INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

GEOMETRY OF DIFFERENTIAL POLYNOMIAL FUNCTIONS, II: ALGEBRAIC CURVES

by

A. BUIUM

PREPRINT SERIES IN MATHEMATICS

No. 1/1990

BUCURESTI

GEOMETRY OF DIFFERENTIAL POLYNOMIAL FUNCTIONS, II: ALGEBRAIC CURVES

by

A. BUIUM*)

January, 1990

*) Department of Mathematics, INCREST, Bd. Pacii 220, 79622 Bucharest, Romania.

GEOMETRY OF DIFFERENTIAL POLYNOMIAL FUNCTIONS. II: ALGEBRAIC CURVES

A. BUIUM

O. INTRODUCTION

This paper is a direct continuation of [B_1]; we shallfreely use the terminology from that paper.

As in [B_1] let $\mathcal{U}$ be an ordinary universal Δ-field of characteristic zero (cf. [K]) with derviation operator δ and field of constants $\mathcal{K}$. For any smooth $\mathcal{U}$-variety X we considered in [B_1] the ring $\mathcal{O}^{\Delta}(X)$ of Δ-polynomial functions on X. This ring has an increasing filtration with subrings $\mathcal{O}^{(n)}(X)$ where for each $n \geq 0$ $\mathcal{O}^{(n)}(X)$ is the ring of Δ-polynomial functions of order at most n (so $\mathcal{O}^{(o)}(X) = \mathcal{O}(X)$). Moreover each ring $\mathcal{O}^{(n)}(X)$ with $n \geq 1$, has a filtration $F^d\mathcal{O}^{(n)}(X)$ with $\mathcal{O}^{(n-1)}(X)$-modules where $F^d\mathcal{O}^{(n)}(X)$ is the space of all Δ-polynomial functions which are locally given by Δ-polynomials in $\mathcal{U}\{y_1,\dots,y_N\}$ (for some N) of order at most n and of degree at most d in the variables $y_1^{(n)},\dots,y_N^{(n)}$ (here and later we write sometimes $x', x'',\dots,x^{(n)},\dots$ insted of $\delta x, \delta^2 x,\dots, \delta^n x,\dots$ whenever x is an element of some Δ-ring). A remarkable easy fact is that the $\mathcal{U}$-linear spaces $F^d\mathcal{O}^{(1)}(X)$ have finite dimension when X is complete. These spaces bare a formal resemblance with pluricannonical systems in algebraic geometry (and indeed as we shall see are related to them at least in the case of curves) so we call them the Δ-pluricanonical spaces of degree d of X. The dimension $N_d = N_d(X)$ of $F^d\mathcal{O}^{(1)}(X)$ will be called the Δ-plurigenus of degree d. It is natural to pick then any basis of $F^d\mathcal{O}^{(1)}(X)$ to produce a Δ-polynomial map

$$\varphi_d : X \longrightarrow \mathcal{U}^{N_d}$$

which we call the Δ-pluricanonical map of degree d and we may hope to discover a "new geometry" of "old varieties" by inspecting these maps. The aim of the present paper is to perform this program in the case of smooth complete curves.

Before stating our results let's define a basic invariant for a smooth complete curve X of genus g over $\mathcal{U}$ which we call the Δ-rank of X and will be denoted by $\mathrm{rank}_{\Delta}(X)$. Let

$$\rho : \mathrm{Der} \longrightarrow H^1(\omega_X^{-1})$$

be the Kodaira-Spencer map associated to $X \longrightarrow \mathrm{Spec}\, \mathcal{U}$ (ω_X denotes as usual the canonical line bundle on X), consider the Kodaira-Spencer class $\rho(\delta) \in H^1(\omega_X^{-1})$ and let $C : H^0(\omega_X) \longrightarrow H^1(\mathcal{O}_X)$ be the map defined by cup-product with $\rho(\delta)$. By definition we let $\mathrm{rank}_{\Delta}(X)$ be the rank of the linear map C. So we always have $0 \leq \mathrm{rank}_{\Delta}(X) \leq g$. If X descends to $\mathcal{K}$ then $\rho(\delta) = 0$ so $\mathrm{rank}_{\Delta}(X) = 0$; conversely if X is nonhyperelliptic and $\mathrm{rank}_{\Delta}(X) = 0$ then using Max Noether's theorem (implying surjectivity of $S^2H^0(\omega_X) \longrightarrow H^0(\omega_X^{\otimes 2}) = H^1(\omega_X^{-1})^0$) we see that $\rho(\delta) = 0$ hence X descends to $\mathcal{K}$ [B_2]. Of course the number $\mathrm{rank}_{\Delta}(X)$ equals the Δ-rank of the Jacobian J(X) as defined in [B_1]; so using the results in [B_1], section 7 we see that equality $\mathrm{rank}_{\Delta}(X) = g$ holds for X "Δ-generic" i.e. for X lying outside a certain proper Δ-closed subset of the moduli space $\mathcal{M}_g$ of smooth curves of genus g.

Here are our main results (in which X is a smooth complete curve over $\mathcal{U}$ of genus $g \geq 2$):

THEOREM 1. If X does not descend to $\mathcal{K}$ then for d sufficiently large φ_d is a Δ-closed embedding. Moreover $N_1(X) = g + 1 - \mathrm{rank}_{\Delta}(X)$. If in addition X is non-hyperelliptic with $\mathrm{rank}_{\Delta}(X) = g$ then φ_d is a Δ-closed embedding for $d \geq 3$, we have $N_1(X) = 1$ and $N_d(X) = (g - 1)(d^2 - 1)$ for $d \geq 2$.

