APPROXIMATION FOR GOURSAT PROBLEM OF HYPERBOLIC CONTROLLED STOCHASTIC DIFFERENTIAL EQUATIONS

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1. Introduction

The problem we are going to consider is intimately related to those studied in [1] and [2] for controlled diffusion equations. What is shown in [1] and [2] regarding the approximation can be resumed to the following. When the drift part in a controlled diffusion equation depends linearly on control functions then the solution is not continuous with respect to control functions using the uniform convergence topology and as a measure of this discontinuity any term from a Lie algebra can be added to the limiting equation. A similar fact appears for hyperbolic controlled stochastic differential equation but this time the previous Lie algebra has to be replaced by an algebra generated using a symmetric bracket and it is determined by the even dimension of the "time" parameter $t = (t^1, t^2)$. As in one-dimensional case (see [2]) it can be useful for getting the existence of bounded or periodic solutions. It is our conviction that dealing with integral equations of Volterra type for which the "time" parameter has an odd dimension we refind the Lie algebra obtained in the one dimensional case and when the dimension is even we get correpondingly the algebra in the two dimensional case.

2. Formulation of the problem and main result

In short the problem can be stated as follows.

Having given a Goursat problem for a nonlinear second order hyperbolic system

$$\frac{\partial^2 x}{\partial t^1 \partial t^2} = f_0(t,x), \quad x \in R^n, \quad t = (t^1, t^2) \in R_+ \times R_+$$

$$x(t^1, 0) = x_1(t^1), \quad x(0, t^2) = x_2(t^2), \quad x_1(0) = x_2(0)$$

we are interested in the behaviour of the solution when the right hand side f_o is perturbed by a control part $\sum_{i=1}^{m} u_i(t)g_i(t,x)$, $g_i \in \mathbf{R}^n$, $u_i \in \mathbf{R}$.

Generally the dependence of the solution $x(\cdot)$ on the continuous function $\tilde{u}(\cdot) = (\tilde{u}_1(\cdot),\ldots,\tilde{u}_m(\cdot))$ on a fixed compact rectagle $[0,T]$, $T = (T^1,T^2)$, is not continuous in the topology of uniform convergence. A measure of this discontinuity is determined by the fact that the bracket $\{g_i,g_j\}(t,x) = ((\partial g_i/\partial x)g_j + (\partial g_j/\partial x)g_i)(t,x)$, $t \in [0,T]$, $x \in R^n$, is not vanishing and any linear combination of such brackets added to the original system generate a solution $x(\cdot)$ which is approximated uniformly on $[0,T]$ by a sequence of solutions $x^h(\cdot)$, $h > 0$, of

$$\frac{\partial^2 x}{\partial t^1 \partial t^2} = f_o(t,x) + \sum_{i=1}^{m} u_i^h(t)g_i(t,x), \quad x^h(t^1,0) = x_1(t^1), \quad x^h(0,t^2) = x_2(t^2)$$

where $\lim_{h \to 0} \max_{t \in [0,T]} |\tilde{u}_i^h(t)| = 0$, $i = 1,\ldots,m$, $\tilde{u}_i(t) = \iint_0^t u_i(s)ds$.

Any hyperbolic system of the form

$$(\partial^2 S/\partial t^2) - 1/c^2(\partial^2 S/\partial \tau^2) + A_1(\partial S/\partial \tau) + A_2(\partial S/\partial t) = f_o(t,\tau,S), \quad S(t,\tau) \in R^n,$$

with commuting matrices A_1, A_2, $(A_1A_2 = A_2A_1)$ can be reduced by a transformation $(t,\tau,S(t,\tau)) \to (t^1,t^2,\tilde{S}(t^1,t^2))$ to the above form.

It will appear as a special case in a Goursat problem for a stochastic hyperbolic system which we are going to state as follows.

Denote $C_b^{r,s}(R^2 \times R^n)$ the space consisting of continuous functions $h(t,x)$ which are piecewice r-differentiable with respect to t and s-differentiable with respect to x which are bounded along with the assumed partial derivatives.

We are given a two parameter Wiener process $w(t)$, $t = (t^1,t^2)$, $t^i \geq 0$, $w(t) \in R^d$, on the probability space $\{\Omega, \mathcal{F}, P\}$, and the differentiable functions $f_i \in C_b^{1,1}(R^2 \times R^n)$, $i = 0,1,\ldots,d$, $g_j \in C_b^{4,\infty}(R^2 \times R^n)$, $j = 1,\ldots,m$.

Write $S(g_1,\ldots,g_m)$ for the algebra over reals generated by $g_1,\ldots,g_m$ using the symmetric bracket

$$\{g_i,g_j\} = ((\partial g_i/\partial x)g_j + (\partial g_j/\partial x)g_i)(t,x), \quad x \in R^n, \ t \geq 0.$$

For any $g(t,x) = \sum_{j=1}^{L} v_j(t,x)h_j(t,x)$, where $h_j \in S(g_1,\dots,g_m)$ and $v_j \in C_b^{4,4}(R^2 \times R^n)$, there exists a unique solution $x(t,\omega)$, of the Goursat problem which is continuous in t for each ω, and measurable in ω for each t, such that

$$1) \qquad x(t) = x_1(t^1) + x_2(t^2) - x_o + \iint_0^t [f_o(s,x(s)) + g(s,x(s))]ds +$$

$$\sum_1^d \iint_0^t f_i(s,x(s))w_i(ds); \quad t = (t^1,t^2) \geq 0, \quad s = (s^1,s^2)$$

where $x_1(\cdot), x_2(\cdot) \in C^1(R_+)$, $x_1(0) = x_2(0) = x_o$, and

$$E \max_{t \in [0,T]} |x(t,\omega)|^2 \leq \text{const.},$$

where "E" stands for expectation.

Along with (1) we consider the system

$$2) \qquad y(t) = x_1(t^1) + x_2(t^2) - x_o + \iint_0^t [f_o(s,y(s)) + \sum_1^m u_j(s,y(s))g_j(s,y(s))]ds +$$

$$\sum_1^d \iint_0^t f_i(s,y(s))w_i(ds).$$

where f_i, g_j, $x_1(\cdot)$, $x_2(\cdot)$, are the given functions and $u_j \in C_b^{3,3}(R^2 \times R^n)$ are to be found such that the solution in (2) to approximate the solution $x(\cdot)$ in (1).

