# ON THE OPERATORIAL NEVANLINNA-PICK PROBLEM

by

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## I. INTRODUCTION

The main purpose of this paper is to present a Schur-Nevanlinna type algorithm for the operatorial Nevanlinna-Pick problem tend to render explicit the Pick criterium as a natural continuation of the operatorial Schur algorithm.

The operatorial Nevanlinna-Pick problem can be solved with the Sarason-Nagy-Foias theorem ([13], [17]) and the classical criterium of Pick can be easily generalized in the operatorial setting ([5],[14]). On the other hand a description of the matricial case was considered in [4] and [8].

Let us now give a short description of the results obtained in this paper. In Section 2 we remind the classical scalar Schur-Nevanlinna algorithm and the method of Weyl circles ([11],[12], [16],[8]).

In Section 3 we consider the operatorial setting of the results presented in the previous section. We establish an operatorial Schur type algorithm which gives us the solutions of the Nevanlinna-Pick problem extending the results from [3],[4]. In the sequel we derive the method of Weyl circles describing more precisely the structure of the solutions.

Section 4 describes the Pick criterium in our operatorial setting reflecting the close connection between Pick matrices of the operatorial Nevanlinna-Pick problem and Toeplitz matrices in the operatorial Schur problem.

## 2. PRELIMINARIES

Let $\mathbb{D} = \left\{ z \in \mathbb{C} \mid |z| \leq 1 \right\}$ and $\mathcal{B} = \left\{ f \in H_\infty(\mathbb{D}) \mid \|f\|_\infty \leq 1 \right\}$
The following interpolation problem for functions in $\mathcal{B}$ it is known as the Nevanlinna-Pick problem (NPP). There is also another variant of this problem for analitic functions on D with Re $f \geqslant 0$. Such problems of interpolationsare of great interest in system theory as broad-band matching, cascade transformations, etc.

(NPP) Given any set of distinct points $\{z_n\}_{n\geq 1} \subset \mathbb{D}$ and $\{w_n\}_{n\geq 1} \subseteq$ D let us consider the question of the existence of a class $\mathcal{B}$ function f satisfying :

$$f(z_n) = w_n$$

The main result is the following Pick criterion (PC)

(PC) Let $$P_n = \left[ \frac{1-\bar{w}_k w_l}{1-\bar{z}_k z_l} \right]_{k,l=1}^{n}$$

A necessary and sufficient condition for the solvability of (NPP) is that all $P_n$ $(n \geqslant 1)$ are non negative definite.

We shall present in the sequel the Schur-Nevanlinna algorythm associated with the preceding (NPP).

Let $b_n(z) = \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1-\bar{z}_n z}$ , $n \geq 1$. We may assume all $z_n \neq 0$ $(n \geq 1)$, otherwise this can be easily realised with a suitable Möbius transformation . We define:

$$f_1(z) = f(z) \qquad g_1 = f_1(z_1)$$

$$(2.1) \qquad f_{n+1}(z) = \frac{1}{b_n(z)} \cdot \frac{f_n(z) - g_n}{1 - \bar{g}_n f_n(z)} \qquad \text{where } g_n = f_n(z_n)$$

It is not difficult to check that ,

the sequence $\{g_n\}_{n\geq 1}$ has the property that $|g_n| \leq 1$, $(n \geq 1)$ and if there is some $n_o \in N$ such that $|g_{n_o}| = 1$ then $g_n = 0$ for $n > n_o$ . In this case we obtain a finite sequence $f_1 = f, f_2, \ldots, f_{n_o} = g_{n_o}, f_{n_o+1} = 0$ and the Schur-Nevanlinna algorithm has a rational function as its unique solution ([11], [16]).