THEOREM 2. If X descends to $\mathcal{K}$ then for any $d \geq 1$ we have $N_d(X) = (g-1)d^2 + 2$ and φ_d sends $X_{\mathcal{K}}$ to a point. If X is non-hyperelliptic φ_d is injective outside $X_{\mathcal{K}}$ for $d \geq 1$.

REMARKS. 1) The notion of Δ-closed embedding appearing in Theorem 1 above will be defined in section 1. In any case a Δ-closed embedding is in particular injective and has a Δ-closed image.

2) It might seem odd to an algebraic geometer to see a projective curve embedded into an affine space (even if this is done by Δ-polynomial rather than by regular functions which is of course impossible). This phenomenon is different in nature from that discovered by P. Cassidy that for any projective $\mathcal{U}$-variety X which descends to $\mathcal{K}$, $X_{\mathcal{K}}$ appears as a Δ-closed subset of some $\mathcal{U}^N$. For here we embed the full set of $\mathcal{U}$-points of a variety and we do it in case the variety does not descend to $\mathcal{K}$!

3) Theorem 1 fails for genus $g = 0$ and $g = 1$. Recall from [B_1] that $\mathbf{P}^1$ carries no non-constant Δ-polynomial functions (of any order!) while for any elliptic curve, and more generally for any abelian variety A, all Δ-polynomial functions (of arbitrary order) must factor through the factor group $A/A^{\#}$ (N.B.: $A^{\#}$ is never trivial !); in fact Δ-polynomial functions on A define an injective map from $A/A^{\#}$ to some affine space.

4) It follows from [B_1] that if A is a principally polarized abelian $\mathcal{U}$-variety which is "Δ-generic" in the moduli space $\mathcal{A}_g$ ($g = \dim A$) then all Δ-pluricanonical maps φ_d are trivial (i.e., $N_d(A) = 1$, equivalently φ_d maps A to a point in $\mathcal{U}$). This fails for A not "Δ-generic": if A descends to $\mathcal{K}$ for instance, then $\varphi_1 : A \to \mathcal{U}^{g+1}$ is precisely Kolchin's logarithmic derivative (cf. [K]) $\ell\delta : A \longrightarrow L(A) \simeq \mathcal{U}^g$ followed by an affine embedding $\mathcal{U}^g \subset \mathcal{U}^{g+1}$.

5) There are some features of our Δ-pluricanonical maps which make them behave quite differently from those in algebraic geometry. For instance, due to the fact that the constant functions (i.e. those in $\mathcal{U}$) belong to any Δ-pluricanonical space it follows that the image of any Δ-pluricanonical map $\varphi_d : X \longrightarrow \mathcal{U}^{N_d}$ is contained in a hyperplane of $\mathcal{U}^{N_d}$ not passing through the origin. Note also that if $d < e$ are two integers then φ_d is obtained by composing φ_e with a linear projection $\mathcal{U}^{N_e} \to \mathcal{U}^{N_d}$.

6) One should be able to generalize perhaps the above results from curves of genus $g \geq 2$ to varieties with ample cotangent bundle [MD].

The paper is organized as follows. In section 1 we present some generalities on Δ-pluricanonical maps. The following two sections are devoted to the proof of the two theorems above respectively.

1. -PLURICANONICAL MAPS

(1.1) Let X be a smooth $\mathcal{U}$-variety. We defined in [B_1] the sheaf $\mathcal{O}_X^\Delta$ of Δ-polynomial functions on X and its subsheaves $\mathcal{O}_X^{(n)}$ of Δ-polynomial functions of order at most n. The latter form an increasing filtration of $\mathcal{O}_X^\Delta$ with $\mathcal{O}_X^{(o)} = \mathcal{O}_X$ and $\bigcup_{n\geq 0} \mathcal{O}_X^{(n)} = \mathcal{O}_X^\Delta$. Moreover, if

$$\longrightarrow X^{n+1} \xrightarrow{(f^{n+1}, \delta)} X^n \xrightarrow{(f^n, \delta)} \cdots \longrightarrow X^o = X \longrightarrow X^{-1} = \operatorname{Spec} \mathcal{U}$$

is the infinite prolongation sequence associated to X as in [B_1] (3.1) then we proved that $\mathcal{O}^{(n)}(X) = \mathcal{O}(X^n)$ for all $n \geq 0$; since the infinite prolongation sequence above is "compatible" with restriction to Zariski open subsets, we get $\mathcal{O}_X^{(n)} = \mathcal{O}_{X^n}$ (here and later we shall abusively write $\mathcal{O}_{X^n}$ instead of $\pi_{n*}\mathcal{O}_{X^n}$ where $\pi_n : X^n \longrightarrow X$ is the canonical projection !). Now define for each $d \geq 0$ and $n \geq 0$ the sheaf $F^d\mathcal{O}_X^{(n)}$ by assigning to each open set $X_o \subset X$ the set of all functions $f : X_o \longrightarrow \mathcal{U}$ such that for each $x_o \in X_o$ there is a Zariski affine neighbourhood X_1 of x_o in X_o, a closed embedding $X_1 \subset \mathcal{U}^N$ and a Δ-polynomial $F \in \mathcal{U}\{y_1,\dots,y_N\}$ of order at most n, which viewed as a polynomial in $y_1^{(n)},\dots,y_N^{(n)}$ has degree at most d, such that $f(x) = F(x)$ for all $x \in X_1$. Clearly $F^d\mathcal{O}_X^{(n)}$ form a filtration of $\mathcal{O}_X^{(n)}$ with $\mathcal{O}_X^{(n-1)}$-modules such that $(F^d\mathcal{O}_X^{(n)})(F^e\mathcal{O}_X^{(n)}) \subset F^{d+e}\mathcal{O}_X^{(n)}$ for any integers d, e in particular the sheaf