Theorem 2.

Let $x(\cdot)$ be the solution in (1). Then there exists $\{u_j^h\}_{h>0} \subset C_b^{3,3}(R^2 \times R^n)$, $j = 1,\dots,m$, such that the corresponding solution $x^h(\cdot)$ in (2) fulfils

$$E \max_{t \in I} |x^h(t) - x(t)|^2 \leq C \max(h^1,h^2),$$

where $I = [0,T^1] \times [0,T^2]$, $T^i > 0$ and the constant C depends on $T = (T^1,T^2)$, but doesn't depend on $h = (h^1,h^2)$.

Remark 1.

The boundedness of the functions f_i, g_j in the hypothesis of the Theorem 2 is not essential. It can be relaxed assuming a linear growth condition in x uniformly with respect to $t \in I$, and the conclusion is replaced by there exists a sequence $\{h_n\}_n \downarrow 0$ such that $\lim_{n\to\infty} E \max_{t\in I} |x_n(t) - x(t)|^2 = 0$, where $x_n(t) = x^{h_n}(t)$, but we assume additionally that $|(\partial^k g_i/\partial x^k)(t,x)| \le C/(1+|x|)$, $k \ge 2$, $i = 1,\dots,m$, $t\in[0,T]$, $x \in R^n$.

3. Some auxiliary results and proof of Theorem 2

As a first step in proving Theorem 2 we consider $m = 2$. We are given $f_i \in C_b^{1,1}(R^2 \times R^n)$, $g,\ell \in C_b^{4,4}(R^2 \times R^n)$, $i = 0,1,\dots,d$, and a two parameter Wiener process $w(t) \in R^d$.

Let $T^1, T^2 > 0$ be fixed and $h^i = T^i/N$, N natural. Define $p(s,h) = p_1(s^1,h^1)p_2(s^2,h^2)$, $q(s,h) = q_1(s^1,h^1)q_2(s^2,h^2)$ where $p_i(s^i,h^i) = p_i((s^i - kh^i)/h^i)$, $q_i(s^i,h^i) = q_i((s^i - kh^i)/h^i)$, for $s^i \in [kh^i,(k+1)h^i]$, $k = 0,1,\dots,N-1$, and $p_i,q_i : [0,1] \to R$ are polynomials fulfilling

$$p_i(0) = p_i(1) = q_i(0) = q_i(1) = 0, \text{ and} \int_0^1 p_i(\tau)d\tau = \int_0^1 q_i(\tau)d\tau = 0,$$
$$\int_0^1 p_i(r)dr \int_0^r q_i(\tau)d\tau = 1, \ i = 1,2.$$

The corresponding equations are the following

$$3) \qquad x(t) = x_1(t^1) + x_2(t^2) - x_0 + \iint_0^t \{g,\ell\}(s,x(s))ds + \mathcal{L}(0;t;x(\cdot))$$

$$4) \quad x^h(t) = x_1(t^1) + x_2(t^2) - x_0 + \iint_0^t [p(s,h)g(s,x^h(s)) + q(s,h)\ell(s,x^h(s))]ds + \mathcal{L}(0;t;x^h(\cdot)),$$

where

$$5) \qquad \mathcal{L}(t_0;t;x(\cdot)) = \iint_{t_0}^t f_0(s,x(s))ds + \sum_1^d \iint_{t_0}^t f_i(s,x(s))w_i(ds).$$

Theorem 1.

Let $x(\cdot)$ and $x^h(\cdot)$ be fulfilling (3) and respectively (4). Then

$$E \max_{t\in I} |x(t) - x^h(t)|^2 \le C \max(h^1,h^2), \ I = [0,T^1] \times [0,T^2].$$

In order to prove this result we need the following lemmas. Denote

$$\square_{00} x = x(h) - x(0,h^2) - x(h^1,0) + x(0)$$

$$6) \qquad M_{11} = \sum_1^d \Big\{ \iint_0^h [p(t,h)(\partial g/\partial x)(0,x(0)) + q(t,h)(\partial \ell/\partial x)(0,x(0))] \cdot$$

$$\cdot \Big(\iint_0^t f_i(s,x(s))w_i(ds)\Big)dt + \iint_0^h [f_i(t,x(t)) - f_i(t,x^h(t))]w_i(dt)\Big\}.$$

Lemma 1.

Let $x(\cdot)$ and $x^h(\cdot)$ be fulfilling (3) and respectively (4). Then

$$\Box_{00}x - \Box_{00}x^h = x(h) - x^h(h) = h^1h^2R + M_{11}$$

$$E\max_{t\in I}|x(t) - x^h(t)|^2 \le Ch^1h^2, \quad E|M_{11}|^2 \le C(h^1h^2)^2, \quad E|R|^2 \le Ch^1h^2$$

where $I = [0,h^1] \times [0,h^2]$.

Proof. Denote $x^h(\cdot) = y(\cdot)$ and using (3) and (4) we get

$$\Box_{00}y = \iint_0^h [p(t,h)g(t,y(t)) + q(t,h)\ell(t,y(t))]dt + \mathcal{L}(0;h;y(\cdot))$$

$$\Box_{00}x = \iint_0^h \{g,\ell\}(t,x(t))dt + \mathcal{L}(0;h;x(\cdot))$$

and developing g and ℓ it follows

7) $$U_1 = \iint_0^h [p(t,h)g(t,y(t)) + q(t,h)\ell(t,y(t))]dt =$$

$$= \iint_0^h (p(t,h)g'(z(0)) + q(t,h)\ell'(z(0)))(z(t) - z(0))dt +$$

$$+ \frac{1}{2}\iint_0^h (p(t,h)g''(z(0)) + q(t,h)l''(z(0)))(z(t) - z(0))(z(t) - z(0))dt +$$

$$+ \frac{1}{3!}\iint_0^h (p(t,h)g'''(z(0)) + q(t,h)l'''(z(0)))(z(t) - z(0))(z(t) - z(0))(z(t) - z(0))dt +$$

$$+ \frac{1}{4!}\iint_0^h (p(t,h)g^{IV}(z_\theta(t)) + q(t,h)l^{IV}(z_\theta(t)))(z(t) - z(0))\dots(z(t) - z(0))dt =$$