Now using (2.1) we have :

$$(2.2)\qquad f(z) = \frac{b_n(z)\,\widetilde{A}_n(z)\,f_{n+1}(z) + \widetilde{B}_n(z)}{b_n(z)\,B_n(z)\,f_{n+1}(z) + A_n(z)}$$

Where $A_n, B_n, \widetilde{A}_n, \widetilde{B}_n$ are rational functions given by the following formulae:

$$A_1(z)=1\ ,\ B_1(z)=\overline{g_1}\ ,\ \widetilde{A}_1(z)=1\ ,\ \widetilde{B}_1(z)=g_1$$

$$(2.3)\qquad \begin{aligned} A_{n+1}(z)&=A_n(z) + g_{n+1}b_n(z)\,B_n(z)\\ B_{n+1}(z)&=\overline{g}_{n+1}A_n(z) + b_n(z)\,B_n(z)\\ \widetilde{A}_{n+1}(z)&=B_n(z)\,\widetilde{A}_n(z) + \overline{g}_{n+1}\widetilde{B}_n(z)\\ \widetilde{B}_{n+1}(z)&=b_n(z)g_{n+1}\widetilde{A}_n(z) + \widetilde{B}_n(z) \end{aligned}$$

Remark : 1. As a consequence of (2.1) we obtain :

$$w_{n+1} = \frac{b_n(z_{n+1})\,\widetilde{A}_n(z_{n+1})\,g_{n+1} + \widetilde{B}_n(z_{n+1})}{b_n(z_{n+1})\,B_n(z_{n+1})\,g_{n+1} + A_n(z_{n+1})}$$

2. We can use a matricial transcription of (2.3) :

$$(2.4)\qquad \begin{bmatrix} A_{n+1}(z) & \widetilde{B}_{n+1}(z) \\ B_{n+1}(z) & \widetilde{A}_{n+1}(z) \end{bmatrix} = \begin{bmatrix} 1 & b_n(z)g_{n+1} \\ g_{n+1} & b_n(z) \end{bmatrix} \begin{bmatrix} A_n(z) & \widetilde{B}_n(z) \\ B_n(z) & \widetilde{A}_n(z) \end{bmatrix}$$

By taking determinants in both sides of (2.4) we also derive:

$$(2.5)\qquad A_n(z)\widetilde{A}_n(z) - B_n(z)\widetilde{B}_n(z) = \prod_{k=1}^{n} b_{k-1}(z)(1-|g_k|^2),$$

where $b_o(z) = 1$

Finaly using (2.2) and (2.5) we obtain :

$$f(z) - \frac{\widetilde{B}_n(z)}{A_n(z)} = \frac{b_n(z)\left[A_n(z)\widetilde{A}_n(z) - B_n(z)\widetilde{B}_n(z)\right] f_{n+1}(z)}{A_n(z)\left[b_n(z)B_n(z)f_{n+1}(z) + A_n(z)\right]} =$$

$$= \prod_{k=1}^{n} b_k(z)(1-|g_k|^2)\ \frac{f_{n+1}(z)}{A_n(z)\left[b_n(z)B_n(z)f_{n+1}(z) + A_n(z)\right]}\ .$$

$A_n, B_n, \widetilde{A}_n, \widetilde{B}_n$ are uniquely determined by $\{g_k\}^n$ and $f$ will be uniquely determined by its parameters when $\left\{\frac{B_n}{A_n}\right\}_{n\geq 1}$ converges uniformly to f on compact subsets of D. (this is the case when $\sum_{n\geq 1}(1-|z_n|)=\infty$).

Let now z D and define :

(2.6) $\mathcal{W}_n(z) = \{ Z \in \mathbb{C} \mid Z = f(z)$ , where f runs through the set of partial solutions of (NPP) with initial dates $\{z_k, w_k\}_{k=1}^n \}$ .

By (2.2) and (2.6) we deduce :

$$\mathcal{W}_n(z) = \left\{ Z\in \mathbb{C} \,\middle|\, Z= \frac{b_n(z)\,\widetilde{A}_n(z)\, f_{n+1}(z) + \widetilde{B}_n(z)}{b_n(z)\, B_n(z)\, f_{n+1}(z) + A_n(z)}, f_{n+1}\in \mathcal{B}\right\}$$

A straight forward computation shows us that :

