SOME RESULTS FOR MATRICES ASSOCIATED WITH PERFECT ELIMINATION BIPARTITE GRAPHS

by

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ABSTRACT

We obtain a formula for the determinant of a nonsingular matrix in terms of the determinants of some of its principal minors in the case when a perfect elimination bipartite graph is associated to its inverse. As a consequence we obtain the main result of [5]. This technique permits also to obtain a counterexample to a conjecture from [5].

INTRODUCTION

The connections between the determinant of a matrix and the graph associated to its inverse has been pointed out in many papers from which we mention here [1], [4], and [5]. These results have applications in computer science, in solving linear systems and other fields.

In [1], using perfect Gaussian elimination it is obtained a formula for the determinant of a nonsingular matrix having associated to its inverse a chordal graph, in terms of the determinants of some of its principal minors. It is also proved that after a cancellation process this formula leads to the formula from the earlier paper [4]. These results are concerning with symmetric zero-pattern of the inverse.

Applying a graph theoretical result from [9] concerning the bipartite graph model of perfect Gaussian elimination, the goal of Section 3 is to obtain a determinantal formulae for some matrices with asymmetric zero-pattern of its inverse.

As consequences we mention in Section 4 the main result from [5] and some determinantal formulas from [1], [3], [5] and [7]. Our technique permits to find a counterexample to a conjecture from [5].

2. PRELIMINARIES

For all terminology and results concerning graph theory we follow here the book [9]. Let $G = (V,E)$ be an undirected graph with vertex set $V = \{1,2,\dots,n\}$ and edge set E, a symmetric irreflexive binary relation on V. We denote by $Adj(v)$ the adjacency set of v, i.e. $w \in Adj(v)$ iff $(v,w) \in E$. Given a subset $A \subseteq V$, we define the subgraph induced by A by $G_A = (A,E_A)$ where $E_A = \{(x,y) \in E \mid x \in A \text{ and } y \in A\}$.

The complete graph is the graph with the property that every pair of distinct vertices is adjacent. A subset $A \subseteq V$ is a clique if it induces a complete subgraph.

A graph G is chordal (or triangulated) if every cycle of length stricly greater than 3 possesses a chord, i.e.. an edge joining two nonconsecutive vertices of the cycle.

A basic fact ([9], Th. 4.1) is that every chordal graph has a perfect vertex elimination scheme (or perfect scheme), i.e. an ordering $\sigma = [v_1,v_2,\dots,v_n]$ of the vertices of G such that each set:

$$(2.1) \qquad S_k = \{v_j \in Adj(v_k) \mid j > k\}$$

is a clique. If we say that a vertex v of G is simplicial when $Adj(v)$ is a clique, then σ is a perfect scheme iff each v_k is simplicial in the induced graph $G_{\{v_k,v_{k+1},\dots,v_n\}}$.

We say that an undirected graph G is a graph for the matrix $M = (m_{ij})_{1 \le i,j \le n}$ (in the combinatorial symmetric sense) if for every $i \neq j$, $(i,j) \notin E$ implies $m_{ij} = m_{ji} = 0$.

For $\alpha, \beta \subseteq \{1,2,\dots,n\}$ we denote by $M(\alpha|\beta)$ the submatrix of M lying in the rows α and columns β and for $\alpha = \beta$ by $M(\alpha)$ the principal submatrix subordinate to the index set α.

If R is a nonsingular matrix and $G = (V,E)$ is chordal and it is a graph for R^{-1} then for every $\sigma = [v_1,v_2,\dots,v_n]$ a perfect scheme for G, by the convention $\det R(\phi) = 1$ in [1] it is proved the following formula:

$$(2.2) \qquad \det R = \prod_{k=1}^{n} \det R(\{v_k\} \cup S_k)/\det R(S_k)$$

with S_k defined by (2.1) and provided that the terms of the denominator are nonzero.

By a cancellation process as in [1] we obtain the determinantal formulae from [4].

We want now to generalize the formulae (2.2) for matrices with asymmetric zero-pattern of its inverse.