$$= T_1 + \frac{1}{2}T_2 + \frac{1}{3!}T_3 + \frac{1}{4!}T_4 = T_1 + R_1$$

where $z(t) = (t,y(t))$, $g' = \partial g/\partial z, \dots, g^{IV} = \partial^4 g/\partial z^4$

$$z_\theta(t) = [z(0) + \theta(z(t) - z(0))].$$

On the other hand

8) $$E|y(t) - x(0)|^8 \le C(h^1h^2)^4, \ (\forall)\ t\in[0,h^1] \times [0,h^2]$$

and using (3) and (8) we write T_1 in (7) as

9) $$T_1 = \iint_0^h p(t,h)dt \iint_0^t q(\tau,h)g'(z(0))l(z(0))d\tau +$$

$$\iint_0^h q(t,h)dt \iint_0^t p(\tau,h)l'(z(0))g(z(0))d\tau + \tilde{M}_{11} + h^1h^2R$$

where

10) $$\tilde{M}_{11} = \sum_1^m \iint_0^h [p(t,h)g'(z(0)) + q(t,h)l'(z(0))] \iint_0^t f_i(s,y(s))w_i(ds)$$

and

11) $$E|R|^2 \leq Ch^1h^2, \quad E|\tilde{M}_{11}|^2 \leq C(h^1h^2)^2.$$

Since

12) $$E|x(t) - y(t)|^8 \leq C(h^1h^2)^4, \quad (\forall)\ t \in [0,h^1] \times [0,h^2],$$

we rewrite T_1 in (9) as

13) $$T_1 = \iint_0^h \{g,l\}(t,x(t))dt + \tilde{M}_{11} + h^1h^2\bar{R}$$

where $E|\bar{R}|^2 \leq C(h^1h^2)$.

Using (8) in (7) we obtain

14) $$\check{R}_1 = h^1h^2\tilde{R}, \quad E|\tilde{R}|^2 \leq Ch^1h^2$$

and finally (7) can be written as

15) $$U_1 = \iint_0^h \{g,l\}(t,y(t))dt + h^1h^2\check{R}_1 + \tilde{M}_{11}$$

where $E|\tilde{R}_1|^2 \leq Ch^1h^2$ and $\tilde{M}_{11}$ is defined in (10).

Using (15) and (12) we get

16) $$x(h) - y(h) = \Pi_{00}x - \Pi_{00}y = h^1h^2R + M_{11}$$

where M_{11} is defined in (6) and

$$R = \tilde{R}_1 + \iint_0^h (f_o(t,x(t)) - f_o(t,y(t)))dt.$$

Denote $\psi(t) = E \max_{\tau \leq t}|x(\tau) - y(\tau)|^2$

and using (12) in (3) and (4) it follows

17) $$\psi(t) \leq C(h^1h^2) \quad (\forall)\ t \in [0,h^1] \times [0,h^2].$$

Combining (16) and (17) we get $E|M_{11}|^2 \leq C(h^1h^2)^2$, $E|R|^2 \leq Ch^1h^2$ and the proof is complete.

Denote $\square_{10}x = x(2h^1,h^2) - x(h^1,h^2) - x(2h^1,0) + x(h^1,0)$, $t_1 = (h^1,0)$, $t_2 = (2h^1,h^2)$ $x^h(t) = y(t)$ and

18) $$M_{21} = \sum_{i=1}^{d} \iint_{t_1}^{t^2} [f_i(t_1y(t)) - f_i(t,x(t))]w_i(dt) + \iint_{t_1}^{t^2} [p(t,h)g'(z(t_1)) +$$

$$+ q(t,h)l'(z(t_1))] \iint_{t_1}^{t} f_i(z(s))w_i(ds))$$

Lemma 2.

Let $x^h(\cdot)$ and $x(\cdot)$ be fulfilling (3) and respectively (4).

Then $E \max_{t \in I} |x^h(t) - x(t)|^2 \leq C(h^1h^2 + (h^1h^2)^2)$, $I = [h^1,2h^1] \times [0,h^2]$

and $\square_{10}x^h - \square_{10}x = x^h(2h^1,h^2) - x(2h^1,h^2) - (x^h(h^1,h^2) - x(h^1,h^2)) =$

$$= h^1h^2R + M_{21}$$

where $E|R|^2 \leq Ch^1h^2$, $E|M_{21}|^2 \leq C(h^1h^2)^2$.

Proof.

$$\square_{10}y = \iint_{t_1}^{t_2} [p(t,h)g(t,y(t)) + q(t,h)l(t,y(t))]dt + \mathcal{L}(t_1;t_2\,;\,y(\cdot))$$

$$\square_{10}x = \iint_{t_1}^{t_2} \{g,l\}(t,x(t))dt + \mathcal{L}(t_1;t_2\,;\,x(\cdot)).$$

Similarly as in Lemma 1 (see (7)) we get

19) $$U_2 = \iint_{t_1}^{t_2} (pg + ql)dt = \iint_{t_1}^{t_2} [p(t,h)g'(z(t_1)) + q(t,h)l'(z(t_1))](z(t) - z(h))dt +$$

$$+ \frac{1}{2}T_2 + \frac{1}{3!}T_3 + \frac{1}{4!}T_4 = T_1 + R_1$$

where $z(0)$ in (7) is repaced by $z(t_1) = (t_1,y(t_1))$.

Using (3) we write

20) $$y(t) - y(t_1) = y(t^1,t^2) - y(h^1,t^2) + y(h^1,t^2) - y(h^1,0) =$$

$$= x_1(t^1) - x_1(h^1) + y(h^1,t^2) - y(h^1,0) + \iint_{t_1}^{t} (pg + ql)dz + \mathcal{L}(t_1;t\,;\,y(\cdot))$$

$t \in [h^1, 2h^1] \times [0, h^2]$

21) $x_1(t^1) - x_1(h^1) = \eta(h^1 h^2), \quad t' \in [h^1, 2h^1],$

$$y(h^1,t^2) - y(h^1,0) = x_2(t^2) - x_2(0) + \int_0^{h^1}\int_0^{t^2} (pg + ql)d\tau + \alpha(0\, ;\, (h^1,t^2)\, ;\, y(\cdot)) =$$

$$= \eta(h^1 h^2)$$

where η fulfils $E\,|\eta(r)|^8 \leq Cr^4$.

Using (21) in (20) we obtain

22) $E|y(t) - y(t_1)|^8 \leq C(h^1 h^2)^4, \quad t \in [h^1, 2h^1] \times [0, h^2]$

and R_1 in (19) can be rewritten as

23) $R_1 = h^1 h^2 R$ where $E|R|^2 \leq Ch^1 h^2$.