(2.7) $\mathcal{W}_n(z) = \{ Z \subset \mathbb{C} \mid |Z - a_n(z)| \leq |r_n(z)| \}$ where :

$$a_n(z)=\frac{\overline{A_n}(z)\,\widetilde{B}_n(z) - |b_n(z)|^2\, \overline{B_n}(z)\,\widetilde{A}_n(z)}{A_n(z)^{\;2} - b_n(z)^{\;2}\quad B_n(z)^{\;2}},$$

$$r_n(z)= \frac{b_n(z)\left[A_n(z)\,\widetilde{A}_n(z) - B_n(z)\,\widetilde{B}_n(z)\right]}{|A_n(z)|^2 - |b_n(z)|^2\,|B_n(z)|^2} =$$

$$= \frac{\prod_{k=1}^{n} b_k(z)\quad (1-|g_k|^2)}{|A_n(z)|^2 - |b_n(z)|^2\,|B_n(z)|^2}.$$

Remark : 1. It is easy to see that $|A_n(z)|^2 - |b_n(z)|^2\; |B_n(z)|^2 \neq 0$

2. Clearly $\mathcal{W}_{n+1}(z) \subseteq \mathcal{W}_n(z)$ and if we denote by $\mathcal{W}(z) = \bigcap_{n\geq 1}\mathcal{W}_n(z)$ (which is a nonvoid set) then any solution f of the (NPP) satisfies $f(z) \in \mathcal{W}(z)$.

# 3. OPERATORIAL SCHUR-NEVANLINNA ALGORITHM AND THE METHOD OF WEYL CIRCLES

We shall consider in the sequel the operatorial variant of the algorithm described in the preceding section. Let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \ldots$ be Hilbert spaces and $\mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ the set of analytical and contractive operatorial valued functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$

(NPP) - operatorial case : given any set of distinct points $\{z_n\}_{n\geq 1} \subset D$ and any set of contractions $\{W_n\}_{n\geq 1} \subset \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ let us consider the question of the existence of a class $\mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ function F satisfying : $F(z_n) = W_n$ , $n \geq 1$

Using Sarason-Nagy-Foias theorem we easily obtain the same Pick type criterium for the existence of the solutions of the operatorial (NPP) (details in [15], [5], [14] ).

Now let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction and denote as usual:

$$(3.1) \qquad D_T = (I-T^*T)^{\frac{1}{2}} \qquad \mathcal{D}_T = \overline{D_T \mathcal{H}_1}$$

the defect operators and the defect spaces of T.

We define :

$$(3.2) \qquad J_T(z) = \begin{bmatrix} T & zD_{T^*} \\ D_T & -zT^* \end{bmatrix} ; \quad \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{D}_{T^*} \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_2 \\ \oplus \\ \mathcal{D}_T \end{matrix}$$

Remark: 1. It is easy to see that $J_T$ is inner from both sides([17])

2. For z=1, $J_T(1) = J(T)$ is the elementary rotation of T ([6],[17]).

In the operatorial setting is is useful to replace Möbius transformations by the so called cascade transformations.

Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{K}_1, \mathcal{H}_2 \oplus \mathcal{K}_2)$ be a contraction and for an arbitrary contraction $X \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1)$ we define the cascade transformation:

$$(3.3) \qquad C_S(X) = A + BX(I - DX)^{-1}C$$

when ever the inverse of (I - DX) exists. One immediately verifies that $C_S(X)$ is a contraction. In a simmilar way we can define cascade transformations for operator valued functions.

We have the following result ([3]) :

THEOREM 1. For any $H \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ the equation $H = C_{J_T}(F)$ has a unique solution $F \in \mathcal{S}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ where $T = H(0)$.

Let us consider in the sequel :

$$(3.4) \qquad J_T^k(z) = \begin{bmatrix} T & b_k(z)D_{T^*} \\ D_T & -b_k(z)T^* \end{bmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathfrak{D}_{T^*} \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_2 \\ \oplus \\ \mathfrak{D}_T \end{matrix}$$

where $b_k(z) = \dfrac{\bar{z}_k}{|z_k|} \dfrac{z_k - z}{1 - \bar{z}_k z}$ , $k \geq 0$

and the following cascade transformation of $F \in \mathcal{S}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ defined by:

$$(3.5) \qquad C_{J_T^k}(F)(z) = T|_{\mathfrak{D}_T} + b_k(z)D_{T^*}F(z)\left[I + b_k(z)T^*F(z)\right]^{-1} D_T|_{\mathfrak{D}_T}$$