A graph $G = (V,E)$ is called bipartite iff $V = X + Y$ and every edge from E has an endpoint in X and one in Y. We denote it by $G = (X,Y,E)$. An edge $e = xy$ of G is called bisimplicial if $\mathrm{Adj}(x) + \mathrm{Adj}(y)$ induces a complete bipartite subgraph of G. Let $\varphi = [x_1y_1, x_2y_2, \ldots, x_ny_n]$ be a sequence of nonadjacent edges of the bipartite graph $G = (X,Y,E)$ with card X = card $Y = n$. We say that φ is a perfect edge elmination scheme of G if each edge x_ky_k is bisimplicial in the induced graph $G_{\{x_k,\ldots x_n\}+\{y_k,\ldots,y_n\}}$.

We call the graphs admiting such a scheme as perfect elimination bipartite graphs.

We say that the bipartite graph $G = (X,Y,E)$ with $X = Y = N = \{1,2,\ldots,n\}$ is a graph for the matrix $M = (m_{ij})_{1\le i,j\le n}$ (in the combinatorial asymmetric sense) if every $x \in X, y \in Y, (x,y) \notin E$ implies $m_{xy} = 0$.

In order to make no confusion we denote the edges of a bipartite graph $H = (X,Y,E)$ by vw assuming that $v \in X$ and $w \in Y$.

As it is mentioned in [9], Thm. 12.1, if a matrix M is associated to a perfect elimination bipartite graph it can be reduced by Gaussian elimination to a matrix having only one nonzero element on each row and column without ever changing a zero entry (even temporarily) to a nonzero by choosing to act as pivots the elements on the positions x_ky_k, $\varphi = [x_1y_1, \ldots, x_ny_n]$ being a perfect edge elimination scheme for G. Throughout this paper we shall assume that arithmetic coincidence does not cause zeros on positions we want to choose as pivots.

3. THE MAIN RESULT

The purpose of this section is to obtain a formula of type (2.2) for the determinant of nonsingular matrices having a perfect elimination bipatite graph associated to their inverse.

Let $G = (X,Y,E)$ be a perfect elimination bipartite graph and R a nonsingular matrix such that G is a graph for the matrix $M = R^{-1}$ and $[x_1y_1, \ldots, x_ny_n]$ a perfect edge elimination scheme for G. Let denote for $k = 1,\ldots,n$ by

$$X_k = \{ y_j \in Adj(x_k) | j > k \}$$

$$(3.1) \quad Y_k = \{ x_j \in Adj(y_k) | j > k \}$$

$$Z_k = \{ y_1, \ldots, y_{k-1} \} \cup \{ y_j \notin Adj(x_k) | j > k \}$$

$$U_k = \{ x_1, \ldots, x_{k-1} \} \cup \{ x_j \notin Adj(y_k) | j > k \}$$

such that $N = \{x_k\} \cup Y_k \cup U_k = \{y_k\} \cup X_k \cup Z_k$.

Our first result is the following:

LEMMA 3.1. In the above conditions, after reducing the matrix M by Gaussian elimination by pivoting on positions $x_1y_1, \ldots, x_ny_n$, we obtain a matrix $D = (d_{ij})_{1 \le i,j \le n}$ with the only nonzero elements $d_{x_ky_k}$, $k = 1, \ldots, n$ given by

$$(3.2) \quad d_{x_ky_k} = (-1)^{s_k+t_k}(\det R^{-1}(\{x_k\} \cup \alpha_k | \{y_k\} \cup Z_k)/\det R^{-1}(\alpha_k | Z_k)) =$$

$$= (-1)^{s_k'+t_k'}(\det R^{-1}(\{x_k\} \cup U_k | \{y_k\} \cup \beta_k)/\det R^{-1}(U_k | \beta_k))$$

provided that the terms of denominators are nonzero, where $\alpha_k = \{x_1, \ldots, x_{k-1}\} \cup \alpha_k'$, $\beta_k = \{y_1, \ldots, y_{k-1}\} \cup \beta_k'$ with $\alpha_k' \subseteq \{x_{k+1}, \ldots, x_n\}$, $\beta_k' \subseteq \{y_{k+1}, \ldots, y_n\}$, arbitrary sets such that $\operatorname{card} \alpha_k = \operatorname{card} Z_k$ and $\operatorname{card} \beta_k = \operatorname{card} U_k$, s_k and t_k (respective s_k' and t_k') being the rows and columns of x_ky_k in the matrix $R^{-1}(\{x_k\} \cup \alpha_k | \{y_k\} \cup Z_k)$ (respective $R^{-1}(\{x_k\} \cup U_k | \{y_k\} \cup \beta_k)$).