For T_1 we repeat the computation in Lemma 1 getting

24) $T_1 = \iint_{t_1}^{t_2} \{g, f\}(t, x(t))dt + h^1 h^2 \widetilde{R} + \widetilde{M}_{21}$

where $E|\widetilde{R}|^2 \leq Ch^1 h^2$, $E|\widetilde{M}_{21}|^2 \leq C(h^1 h^2)^2$ and

25) $\widetilde{M}_{21} = \sum_1^d \iint_{t_1}^{t_2} [p(t,h)g'(z(t_1)) + q(t,h)f'(z(t_1))] \iint_{t_1}^{t} f_i(z(t))w_i(dt).$

Using (23) and (24) in (19) it follows

26) $\square_{10} y = \iint_{t_1}^{t_2} \{g, f\}(t, x(t))dt + \alpha(t_1;t_2;y(\cdot)) + h^1 h^2 \widetilde{R} + \widetilde{M}_{21}$

and

27) $\square_{10} y - \square_{10} x = h^1 h^2 \widetilde{R} + \widetilde{M}_{21} + \alpha(t_1;t_2;x(\cdot)) - \alpha(t_1;t_2;y(\cdot))$

where $E(\widetilde{M}_{21}/M_{11}) = 0$ ($E(X/Y)$ stands for conditioned expectation). To estimate $\alpha(t_1;t_2;y(\cdot)) - \alpha(t_1;t_2;x(\cdot))$ is necessary to evaluate $E \max_{t \in I} |y(t) - x(t)|^2$.

By definition

28) $y(t) - x(t) = (y(h^1,t^2) - x(h^1,t^2)) + \iint_{t_1}^{t} [pg + ql)ds -$

$$- \iint_{t_1}^{t} \{g, f\}(s, x(s))ds + (\alpha(t_1;t;y(\cdot)) - \alpha(t_1;t;x(\cdot))) = L_1 + L_2 + L_3$$

and using Lemma 1 we have

29) $$E \max_{t \in I} |L_1|^2 \leq C(h^1h^2), \quad I = [0,h^1] \times [0,h^2].$$

Similarly we get

30) $$E \max_{t \in I} |L_2|^2 \leq C(h^1h^2)^2, \quad I = [h^1,2h^1] \times [0,h^2]$$

31) $$E \max_{t \in I} |L_3|^2 \leq h^1h^2 \, E \max_{t \in I} |y(t) - x(t)|^2, \quad I = [h^1,2h^1] \times [0,h^2].$$

On the other hand, from (28) via (29)-(31) we obtain

32) $$E \max_{t \in I} |y(t) - x(t)|^2 \leq C(h^1h^2 + (h^1h^2)^2), \; I = [h^1,2h^1] \times [0,h^2],$$

and finally

33) $$\square_{10}y - \square_{10}x = h^1h^2R + M_{21}$$

where $E|R|^2 \leq C(h^1h^2)$, $E(M_{21}/M_{11}) = 0$, $E|M_{21}|^2 \leq C(h^1h^2)^2$

34) $$M_{21} = \tilde{M}_{21} + \sum_1^d \iint_{t_1}^{t_2} [f_i(t,y(t)) - f_i(t_1,x(t))]w_i(dt).$$

The proof is complete.

Denote

35) $$\square_{10}x = x[(i+1)h^1,h^2) - x(ih^1,h^2) - x((i+1)h^1,0) + x(ih^1,0)$$

36) $$M_{i+1,1} = \sum_{j=1}^d \iint_{t_{i+1}}^{t_{i+2}} [p(t,h)g'(z(t_{i+1})) + q(t,h)f'(z(t_{i+1}))] \iint_{t_{i+1}}^{t} f_j(z(s))w_j(ds) +$$
$$+ \sum_{j=1}^d \iint_{t_{i+1}}^{t_{i+2}} [f_j(z(t)) - f_j(t,x(t))]w_j(dt)$$

where $((i+1)h^1,0) = t_{i+1}$, $((i+2)h^1,h^2) = t_{i+2}$, $z(t) = (t,y(t))$.

It holds

Lemma 3.

Let $x^h(\cdot)$ and $x(\cdot)$ be fulfilling (3) and (4). Then

$$\square_{i+1,0}x^h - \square_{i+1,0}x = x^h((i+2)h^1,h^2) - x((i+2)h^1,h^2) - (\square_{i0}x - \square_{i,0}y) =$$
$$= h^1h^2R + M_{i+1,1}$$

$$E \max_{t \in I} |x^h(t) - x(t)|^2 \leq C[h^1h^2 + (ih^1)^2(h^2)^2]$$

where $E|R|^2 \leq Ch^1h^2$, $E(M_{i+1,1}/M_{j,1}) = 0$, $j = 1,2,\dots,i$.

$$E|M_{i+1,1}|^2 \leq C(h^1h^2)^2, \quad I = [(i+1)h^1, (i+2)h^1] \times [0,h^2]$$

Denote

$$\square_{0i}x = x(h^1,(i+1)h^2) - x(0,(i+1)h^2) - x(h^1,ih^2) + x(0,ih^2).$$

Along with Lemma 3 it holds

Lemma 4.

Let $x^h(\cdot)$ and $x(\cdot)$ be fulfilling (3) and (4). Then

$$\square_{0,i+1}x^h - \square_{0,i+1}x = x^h(h^1,(i+2)h^2) - x(h^1,(i+2)h^2 - (\square_{0,i}x^h - \square_{0,i}x) =$$

$$= h^1h^2R + M_{1,i+2}$$

$$E\max_{t\in I}|x^h(t) - x(t)|^2 \leq C[h^1h^2 + (h^1(ih^2))^2], \quad I = [0,h^1] \times [(i+1)h^2, (i+2)h^2]$$

where $E|R|^2 \leq Ch^1h^2$, $E|M_{1,i+1}|^2 \leq C(h^1h^2)^2$, $E(M_{1,i+2}/M_{1,j}) = 0$.