Remark: 1. Clearly $C_{J_T^k}(F)$ makes sense because for $|z| < 1$ , $|b_k(z)| < 1$ and as $T$ , $F(z)$ are contractions $I + b_k(z)T\ F(z)$ is invertibile

2. Since a short computation shows us that :

$$I - C_{J_T^k}(F)(z)^* C_{J_T^k}(F)(w) =$$

$$= D_T\left[I + \bar{b}_k(z)TF^*(z)\right]^{-1}\left[I - b_k(z)b_k(w)F^*(z)F(w)\right]\left[I + b_k(w)T^*F(w)\right]^{-1}D_T|_{\mathfrak{D}_T}$$

we have : $C_{J_T^k}(F) \in \mathcal{S}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ .

Let $G \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ and let us define :

$$\frac{\bar{z}_0}{|z_0|} H(z) = G\left(\frac{z_0 - z}{1 - \bar{z}_0 z}\right) \qquad \text{with } H \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$$

Clearly $\dfrac{\bar{z}_0}{|z_0|} H(0) = G(z_0)$ .

Denoting $T = G(z_0) \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $T' = H(0) \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ we have $T' = \dfrac{|z_0|}{\bar{z}_0} T$ and as $I - T'^* T' = I - T^* T$ and $I - T'T'^* = I - TT^*$

$$D_{T'} = D_T \quad , \quad D_{T'} = D_T \quad , \quad D_{T'} = D_T \; , \quad D_{T'} = D_T \; .$$

Let us consider the equation $H = C_{J_{T'}}(F)$ and for $z = \frac{z_o - Z}{1-\bar{z}_o Z}$, $Z \in \mathbb{D}$ we have :

$$H\left(\frac{z_o-Z}{1-\bar{z}_o Z}\right) = \frac{|z_o|}{z_o} T\Big|_{\mathcal{D}_T} + \frac{z_o - Z}{1-\bar{z}_o Z} D_T F\left(\frac{z_o-Z}{1-\bar{z}_o Z}\right)\left[I + \frac{z_o-Z}{1-\bar{z}_o Z}\frac{|z_o|}{z_o} T F \frac{z_o-Z}{1-\bar{z}_o Z}\right]^{-1} D_T\Big|_{\mathcal{D}_T}$$

Denoting $F_1(Z) = F\left(\frac{z_o-Z}{1-\bar{z}_o Z}\right)$ we obtain with Theorem 1 :

<u>PROPOSITION 2</u>. For any $G \in \mathcal{S}(\mathcal{H}_1,\mathcal{H}_2)$ the equation $G = C_{J_T^o}(F_1)$ has a unique solution $F_1 \in \mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$ where $T = G(z_o)$.

The main object used for the description of the Schur-Nevanlinna algorithm is the notion of Schur sequence. A sequence of contractions $\{\Gamma_n\}_{n\geq 1}$ is called Schur sequence if $\Gamma_1 \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ and for $n \geq 2$, $\Gamma_n : \mathcal{D}_{\Gamma_{n-1}} \to \mathcal{D}_{\Gamma^*_{n-1}}$.

Let $F \in \mathcal{S}(\mathcal{H}_1,\mathcal{H}_2)$ solution of the operatorial (NPP) with the initial dates $\{z_n, W_n\}_{n\geq 1}$. Looking forward to Proposition 2 we shall define an algorithm similar with the one in section 2 :

$$F_1(z) = F(z) \qquad , \qquad \Gamma_1 = F_1(z_1) = W_1$$

$F_2$ is defined by the equation :

$$F_1 = C_{J^1_{\Gamma_1}}(F_2) \; , \; \Gamma_2 = F_2(z_2) : \mathcal{D}_{\Gamma_1} \to \mathcal{D}_{\Gamma_1^*} \text{ is a contraction}$$
$$F_2 \in \mathcal{S}(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_1^*})$$

By induction we implicitely define $F_{n+1}$ :

(3.6) $$F_n = C_{J^n_{\Gamma_n}}(F_{n+1}) \; , \; \Gamma_{n+1} = F_{n+1}(z_{n+1}) : \mathcal{D}_{\Gamma_n} \to \mathcal{D}_{\Gamma_n^*} \text{ is a contraction}$$
$$F_{n+1} \in \mathcal{S}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n^*})$$

