ON SIMPLE GERMS WITH NON-ISOLATED SINGULARITIES

by

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§1. Introduction .

Let $\mathcal{O}=\mathcal{O}_n$ denote the local ring of germs of analytic functions $f:(\mathbb{C}^n,0)\to\mathbb{C}$ and m its maximal ideal . For an analytic germ $f\in\mathcal{O}$ we denote by J_f its Jacobi ideal , namely $J_f=\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$. For an ideal $I\subseteq\mathcal{O}$ we consider as in [8], [9] :

- the primitive ideal $\int I$, defined by $\int I=\{f\in\mathcal{O} \mid (f)+J_f\subseteq I\}$; we have $I^2\subseteq\int I\subseteq I$;

- the group $\mathcal{D}_I$ of local analytic isomorphisms $h:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ such that $h^*(I)=I$; it is a subgroup of the group of all germs of local analytic isomorphisms of $(\mathbb{C}^n,0)$.

$\mathcal{D}_I$ acts on $\int I$ and we shall consider the $\mathcal{R}_I$ (right-equivalence) relation on $\int I$.

In the next section we prove the following

Theorem 1. Let $I\subseteq\mathcal{O}$ be a radical ideal defining a germ of a quasihomogeneous complete intersection in $(\mathbb{C}^n,0)$ with isolated singularity . Suppose that there exist $\mathcal{R}_I$-simple germs in $\int I$. Then in some coordinates $(z_1, \dots, z_n)$ of $(\mathbb{C}^n,0)$ we have either

a) there exists $k\in\{1, \dots, n\}$ such that $I=(z_1, \dots, z_k)$, or

b) there exists $k\in\{1, \dots, n\}$ and a quasihomogeneous isolated singularity $g=g(z_1, \dots, z_k)\in\mathcal{O}_k$ such that $I=(g, z_{k+1}, \dots, z_n)$.

A. Némethi has proved a similar result in [7] for the case when $I=(f^s)$ where $s\geq 1$ and $f\in\mathcal{O}$ is an isolated singularity . When n=3 , D. Siersma has considered a similar problem for the inner modality (see [12]) .

In the last section we derive the list of $\mathcal{R}_I$-simple germs for $I=(z_1, z_2)$.

§2. Proof of Theorem 1 .

We recall from [8], [9] that for an ideal $I\subseteq\mathcal{O}$ and for $f\in\int I$, the tangent space at f to the $\mathcal{R}_I$-orbit of f is defined by

$$T_I(f)=\left\{\eta(f) \;\middle|\; \eta=\sum_{j=1}^{n}\eta_j\frac{\partial}{\partial z_j} \text{ with } \eta(I)\subseteq I \text{ and } \eta_j\in m \text{ for } j=1,\dots,n\right\}$$

and the I-codimension of f is

$$c_I(f)=\dim_{\mathbb{C}}\frac{\int I}{T_I(f)} \quad .$$

Let f_1 , ... , f_p be a minimal set of quasihomogeneous generators of I . Let q be the dimension of the $\mathbb{C}$-vector space $I+m^2/m^2$. If q=p , we have a) with k=q=p .

Suppose that $q<p$. Using a linear change of coordinates , we can assume , without altering the quasihomogeneity of f_1 , ... , f_p , that $f_j(z)=z_j$+higher monomials not containing z_j , for j=1,...,q (we assume that the weights of the coordinates are positive) . Thus , we can consider , by substracting suitable multiples of f_1 , ... , f_q , if necessary , that f_{q+1} , ... , f_p are quasihomogeneous polynomials , not depending on z_1 , ... , z_q . It follows that , in a suitable system z of coordinates , the ideal I is generated by $f_1=z_1$, ... , $f_q=z_q$, f_{q+1} , ... , f_p , where f_{q+1} , ... , $f_p\in m^2$ are quasihomogeneous polynomials depending only on z_{q+1} , ... , z_n .

Since there exist $\mathcal{R}_I$-simple germs in $\int I$, we can find $f\in\int I$ such that $c_I(f)=0$. (The $\mathcal{R}_I$-simple germs are defined similarly with the simple isolated singularities ; see for example [2] or [4] .) From [8], [9], we have $\int I=I^2$ and we can write $f=\sum_{i,j=1}^{p} g_{ij}f_if_j$, with $g_{ij}=g_{ji}$. Let r be the rank of the matrix $(g_{ij}(0))_{i,j=1,q}$. Then r is also the rank of the Hessian matrix evaluated in 0 , $\left(\frac{\partial^2 f}{\partial z_i\partial z_j}(0)\right)_{i,j=1,n}$. As in the proof of Morse Lemma (see for example [6]) we can obtain a system $\tilde{z}$ of coordinates , with $\tilde{z}_j=z_j$ for $j>q$, such that I is generated by $\tilde{f}_1=\tilde{z}_1$, ... , $\tilde{f}_q=\tilde{z}_q$, $\tilde{f}_{q+1}=f_{q+1}$, ... , $\tilde{f}_p=f_p$ and such that