In proving Lemmas 3 and 4 we need

Lemma 5

It holds

$$\int_0^{t'}\int_{kh^2}^{t^2} [p(s,h)g(s,y(s)) + q(s,h)l(s,y(s))]ds = \eta(h^2), \quad E|\eta(r)|^2 \leq Cr, \quad h = (h^1, h^2)$$

Proof. Proof $(\forall)$ $t^2 \in [kh^2, (k+1)h^2]$, $t^1 \in [0,T^1]$, $k = 0,1,\dots,N-1$.

Let $t^1 \in [ih^1, (i+1)h^1]$ for some $i \in \{0,1,\dots,N-1\}$ and we write

$$\int_0^{t^1}\int_{kh^2}^{t^2}(pg + ql)ds = \int_0^{ih^1}\int_{kh^2}^{t^2}(pg + ql)ds + \int_{ih^1}^{t^1}\int_{kh^2}^{t^2}(pg + ql)ds = I + II$$

It is easily seen that $II = \eta(h^1h^2)$ and we shall prove that $I = \eta(h^2)$. For $i = 1$, it requires $I = \eta(h^2)$ for $k = 0,1,\dots,N-1$. We have

$$I = \int_0^{h^1}\int_{kh^2}^{t^2}(pg + ql)ds = \int_0^{h^1}\int_{kh^2}^{t^2} < p(s,h)\int_0^1 g''_z(z_1(\theta))d\theta + q(s,h)\int_0^1 l'_z(z_1(\theta))d\theta,$$

$$(s^1, y(s) - y(0,s^2)) > ds,$$

where $z = (s^1,y)$, $z_1(\theta) = (\theta s^1, s^2, y(0,s^2) + \theta(y(s) - y(0,s^2))$, and

$$y(s) - y(0,s^2) = x_1(s^1) - x_1(0) + \int_0^{s^1}\int_0^{s^2}(pg + ql)dt + \mathcal{L}(0,s;y(\cdot)) =$$

$$= \eta(h^1) + \int_0^{s^1}\int_0^{kh^2}(pg + ql)dt, \quad s^2 \in [kh^2, t^2]$$

We have to prove that

$$U_k = \int_0^{s^1}\int_0^{kh^2} (pg + ql)dt = kh^2\eta(h^1), \ (\forall)\ k = 1,2,\ldots,N-1$$

For $k = 1$, it is obvious that $U_1 = h^2\eta(h^1)$, and assuming that $U_{k-1} = (k-1)h^2\eta(h^1)$ we get

$$U_k = (k-1)h^2\eta(h^1) + \int_0^{s^1}\int_{(k-1)h^2}^{kh^2} (pg + ql)dt$$

and since

$$y(t) - y(t^1, (k-1)h^2) = x_2(t^2) - x_2((k-1)h^2) + \int_0^{t^1}\int_{((k-1)h^2}^{t^2} (pg + ql)ds + \mathcal{L}(0,(k-1)h^2; t; y(\cdot)) =$$

$$= \eta(h^1h^2)$$

we can rewrite the second term in U_k as I for $i = 1$, obtaining that

$$\int_0^{s^1}\int_{(k-1)h^2}^{kh^2} (pg + ql)dt = h^2\eta((h^1)^2)$$

Therefore $U_k = kh^2\eta(h^1)$, $k = 0,1,\ldots,N-1$, and $y(s) - y(0,s^2) = \eta(h^1)$ which implies $I = \eta(h^2)$ for $i = 1$. Assume $I = \eta(h^2)$ up to $i-1$ and we shall prove it for i.

By definition

$$I = \sum_{j=0}^{i-1}\int_{jh^1}^{(j+1)h^1}\int_{kh^2}^{t^2} (pg + ql)ds = \sum_{j=0}^{i-1} T_j$$

and T_j can be rewritten as

$$T_j = \int_{kh^2}^{t^2}\int_{jh^1}^{(j+1)h^1} \langle p(s,h)\int_0^1 g'_z(z_1(\theta))d\theta + q(s,h)\int_0^1 l'_z(z_1(\theta))d\theta\ ,\ (s^1 - jh^1, y(s) - y(jh^1, s^2))\rangle ds$$

where $z = (s^1, y)$, $z_1(\theta) = [jh^1 + \theta(s^1 - jh^1), s^2, y(jh^1, s^2) + \theta(y(s) - y(jh^1, s^2))]$

and we have

$$y(s) - y(jh^1, s^2) = y(s^1, kh^2) - y(jh^1, kh^2) + \int_{jh^1}^{s^1}\int_{kh^2}^{s^2}(pg + ql)dt + \mathcal{L}(jh^1, kh^2; s; y(\cdot)) =$$

$$= y(s^1, kh^2) - y(jh^1, kh^2) + \eta(h^1h^2),$$

$$y(s^1, kh^2) - y(jh^1, kh^2) = x_1(s^1) - x_1(jh^1) + \int_0^{kh^2}\int_{jh^1}^{s^1} (pg + ql)dt + \mathcal{L}(jh^1, 0; s^1, kh^2; y(\cdot))$$

for $s^1 \in [jh^1, (j+1)h^1]$, $s^2 \in [kh^2, t^2]$

To complete the proof it requires

$$\int\int^{kh^2\ s^1} \ldots$$

which is implied by

$$R_m = \int_{mh^2}^{(m+1)h^2} \int_{jh^1}^{s^1} (pg + ql)dt = h^2 \eta(h^1), \ m = 0,1,\dots,k-1.$$

We rewrite R_m as

$$R_m = \int_{jh^1}^{s^1} \int_{mh^2}^{(m+1)h^2} \langle p(t,h)\int_0^1 g'_z(z_1(\theta))d\theta + q(t,h)\int_0^1 l'_z(z_1(\theta))d\theta, (t^2 - mh^2, y(t) -$$

$$-y(t^1,mh^2)) \rangle dt$$

where

$$y(t) - y(t^1,mh^2) = y(jh^1,t^2) - y(jh^1,mh^2) + \int_{jh^1}^{t^1}\int_{mh^2}^{t^2} (pg + ql)ds + \mathcal{L}(jh^1,ih^2;t;y(\cdot))$$

and

$$y(jh^1,t^2) - y(jh^1,mh^2) = x_2(t^2) - x_2(mh^2) + \int_0^{jh^1}\int_{mh^2}^{t^2} (pg + ql)ds + \mathcal{L}(0,mh^2;jh^1,t^2;y(\cdot)).$$

In the last equation $j \leq i-1$, and the induction argument insures that $y(jh^1,t^2) - y(jh^1,mh^2) = \eta(h^2)$ for any $m = 0,1,\dots,k-1$, $k = 1,\dots,N-1$ and $y(t) - y(t^1,mh^2) = \eta(h^2)$, $R_m = h^2\eta(h^1)$. The proof is complete.