$Gr_F \mathcal{O}_X^{(n)}$ has a natural structure of sheaf of graded rings. Moreover note that $F^d\mathcal{O}_X^{(n)}$ is the subsheaf of $\mathcal{O}_X^{\Delta}$ generated by products of the form $b_o b_1' \dots b_e'$ for $e \leq d$ where $b_o, b_1, \dots, b_e \in \mathcal{O}_X^{(n-1)}$ (because if $F \in \mathcal{U}\{y_1,\dots,y_N\}$ is a Δ-polynomial of order at most $n - 1$ then F' has degree at most one as a polynomial in $y_1^{(n)},\dots,y_N^{(n)}$) so $F^d\mathcal{O}_X^{(n)}$ can be constructed directly from the sheaf of Δ-rings $\mathcal{O}_X^{\Delta}$ and its subsheaf $\mathcal{O}_X$ by an obvious inductive procedure.

A particular role will be played by the filtration $F^d\mathcal{O}^{(n)}(X)$ on $\mathcal{O}^{(n)}(X)$ defined by $F^d\mathcal{O}^{(n)}(X) = H^o(X, F^d\mathcal{O}_X^{(n)})$; one can associate to it the graded ring $Gr_F \mathcal{O}^{(n)}(X)$.

(1.2) Let Z be any Noetherian separated scheme, E a locally free coherent sheaf on Z and P the $\mathbf{V}(\check{E})$-torsor on Z in the Zariski topology corresponding to some class $\eta \in H^1(Z,\check{E})$ (recall that we put $\mathbf{V}(\check{E}) = \operatorname{Spec} S(E)$). We shall write in what follows abusively $\mathcal{O}_P$ instead of $\pi_* \mathcal{O}_P$ where $\pi : P \longrightarrow Z$ is the canonical projection. Then there is a natural filtration $F^d\mathcal{O}_P$ on $\mathcal{O}_P$ with locally free coherent $\mathcal{O}_Z$-modules such that $F^o\mathcal{O}_P = \mathcal{O}_Z$, $\bigcup_{d\geq 0} F^d\mathcal{O}_P = \mathcal{O}_P$ and $(F^d\mathcal{O}_P)(F^e\mathcal{O}_P) \subset F^{d+e}\mathcal{O}_P$ $(d,e \geq 0)$ and there is a natural isomorphism of sheaves of graded algebras $Gr_F \mathcal{O}_P \simeq S(E)$. The construction of $F^d\mathcal{O}_P$ is the following. Cover Z with affine open sets Z_i and let $\eta_{ij} \in H^1(Z_i \cap Z_j, \check{E})$ represent η. Then P is defined by glueing $\operatorname{Spec} S(E/_{Z_i})$ via the $\mathcal{O}_Z$-isomorphisms

$$\varphi_{ij} : S(E/_{Z_i \cap Z_j}) \longrightarrow S(E/_{Z_i \cap Z_j})$$

$$\varphi_{ij}(e) = e + \langle e, \eta_{ij} \rangle, \quad e \in E/_{Z_i \cap Z_j}$$

Now $S(E/_{Z_i})$ has a natural filtration defined by $F^dS(E/_{Z_i}) = \bigoplus_{k=0}^{d} S^k(E/_{Z_i})$. These filtrations clearly glue together via φ_{ij} to give a filtration $F^d\mathcal{O}_P$. Moreover, if $\sigma \in F^dS(E/_{Z_i \cap Z_j})$ then one checks that $\varphi_{ij}(\sigma) - \sigma \in F^{d-1}S(E/_{Z_i \cap Z_j})$. This provides the isomorphism $Gr_F^d \mathcal{O}_P \simeq S^d(E)$. In particular if we define a filtration on

$\mathcal{O}(P)$ by $F^d\mathcal{O}(P) = H^o(Z, F^d\mathcal{O}_P)$ then $\bigcup_{d\geq 0} F^d\mathcal{O}(P) = \mathcal{O}(P)$ and the exact sequence

$$0 \to F^{d-1}\mathcal{O}_P \longrightarrow F^d\mathcal{O}_P \longrightarrow S^d(E) \longrightarrow 0$$

provides a natural injective map of graded algebras

$$\mathrm{Gr}_F\,\mathcal{O}(P) \longrightarrow \bigoplus_{d\geq 0} H^o(Z, S^d(E)) .$$

Note that $F^d\mathcal{O}_P$ is the subsheaf of $\mathcal{O}_P$ generated by all products of the form $f_1 \dots f_e$ where $e \leq d$, $f_1, \dots, f_e \in F^1\mathcal{O}_P$.