(1) $\quad f=\tilde{z}_1^2+\dots+\tilde{z}_r^2+\sum_{i,j=r+1}^{p}\tilde{g}_{ij}\tilde{f}_i\tilde{f}_j$,

with $\tilde{g}_{ij}=\tilde{g}_{ji}$ and with $\tilde{g}_{ij}\tilde{f}_i\tilde{f}_j\in m^3$. It is easy to see that for any $i,j\geq r+1$, there exists $h_{ij}=h_{ij}(\tilde{z}_{r+1}$, ... , $\tilde{z}_n)$ with $h_{ij}\tilde{f}_i\tilde{f}_j\in m^3$ and such that for any $k\in\mathbb{N}$, $k\geq 2$, we have that f is $\mathcal{R}_I$-equivalent to $\tilde{z}_1^2+\dots+\tilde{z}_r^2+\sum_{i,j=r+1}^{p}h_{ij}\tilde{f}_i\tilde{f}_j+$

$+\sum_{i,j=r+1}^{p}\varphi_{ij}f_if_j$, for some $\varphi_{ij}\in(\tilde{z}_1, \dots, \tilde{z}_r)^k$. Since $c_{\mathbb{I}}(f)<\infty$, f is $\mathbb{I}$-finitely determined (see [8], [9]) . Hence we can assume that in (1) the germs $\tilde{g}_{ij}$ do not depend on $\tilde{z}_1, \dots, \tilde{z}_r$.

We shall write in the sequel z for $\tilde{z}$, f_j for $\tilde{f}_j$ and g_{ij} for $\tilde{g}_{ij}$. Since $c_{\mathbb{I}}(f)=0$, we must have $T_{\mathbb{I}}(f)=\int \mathbb{I}=\mathbb{I}^2$; we prove that this equality implies that r=q=p-1 .

Let $\Theta_{\mathbb{I}}=\{\eta=\sum_{j=1}^{n}\eta_j\frac{\partial}{\partial z_j} \mid \eta(\mathbb{I})\subseteq \mathbb{I}\}$ be the $\mathcal{O}$-module of logarithmic vector fields for I . Since $I=(f_1, \dots, f_p)$ is a reduced quasihomogeneous complete intersection in $(\mathbb{C}^n,0)$ with isolated singularity , the $\mathcal{O}$-module $\Theta_{\mathbb{I}}$ is generated by the following vector fields (see for example [3]) :

(A) $f_i\frac{\partial}{\partial z_j}$, where i=1,...,p and j=1,...,n ;

(B) the "trivial vector fields"

$$\begin{vmatrix} \frac{\partial}{\partial z_{i_1}} & \cdots & \frac{\partial}{\partial z_{i_{p+1}}} \\ \frac{\partial f_1}{\partial z_{i_1}} & \cdots & \frac{\partial f_1}{\partial z_{i_{p+1}}} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_p}{\partial z_{i_1}} & \cdots & \frac{\partial f_p}{\partial z_{i_{p+1}}} \end{vmatrix}$$

for all (p+1)-tuples $(i_1, \dots, i_{p+1})$ satisfying $1\leq i_1\leq i_2\leq\dots\leq i_{p+1}\leq n$;

(C) the Euler vector field $E=\sum_{j=1}^{n} w_jz_j\frac{\partial}{\partial z_j}$, where $w_1, \dots, w_n$ are the weights of the coordinates .

It is clear that $T_I(f)=\Theta_I(f)$.

We recall that $f_j=z_j$ for $1\leq j\leq q$ and $f_{q+1}, \dots, f_p\in m^2$ do not depend on $z_1, \dots, z_q$. Also we recall that $f=z_1^2+\dots+z_r^2+\sum_{i,j=r+1}^{p} g_{ij}f_if_j$ with $g_{ij}=g_{ji}$ not depending on $z_1,\dots,z_r$ and with $g_{ij}f_if_j\in m^3$.

Suppose first that $r<q$. Then a moment thought will convince us that for any $\eta\in\Theta_{\mathbb{I}}$, if we consider the expansion of $\eta(f)$ in a power series , then the coefficient of z_q^2 is zero . Hence $z_q^2\notin T_I(f)=\mathbb{I}^2$, a contradiction . It follows that r=q .