With respect to the proof of Lemmas 3 and 4 the following remark is in order.

Remark 2.

In the Lemmas 1-4 the following estimate

$$\int_{ih^1}^{(i+1)h^1}\int_0^{h^2} [p(t,h)g(t,y(t)) + q(t,h)l(t,y(t))]dt = T_1 + \frac{1}{2}T_2 + \frac{1}{3!}T_3 + \frac{1}{4!}T_4 =$$

$$= \int_{ih^1}^{(i+1)h^1}\int_0^{h^2} \{g,l\}(t,x(t))dt + h^1h^2R + M_{i1}$$

is based on the presence of $y(t) - y(ih^1,0)$ in the terms $T_1,\dots,T_4$. Since

$$y(t) - y(ih^1,0) = y(t^1,t^2) - y(ih^1,t^2) + y(ih^1,t^2) - y(ih^1,0) = U_1 + U_2$$

and

$$U_1 = y(t) - y(ih^1,t^2) = x_1(t^1) - x_1(ih^1) + \int_{ih^1}^{t^1}\int_0^{t^2} (pg + ql)(s,y(s))ds +$$

$$\mathcal{L}(ih^1,0;t;y(\cdot))$$

$$U_2 = y(ih^1,t^2) - y(ih^1,0) = x_2(t^2) - x_2(0) + \int_0^{ih^1}\int_0^{t^2} (pg + ql)(s,y(s))ds +$$

$$+\mathcal{L}(0;ih^1,t^2;y(\cdot)), \quad t \in [ih^1,(i+1)h^1] \times [0,h^2].$$

We notice that U_1 insures the passing to the term $\{g,l\}$ while U_2 will give a term of the type h^1h^2R, with $E|R|^2 \le C(h^1h^2)$.

When U_2 is not combined with U_1 in T_1, T_2 and T_3 its contribution doesn't count and in T_4 using Lemma 5 it gives a term of the form $C(h^2)^2 \cong Ch^1h^2 \cong E[\eta(h^1h^2)]^2$

Denote $x^h(\cdot) = y(\cdot)$, $z(t) = (t,y(t))$, $t_{ij} = (ih^1, jh^2)$, and

37) $$\square_{ij}x = x((i+1)h^1, (j+1)h^2) - x(ih^1,(j+1)h^2) - x((i+1)h^1, jh^2) + x(ih^1, jh^2)$$

38) $$M_{i,j} = \sum_{k=1}^{d}\left\{\iint_{t_{ij}}^{t_{i+1,j+1}} [p(t,h)g'(z(t_{ij})) + q(t,h)l'(z(t_{ij}))]\iint_{t_{ij}}^{t} f_k(z(s))w_k(ds) + \right.$$
$$\left. + \iint_{t_{ij}}^{t_{i+1,j+1}} [f_k(z(t)) - f_k(t,x(t))]w_k(dt)\right\}$$

Lemma 6.

Let $x^h(\cdot)$, $x(\cdot)$ be fulfilling (3) and (4). Then

$$\square_{ij}x^h - \square_{ij}x = h^1h^2R + M_{i+1,j+1}$$

$$E\max_{t\in I}|x^h(t) - x(t)|^2 \le C(h^1h^2), \quad E\max_{t\in I}|x(t) - x(t_{ij})|^2 \le Ch^1h^2$$

where $I = [ih^1,(i+1)h^1] \times [jh^2, (j+1)h^2]$, $E|R|^2 \le Ch^1h^2$ and it holds $E(M_{ij}/M_{lk}) = 0$ either $l < i$, or $k < j$.

Proof.

For $j = 0$, and $i = 0$, the Lemma 6 is contained in Lemma 3, and respectively in Lemma 4. Let the Lemma 6 be fulfilled for (i,j).

We have to prove the following

$$\square_{i+1,k}y - \square_{i+1,k}x = h^1h^2R + M_{i+2,k} \quad (\forall)\, k$$

$$E\max_{t\in I}|y(t) - x(t)|^2 \le Ch^1h^2, \text{ where } I = [(i+1)h^1, (i+2)h^1] \times [kh^2,(k+1)h^2]$$

$E|R|^2 \le Ch^1h^2$, $E(M_{i+2,k}/M_{lp}) = 0$, $(\forall)$ l,k fulfilling either $l < i+2$, or $p \not< k$.

By definition

39) $$\square_{i+1,k}y = y((i+2)h^1, (k+1)h^2) - y((i+1)h^1,(k+1)h^2) - y((i+2)h^1,kh^2) +$$

$$+ y((i+1)h^1,kh^2) = \iint_{t_1}^{t_2} (pg + ql)dt + \mathcal{L}((i+1)h^1, kh^2; (i+2)h^1, (k+1)h^2 ; y(\cdot))$$

40) $$\square_{i+1,k}x = \iint_{t_1}^{t_2} \{g,l\}(s,x(s))ds + \mathcal{L}((i+1)h^1,kh^2 ; (i+2)h^1,(k+1)h^2 ; x(\cdot))$$

where $t_1 = ((i+1)h^1,kh^2)$, $t_2 = ((i+2)h^1,(k+1)h^2)$, and

41) $$U = \iint_{t_1}^{t_2} (pg + ql)dt = \iint_{t_1}^{t_2} [p(t,h)g'(z(t_1)) + q(t,h)l'(z(t_1))](z(t) - z(t_1))dt +$$

$$\frac{1}{2}\iint_{t_1}^{t_2} [p(t,h)g''(z(t_1)) + q(t,h)l''(z(t_1))]z(t) - z(t_1),\ z(t) - z(t_1)) +$$

$$+ \frac{1}{3!}\iint_{t_1}^{t_2} [p(t,h)g'''(z(t_1)) + q(t,h)l'''(z(t_1))]z(t) - z(t_1),\ z(t) - z(t_1),\ z(t) - z(t_2))dt +$$

$$+ \frac{1}{4!}\iint_{t_1}^{t_2} [p(t,h)g\ \ (z_\theta(t)) + q(t,h)l\ \ (z_\theta(t_1))]z(t) - z(t_1),\dots,z(t) - z(t_1))dt =$$

