an unilateral contact problem known as Signorini problem with non-local friction [6], [7], whose variational formulation is a quasi-variational inequality as formulated in Sect.2. In Sect.4.1 we apply the results of Sects.2 and 3 to this problem. We also derive, in Sect.4.2, two error estimates of the finite element approximation of this problem. An analysis of some numerical results is made in Sect.4.3.

2. A General Quasi-Variational Inequality

Let $(V,\|\cdot\|)$ a real reflexive Banach space, $(V',\|\cdot\|')$ its dual and $(\cdot,\cdot)$ the duality product between V' and V. Let $K\subset V$ be a non-empty convex closed subset of V.

Let us consider an operator $A:V\longrightarrow V'$ such that:

$$(Au-Av,u-v)\geqslant m\|u-v\|^2 \quad ,\forall\, u,v\in V \,, \tag{2.1}$$

$$\|Au-Av\|'\leqslant M\|u-v\|^2 \quad ,\quad \forall\, u,v\in V \,, \tag{2.2}$$

where m and M are positive constants.

Let $j:V\times V\longrightarrow(-\infty,+\infty]$ satisfying the following conditions:

$\forall\, u\in V$, $j(u,\cdot)$ is convex, continuous in its effective domain and proper on K , (2.3)

$$|j(u_1,v_2)+j(u_2,v_1)-j(u_1,v_1)-j(u_2,v_2)|\leqslant$$
$$\leqslant c\|u_1-u_2\|\cdot\|v_1-v_2\| \,,\forall u_1,u_2,v_1,v_2\in K \tag{2.4}$$

where c is a positive constant.

We consider the following quasi-variational inequality: find $u\in K$ such that

$$(Au,v-u)+j(u,v)-j(u,u)\geqslant L(v-u) \quad ,\forall\, v\in K \tag{2.5}$$

where $L\in V'$ is given.

In order to justify the application of an algorithm of Bensoussan-Lions type to (2.5), we shall prove the following theorem concerning the existence and uniqueness of solutions of quasi-variational inequalities.

Theorem 2.1. Let conditions (2.1)-(2.4) above hold with $c<m$. Then there exists a unique solution of the problem (2.5).

Proof: Let us denote by S the mapping $S:K \longrightarrow K$ which associates to every $w \in K$ the solution, that there exists and is unique (see, e.g.[3]), of the following variational inequality of the second kind:

$$(A(Sw),v-Sw)+j(w,v)-j(w,Sw) \geqslant L(v-Sw) \quad , \forall v \in K. \qquad (2.6)$$

We remark that the set of fixed points of S coincides with the set of solutions of the inequality (2.5). Therefore the question of the existence and uniqueness of solutions of (2.5) reduces to the existence and uniqueness of fixed points of S.

Now we show that S is a contraction. Indeed, for $w_1, w_2 \in K$ arbitrarily, from (2.6) we obtain by using (2.1) and (2.4):

$$\| Sw_1 - Sw_2 \| \leqslant k \| w_1 - w_2 \| \qquad (2.7)$$

where $k = \frac{c}{m} < 1$.

Hence the mapping S has a unique fixed point. Therefore there exists a unique solution of (2.5).

We define the sequence: for $u^o \in K$ chosen arbitrarily, we put $u^n = Su^{n-1}$ i.e. u^n satisfies the inequality:

$$(Au^n, v-u^n)+j(u^{n-1},v)-j(u^{n-1},u^n) \geqslant L(v-u^n) \quad , \forall v \in K. \qquad (2.8)$$

Using (2.7) we obtain:

$$\| u^n - u \| \leqslant k^n \| u^o - u \| \leqslant C \cdot k^n \qquad (2.9)$$

where u is the unique solution of the inequality (2.5) and C is a positive constant independent of n.

3. Internal Approximation and Convergence Result

We shall give an internal approximation of the quasi-variational inequality (2.5) [10].

We consider a parameter h converging to 0 and a family $(V_h)_h$ of finite dimensional subspaces of V.

Let $(K_h)_h$ be a family of non-empty convex closed

subsets of V_h which approximates K in the sense that:

(i) $\forall\, v \in K, \exists\, r_h v \in K_h$ such that $r_h v \longrightarrow v$ strongly in V,
(ii) $\forall\, v_h \in K_h$ with $v_h \rightharpoonup v$ weakly in V, then $v \in K$.

We shall approximate the functional $j(\cdot\,,\cdot)$ by the family $(j_h)_h$ which, for every $u \in V$, satisfies the following conditions [9]:

$$\left.\begin{array}{l}\text{for each } h,\ j_h(u,\cdot): V_h \longrightarrow (-\infty,+\infty] \text{ is convex and lower} \\ \text{semicontinuous}\end{array}\right\} \quad (3.1)$$

$$\left.\begin{array}{l}\text{the family } (j_h(u,\cdot))_h \text{ is uniformly proper in } h \text{ i.e.} \\ \exists\, \lambda=\lambda(u) \in V',\ \exists\, \mu=\mu(u) \in R \text{ such that} \\ \qquad j_h(u,v_h) \geqslant \lambda(v_h)+\mu,\ \forall\, v_h \in V_h,\ \forall\, h.\end{array}\right\} \quad (3.2)$$

$$\left.\begin{array}{l}\text{for every } v_h \in V_h \text{ with } v_h \rightharpoonup v \text{ weakly in V then} \\ \qquad \liminf\limits_{h\to 0} j_h(u,v_h) \geqslant j(u,v)\end{array}\right\} \quad (3.3)$$

$$\liminf_{h\to 0} j_h(u,r_h v) = j(u,v),\ \forall\, v \in K. \quad (3.4)$$

Furthermore we assume that, for every h, we have:

$$|j_h(u_h^1,v_h^2)+j_h(u_h^2,v_h^1)-j_h(u_h^1,v_h^1)-j_h(u_h^2,v_h^2)| \leq$$
$$\leq c\|u_h^1-u_h^2\|\cdot\|v_h^1-v_h^2\|,\ \forall\, u_h^1,u_h^2 \in K_h,\ \forall\, v_h^1,v_h^2 \in K_h. \quad (3.5)$$