Proof. Since $Z_k = \{ y_1, \ldots, y_{k-1} \} \cup \{ y_j \notin Adj(x_k) | j > k \}$, after performing partial Gaussian elimination in the matrix $R^{-1}(\{x_k\} \cup \alpha_k | \{ y_k\} \cup Z_k)$, (in which we keep the same indices as in R^{-1}) by pivoting on the positions $x_1y_1, \ldots, x_{k-1}y_{k-1}$ we obtain a matrix having on the rows $x_1, \ldots, x_{k-1}$ and on the columns $y_1, \ldots, y_{k-1}$ exactly one nonzero element and since no zero element is changed in a nonzero, all elements in the positions x_ky_s with $y_s \in \{ y_j \notin Adj(x_k) | j > k \}$ are zero.

Performing the same operations in the matrix $R^{-1}(\alpha_k | Z_k)$ we obtain the same matrix as before but without its x_k-row and y_k-column. Dividing the determinants of these two matrices we obtain the first equality in (3.2). The second one is obtained in the same way.

$\square$

THEOREM 3.2. The elements $d_{x_k y_k}$ can be obtained from R by the formulas:

$$(3.3)\quad d_{x_k y_k} = (-1)^{s_k+t_k+x_k+y_k} \det R(X_k \mid \gamma_k) / \det R(\{y_k\} \cup X_k \mid \{x_k\} \cup \gamma_k) =$$

$$= (-1)^{s_k'+t_k'+x_k+y_k} \det R(\delta_k \mid Y_k) / \det R(\{y_k\} \cup \delta_k \mid \{x_k\} \cup Y_k).$$

where $\gamma_k \subseteq \{x_{k+1},\dots,x_n\}$ and $\delta_k \subseteq \{y_{k+1},\dots,y_n\}$ are arbitrary sets with card γ_k = card X_k and card δ_k = card Y_k.

Proof. Using the Jacobi identity (see, e.g. [10] p.21) we have for $\alpha, \beta \subseteq N$:

$$\det R^{-1}(\alpha \mid \beta) = (-1)^u \det R(C_\beta \mid C_\alpha) / \det R$$

where C_α and C_β are the complementary sets of α and β in N and $u = \sum_{i\in\alpha} i + \sum_{j\in\beta} j$, the formulas (3.3) are obtained directly from (3.2).

■

COROLLARY 3.3. The determinant of R can be obtained as:

$$(3.4)\quad \det R = \operatorname{sgn}\theta \Big/ \prod_{k=1}^{n} d_{x_k y_k}$$

with $d_{x_k y_k}$ given by (3.3) and θ is the permutation in which y_k corresponds to x_k.

EXAMPLE. Let consider

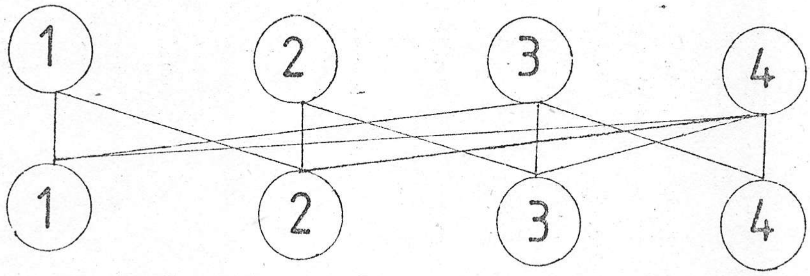
$$R = \begin{bmatrix} 3/5 & -4/5 & -4/5 & 2/5 \\ 1/5 & 2/5 & 2/5 & -1/5 \\ 2/5 & -1/5 & 4/5 & -2/5 \\ -1/5 & -2/5 & -7/5 & 6/5 \end{bmatrix}, \text{ with}$$

$$R^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

In the combinatorial symmetric sense only the complete graph is a graph for R^{-1} and so

(2.2) leads to det R = det R.