It will useful for us to give an alternative description of the filtration $F^d\mathcal{O}(P)$ defined above. Consider the extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow E \longrightarrow 0$$

corresponding to $\eta \in H^1(Z, \check{E}) = \mathrm{Ext}^1(E, \mathcal{O}_Z)$. Explicitely $\mathcal{E}$ is obtained by glueing the sheaves $\mathcal{O}_{Z_i} \oplus E/_{Z_i}$ via the linear automorphisms of $\mathcal{O}_{Z_i \cap Z_j} \oplus E/_{Z_i \cap Z_j}$ defined by the matrices $\begin{pmatrix} 1 & 0 \\ \eta_{ij} & 1 \end{pmatrix}$; using this description of $\mathcal{E}$ we see (compare with [MD]) that P identifies with the complement in $\mathbf{P}(\mathcal{E})$ of the divisor $D = \mathbf{P}(E) \in |\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$ and the filtration $F^d\mathcal{O}(P)$ corresponds to the filtration of $\mathcal{O}(\mathbf{P}(\mathcal{E}) \setminus D)$ given by the vector spaces $H^o(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(dD))$ of all regular functions on $\mathbf{P}(\mathcal{E}) \setminus D$ with poles of order at most d along D. Recall also that we have an identification $H^o(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(dD)) = H^o(Z, S^d(\mathcal{E}))$.

(1.3) Recall from $[B_1]$ that, in notations of (1.1) we have that for each $n \geq 1$, $X^n \longrightarrow X^{n-1}$ is a torsor under the relative tangent bundle $V(T_{X^{n-1}/X^{n-2}})$, corresponding to the Kodaira-Spencer class $\rho(\delta)$ where $\rho : \mathrm{Der}(\mathcal{O}_{X^{n-2}}, f_*^{n-1}\mathcal{O}_{X^{n-1}}) \longrightarrow H^1(T_{X^{n-1}/X^{n-2}})$. We claim that under the identifications $\mathcal{O}_{X^n} = \mathcal{O}_X^{(n)}$ and $\mathcal{O}_{X^{n-1}} = \mathcal{O}_X^{(n-1)}$ the filtration $F^d\mathcal{O}_X^{(n)}$ defined in (1.1) coincides with the filtration $F^d\mathcal{O}_{X^n}$ defined in (1.2) (here a slight confusion may arrise from our abuse of notation making $\mathcal{O}_P$ in (1.2) a sheaf on X!. Indeed we only defined in (1.2) a filtration on the direct image of $\mathcal{O}_{X^n}$ on X^{n-1}, but this filtration induces in a natural way a filtration on the direct image of $\mathcal{O}_{X^n}$ on X). To check the claim note that as remarked in [B_1] (1.6) for any affine subset U of X, $F^1\mathcal{O}_{X^n}(U)$ is generated as an $\mathcal{O}_{X^{n-1}}(U)$-module by $\delta\mathcal{O}_{X^{n-1}}(U)$, hence by (1.2) $F^d\mathcal{O}_{X^n}$ is generated as a subsheaf of $\mathcal{O}_{X^n}$ by elements of the form $b_o(\delta b_1)\dots(\delta b_e)$ with $e \leq d$, $b_o, b_1, \dots, b_e \in \mathcal{O}_{X^{n-1}}$. In view of the similar remark about $F^d\mathcal{O}_X^{(n)}$ made in (1.1) we conclude that $F^d\mathcal{O}_X^{(n)} = F^d\mathcal{O}_{X^n}$. We get then that $F^d\mathcal{O}^{(n)}(X) = F^d\mathcal{O}(X^n)$ for $d \geq 0$.

As a consequence of the above discussion we get:

(1.4.) PROPOSITION. Assume X is a smooth $\mathcal{U}$-variety. Then there is a natural injective map of graded algebras

$$\mathrm{Gr}_F\,\mathcal{O}^{(1)}(X) \longrightarrow \bigoplus_{d\geq 0} H^o(X, S^d\Omega_{X/\mathcal{U}})$$

In particular, if X is complete, $\dim_{\mathcal{U}} F^d\mathcal{O}^{(1)}(X) < \infty$ for all $d \geq 0$.

(1.5) Now we can make the definitions formulated in the Introduction. We call $F^d\mathcal{O}^{(1)}(X)$ the Δ-pluricanonical spaces of X, their dimension $N_d(X)$ will be called the Δ-plurigenera and choosing any basis $b_1, \dots, b_{N_d}$ of $F^d\mathcal{O}^{(1)}(X)$ we call the map $\varphi_d : X \longrightarrow \mathcal{U}^{N_d}$ with components $(b_j)_j$ the Δ-pluricanonical map of degree d. Since $1 \in F^d\mathcal{O}^{(1)}(X)$, if $\lambda_1, \dots, \lambda_{N_d} \in \mathcal{U}$ are such that $\sum \lambda_j b_j = 1$ then we see that $\varphi_d(X)$ is contained in the affine hyperplane in $\mathcal{U}^{N_d}$ of equation $\sum \lambda_j x_j = 1$ where $x_1, \dots, x_{N_d}$ are affine coordinates in $\mathcal{U}^{N_d}$.