Now, we formulate the following discrete problem: find $u_h \in K_h$ such that:

$$(Au_h, v_h-u_h)+j_h(u_h,v_h)-j_h(u_h,u_h) \geqslant L(v_h-u_h) \quad ,\forall\, v_h \in K_h \quad (3.6)$$

Further on we assume that the conditions of Theorem 2.1 hold. Applying similar arguments as in Sect.2, it results that the mapping $S_h: K_h \longrightarrow K_h$ which associates to every $w_h \in K_h$ the element $S_h w_h \in K_h$ defined by:

$$(A(S_h w_h), v_h-S_h w_h)+j_h(w_h,v_h)-j_h(w_h,S_h w_h) \geqslant L(v_h-S_h w_h),\forall\, v_h \in K_h$$

is a contraction. Therefore we have the following result:

Proposition 3.1. The discrete quasi-variational inequality (3.6) has a unique solution $u_h \in K_h$.

Let $(u_h^o)_h$ be an uniformly bounded sequence such that $u_h^o \in K_h$. From Proposition 3.1 it follows that we may define the sequence:

$$u_h^n = S_h u_h^{n-1} \quad , \; n \geq 1 \; , \tag{3.7}$$

hence we have:

$$\| u_h^n - u_h \| \leq k^n \| u_h^o - u_h \| \leq C \cdot k^n \tag{3.8}$$

where u_h is the solution of (3.6) and C is a positive constant independent of h and n.

In order to obtain the convergence of $(u_h)_h$ to u we define an auxiliary sequence of problems: for $w_h^o \in K_h$ given such that $(w_h^o)_h$ is uniformly bounded, we denote by $w_h^n \in K_h$ the solution, that there exists and is unique, of the problem:

$$(Aw_h^n, v_h - w_h^n) + j_h(u^{n-1}, v_h) - j_h(u^{n-1}, w_h^n) \geq L(v_h - w_h^n) \quad , \forall v_h \in K_h \tag{3.9}$$

where $u^{n-1} \in K$ is defined by (2.8).

We remark that (3.9) is an internal approximation of the elliptic variational inequality of the second kind (2.8). So, we have the following result [9].

Proposition 3.2. The sequence $(w_h^n)_h$ defined by (3.9) approximates the solution u^n of (2.8) in the sense:

$$w_h^n \longrightarrow u^n \text{ strongly in } V \text{ as } h \longrightarrow o.$$

We are now prepared to prove the main result of this paper.

Theorem 3.1. Let u and u_h be the solution of (2.5) and (3.6), respectively. Then:

$$u_h \longrightarrow u \text{ strongly in } V \text{ as } h \longrightarrow o.$$

Proof: We observe that we have:

$$\| u_h - u \| \leq \| u_h - u_h^n \| + \| u_h^n - u^n \| + \| u^n - u \| \quad , \forall \, n \geq o. \tag{3.10}$$

In order to estimate the second term in the right-hand side of (3.10) we first deduce from the definition of u_h^n and w_h^n that:

$$m\,\|u_h^n-w_h^n\|^2 \leq (Aw_h^n-Au_h^n, w_h^n-u_h^n) \leq$$
$$\leq j_h(u^{n-1},u_h^n)+j_h(u_h^{n-1},w_h^n)-j_h(u^{n-1},w_h^n)-j_h(u_h^{n-1},u_h^n),$$

from which, using (3.5), we obtain:

$$\|u_h^n-w_h^n\| \leq \|u_h^{n-1}-u^{n-1}\|. \tag{3.11}$$

By choosing $w_h^o=u_h^o$, we shall prove by recurrence that:

$$\|u_h^n-u^n\| \leq \sum_{i=o}^{n} \|w_h^i-u^i\| \quad , \ \forall\, n \geqslant 0. \tag{3.12}$$

Indeed, for n=o the result is evident. If we suppose that (3.12) holds for n-1, then we have:

$$\|u_h^n-u^n\| \leq \|u_h^n-w_h^n\| + \|w_h^n-u^n\| <$$
$$< \|u_h^{n-1}-u^{n-1}\| + \|w_h^n-u^n\| \leq \sum_{i=o}^{n} \|w_h^i-u^i\|$$

where we have used (3.11). It follows that (3.12) holds for every $n \geqslant 0$.