In combinatorial asymmetric sense R^{-1} is associated with the folloowing perfect elimination bipartite graph:

G = 

With $\varphi = [34, 11, 22, 43]$ a perfect edge elimination scheme.

Choosing this pivots we have the following reduction of R^{-1}:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & \boxed{1} \\ -1 & 1 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \boxed{-5/2} & 0 \end{bmatrix}$$

In this case $X_1 = \{1,3\}$. Since $x_1 = 3$ we have for δ_1 the possibilities $\{1,2\}$, $\{1,4\}$ and $\{2,4\}$ so

$$d_{34} = -\frac{\det R(\{1,3\} \mid \{1,2\})}{\det R(\{1,3,4\} \mid \{1,2,3\})} = -\frac{\det R(\{1,3\} \mid \{2,4\})}{\det R(\{1,3,4\})}$$

$$= -\frac{\det R(\{1,3\} \mid \{2,4\})}{\det R(\{1,3,4\} \ \{2,3,4\})} = 1$$

Since $Y_1 = \{4\}$ and $y_1 = 4$, for δ_1 we have the possibilities $\{1\}$, $\{2\}$, $\{3\}$ and so:

$$d_{34} = -\frac{r_{14}}{\det R(\{1,4\} \mid \{3,4\})} = -\frac{r_{24}}{\det R(\{2,4\} \mid \{3,4\})} = -\frac{r_{34}}{\det R(\{3,4\})} = 1$$

since $X_2 = \{2\}$, $Y_2 = \{4\}$ we have for d_{11} the following posibilities:

$$d_{11} = \frac{r_{24}}{\det R(\{1,2\} \mid \{1,4\})} = \frac{r_{22}}{\det R(\{1,2\})} = \frac{r_{34}}{\det R(\{1,3\} \mid \{1,4\})} = 1$$

Since $X_3 = \{3\}$, $Y_3 = \{4\}$ we have:

$$d_{22} = \frac{r_{34}}{\det R(\{2,3\} \mid \{2,4\})} = 2,$$

and finally:

$$d_{43} = \frac{1}{r_{34}} = -\frac{5}{2}.$$

Thus det R can be obtained by computing only 2-by-2 determinants.

4. APPLICATIONS

In this section we apply formula (3.3) to obtain the main result of [5]. We give also a counterexample to a conjecture from [5].

It is known ([9], Th. 4.8) that a graph $G = (N,E)$, $N = \{1,2,\ldots,n\}$ is chordal iff there exists a tree $T = (\mathcal{V}(T), \mathcal{E}(T))$ with node set $\mathcal{V}(T) = \{V_1, V_2,\ldots,V_m\}$ where $V_1, V_2,\ldots,V_m$ are the maximal cliques of G and the edge set $\mathcal{E}(T)$, verifying the intersection property:

(4.1) $\quad V_i \cap V_j \subseteq V_k$ whenever V_k lies on a path from V_i to V_j in T

In this case G is the intersection graph of T. We call T a tree for G. In general T is not uniquely determined by G.

Consider now a fixed chordal graph $G = (N,E)$ and T a tree for G. Consider as in [5] an orientation D on the edges of T. There are 2^{m-1} such orientations on T.

DEFINITION 4.1.

We say that a n-by-n matrix M has a nonzero-pattern allowed by the pair (T,D) if whenever $m_{ij} \neq 0$ then either:

i) $\{i,j\} \subseteq V_k$ for some $k = 1,\ldots,m$ or

ii) there is a path $(V_{k_1}, V_{k_2},\ldots,V_{k_p})$ in D such that $i \in V_{k_1}$ and $j \in V_{k_p}$.