(1.6) A Δ-polynomial map $\varphi : X \longrightarrow Y$ between two smooth $\mathcal{U}$-varieties will be called a Δ-closed immersion if the morphism of D-schemes $X^\infty \longrightarrow Y^\infty$ corresponding to it (cf. [B_1] (3.6)) is a closed immersion. In particular such a map as above is injective and its image $\varphi(X)$ is Δ-closed in Y. The composition of two Δ-closed immersions is again a Δ-closed immersion. Note also that if

$j : X \longrightarrow Y$ is a closed immersion of $\mathcal{U}$-varieties then it is also a Δ-closed immersion; this follow from $[B_1]$ (3.4).

(1.7.) A remark which will be useful in what follows is that if R is any ring on which an increasing filtration $(F^dR)_{d\geq 0}$ is given such that $\bigcup_{d\geq 0} F^dR = R$ and $(F^dR)(F^eR) \subset F^{d+e}R$ for all $d,e \geq 0$ and if $Gr_F R$ is generated as a ring by $\bigoplus_{k\leq d} Gr_F^k R$ then R is generated as a ring by F^dR.

(1.8) We will also need the following easy remark. Assume we are in the situation of (1.1). Then for each $n \geq 1$ there exists a Δ-polynomial map: $\nabla_n : X \longrightarrow X^n$ which is a Δ-closed immersion and which is a section for the canonical projection $X^n \longrightarrow X$ such that for any regular function $\tilde{f} \in \mathcal{O}(X^n)$, $\tilde{f} : X^n \rightarrow \mathcal{U}$, the composition $\tilde{f} \circ \nabla_n : X \rightarrow X^n \rightarrow \mathcal{U}$ is precisely the Δ-polynomial map $f \in \mathcal{O}^{(n)}(X)$ corresponding to $\tilde{f}$ under the identification $\mathcal{O}^{(n)}(X) = \mathcal{O}(X^n)$. To check this recall from $[B_1]$, section 3 that by adjunction $\mathrm{Hom}_{\mathcal{U}\text{-sch}}(X^{\infty !}, X^n) = \mathrm{Hom}_{D\text{-sch}}(X^{\infty}, (X^n)^{\infty})$ so to the canonical projection $(X^{\infty})^! \rightarrow X^n$ there corresponds a morphisms of $\mathbb{D}$-schemes $X^{\infty} \longrightarrow (X^n)^{\infty}$ (which is a section for $(X^n)^{\infty} \longrightarrow X^{\infty}$ hence is a closed immersion of schemes). Finally, to $X^{\infty} \longrightarrow (X^n)^{\infty}$ there corresponds by $[B_1]$, section 3, a Δ-polynomial map $\nabla_n : X \longrightarrow X^n$ which satisfies the desired properties.

2. CURVES WHICH DO NOT DESCEND TO $\mathcal{K}$.

In this section we prove Theorem 1 from the Introduction. Throughout this (and the following) section we keep notations from section 1, especially from (1.1), (1.4), (1.5).

(2.1) LEMMA. Let X be a smooth complete curve over $\mathcal{U}$ of genus $g \geq 2$ which does not descend to $\mathcal{K}$. Then X^1 is affine.

PROOF. Let $\mathcal{E}$ be the vector bundle on X defined by taking the extension

$$(*) \qquad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \omega_X \longrightarrow 0$$

corresponding to the class $\rho(\delta) \in H^1(\omega_X^{-1}) \simeq \mathrm{Ext}^1(\omega_X, \mathcal{O}_X)$. By (1.2) and (1.3) X^1 identifies with $\mathbf{P}(\mathcal{E}) \setminus \mathbf{P}(\omega_X)$. Since X does not descend to $\mathcal{K}$, $\rho(\delta) \neq 0$ so the exact sequence (*) does not split. By a result of Giesekr [G], $\mathcal{E}$ is ample. But $\mathbf{P}(\omega_X) \in |\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$ so $\mathbf{P}(\omega_X)$ is ample on $\mathbf{P}(\mathcal{E})$, hence X^1 is affine.

REMARK. This geometric situation is analogue to that in [MD]; this suggests that a generalisation should be possible to varieties with ample cotangent bundle.

(2.2) COROLLARY. Let X be as in (2.1). Then $\mathcal{O}^{(1)}(X)$ is generated as an $\mathcal{U}$-algebra by $F^d\mathcal{O}^{(1)}(X)$ for $d \gg 0$; in particular $\varphi_d : X \longrightarrow \mathcal{U}^{N_d}$ is a Δ-closed immersion for $d \gg 0$ and $\varphi_d(X)$ is contained in a Zariski closed smooth surface in $\mathcal{U}^{N_d}$. Moreover $\mathcal{O}^{\Delta}(X)$ is generated as a Δ-$\mathcal{U}$-algebra by $\mathcal{O}^{(1)}(X)$ hence it is Δ-finitely generated.