We readily deduce from (3.6) :

(3.7) $$F = C_{J^1_{\Gamma_1}}(C_{J^2_{\Gamma_2}} \ldots (C_{J^n_{\Gamma_n}}(F_{n+1}))\ldots).$$

It is easy to check that for two contractions $S_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$ we have the relations:

(3.8) $$C_{S_1}(C_{S_2}(X)) = C_{S_1 * S_2}(X)$$

where $*$ is the Redheffer product defined by :

$$S_1 * S_2 = \begin{bmatrix} A_1+S_1A_2(I-D_1A_2)^{-1}C_1 & B_1A_2(I-D_1A_2)^{-1}D_1B_2+B_1B_2 \\ C_2(I-D_1A_2)^{-1}C_1 & C_2(I-D_1A_2)^{-1}D_1B_2+D_2 \end{bmatrix}$$

whenever the inverse of $(I-D_1A_2)$ exists.

Then (3.7) and (3.8) give us :

$$(3.9) \qquad F = C_{J^1_{\Gamma_1} * J^2_{\Gamma_2} * \ldots * J^n_{\Gamma_n}}(F_{n+1}).$$

Considering the matricial functions :

$$(3.10) \qquad \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = J^1_{\Gamma_1} * J^2_{\Gamma_2} * \ldots * J^n_{\Gamma_n} \qquad \text{we have}$$

$$(3.11) \qquad F = C_{\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}}(F_{n+1}) \qquad \text{where } A_n, B_n, C_n, D_n \text{ satisfy :}$$

$$A_1(z)=\Gamma_1 \;,\quad B_1(z)=b_1(z)D_{\Gamma_1^*} \;,\quad C_1(z)=D_{\Gamma_1} \;,\quad D_1(z)=-b_1(z)\Gamma_1^*$$

$$A_{n+1}(z)= A_n(z)+B_n(z)\Gamma_{n+1}\left[I-D_n(z)\Gamma_{n+1}\right]^{-1}C_n(z)$$

$$B_{n+1}(z)= b_{n+1}(z)B_{n+1}\Gamma_{n+1}\left[I-D_n(z) \;_{n+1}\right]^{-1}D_nD_{\Gamma^*_{n+1}} +$$

$$(3.12) \qquad\qquad + b_{n+1}(z)B_n(z)D_{\Gamma^*_{n+1}}$$

$$C_{n+1}(z)=D_{\Gamma_{n+1}}\left[I-D_n(z)\Gamma_{n+1}\right]^{-1}C_n(z)$$

$$D_{n+1}(z)=b_{n+1}(z)D_{\Gamma_{n+1}}\left[I-D_n(z)\Gamma_{n+1}\right]^{-1}D_n(z)D_{\Gamma^*_{n+1}} - b_{n+1}(z)\Gamma^*_{n+1}$$

THEOREM 3. The collection of partial solutions of the operatorial (NPP) with initial dates $\{z_k, W_k\}_{k=1}^n$ is contained in the set of contractive operator valued functions given by the formulae :

$$F = C_{\begin{bmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{bmatrix}}(F_{n+1}) \qquad \text{where } F_{n+1} \in \mathcal{S}(\Delta_{\Gamma_n}, \Delta_{\Gamma_n^*})$$

and where $A_n, B_n, C_n, D_n$ are given by (3.6) (3.12) and :

$$W_k = C_{\begin{bmatrix} A_{k-1}(z_k) & B_{k-1}(z_k) \\ C_{k-1}(z_k) & D_{k-1}(z_k) \end{bmatrix}}(\Gamma_k) \qquad , \quad k=1,2,\ldots,n$$

Remark: 1. $\{\Gamma_n\}_{n\geq 1}$ will be a so called Schur (-Nevanlinna) sequence.

2. Similar relations can be obtained in the operatorial (NPP) with infinite initial dates.

Obviously $\{\Gamma_n\}_{n\geq 1}$ is uniquely determined by F.