To a matrix M with a nonzero-pattern allowed by the pair (T,D) we associate first the chordal graph $G = (N,E)$, the intersection graph of T having the property that $m_{ij} \neq 0$ and $m_{ji} \neq 0$ implies $(i,j) \in E$, and second the bipatite graph $H = (X,Y,F)$, $X = Y = N$ with the property that $ij \in F$ iff one of the condition i) or ii) from Definition 4.1 is satisfied.

We construct by the aid of T a perfect scheme $\sigma = [v_1,v_2,\ldots,v_n]$ of G in the following way:

- choose an extremal node set V_s of T which must contain a simplicial vertex v_1 of G.

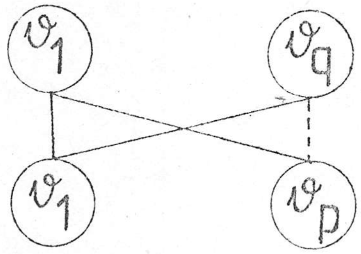
If $V_s\setminus\{v_1\}$ is a maximal clique in $G_{\{v_2,\ldots,v_n\}}$ then a tree T' of $G_{\{v_2,\ldots,v_n\}}$ is obtained by replacing in T, V_s with $V_s\setminus\{v_1\}$. Consider also D' the orientation on T' induced by D.

- If $V_s\setminus\{v_1\}$ is not maximal in $G_{\{v_2,\ldots,v_n\}}$ then T' is obtained by deleting V_1 and its edge from T.

- Continue now by choosing v_2 from an extremal node set of T', and so on, still we obtain $\sigma=[v_1,v_2,\ldots,v_n]$.

LEMMA 4.2. For σ constructed as above, $\varphi=[v_1v_1,v_2v_2,\ldots,v_nv_n]$ is a perfect edge elimination scheme for H.

Proof. First we prove that v_1v_1 is bisimplicial. If v_1v_p, $v_qv_1 \in F$ for $p,q \geq 2$, since $v_1\in V_s$ and V_s is an extremal node set in T we have that $v_p\in V_s$ or $v_q\in V_s$. Assume $v_q\in V_s$. If $v_p\in V_s$ it is clear $v_qv_p \in F$. If $v_p \in V_t$, $t\neq s$, since $v_1v_p \in F$ there



exists a path in D from V_s to V_t and since $v_q\in V_s$ we have that $v_qv_p \in F$ and so v_1v_1 is bisimplicial. We obtain the same fact if we assume $v_q\in V_t$, $t\neq s$. Using now the tree (T',D') we obtain that v_2v_2 is bisimplcial in $H_{\{v_2,\ldots,v_n\}+\{v_2,\ldots,v_n\}}$, and so on as we obtain the desired result. ∎

In the particulary case of the graph H and perfect edge elimination scheme constructed above, we have for each $k=1,\ldots,n$, $X_k=S_k$ or $Y_k=S_k$ where S_k is given by (2.1) for G and σ, X_k, Y_k are given by (3.1) for H and φ, with $x_k=v_k$ and $y_k=v_k$.

By choosing in the formula (3.3), $\gamma_k=S_k$ or $\delta_k=S_k$ and replacing in the formula (3.4), we obtain the following:

COROLLARY 4.3. If R is a nonsingular n-by-n matrix which inverse has a nonzero pattern allowed by (T,D), then:

$$(4.2)\quad \det R = \prod_{k=1}^{n} \frac{\det R(\{v_k\} \cup S_k)}{\det R(S_k)}$$

provided the terms in the denominator are nonzero.

After a cancellation process as in Proposition 3.5 from [1], we obtain Theorem 5.1. from [5] as a Corollary of Theorem 3.2:

COROLLARY 4.4. In the same condition as above:

$$(4.3)\quad \det R = \frac{\prod_{k=1}^{m} \det R(V_k)}{\prod_{\{V_i,V_j\} \in \mathcal{E}(T)} \det R(V_i \cap V_j)}$$

It is clear that the determinant formulas from [4] and [1] for the case of symmetric zero-pattern of the inverse are consequences of Corollary 4.4 and Corollary 4.3.