$$= T_1 + T_2 + T_3 + T_4$$

where $z(t) = (t,y(t))$

$$y(t) - y(t_1) = y(t^1,t^2) - y((i+1)h^1,t^2) + y((i+1)h^1,t^2) - y((i+1)h^1,kh^2) = U_1 + U_2$$

and

$$U_1 = x_1(t^1) - x_1((i+1)h^1 + \int_{(i+1h^1}^{t^1}\int_0^{t^2} (pg + ql)ds + \mathcal{L}((i+1)h^1,0;t;y(\cdot))$$

$$U_2 = x_2(t^2) - x_2(kh^2) + \int_0^{(i+1)h^1}\int_{kh^2}^{t^2} (pg + ql)ds + \mathcal{L}(0,kh^2;(i+1)h^1,t^2;y(\cdot))$$

for $t \in [t_1,t_2] = [(i+1)h^1, (i+2)h^1] \times [kh^2,(k+1)h^2]$.

Using Lemma 5 it follows that U_1 and U_2 can be rewritten as

42) $$U_1 = \eta(h^1), \quad U_2 = \eta(h^2)$$

and combining (42) with the transformation of T_1 in (41) as in Lemma 1 (see (7) and (15)) we get

43) $$U = \iint_{t_1}^{t_2} \{g,1\}(t,x(t))dt + h^1h^2R + \tilde{M}_{t_1,t_2}$$

$$E|R|^2 \le Ch^1h^2, \quad E|\tilde{M}_{t_1,t_2}|^2 \le C(h^1h^2)^2.$$

Adding (40) and (43) we get finally

44) $$\Box_{i+1,k}y - \Box_{i+1,k}x = h^1h^2R + M_{i+2,k},$$

$$E \max_{t\in I}|y(t) - x(t)|^2 \le Ch^1h^2, \quad E|R|^2 \le Ch^1h^2, \text{ where } I = [t_1,t_2],$$

and the proof is complete.

Now we are in position to prove Theorem 1. Denote $x^h(\cdot) = y(\cdot)$.

Proof of Theorem 1.

For $t \in \{0,h^1,2h^1,\dots,N_1h^1\} \times \{0,h^2,\dots,N_2h^2\} = H$, $t = (k_1h^1, k_2h^2)$, we have

$$y(t) - x(t) = \sum_{i\le k_1, j\le k_2} \Box_{ij}y - \Box_{ij}x$$

$$\max_{t\in H}|y(t) - x(t)|^2 \le \left|\sum_{i=0,j=0}^{N_1-1,N_2-1} \Box_{ij}y - \Box_{ij}x\right|^2$$

and using Lemma 6 it follows

43) $$\left(E \max_{t\in H}|y(t) - x(t)|^2\right)^{\frac{1}{2}} \le \left\{E\left|\sum_{i=0,j=0}^{N_1-1,N_2-1} (\Box_{ij}y - \Box_{ij}x)\right|^2\right\}^{\frac{1}{2}} \le C\sqrt{h^1h^2}$$

Denote $\gamma(r)$ a random variable which fulfils $(E|\gamma(r)|^2)^{\frac{1}{2}} \le Cr$, $r = \max(h^1,h^2)$.

Let $t \in I_{ij} = [ih^1,(i+1)h^1] \times [jh^2,(j+1)h^2]$. By definition of solution we have

44) $$y(t) = x_1(t^1) + x_2(t^2) - x_o + \iint_0^t (pg + ql)dt + \alpha(0;t;y(\cdot))$$

and

$$U_t = \iint_0^t (pg + ql)ds = \int_0^{ih^1} \int_{jh^2}^{t^2} (pg + ql)ds + \int_{ih^1}^{t^1} \int_0^{t^2} (pg + ql)ds + \iint_0^{t_{ij}} (pg + ql)dt$$

where $t_{ij} = (ih^1,jh^2)$.

Using Lemma 5 and (43) we get

45) $$U_t = \gamma(\sqrt{r}) + y(t_{ij}) - x_1(t_i^1) - x_2(t_j^2) + x_o - \mathcal{L}(0;t_{ij};y(\cdot)) =$$

$$= \gamma(\sqrt{r}) + \gamma(r) + x(t_{ij}) - x_1(t_i^1) - x_2(t_j^2) + x_o - \alpha(0;t_{ij};y(\cdot)) =$$

$$= \gamma(\sqrt{r}) + \gamma(r) + \int\int_0^{t_{ij}} \{g,l\}(s,x(s))ds + \alpha(0;t_{ij};x(\cdot)) - \alpha(0;t_{ij};y(\cdot))$$

Combining (44) and (45) it follows

46) $$y(t) - x(t) = \gamma(\sqrt{r}) + 2\gamma(r) + \alpha(0;t;y(\cdot)) - \mathcal{L}(0;t;x(\cdot)) +$$

$$+ \alpha(0;t_{ij};x(\cdot)) - \alpha(0;t_{ij};y(\cdot))$$

and finally

47) $$E \max_{t\in[0,\tau]} |y(t) - x(t)|^2 \leq C_1 r + C_2 \int\int_0^{\tau} E \max_{t\in[0,s]} |y(t) - x(t)|^2 ds$$

The inequality (47) implies the same conclusion as in the one dimensional case (see remark below)

48) $$E \max_{t\in[0,T]} |y(t) - x(t)|^2 \leq Cr, \quad r = \max(h^1,h^2)$$

and the proof is complete.

Remark 3.

Denote $E \max_{t\in[0,\tau]} |y(t) - x(t)|^2 = \varphi(\tau)$ and assume

$$\varphi(\tau) \leq K + L\int\int_0^{\tau} \varphi(t)dt, \quad (\forall)\ \tau\in[0,T], \text{ where } K,L \geq 0.$$

Let $\nu(\cdot)$ be the unique solution of

$$\nu(\tau) = K + L\int\int_0^{\tau} \nu(t)dt, \quad \tau\in[0,T]$$

and it fulfils the following estimate $|\nu(t)| \leq C_1 K$.

$(\forall)\ t\in[0,T]$, where $C_1 > 0$ is a constant.

Denote $u(t) = \varphi(t) - \nu(t)$, and we have

$$u(s) \leq L\int\int_0^{s} u(t)dt \quad \text{or}$$

$$u(s) = L \iint_0^s u(t)dt + v_0(s), \text{ where } v_0(t) \leq 0,\ t \in [0,T],$$

is a fixed continuous functions.