We look now for $f_{q+1}^2, \dots, f_p^2$. It is easy to see that if $\eta\in\Theta_I$ is one of the generators from (A) or (B) , then $\eta(f)$ belongs to the ideal

$L=m\cdot(f_{q+1}, \ldots, f_p)^2+(z_1, \ldots, z_q)\cdot(f_{q+1}, \ldots, f_p)+(z_1, \ldots, z_q)^2$.
On the other hand , for any germ $g\in m$ we have also $(gE)(f)\in L$. Thus $\theta_I(f)=L+\mathbb{C}\cdot E(f)$. If $p-q\geqslant 2$ we have the uniqueness of the weights $w_{q+1}, \ldots, w_n$ (see for example [4]) , hence $f^2_{q+1}, \ldots, f^2_p$ can not belong simultaneously to $\theta_I(f)$, in contradiction with the equality $I^2=\theta_I(f)$. It follows that $q+1=p$. The theorem is proved .

§3. The simple germs for $I=(z_1, z_2)$.

D. Siersma has found the $\mathcal{R}_I$-simple germs when $I=(z_1, \ldots, z_{n-1})\subseteq \mathcal{O}$ in [10] and for $I=(z_1z_2, z_3, \ldots, z_n)\subseteq\mathcal{O}$ in [12] . For the case when $I=(z_1)\subseteq\mathcal{O}$, the list of $\mathcal{R}_I$-simple germs follows from the work of V. I. Arnold [1] (see for example [13]) .

In the sequel we derive the list of $\mathcal{R}_I$-simple germs for $I=(z_1, z_2)$. We shall suppose that $n\geqslant 4$ and we shall consider only germs $f\in I^2$ with $j^2f=0$. (The simple germs $f\in I^2$ with $j^2f\neq 0$ are suspensions of those in [13] .)

We use the following classical lemma :

<u>Lemma</u> . Let $f_t=f+t\cdot\phi\in I^2$ be a family of germs , with $t\in\mathbb{R}$.

a) If $\phi\in\mathcal{T}(f_t)$ for every $t\in\mathbb{R}$, then , for any $t\in\mathbb{R}$, f_t is $\mathcal{R}_I$-equivalent with f_0 .

b) If $\phi\notin\mathcal{T}(f_t)$ for every $t\in\mathbb{R}$, then , for any $t\in\mathbb{R}$, f_t is not $\mathcal{R}_I$-simple .

If we denote the coordinates $z_3, \ldots, z_n$ by $u_1, \ldots, u_{n-2}$ and the Milnor number of an isolated singularity g by $\mu(g)$, we have the following :

<u>Theorem 2</u> . Let $I=(z_1, z_2)\subseteq\mathcal{O}$ and $f\in I^2$ with $j^2f=0$. Then f is $\mathcal{R}_I$-simple if and only if f is $\mathcal{R}_I$-equivalent to a germ in the following table .

	Normal form of f	$c_I(f)$	Conditions
I_n	$u_1z_1^2+u_2z_2^2+u_3z_1z_2$	3	$n\geqslant 5$
I_4	$u_1z_1^2+u_2z_2^2$	3	$n=4$
II	$u_1z_1^2+u_2z_2^2+z_1z_2\cdot g(u_3, \dots, u_{n-2})$	$n-2+\mu(g)$	$n\geqslant 5$; $g\in$ A-D-E
III	$u_1z_1z_2+u_2z_1^2+z_2^2(z_2+u_2^k+u_3^2+\dots+u_{n-2}^2)$	$k+n-2$	$n\geqslant 4$; $k\geqslant 2$
	$u_1z_1z_2+u_2z_1^2+z_2^2(z_2+u_2u_3+u_3^k+u_4^2+\dots+u_{n-2}^2)$	$k+n-2$	$n\geqslant 4$; $k\geqslant 3$
	$u_1z_1z_2+u_2z_1^2+z_2^2(z_2+u_2^2+u_3^3+u_4^2+\dots+u_{n-2}^2)$	$n+2$	$n\geqslant 4$
IV	$u_1z_1z_2+u_2z_1^2+z_2^2(z_2^k+u_2^2+\dots+u_{n-2}^2)$	$k+n-1$	$n\geqslant 4$; $k\geqslant 2$
Va	$u_1z_1z_2+u_2z_1^2+z_2^2(z_2^2+u_2^3+u_3^2+\dots+u_{n-2}^2)$	$n+3$	$n\geqslant 4$
Vb	$u_1z_1z_2+u_3z_1^2+z_2^2(z_2^2+u_2^3+u_3^2+\dots+u_{n-2}^2)$	$n+4$	$n\geqslant 4$
VI	$u_1z_1z_2+u_2z_1^2+z_2^2(z_2u_2+u_2^k+u_3^2+\dots+u_{n-2}^2)$	$k+n-1$	$n\geqslant 4$; $k\geqslant 3$
VI^3	$u_1z_1z_2+u_3z_1^2+z_2^2(z_2u_2+u_2^3+u_3^2+\dots+u_{n-2}^2)$	$n+4$	$n\geqslant 4$

Proof . If $f\in I^2$ has $j^2f=0$, then $j^3f=u_1Q_1(z_1, z_2)+\dots+u_{n-2}Q_{n-2}(z_1, z_2)+C(z_1, z_2)$, where $Q_1, \dots, Q_{n-2}$ are quadrics and C is a cubic in z_1 , z_2 . We suppose that $c_I(f)<\infty$. Hence f is $\mathcal{R}_I$-equivalent to a jet j^kf for sufficiently large k (see [9]) .