Formulas (4.2) and (4.3) permit also to obtain the maximum determinant over the determinants of all positive completions of a partial positive matrix associated with a chordal graph (see [6] and [2]).

In [3], [8] and [7] it is proved that in some conditions, a partial matrix can be uniquely completed to an invertible matrix with the property that its inverse has 0 in the positions corresponding to the unspecified entries of the initial partial matrix. Formulas (4.2) and (4.3) permit the computation of the determinant of this completions.

We analyse now the converse question. First we recall some definitions from [5].

Let us consider the index sets $V_1, \ldots, V_m \subseteq N$ with the property:

$$(4.4)\quad \bigcup_{k=1}^{m} V_k = N.$$

DEFINITION 4.5. If $V_1, V_2, \ldots, V_m \subseteq N$ are index sets satisfying (4.4) and $Z \subseteq N \times N$, we say that Z lies outside the profile $V_1,\dots,V_m$ if $Z \cap [\bigcup_{k=1}^{m} (V_k \times V_k)] = \emptyset$

DEFINITION 4.6. If $Z \subset N \times N$, we say that the n-by-n matrix M has a nonzero pattern allowed by Z if $m_{rs} = 0$ for all $(r,s) \in Z$. Let $\mathcal{A}_Z$ be the set of all n-by-n matrices with nonzero pattern allowed by Z.

DEFINITION 4.7. Given a directed tree (T,D) let Z(T,D) be the set of all $(r,s) \in N \times N$ satisfying neither i) nor ii) of Definition 4.1.

DEFINITION 4.8. Let $V_1,\dots,V_m \subseteq N$ be index sets satisfying (4.4) and let T_1 and T_2 be distinct trees with node sets $V_1,\dots,V_m$. We say that T_1 and T_2 are equivalent if the two collections $\{V_i \cap V_j : \{V_i,V_j\} \in \mathcal{E}(T_1)\}$ and $\{V_i \cap V_j : \{V_i,V_j\} \in \mathcal{E}(T_2)\}$ are identical.

In [5] it is conjuecture the following:

CONJECTURE 4.9. Let $V_1,\dots,V_m \subseteq N$ be index sets satisfying (4.4) and T a tree with node set $V_1,\dots,V_m$. Let $Z \subseteq N \times N$ lie outside of the profile of $V_1,\dots,V_m$, and assume that:

$$(4.5) \quad \prod_{k=1}^{m} \det R(V_k) = (\det R) \cdot \prod_{\{V_i,V_j\} \in \mathcal{E}(T)} \det R(V_i \cap V_j)$$

for all nonsingular matrices R for which $R^{-1} \in \mathcal{A}_Z$. Then T satisfies the intersection property (4.1). Furthermore there is a tree T' equivalent to T and an orientation D on T' such that $Z \subseteq Z(T',D)$.

It is proved in [5] that in this case T satisfies intersection property (4.1). We give now a counterexample to the conjecture 4.9.

For n = 6, consider $V_1 = \{1,2\}$, $V_2 = \{2,3,4\}$, $V_3 = \{2,4,5\}$, $V_4 = \{2,5,6\}$ and the tree T:

$$V_1 - V_2 - V_3 - V_4$$

and $Z = \{(3,1),(3,6),(4,1),(4,6),(5,1),(5,3),(6,1),(6,3)\}$. Let consider an invertible R with $R^{-1} \in \mathcal{A}_Z$. R^{-1} has the zero pattern:

$$\begin{bmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ 0 & x & x & x & x & 0 \\ 0 & x & x & x & x & 0 \\ 0 & x & 0 & x & x & x \\ 0 & x & 0 & x & x & x \end{bmatrix}$$