By (3.8) and (2.9), for $\varepsilon > 0$ given, there exists $N_\varepsilon > 0$ such that:

$$\|u_h^n-u_h\| + \|u^n-u\| \leq \frac{\varepsilon}{2} \quad , \ \forall\, n \geqslant N_\varepsilon. \tag{3.13}$$

Choosing $n=N_\varepsilon$ in (3.10) we have by using (3.12) and (3.13):

$$\|u_h-u\| \leq \frac{\varepsilon}{2} + \sum_{i=o}^{N_\varepsilon} \|w_h^i-u^i\| \ . \tag{3.14}$$

But, from Proposition 3.2 it results that, for every i, there exists $H_\varepsilon^i > 0$ such that:

$$\|w_h^i-u^i\| \leqslant \frac{\varepsilon}{2(N_\varepsilon+1)} \quad , \ \forall\, h \leqslant H_\varepsilon^i. \tag{3.15}$$

Concluding, from (3.14) and (3.15), for $\varepsilon > 0$ given, there exists $H_\varepsilon = \min\{H_\varepsilon^i, i=0,\ldots,N_\varepsilon\}$ such that:

$$\|u_h-u\| \leq \varepsilon \quad , \ \forall\, h \leq H_\varepsilon \ ,$$

hence $u_h \to u$ strongly in V as $h \to 0$.

4. Application to the Signorini Problems with Friction

4.1. Setting of the Problem and its Approximate

We shall consider the problem of finding the field of displacements in a linearly elastic body which is in unilateral contact with a rigid support following a non-local friction law [7], [13].

In order to give a variational formulation of this problem, let Ω be the bounded Lipschitzian domain in R^p, $p=2,3$, occupied by the body in the initial unstressed state. Let us denote by Γ the boundary of Ω and let $\Gamma_0, \Gamma_1, \Gamma_2$ be open and disjoint parts of Γ such that $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and $\Gamma_2 \neq \emptyset$.

We introduce the following space:

$$V = \left\{ v \in [H^1(\Omega)]^p;\ v=0 \text{ a.e. on } \Gamma_0 \right\}, \tag{4.1}$$

which is a Hilbert space with the scalar product of $[H^1(\Omega)]^p$. We shall denote by $\|\cdot\|$ its associated norm.

We shall use the following notations for the normal and tangential components of the displacements and of the stress vector, respectively:

$$u_n = u_i n_i, \quad u_{ti} = u_i - u_n n_i,$$
$$\sigma_n = \sigma_{ij} n_i n_j, \quad \sigma_{ti} = \sigma_{ij} n_j - \sigma_n n_i,$$

where $n=(n_i)$ is the outward normal unit vector on Γ.

If we denote by K the following non-empty closed convex subset of V:

$$K = \left\{ v \in V;\ v_n \leq 0 \text{ a.e. on } \Gamma_2 \right\} \tag{4.2}$$

then it is known [8], [13] that a variational formulation of the Signorini problem with non-local friction law is as follows: find $u \in K$ such that

$$a(u, v-u) + j(u,v) - j(u,u) \geq L(v-u), \ \forall v \in K \tag{4.3}$$

where

$$a(u,v)=\int_{\Omega}\sigma_{ij}(u)\varepsilon_{ij}(v)\,dx \tag{4.4}$$

$$j(u,v)=\int_{\Gamma_2}\mu|R(\sigma_n(Pu))||v_t|\,ds \tag{4.5}$$

$$L(v)=\int_{\Omega}f_iv_i\,dx+\int_{\Gamma_1}t_iv_i\,ds \tag{4.6}$$

where $\varepsilon_{ij}(v)=\frac{1}{2}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})$ is the strain tensor, $\sigma_{ij}(v)$ is the stress tensor related to $\varepsilon_{ij}(v)$ by means of generalized Hooke's law:

$$\sigma_{ij}(v)=a_{ijkh}\varepsilon_{kh}(v) \quad \text{in } \Omega,$$

$\mu\in L^{\infty}(\Gamma_2)$ is the coefficient of friction such that $\mu\geqslant 0$ a.e. on Γ_2, R is a linear and continuous mapping from $H^{-\frac{1}{2}}(\Gamma)$ to $L^2(\Gamma_2)$ where $H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H^{\frac{1}{2}}(\Gamma)$, $f\in[L^2(\Omega)]^p$ is the body force, $t\in[H^{\frac{1}{2}}_o(\Gamma_1)]^p$ is the prescribed surface traction where $H^{\frac{1}{2}}_o(\Gamma_1)$ is the space of all functions w of $H^{\frac{1}{2}}(\Gamma)$ such that $w/_{\Gamma\setminus\Gamma_1}=0$ provided with the norm of $H^{\frac{1}{2}}(\Gamma)$. We also have denoted by P the projection map of V onto K_o where

$$K_o=\left\{v\in V;\ a(v,\varphi)=f(\varphi)\ ,\forall\varphi\in[\mathcal{D}(\Omega)]^p\right\},$$

$$f(\varphi)=\int_{\Omega}f_i\varphi_i\,dx.$$

We remark that if u is the solution of (4.3) then Pu=u.

In the above notations we have used the summation convention.

Suppose that the elasticity coefficients of the body a_{ijkh} satisfy the symmetry condition:

$$a_{ijkh}=a_{jikh}=a_{khij}$$

and that $a_{ijkh}\in C^1(\bar{\Omega})$ where $C^1(\bar{\Omega})$ denotes the once continuously differentiable functions on $\bar{\Omega}$.

Suppose in addition that $mes(\Gamma_o)>0$ from which it follows, by Korn's inequality, that the bilinear and continuous form $a(\cdot,\cdot)$ is V-elliptic i.e. there exists

$m > 0$ such that

$$a(v,v) \geq m\|v\|^2, \forall v \in V . \tag{4.7}$$

It is easy to verify that j defined by (4.5) satisfies (see, e.g. [5]):

$$|j(u_1,v_2)+j(u_2,v_1)-j(u_1,v_1)-j(u_2,v_2)| \leq$$
$$\leq c_0\|\mu\|_{L^\infty(\Gamma_2)}\|u_1-u_2\|\cdot\|v_1-v_2\| , \forall u_1,u_2,v_1,v_2 \in \underline{K}. \tag{4.8}$$

Theorem 4.1. Let V, K, a, j, and L defined by (4.1), (4.2), (4.4)-(4.6). Then there exists $\mu_1 > 0$ such that for every $\mu \geq 0$ with $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$ the inequality (4.3) has a unique solution.