PROOF. Since $\mathcal{O}(X^1) = \mathcal{O}^{(1)}(X) = \bigcup_{d \geq 0} F^d\mathcal{O}^{(1)}(X)$ is finitely generated by (2.1), $F^d\mathcal{O}^{(1)}(X)$ generates $\mathcal{O}^{(1)}(X)$ for $d \gg 0$. Let $b_1,\dots,b_{N_d}$ be an $\mathcal{U}$-basis of $F^d\mathcal{O}^{(1)}(X)$ (where $d \gg 0$) and let $\tilde{b}_1,\dots,\tilde{b}_{N_d} \in F^d\mathcal{O}(X^1)$ be the corresponding elements under the identification from (1.3). Call $\tilde{\varphi}_d : X^1 \longrightarrow \mathcal{U}^{N_d}$ the morphism of $\mathcal{U}$-varieties with components $(b_j)_j$. By (1.8) we have $\varphi_d = \tilde{\varphi}_d \circ \nabla_1$; since ∇_1 and $\tilde{\varphi}_d$ are Δ-closed immersions so is φ_d. We also get that $\varphi_d(X) \subset \tilde{\varphi}_d(X^1)$ the latter being a smooth surface (isomorphic to X^1). Finally since X^1 is affine, X^2 is also affine and $\mathcal{O}(X^2)$ is generated as an $\mathcal{O}(X^1)$-algebra by $\delta\mathcal{O}(X^1)$ (cf. [B_1] (1.6)). By induction we get that $\mathcal{O}^{\Delta}(X)$ is generated as a Δ-$\mathcal{U}$-algebra by $\mathcal{O}(X^1)$ and the Corollary is proved.

In the statement below (and in its proof) we shall use the following notation. Let X be any smooth complete curve over $\mathcal{U}$ and let $\rho(\delta) \in H^1(\omega_X^{-1})$ be its Kodaira-Spencer class. Then we let for any $m \geq 1$

$$C_m : H^0(\omega_X^{\otimes m}) \to H^1(\omega_X^{\otimes(m-1)})$$

be the cup product with $\rho(\delta)$. Note that the target space of C_m vanishes for $m \geq 3$ so the only relevant values for m are $m = 1$ (in this case $C_1 = C$ from the Introduction) and $m = 2$.

(2.3) LEMMA. Let X be as in (2.1). Then (under the map from (1.4)):

$$Gr_F^1 \mathcal{O}^{(1)}(X) = \operatorname{Ker} C_1$$

in particular $N_1(X) = g + 1 - \operatorname{rank}_\Delta(X)$. Assume moreover that $\operatorname{rank}_\Delta(X) = g$ (equivalently $Gr_F^1 \mathcal{O}^{(1)}(X) = 0$, equivalently C_1 is an isomorphism). Then C_2 is surjective and we have

$$Gr_F^2 \mathcal{O}^{(1)}(X) = \operatorname{Ker} C_2$$

$$Gr_F^d \mathcal{O}^{(1)}(X) = H^0(\omega_X^{\otimes d}) \text{ for } d \geq 3$$

PROOF. Let's come back to the notations from the proof of (2.1).

We dispose of exact sequences ($d \geq 1$)

$$0 \to S^{d-1}(\mathcal{E}) \to S^d(\mathcal{E}) \to \omega_X^{\otimes d} \to 0$$

which give rise to exact sequences

(*) $$0 \to H^0(S^{d-1}(\mathcal{E})) \to H^0(S^d(\mathcal{E})) \to H^0(\omega_X^{\otimes d}) \xrightarrow{\partial_d}$$

$$\xrightarrow{\partial_d} H^1(S^{d-1}(\mathcal{E})) \to H^1(S^d(\mathcal{E})) \xrightarrow{\gamma_d} H^1(\omega_X^{\otimes d}) \to 0$$

Making $d = 1$ in (*) and noting that $\partial_1 : H^0(\omega_X) \to H^1(\mathcal{O}_X)$ is nothing but our map C_1 we get using (1.2) and (1.3) that $Gr_F^1 \mathcal{O}^{(1)}(X) = \operatorname{Ker} C_1$. Assume now C_1 is an isomorphism. From the sequence (*) for $d = 1$ we get that $\gamma_1 : H^1(\mathcal{E}) \to H^1(\omega_X) \simeq \simeq \mathcal{U}$ is an isomorphism. Making $d = 2$ in (*) and noting that we have a commutative diagram

$$\begin{tikzcd}
H^0(\omega_X^{\otimes 2}) \arrow[rr, "C_2"] \arrow[dr, "\partial_2"'] & & H^1(\omega_X) \\
& H^1(\mathcal{E}) \arrow[ur, "\gamma_1"'] &
\end{tikzcd}$$

we see that $\operatorname{Ker} C_2 = \operatorname{Ker} \partial_2$. On the other hand the map C_2 composed with the trace map $H^1(\omega_X) \xrightarrow{\sim} \mathcal{U}$ is easily identified as an element of the dual space $H^0(\omega_X^{\otimes 2})^0 \simeq$ $\simeq H^1(\omega_X^{-1})$ with $\rho(\delta)$. Consequently ∂_2 is surjective. Since $H^1(\omega_X^{\otimes 2}) = 0$ we get that $Gr_F^2 \mathcal{O}^{(1)}(X) = \operatorname{Ker} C_2$ and that $H^1(S^2(\mathcal{E})) = 0$. Finally, taking $d \geq 3$ in $(*)$ we easily get by induction that $H^1(S^d(\mathcal{E})) = 0$ and that $Gr_F^d \mathcal{O}^{(1)}(X) = H^0(\omega_X^{\otimes d})$ for $d \geq 3$.