For the converse :

$$F(z) - A_n(z) = C_{\begin{bmatrix} A_{n-1}(z) & B_{n-1}(z) \\ C_{n-1}(z) & D_{n-1}(z)\end{bmatrix}}(F_n)(z) - C_{\begin{bmatrix} A_{n-1}(z) & B_{n-1}(z) \\ C_{n-1}(z) & D_{n-1}(z)\end{bmatrix}}(\Gamma_n)(z) =$$

$$= B_{n-1}(z)\left\{F_n(z)\left[I-D_{n-1}(z)F_n(z)\right]^{-1} - \Gamma_n\left[I-D_{n-1}(z)\Gamma_n\right]^{-1}\right\}C_{n-1}(z).$$

But from (3.12) we obtain :

$$B_n(z) = \prod_{k=1}^{n} b_k(z)\widehat{B}_n(z) \quad \text{with } \widehat{B}_n(z) \in \mathcal{S}(D_{\Gamma_n^{*}}, \mathcal{H}_2)$$

It follows :

(3.13) $\quad F(z) - A_n(z) = \prod_{k=1}^{n} b_k(z)\widehat{F}_n(z)$ where $\widehat{F}_n$ is a bounded operator valued function.

As a consequence of (3.13) $\{A_n\}_{n\geq 1}$ converges uniformely to F on compact subsets of D -in the uniform norm- iff $\prod_{k=1}^{n} b_k(z)$ converges uniformely to 0 on compact subsets of D. This is the case iff $\sum_{n\geq 1}(1-|z_n|)=\infty$.

In this case F is uniquely determined by $\{\Gamma_n\}_{n\geq 1}$ as $\{\Gamma_n\}_{n\geq 1}$ uniquely determines $\{A_n\}_{n\geq 1}$.

We shall restict our study to the nondegenerate case, when we have at least a solution of the operatorial (NPP), to present the method Weyl circles. For simplicity of notations we consider $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{S}(\mathcal{H}) = \mathcal{S}(\mathcal{H},\mathcal{H})$.

Let $\mathbb{W}_n(z) = \{ Z \in \mathcal{L}(\mathcal{H}) \mid Z = F(z)$ where F runs through the collection of the solutions of the operatorial (NPP) with initial dates $\{z_k, W_k\}_{k=1}^{n} \}$

THEOREM 4. We have the following representation for the elements of $\mathbb{W}_n(z)$.

(3.14) $$Z = \left[A_n(z) + B_n(z)\, D_n^{*} D^{-2}_{D_n^*}(z) C_n(z)\right] +$$

$$+ \left[B_n(z)\, D^{-2}_{D_n}(z)\, B_n^{*}(z)\right]^{\frac{1}{2}} E_n X_n F_n \left[C_n^{*}(z) D^{-2}_{D_n} C_n(z)\right]^{\frac{1}{2}}$$

where $E_n, F_n, X_n \in \mathcal{L}(\mathcal{H})$ are properly defined and $A_n, B_n, C_n, D_n$ are given by (3.12).

Proof : As a consequence of (3.11) we have :

$$\mathbb{W}_n(z) = \left\{ Z \in \mathcal{L}(\mathcal{H}) \;\middle|\; Z = C_{\begin{bmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{bmatrix}}(W)(z) \quad \text{where } W \in \mathcal{S}(\Delta_{\Gamma_n}, \Delta_{\Gamma_n^*}) \right\}$$

And so:

(3.15) $$Z = A_n(z) + B_n(z)\, W(z)\left[I - D_n(z)\, W(z)\right]^{-1} C_n(z)$$

Let us consider the relation :

$$Y = W(z)\left[I - D(z)\, W(z)\right]^{-1}$$

$$Y\left[I - D(z)\, W(z)\right] = W(z)$$

$$Y = \left[I + Y\, D(z)\right] W(z) \qquad \text{so :}$$

$$YY^{*} = \left[I + Y\, D(z)\right] W(z)\, W^{*}(z) \left[I + D^{*}(z)\, Y^{*}\right] \leq \left[I + YD(z)\right]\left[I + D^{*}(z) Y^{*}\right]$$

$$Y\left[I - D(z) D^{*}(z)\right] Y^{*} - YD(z) - D^{*}(z) Y^{*} - I \leq 0$$

With a short computation we obtain :