Proof: From (4.8) it follows that j satisfies (2.4) with $c = c_0\|\mu\|_{L^\infty(\Gamma_2)}$. Taking $\mu_1 > 0$ such that $\mu_1 < \frac{m}{c_0}$ it results $c < m$ for every $\mu \geq 0$ with $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$.

Let us define $A: V \longrightarrow V'$ by

$$(Au,v) = a(u,v) , \forall u,v \in V. \tag{4.9}$$

We observe that V, K, A, j and L defined by (4.1), (4.2), (4.9), (4.5) and (4.6), respectively, satisfy the conditions of Theorem 2.1 from which the result follows.

We may also apply Proposition 3.1 and Theorem 3.1 for every internal approximation (V_h, K_h, j_h) of (V, K, j) which satisfies the conditions given in Sect. 3, to obtain the following result.

Proposition 4.2. Suppose that $\|\mu\|_{L^\infty(\Gamma_2)} < \frac{m}{c_0}$. Then the approximate quasi-variational inequality of (4.3) has a unique solution $u_h \in K_h$. Moreover, we have $u_h \longrightarrow u$ strongly in V as $h \longrightarrow 0$, where u is the unique solution of (4.3).

4.2. Finite Element Error Estimates

In the following we derive two error estimates for a finite element approximation of (4.3).

Let us begin by making additional assumptions about Γ_0, V_h and K_h. Assume that $\Gamma_0 \in C^\infty$.

Suppose that there exists an operator $\pi_h : V \longrightarrow V_h$ such that:

$$\|\pi_h v - v\| \leq C\, h \|v\|_2 \quad , \tag{4.10}$$

$$\|\pi_h v - v\|_{s,\Gamma} \leq C\, h^{\frac{3}{2}-s} \|v\|_2 \quad , \; s = -\tfrac{1}{2}, 0 \; , \tag{4.11}$$

for every $v \in [H^2(\Omega)]^p \cap V$, where $\|\cdot\|_{s,\Gamma}$ and $\|\cdot\|_2$ denote the norms on $[H^s(\Gamma)]^p$ and $[H^2(\Omega)]^p$, respectively (see, e.g. [1], [12]).

We also suppose that

$$K_h \subset K \quad , \tag{4.12}$$

$$\pi_h u \in K_h \tag{4.13}$$

where u is the solution of (4.3).

Remark. It is known that if V_h is defined by

$$V_h = \left\{ v \in V \cap [C^0(\bar{\Omega})]^p ;\; v/_T \in [P_k]^p, \forall T \in \mathcal{T}_h \right\} ,$$

where $\mathcal{T}_h$ is a regular triangulation on $\bar{\Omega}$ such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$$

and P_k is the space of all polynomials of degree less than k in the variables $x_1, \ldots, x_p$ with $k \geq 1$, then (4.10) and (4.11) are satisfied if $\pi_h v$ denotes, as usual, the V_h-interpolant of the function $v \in V$ [4]. Next, defining K_h as in [11], (4.12) and (4.13) hold.

We consider the following finite element approximation of (4.3): find $u_h \in K_h$ such that

$$a(u_h, v_h - u_h) + j(u_h, v_h) - j(u_h, u_h) \geq L(v_h - u_h) \; , \forall v_h \in K_h . \tag{4.14}$$

Remark. It is easy to verify that $j_h=j$ satisfies the assumptions (3.1)-(3.5).

In the following we denote by u_h the solution of (4.14) and by u the solution of (4.3).

Theorem 4.1. Suppose that the conditions (4.10)-(4.13) hold and that $\|\mu\|_{L^\infty(\Gamma_2)} < \frac{m}{c_o}$. If $u \in [H^2(\Omega)]^p \cap K$ then:

$$\|u_h-u\| \leq C\, h^{3/4}\, \|u\|_2 \tag{4.15}$$

where C is a positive constant independent of h.

Proof: Taking $v=u_h$ in (4.3) and $v_h=\pi_h u$ in (4.14) we obtain:

$$m\|u_h-u\|^2 \leq a(u-u_h,u-u_h) \leq a(u_h-u,\pi_h u-u)+a(u,\pi_h u-u)-$$
$$-L(\pi_h u-u)+j(u,u_h)+j(u_h,\pi_h u)-j(u_h,u_h)-j(u,\pi_h u)+$$
$$+j(u,\pi_h u)-j(u,u). \tag{4.16}$$

Taking into account that $u \in [H^2(\Omega)]^p$ it follows that:

$$a(u,\pi_h u-u)-L(\pi_h u-u)= \int_\Gamma \sigma_{ij}(u)n_j(\pi_h u-u)_i ds- \int_{\Gamma_1} t_i(\pi_h u-u)_i ds \leq$$
$$C_1'(\|\sigma\cdot n\|_{\frac{1}{2},\Gamma}+\|t\|_{\frac{1}{2},\Gamma})\|\pi_h u-u\|_{-\frac{1}{2},\Gamma} \leq C_1\|u\|_2\|\pi_h u-u\|_{-\frac{1}{2},\Gamma} \tag{4.17}$$

where we have used Green's formula, the notation $(\sigma\cdot n)_i= \sigma_{ij}(u)\, n_j$, $i=1,\dots,p$, the relation

$$\|t\|_{\frac{1}{2},\Gamma} \leq \|\sigma\cdot n\|_{\frac{1}{2},\Gamma}$$

and trace theorem.