(2.4) REMARK. From (2.3) one immediately obtains the formula for $N_d(X)$ given in Theorem 1. The only assertion in that Theorem which we did not yet prove is that if X is non-hyperelliptic and of Δ-rank g then φ_d is a Δ-closed embedding for $d \geq 3$. This follows from the discussion in the proof of (2.2) and from the following:

(2.5) LEMMA. Assume X is non-hyperelliptic and $\operatorname{rank}_{\Delta}(X) = g$. Then $\mathcal{O}^{(1)}(X)$ is generated as an $\mathcal{U}$-algebra by $F^3\mathcal{O}^{(1)}(X)$.

Proof. By (1.7) and (2.3) it is sufficient to prove that $Gr_F \mathcal{O}^{(1)}(X)$ is generated as an $\mathcal{U}$-algebra by $\operatorname{Ker} C_2 \oplus H^0(\omega_X^{\otimes 3})$. By Max Noether's theorem the canonical ring $\bigoplus_{n \geq 0} H^0(\omega_X^{\otimes n})$ is a quotient of $S(H^0(\omega_X))$. Since $\operatorname{rank}_{\Delta}(X) = g$, the image of $\rho(\delta) \in H^1(\omega_X^{-1}) \simeq H^0(\omega_X^{\otimes 2})^0$ via the injective map $H^0(\omega_X^{\otimes 2})^0 \longrightarrow$ $\longrightarrow (H^0(\omega_X) \otimes H^0(\omega_X))^0$ obtained by transposing the multiplication is a symmetric bilinear form $b : H^0(\omega_X) \times H^0(\omega_X) \longrightarrow \mathcal{U}$ of maximum rank g hence there exists a basis $x_1, \ldots, x_g$ of $H^0(\omega_X)$ such that $b(x_i, x_j) = \delta_{ij}$ (Kronecker delta) for all i,j. Identifying $S(H^0(\omega_X))$ with the polynomial ring $\mathcal{U}[x_1, \ldots, x_g]$ we see that $Gr_F \mathcal{O}^{(1)}(X)$ is a quotient of the graded subalgebra $R = \bigoplus_{d \geq 0} R_d$ of $\mathcal{U}[x_1, \ldots, x_g]$ where

$$R_o = \mathcal{U}$$

$$R_1 = 0$$

$$R_2 = \Big\{ \text{space of forms } \sum_{i,j} a_{ij} x_i x_j \text{ with } \sum_i a_{ii} = 0 \Big\}$$

$$R_d = \Big\{ \text{space of all forms of degree } d \Big\} \text{ for } d \geq 3.$$

So we shall be done if we prove that R is generated as an $\mathcal{U}$-algebra by R_2 and R_3. Put $u_{ij} = x_i^2 - x_j^2 \in R_2$ and $v_{ij} = x_i x_j \in R_2$. Now R_4 belongs to the $\mathcal{U}$-subalgebra of $\mathcal{U}[x_1,\dots,x_g]$ generated by R_2 because of the following identities (in which i,j,k are distinct indices and if $g \geq 4$ then r is an index distinct from i,j,k):

$$x_i^4 = u_{ij}u_{ik} + v_{ij}^2 + v_{ik}^2 - v_{jk}^2$$

$$x_i^3 x_j = u_{ik}v_{ij} + v_{ki}v_{kj}$$

$$x_i^2 x_j^2 = v_{ij}^2$$

$$x_i^2 x_j x_k = v_{ij}v_{ik}$$

$$x_i x_j x_k x_r = v_{ij}v_{kr}$$

To check that R_5 belongs to the $\mathcal{U}$-algebra generated by R_2 and R_3 note that

$$x_i^5 = u_{ij}x_i^3 + v_{ij}x_i^2 x_j$$

and note that any monomial in the x_i's of degree 5 which is not of the form x_i^5 is of the form $v_{ij}m$ with m a monomial of degree 3. Since the submonoid of the naturals generated by 3, 4 and 5 is the monoid of all naturals $n \geq 3$ it follows that for $d \geq 6, R_d$ is in the algebra generated by R_2 and R_3 and our proof is closed.

(2.6) We close this section by making a remark on characters of Jacobians (this subject deserves a detailed further investigation). Let X be a smooth complete curve of genus g, A = J(X) its Jacobian and $X \longrightarrow A$ the natural embedding. Then one can investigate the restrictions on X of the Δ-polynomial functions on A i.e. the map $\mathcal{O}^{\Delta}(A) \longrightarrow \mathcal{O}^{\Delta}(X)$ and more precisely the maps $\mathcal{O}^{(n)}(A) \longrightarrow \mathcal{O}^{(n)}(X)$. The latter maps need not be surjective: for instance they are not for $n = 1$ and X "Δ-generic" in $\mathcal{M}_g$ (since by [B_1] $\mathcal{O}^{(1)}(A) = \mathcal{U}$ in this case while from Theorem 1 in our paper $\mathcal{O}^{(1)}(X)$ has Krull dimension 2!) We leave open the problem whether $\mathcal{O}^{\Delta}(A) \longrightarrow \mathcal{O}^{\Delta}(X)$ is surjective. Indeed it may happen apriori that $\mathcal{O}^{(n)}(A) \longrightarrow$ $\longrightarrow \mathcal{O}^{(n)}(X)$ is not surjective for some n but that any element of $\mathcal{O}^{(n)}(X)$ lies in the image of $\mathcal{O}^{(m)}(A) \longrightarrow \mathcal{O}^{(m)}(X)$ for some bigger m. Let's remark here that if X is Δ-generic in $\mathcal{M}_g$, the image of $\mathcal{O}^{(2)}(A) \longrightarrow \mathcal{O}^{(2)}(X)$ is not entirely contained in $\mathcal{O}^{(1)}(X)$. Indeed, by [B_1], section 6, $A^2 = A^1 \times L(A)$ if $\mathrm{rank}_{\Delta}(A) = g$; viewing X^2 as embedded into A^2 we may take a fibre Y of the projection $X^2 \longrightarrow X^1$, view it as a subset of the corresponding fibre of $A^2 \longrightarrow A^1$ hence view Y as embedded into L(A). Then choose a linear form α on L(A) which is non-constant on Y. Viewing α as an element of $X_a(L(A)) = X_a(A^2) \subset \mathrm{Ch}_{\Delta}(A)$ we see that α is non-constant on the fibres of $X^2 \longrightarrow X^1$ hence its restriction to X^2 cannot factor through $X^2 \longrightarrow X^1$. Our "remark" is proved.