$$\left\{ Y\left[I - D(z)D^{*}(z)\right]^{\frac{1}{2}} - D^{*}(z)\left[I - D(z)D^{*}(z)\right]^{\frac{1}{2}} \right\}\left\{ Y\left[I - D(z)D^{*}(z)\right]^{\frac{1}{2}} - D^{*}(z)\left[I - D(z)D^{*}(z)\right]^{\frac{1}{2}} \right\}^{*}$$

$$\leq \left[I - D^{*}(z) D(z)\right]^{-1}.$$

We have the factorization:

$$Y\left[I - D(z)D^{*}(z)\right]^{\frac{1}{2}} - D^{*}(z)\left[I - D(z)D^{*}(z)\right]^{\frac{1}{2}} = \left[I - D^{*}(z)D(z)\right]^{-\frac{1}{2}} X, \text{ with } X \in \mathcal{L}(\mathcal{H})$$

It results:

(3.16) $$Y = D(z)\, D^{-2}_{D(z)} + D^{-1}_{D(z)} X D^{-1}_{D^*(z)}$$ , so (3.15) becomes:

$$Z = A_n(z) + B_n(z)\, D_n^{*}(z)\, D^{-2}_{D_n}(z)\, C_n(z) +$$

$$+ B_n(z)\, D^{-1}_{D_n}(z)\, X_n\, D^{-1}_{D_n}(z)\, C_n(z) \quad \text{where } X_n \in \mathcal{L}(\mathcal{H})$$

With another factorization we obtain :

$$Z = \left[A_n(z) + B_n(z)\, D_n(z)\, D_{D_n}^{-2}(z)\, C_n(z)\right] +$$

$$+ \left[B_n(z)\, D_{D_n}^{-2}(z)\, B_n(z)\right]^{\frac{1}{2}} \cdot E_n X_n F_n \cdot \left[C_n(z)\, D_{D_n}^{-2}(z)\, C_n(z)\right]^{\frac{1}{2}}$$

where $E_n, F_n \in \mathcal{L}(\mathcal{H})$ □

Remark: $W_n(z)$ is a closed sphere of "center"

$$A_n(z) + B_n(z)\, D_n(z)\, D_{D_n}^{-2}(z)\, C_n(z)$$

and right and left "radii" given by :

$$R_n^r(z) = C_n(z)\, D_{D_n}^{-2}(z)\, C_n(z)$$

$$R_n^l(z) = B_n(z)\, D_{D_n}^{-2}(z)\, B_n(z)$$

Clearly we have a unique solution when $R_n^r(z) = 0$ or $R_n^l(z) = 0$

## 4. PICK CRITERIA

Considering the Carathéodory class $\mathcal{C}(\mathcal{H})$ of analytic functions on D with positive real part and taking values in $\mathcal{L}(\mathcal{H})$, let us define :

(4.1) $\quad F(z) = (I - G(z))(I + G(z))^{-1}$ where $G \in \mathcal{C}(\mathcal{H})$

This reflects a one to one correspondence between functions in $\mathcal{S}(\mathcal{H})$ with $F(0) = 0$ and the functions $G \in \mathcal{C}(\mathcal{H})$ with $G(0) = I$. Now let :

(4.2) $\quad$ Now let $P_n = \left[\dfrac{I - W_k^* W_l}{1 - \bar{z}_k z_l}\right]_{k,l=0}^{n}$ where $\{z_n\}_{n \geq 0} \subseteq D$

and $\{W_n\}_{n \geq 0} \subset \mathcal{L}(\mathcal{H})$ contractions (we assume $z_0 = 0$, $W_0 = 0_{\mathcal{H}}$)

Let us suppose the inverse of $(I + W_k)$ exists for $k \geq 0$. We consider then the correspondence, closely related with (4.1) given by :

$$Y_k = (I + W_k)^{-1}(I - W_k)$$

(4.3)

$$W_k = (I - Y_k)(I + Y_k)^{-1}$$

For the operatorial (NPP) with data $\{z_k, W_k\}_{k \geq 0}$ and the solutions $F$ such that $F(z_k) = W_k$, $k \geq 0$ we clearly have $G(z_k) = Y_k$.