We also have

$$j(u,\pi_h u)-j(u,u) \leq C_2'\|R(\sigma_n(u))\|_{o,\Gamma_2}\|\pi_h u-u\|_{o,\Gamma_2} \leq$$
$$\leq C_2''\|\sigma_n(u)\|_{-\frac{1}{2},\Gamma}\|\pi_h u-u\|_{o,\Gamma_2} \leq C_2\|u\|_2\|\pi_h u-u\|_{o,\Gamma}\,. \tag{4.18}$$

Note that for simplicity we don't distinguish in writing the norms in $H^{-\frac{1}{2}}(\Gamma)$ and $[H^{-\frac{1}{2}}(\Gamma)]^p$.

Substitution of (4.17), (4.18) and (4.8) into (4.16) yields by the continuity of a:

$$(m-c)\|u_h-u\|^2 \leq (M+c)\|u_h-u\|\|\pi_h u-u\|+C_1\|u\|_2\|\pi_h u-u\|_{-\frac{1}{2},\Gamma}+$$

$$+C_2\|u\|_2\|\pi_h u-u\|_{o,\Gamma}: \qquad (4.19)$$

Using Young's inequality:

$$ab \leq \frac{\varepsilon a^2}{2}+\frac{b^2}{2\varepsilon} \quad , \forall \varepsilon > 0, \ \forall a,b \in R$$

for $\varepsilon < \frac{2(m-c)}{M+c}$, we obtain from (4.19):

$$(m-c-\frac{M+c}{2}\varepsilon)\|u_h-u\|^2 \leq \frac{M+c}{2\varepsilon}\|\pi_h u-u\|^2+C_1\|u\|_2\|\pi_h u-u\|_{-\frac{1}{2},\Gamma}+$$

$$+C_2\|u\|_2\|\pi_h u-u\|_{o,\Gamma} .$$

Therefore, by (4.10) and (4.11), the estimate (4.15) follows.

We will prove that a higher order of approximation in (4.15) can be obtained by choosing a suitable R.

Let the mapping R be given by the convolution:

$$R(\varphi)=\omega*\varphi \quad , \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \qquad (4.20)$$

where $\omega \in \mathcal{D}(-\delta,\delta)$ with $\delta \in R, \delta > 0$, such that $\int_{-\delta}^{\delta} \omega(t)\, dt=1$ (see, e.g. [13]).

It follows that, if the solution u of (4.3) is in $[H^2(\Omega)]^p$ then $R(\sigma_n(u)) \in H^{\frac{1}{2}}(\Gamma)$. So, we have:

$$j(u,\pi_h u)-j(u,u) \leq C_3\|R(\sigma_n(u))\|_{\frac{1}{2},\Gamma}\|\pi_h u-u\|_{-\frac{1}{2},\Gamma} \qquad (4.21)$$

From the definition of the norm on $H^{\frac{1}{2}}(\Gamma)$:

$$\|\psi\|_{\frac{1}{2},\Gamma} = \inf\left\{ \|v\| \ ; \ v \in H^1(\Omega), \psi = \gamma v \right\} ,$$

where γ is the trace operator, we have:

$$\|R(\sigma_n(u))\|^2_{\frac{1}{2},\Gamma} \leq \|\omega*\sigma_n(u)\|^2 =$$

$$= \int_\Omega \left[\int_\Gamma \omega(|x-y|)(\sigma_n(u))(y)\, dy\right]^2 dx +$$

$$+ \sum_{i=1}^{p} \int_\Omega \left[\int_\Gamma \frac{\partial}{\partial x_i}\omega(|x-y|)(\sigma_n(u))(y)\, dy\right]^2 dx \leq$$

$$\leq C_4 \|\sigma_n(u)\|^2_{0,\Gamma} \ . \tag{4.22}$$

Now, from (4.16),(4.17),(4.21) and (4.22) we obtain the following result.

Theorem 4.2. Let the conditions of Theorem 4.1 hold. Suppose in addition that R is given by (4.20) and that

$$\| |\pi_h v - v| \|_{-\frac{1}{2},\Gamma} \leq C\, h^{3/2} \|v\|_2 \ , \ \forall\, v \in [H^2(\Omega)]^p \cap V.$$

Then, if $u \in [H^2(\Omega)]^p \cap K$, we have the following estimate:

$$\| u_h - u \| \leq C\, h \|u\|_2 \ .$$

4.3. Numerical Examples

Let us consider a plane linear elastic body which in the initial unstressed state occupies the domain $\Omega = (0,16)\times(0,8)$. For comparison purposes we have considered problems for the same Ω with different values of tractions and coefficients of friction.

The decomposition of the boundary $\partial\Omega$ into Γ_0, Γ_1 and Γ_2 is given by

$$\Gamma_0 = \{(x_1,x_2) \in R^2;\ 0 < x_2 < 8,\ x_1 = 0\} \ ,$$
$$\Gamma_1 = \Gamma_1' \cup \Gamma_1'' \quad \text{where}$$
$$\Gamma_1' = \{(x_1,x_2) \in R^2;\ 0 < x_1 \leq 16,\ x_2 = 8\} \ ,$$
$$\Gamma_1'' = \{(x_1,x_2) \in R^2;\ 0 < x_2 < 8,\ x_1 = 16\} \ ,$$
$$\Gamma_2 = \{(x_1,x_2) \in R^2;\ 0 < x_1 < 16,\ x_2 = 0\} \ .$$

We suppose that the body is homogeneous isotropic and is characterized by a Young's modulus of $E = 10^6$ and a Poisson's ratio of $\nu = 0,3$. We have considered plane stress problems.