3. CURVES WHICH DESCEND TO $\mathcal{K}$.

In this section we prove Theorem 2 from the Introduction.

(3.1.) LEMMA. Let X be a smooth $\mathcal{U}$-variety which descends to $\mathcal{K}$ ($X = X_o \otimes_{\mathcal{K}} \mathcal{U}$, X_o a $\mathcal{K}$-variety). Let δ^* be the trivial lifting of δ from $\mathcal{U}$ to X and use δ^* to identify X^1 with $\mathbf{V}(T_{X/\mathcal{U}})$. Let $X_o \subset X^1 = \mathbf{V}(T_{X/\mathcal{U}})$ be the zero section of the tangent bundle. Then $\nabla_1^{-1}(X_o) = X_{\mathcal{K}}$.

PROOF. At the level of $\mathcal{U}$-points ∇_1 takes any $\mathcal{U}$-point u: $Z = \mathrm{Spec}\,\mathcal{U} \longrightarrow X$ into the pair (u,D) where $D = \delta \circ u \in \mathrm{Der}(\mathcal{O}_X, u_* \mathcal{O}_Z)$ (we use here the description of $\mathrm{Hom}(\mathrm{Spec}\,\mathcal{U}, X^1)$ given in [B_1] (1.4)). The isomorphism $X^1 \longrightarrow \mathbf{V}(T_{X/\mathcal{U}})$ defined by δ^* is given by $(u,D) \mapsto (u, D - u \circ \delta^*)$. Under this identification $\nabla_1(u) \in X_o$ for some $\mathcal{U}$-point u of X if and only if $\delta \circ u = u \circ \delta^* : \mathcal{O}_X \to \mathcal{U}$. One easily checks that this happen if and only if

$u : \mathrm{Spec}\,\mathcal{U} \longrightarrow X$ factors through $\mathrm{Spec}\,\mathcal{U} \longrightarrow \mathrm{Spec}\,\mathcal{K} \longrightarrow X_o$ and we are done.

REMARK. If in (3.1) we view X as a D-scheme via δ^* then ∇_1 identifies with $\nabla : X \longrightarrow TX$ defined in [B_1] (3.8).

(3.3) Let's prove Theorem 2. We have $\mathcal{O}(X^1) = \mathcal{O}(V(T_{X/\mathcal{U}})) = \bigoplus_{d \geq 0} H^o(\omega_X^{\otimes d})$ which gives the formula for $N_d(X)$. The morphism $X^1 \longrightarrow \mathrm{Spec}\,\mathcal{O}(X^1)$ is nothing but the contraction of the zero section of $V(T_{X/\mathcal{U}})$. Any basis of $F^1\mathcal{O}(X^1)$ provides an embedding of $\mathrm{Spec}\,\mathcal{O}(X^1)$ into an affine space and we conclude exactly as in the proof of (2.2) by using (3.1) above.

(3.4) REMARKS. 1) The image of $\varphi_1 : X \longrightarrow \mathcal{U}^{g+1}$ in Theorem 2 in case X is non-hyperelliptic is contained in the affine cone over the canonical curve $X \subset \mathbf{P}^{g-1}$; this cone lies in $\mathbf{A}^g = \mathcal{U}^g$ and this $\mathcal{U}^g$ lies as a hyperplane not passing through the origin in $\mathcal{U}^{g+1}$ as explained in (1.5).

2) We leave open the problem whether $\mathcal{O}^{\Delta}(X)$ is Δ-finitely generated if X descends to $\mathcal{K}$ (compare with (2.2)).

3) The list of references below has to be completed with the one appearing in [B_1].

REFERENCES

[B_1] **A. Buium**, Geometry of differential polynomial functions, I: algebraic groups.

[B_2] **A. Buium**, Fields of definition of algebraic varieties in characteristic zero, Compositio Math. 61 (1987), 339-352.

[G] **D. Gieseker**, P-ample bundles and their Chern classes, Nagoya Math. J. 43 (1971).

[K] **E. Kolchin**, Differential Algebra and Algebraic Groups, Academic Press, New York, 1973.

[MD] **M. Martin-Deschamps**, Propriétés de déscente des variétés à fibré cotangent ample, Ann. Inst. Fourier 33, 3 (1984), 39-64.

Department of Mathematics
INCREST, Bd.Pacii 220
79622 Bucharest, Romania