Let $\zeta_k = \prod_{l \neq k} (z_k - z_l)^{-1}$ and define $U_n = \left[(I+Y_k)\,\zeta_k\, z_k^l\right]_{k,l=0}^{n}$

With little algebra we obtain that:

$M_n = \frac{1}{2} U_n^* P_n U_n$ is a hermitian Toeplitz matrice

Let us assume that $M_n, n \geq 0$ are extended to an infinite Toeplitz form on $\mathbb{Z}$ denoted by $\mathcal{T}$. The positivity of $\mathcal{T}$ is equivalent with the positivity of $M_n$, $n \geq 0$. In this case we obtain that the positive Toeplitz kernel on $\mathbb{Z}$ is a realization of the semispectral measure $E$ on $\mathbb{T}$, the unit circle:

A semispectral measure on $\mathbb{T}$ is a linear positive map

$$E : \mathcal{C}(\mathbb{T}) \longrightarrow \mathcal{L}(\mathcal{H})$$

where $\mathcal{C}(\mathbb{T})$ denote the set of continuous functiond on $\mathbb{T}$

Then following (4.3) :

$$S_n = E(e_n), \; n \in \mathbb{Z}, \text{ where } e_n(e^{it}) = e^{int} \text{ and } S_{-n} = S_n^*,$$

Let us suppose $P_n$, $n \geq 0$ are nonnegative defined and taking in consideration (4.1) we shall define $G_n \in \mathcal{C}(\mathcal{H})$ that satisfies the finite operatorial (NPP) with data $\{z_k, Y_k\}_{k=0}^{n}$ as follows:

First we consider the semispectral measure :

$$(4.4) \qquad \begin{aligned} &E_n : \mathcal{C}(\mathbb{T}) \longrightarrow \mathcal{L}(\mathcal{H}) \\ &E_n(f) = \tfrac{1}{2} E\left(|m_n|^2 f\right) \quad \text{where } m_n(e^{i\theta}) = \prod_{k=0}^{n} (e^{i\theta} - z_k) \end{aligned}$$

and the family of functions in :

$$(4.5) \qquad \begin{aligned} &g_z : \mathbb{T} \to \mathbb{C} \\ &g_z(e^{i\theta}) = \frac{e^{i\theta} + z}{e^{i\theta} - z} \end{aligned}$$

We define $G_n \in \mathcal{C}(\mathcal{H})$ by :

$$(4.6) \qquad G_n(z) = E_n(g_z), \; z \in \mathbb{D}$$

A short computation shows us that :

$$\frac{G_n^*(z_k) + G_n(z_l)}{1 - \bar{z}_k z_l} = \frac{Y_k^* + Y_l}{1 - \bar{z}_k z_l} \quad , \qquad k,l = 0,1,\dots,n. \tag{4.7}$$

Now it is clear that $G_n(z_k) = Y_k$ , $k = 0,1,\dots,n$.

For the operatorial (NPP) with data $\{z_k, W_k\}_{k \geq 0}$ let us consider the sequence $\{G_n\}_{n \geq 0}$ with $G_n \in \mathcal{C}(\mathcal{H})$ such that $G_n(z_k) = Y_k$, $k = 0, 1, \dots n$ and the related sequence $\{F_n\}_{n \geq 0}$ ( $F_n$ and $G_n$ are related by (4.1) ) with $F_n \in \mathcal{S}(\mathcal{H})$ such that $F_n(z_k) = W_k$ , $k=0,1,\dots,n$. The sequence $\{F_n\}_{n \geq 0}$ has a subsequence converging (weakly) to an operator valued function $F \in \mathcal{S}(\mathcal{H})$ which clearly satisfies the (NPP) with data $\{z_n, W_n\}_{n \geq 0}$.

<u>THEOREM 5</u>. A necessary and sufficient condition for the solvability of the operatorial (NPP) with data $\{z_n; W_n\}_{n \geq 0}$ is that all $P_n$ $(n \geq 0)$ given by (4.2) are nonnegative defined.

<u>Proof</u> : sufficiency was already proved and necesity is a consequence of the Riesz-Hergloz theorem for functions in class $\mathcal{C}(\mathcal{H})$ $\square$

In this way the operatorial Nevanlinna-Pick algorithm and the Pick criteria are intimately related with the operatorial Schur algorithm and the criteria for the solvability of the Schur problem.

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