Let $(\mathcal{T}_h)_h$ be a regular family of triangulations of $\bar{\Omega}$ such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \ .$$

Let us denote by Σ_h the set of nodal points of the triangulation $\mathcal{T}_h$ of $\bar{\Omega}$.

Let V_h, K_h be the finite element approximations of the space V and of the convex set K, respectively, defined by:

$$V_h = \left\{ v_h = (v_h^1, v_h^2) \in [C^0(\bar{\Omega})]^2;\ v_h(a_i) = 0, \forall a_i \in \Gamma_0 \cap \Sigma_h,\ v_h/_T \in [P_1]^2, \forall T \in \mathcal{T}_h \right\},$$

$$K_h = \left\{ v_h = (v_h^1, v_h^2) \in V_h;\ v_h^2(a_i) \geq 0, \forall a_i \in \Gamma_2 \cap \Sigma_h \right\}.$$

In order to solve the discrete problem (4.14) which approximates the given problem (4.3) it suffices, as we have seen in Sect.3, to solve the following sequence of discrete variational inequalities: find $u_h^n \in K_h$ such that

$$a(u_h^n, v_h - u_h^n) + j(u_h^{n-1}, v_h) - j(u_h^{n-1}, u_h^n) \geq L(v_h - u_h^n), \forall v_h \in K_h \quad (4.23)$$

for $u_h^0 \in K_h$ given.

We remark that (4.23) is a Signorini problem with "given friction" which is equivalent with a nondifferentiable optimization problem. For this reason it is advantageous to use the following saddle point formulation [11]: find $(u^n, p^n) \in K \times \Lambda^n$ such that

$$\mathcal{L}^n(u^n, q) \leq \mathcal{L}^n(u^n, p^n) \leq \mathcal{L}^n(v, p^n), \forall v \in K, \forall q \in \Lambda^n \quad (4.24)$$

where

$$\mathcal{L}^n(v,q) = \frac{1}{2} a(v,v) - L(v) + \int_{\Gamma_2} q\, g^n v_t \, ds \ ,$$

$$g^n = \mu |R(\sigma_n(u^{n-1}))| \ ,$$

$$\Lambda^n = \left\{ q \in L^2(\Gamma_2);\ |q| \leq 1 \text{ a.e. on supp } g^n, q = 0 \text{ on } \Gamma_2 \setminus \text{supp } g^n \right\}.$$

For simplicity, we have omitted the subscript h.

We have applied Uzawa's algorithm to resolve the problem (4.24).

Three numerical examples have been solved by the finite element approximation discussed in above, assuming the absence of body forces i.e. f=0. In the first example we take the traction t defined by t=(0,0) on Γ_1 and t=(500,0)

on Γ_1''. In the second example we consider $t=(0,-300)$ on Γ_1' and $t=(500,0)$ on Γ_1'' and in the last example $t=(500,-300)$ on Γ_1' and $t=(0,0)$ on Γ_1''.

We decomposed $\bar{\Omega}$ into 64 triangular finite elements as is shown in figure 1.

Fig.1. The finite element mesh

In all these examples we are particularly interested in showing the influence of the coefficient of friction on the tangential displacements on Γ_2 as is illustrated in figures 2-4.

Fig.2. The tangential displacements in example 1.

Fig.3. The tangential displacements in example 2.

Fig.4. The tangential displacements in example 3.

For this purpose the coefficients of friction were be taken equal 0.2 , 0.4 and 0.6 respectively.

To initialize the process defined by (4.23) we have take u_h^o as being the unique solution of the classical Signorini problem:

$$a(u_h^o, v_h - u_h^o) \geqslant L(v_h - u_h^o) \ , \ \forall \ v_h \in K_h \ ,$$

which corresponds to the case $\mu=0$.

According to expectation, the tangential displacements obtained in example 2 are smaller than those obtained in example 1. Also, in figure 4 one can see the influence of combined tractions on Γ_1' on the tangential displacements on Γ_2.