To verify that:

$$(4.6) \qquad \prod_{k=1}^{4} \det R(V_k) = (\det R) \prod_{\{V_i,V_j\}\in\mathcal{E}(T)} \det R(V_i \cap V_j)$$

by the Jacobi identity is equivalent with:

$$\det B(\{1,3,4,5,6\})\cdot \det B(\{1,3,5,6\})\cdot \det B(\{1,3,4,6\}) =$$

$$= \det B(\{3,4,5,6\})\cdot \det B(\{1,5,6\})\cdot \det B(\{1,3,6\})\cdot \det B(\{1,3,4\}) \text{ for every } B\in \mathcal{A}_Z.$$

Sinbce $b_{31} = b_{41} = b_{51} = b_{61} = 0$, this relation is equivalent with:

$$\det B(\{3,5,6\})\cdot \det B(\{3,4,6\}) = \det B(\{5,6\})\cdot \det B(\{3,6\})\cdot \det B(\{3,4\})$$

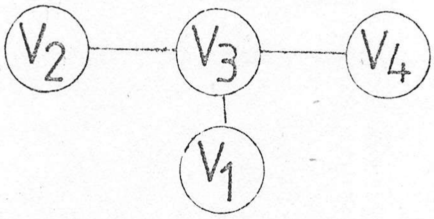
Since $b_{53} = b_{63} = 0$, we have to prove:

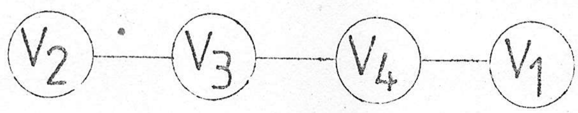
$b_{33}\det B(\{3,4,6\}) = \det B(\{3,6\})\cdot \det B(\{3,4\})$.

This last relation is true since by $b_{36} = b_{46} = 0$, $\det B(\{3,4,6\}) = b_{66}\cdot B(\{3,4\})$ and by $b_{36} = b_{63} = 0$, $\det B(\{3,6\}) = b_{33}\cdot b_{66}$.

Thus (4.6) is verified for every R with $R^{-1}\in \mathcal{A}_Z$.

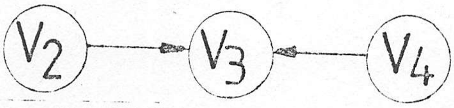
There exist the following two trees equivalent with T, denoted by T' and T'' :



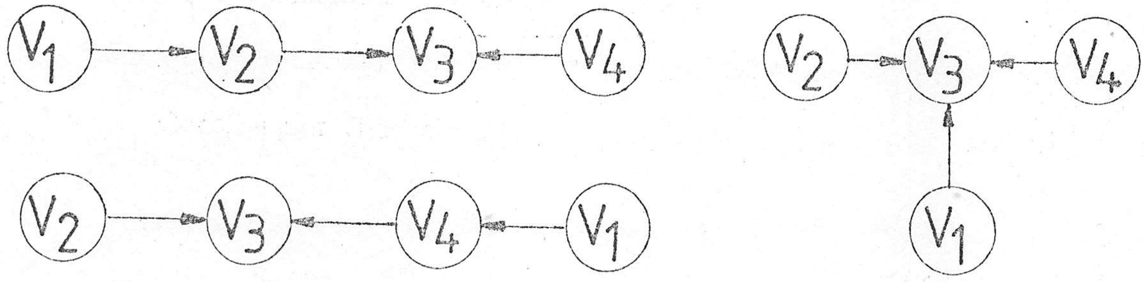


We prove that there is no orientation on any of trees T, T' or T'' such that its set of mandatary zero given by Definition 4.7 is included in Z.

Let assume that there exists an orientation of one of the trees such its mandatory zero is included in Z. Since $(3,5)\notin Z$ and $(6,4)\notin Z$, we must have on the subtree corresponding to the node set V_2, V_3, V_4 the orientation:



Since $(k,1) \in Z$ for $k \geq 3$, the unique edge $\{V_1, V_t\}$ involving V_1 must have the orientation from V_1 to V_t. So we may have the following orientations D, D' and D'' on T, T' respective T'':



But $(1,6) \in Z(T,D)$, $(1,3) \in Z(T',D')$ and $(1,3) \in Z(T'',D'')$ so none of them is included in Z and thus the conjecture is not true.

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