Let V be the $\mathbb{C}$-vector space generated by $Q_1, \dots, Q_{n-2}$ in the vector space of quadrics in z_1 , z_2 .

If dimV=3 then $n\geqslant 5$ and we can find in $\mathcal{D}_I$ a linear isomorphism of $(\mathbb{C}^n,0)$ such that $j^3f=u_1z_1^2+u_2z_2^2+u_3z_1z_2$. It follows by [8] , [9] that f is $\mathcal{R}_I$-equivalent with j^3f (f is a D(1,1)-type germ) and f is $\mathcal{R}_I$-simple .

If dimV≤ 1 then f is not $\mathcal{R}_I$-simple . Namely , any neighbourhood of f contains a germ which is $\mathcal{R}_I$-equivalent to a germ $\tilde{f}=u_1z_1z_2+z_1^3+z_2^3+z_1^2(u_2^2+\dots+u_{n-2}^2)+z_2^2\cdot\varphi(u_2, \dots, u_{n-2})$ where $\varphi\in m^2$. It is easy to see that for any φ , $\tilde{f}$ is not $\mathcal{R}_I$-simple .

If dimV=2 then , using the classification of pencils of quadrics in z_1 , z_2 , we can find in $\mathcal{D}_I$ some linear isomorphisms of $(\mathbb{C}^n,0)$ such that j^3f is one of the following cubics :

$$u_1z_1^2+u_2z_2^2 \quad ; \quad u_1z_1z_2+u_2z_1^2 \quad \text{or} \quad u_1z_1z_2+u_2z_1^2+z_2^3 .$$

When $j^3f=u_1z_1^2+u_2z_2^2$ it follows , directly or using the technique of global transversal from [5] , that f is $\mathcal{R}_{II}$-equivalent to $u_1z_1^2+u_2z_2^2+$ $+z_1z_2g(u_3 , \dots , u_{n-2})$. Now it is easy to see , for $n\geqslant 5$, that f is $\mathcal{R}_{II}$-simple if and only if g is a simple isolated singularity (g is an A-D-E singularity ; see [2] ,or [4] for the normal forms) .

If $j^3f=u_1z_1z_2+u_2z_1^2+z_2^3$, then f is $\mathcal{R}_{II}$-equivalent to $u_1z_1z_2+u_2z_1^2+z_2^2(z_2+$ $+g(u_2 , \dots , u_{n-2}))$ with $g\in m^2$. It is easy to see that f is $\mathcal{R}_{II}$-simple if and only if g is a simple boundary singularity in the sense of Arnold , the boundary being $u_2=0$ (see [1]) .

The most difficult case is when $j^3f=u_1z_1z_2+u_2z_1^2$. In this situation f is $\mathcal{R}_{II}$-equivalent to $u_1z_1z_2+u_2z_1^2+z_2^2h(z_2 , u_2 , \dots , u_{n-2})$ with $h\in m^2$. If h is a simple boundary singularity with respect to $z_2=0$, we change the coordinates such that h becomes the normal form of a B-C-F singularity . Then f is $\mathcal{R}_{II}$-equivalent to $u_1z_1z_2+\varphi(u_2 , \dots , u_{n-2})z_1^2+z_2^2h$, with $\varphi\in m\setminus m^2$, and we obtain the germs in the table by using the lemma and loocking at $j^1\varphi$.

If h is not a simple boundary singularity then f can be deformed to a germ which is $\mathcal{R}_{II}$-equivalent to $u_1z_1z_2+\varphi(u_2 , \dots , u_{n-2})z_1^2+z_2^2h(z_2 , u_2 , \dots , u_{n-2})$ where $\varphi\in m\setminus m^2$ and h is one of the following unimodal boundary singularities (see [2]):

$F_{1,0}$: $z_2^3+az_2u_2^2+u_2^3+u_3^2+\dots+u_{n-2}^2$, $4a^3+27\neq 0$

$K_{4,2}$: $z_2^2+az_2u_2^2+u_2^4+u_3^2+\dots+u_{n-2}^2$, $a^2\neq 4$

or L_6: $z_2u_2+az_2u_3+u_2^2u_3+u_3^3+u_4^2+\dots+u_{n-2}^2$.

Using the lemma with $\tau(f)$ replaced by $\tau(f)+(u_2 , \dots , u_{n-2})^3(z_1)^2$ we obtain that f is not $\mathcal{R}_{II}$-simple .

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