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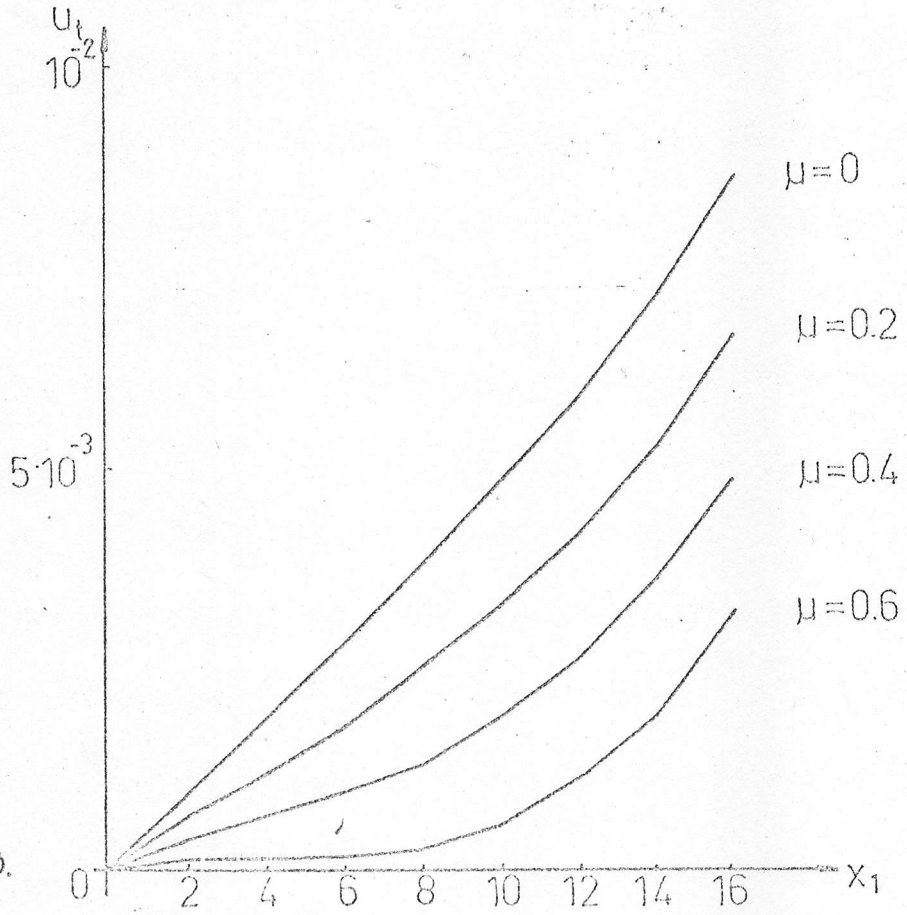
u_1
10^{-2}
$5 \cdot 10^{-3}$
0
2
4
6
8
10
12
14
16
x_1
μ=0
μ=0.2
μ=0.4
μ=0.6

Fig 3.

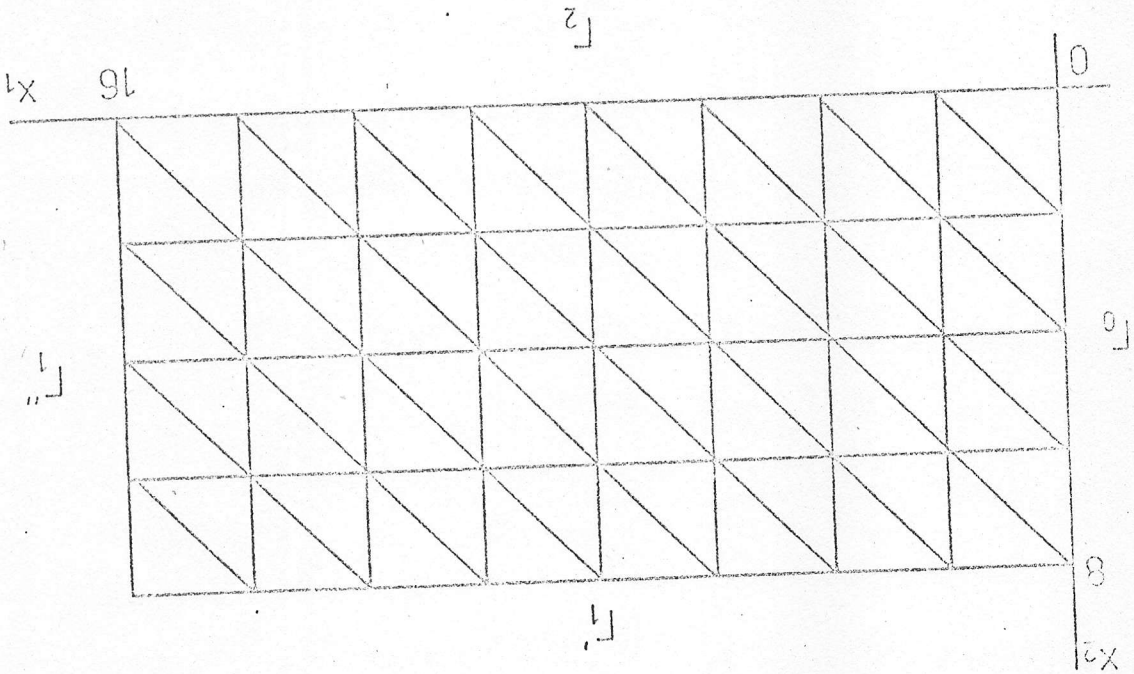
x_2
8
Γ_1'
0
Γ_0
16
x_1
Γ_2

Fig. 4.

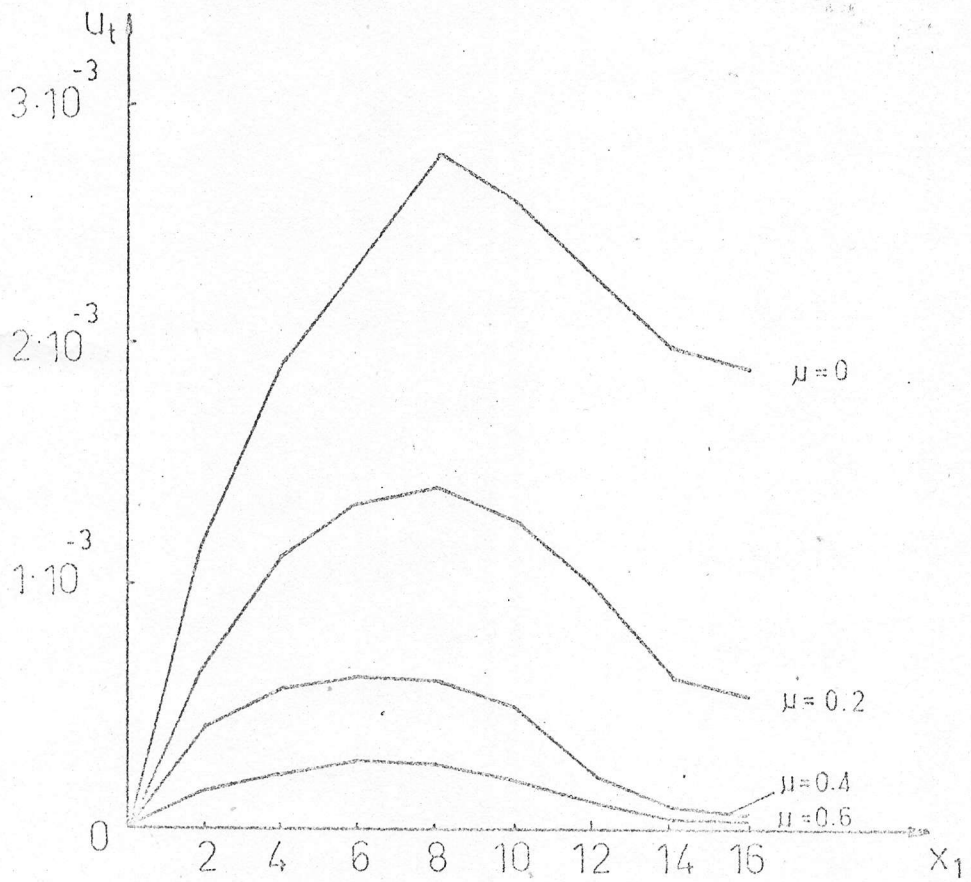
u_t
$3 \cdot 10^{-3}$
$2 \cdot 10^{-3}$
$1 \cdot 10^{-3}$
0
2 4 6 8 10 12 14 16
x_1
$\mu = 0$
$\mu = 0.2$
$\mu = 0.4$
$\mu = 0.6$

Fig. 4

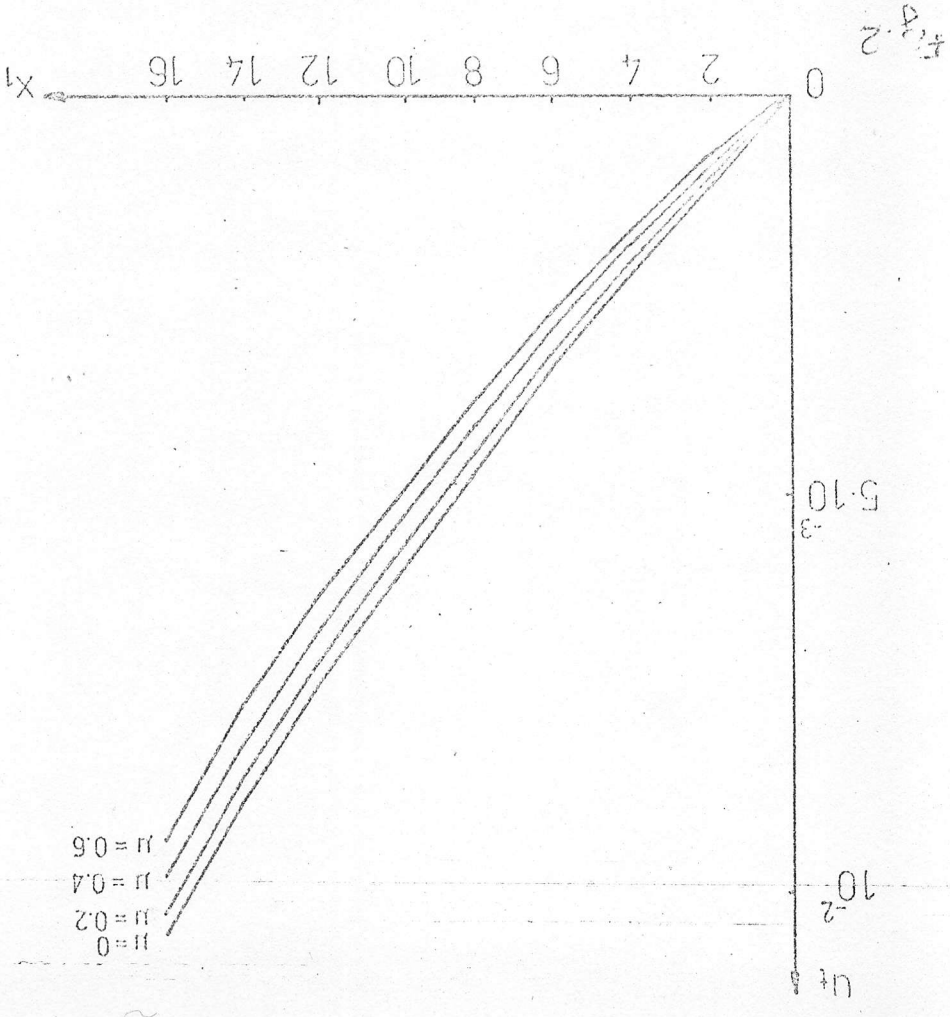
u_t
10^{-2}
$5 \cdot 10^{-3}$
0
2 4 6 8 10 12 14 16
x_1
$\mu = 0$
$\mu = 0.2$
$\mu = 0.4$
$\mu = 0.6$

Fig. 2