The solution of the last equation is unique and by using the standard argument of the successive approximations we get $u(t) = \lim_n u_n(t)$, where $u_0(t) = v_0(t)$, and $u_n(t) \leq 0$ $n \geq 0$, $t \in [0,T]$, which shows that $u(t) = \varphi(t) - \nu(t) \leq 0$ and $\varphi(t) \leq \nu(t) \leq C_1 K\ (\forall)\ t \in [0,T]$.

The result in Theorem 1 will play the main role in proving theorem 2. It is neccessary also to be alble to approximate an equation which contains combinations of symmetric brackets of different lengths by one containing such brackets on disjoint intervals. In this respect, let λ_1, $\lambda_2 : [0,T] \to R_+$ be measurable fulfilling $\lambda_i(t) \geq 0$, $\lambda_1(t) + \lambda_2(t) = 1$, and $f_j, \ell_i \in C^{1,1}(R^2 \times R^n)$, $j = 0,1,\dots,d$, $i = 1,2$, fulfilling $|(\partial \ell_i/\partial x)(t,x))|, |(\partial f_j/\partial x)(t,x)| \leq K$, $(\forall)$ $t \in [0,T]$, $x \in R^n$. Let $T^i > 0$ be fixed and $h = (h^1, h^2)$, $h^i = T^i/N$, $i = 1,2$. Denote $t_{ij} = (ih^1, jh^2)$, $i,j = 0,1,\dots,N-1$. Let $y_\lambda(\cdot)$ be the solution of

$$y(t) = x_1(t^1) + x_2(t^2) - x_0 + \iint_0^t [\lambda_1(s)\ell_1(s,y)(s)) + \lambda_2(s)\ell_2(s,y)(s))]ds + \mathcal{L}(0;t;y(\cdot)) \tag{49}$$

where $\mathcal{L}$ is defined in (5), and let ℓ_λ^h be as follows.

$$\ell_\lambda^h(t,x) = \begin{cases} \ell_1(t,x),\ t \in \Delta_1(i,j),\ x \in R^n \\ \ell_2(t,x),\ t \in \Delta_2(i,j),\ x \in R^n \end{cases} \tag{50}$$

where Δ_1, Δ_2 are disjoint and fulfils $\Delta_1 \cup \Delta_2 = \Delta$,

$$\Delta(i,j) = [t_{ij}^1, t_{ij}^1 + h^1] \times [t_{ij}^2, t_{ij}^2 + h^2], \text{ and}$$

$$\alpha = \text{meas } \Delta_1(i,j) = \iint_{\Delta(i,j)} \lambda_1(\tau)d\tau,\quad 1-\alpha = \text{meas } \Delta_2(i,j) = \iint_{\Delta(i,j)} \lambda_2(\tau)d\tau$$

$$\Delta_1(i,j) = [t_{ij}^1, t_{ij}^1 + \alpha h^1] \times [t_{ij}^2, t_{ij}^2 + h^2]$$

$$\Delta_2(i,j) = [t_{ij}^1 + \alpha h^1, t_{ij}^1 + h^1] \times [t_{ij}^2, t_{ij}^2 + h^2]$$

Lemma 7

Let $y_\lambda(\cdot)$ be the solution in (49) and $l^h_\lambda : I \times R^n \to R^n$ be as in (50). Then $x(\cdot)$ the solution $x^h_\lambda(\cdot)$ of

$$x(t) = x_1(t^1) + x_2(t^2) - x_0 + \iint_0^t l^h_\lambda(\tau, x(\tau))d\tau + \alpha(0;t;x(\cdot))$$

fulfils $E \max_{t\in I} |x^h_\lambda(t) - y_\lambda(t)|^2 \leq C \max(h^1, h^2)$, $I = [0,T]$.

Remark 3.

The conslusion in Lemma 7 remains unchanged if we replace $\lambda_1(\cdot), \lambda_2(\cdot)$, by a finite set $\lambda_1(\cdot), \ldots, \lambda_L(\cdot)$, $\lambda_i(t) \geq 0$, $\sum_{i=1}^{L} \lambda_i(t) = 1$, and $l^h_\lambda(\cdot)$ in (50) is defined accordingly.

Proof of Lemma 7.

The proof follows the same lines as in [1] and we shall give only a scketch. Denote $l_\lambda(t,x) = \lambda_1(t) l_1(t,x) + \lambda_2(t) l_2(t,x)$. It is easily seen that we have the following estimate

$$E \max_{\tau\leq t} | x^h_\lambda(\tau) - y(\tau)|^2 \leq C\Big\{ [\sum_{i,j} (E | \iint_{\Delta(i,j)} [l^h_\lambda(t, x^h_\lambda(t)) - l_\lambda(t, x^h_\lambda(t))]dt|^2)^{\frac{1}{2}}]^2 +$$

$$E \max_{(i,j)} (\iint_{\Delta(i,j)} | l^h_\lambda(t, x^h(t)) - l_\lambda(t, x^h(t)) | dt)^2 + \int_0^t (E \max_{s\leq\tau} |x^h_\lambda(s) - y(s))|^2) d\tau \Big\}$$

and

$$\iint_{\Delta(i,j)} l^h_\lambda(t, x^h_\lambda(t))dt = \iint_{\Delta_1(i,j)} l_1(t, x^h_\lambda(t))dt + \iint_{\Delta_2(i,j)} l_2(t, x^h_\lambda(t))dt =$$

$$= \iint_{\Delta(i,j)} l_\lambda(t, x^h_\lambda(t))dt + h^1 h^2 R(h)$$

where $E|R(h)|^2 \leq C \max(h^1, h^2)$.

On the other hand

$$[E \max_{(i,j)} | \iint_{\Delta(i,j)} [l^h_\lambda(t, x^h_\lambda(t)) - l_\lambda(t, x^h_\lambda(t))]dt|^2]^{\frac{1}{2}} \leq C\sqrt{h^1 h^2} (\iint_{\Delta(i,j)} E(1 + |x^h_\lambda(t)|^2)dt)^{\frac{1}{2}} \leq$$

$$\leq C h^1 h^2$$

Write $\varphi(t) = E \max_{\tau\leq t} |x^h_\lambda(\tau) - y(\tau)|^2$ and we get

$$\varphi(t) \leq E|R(h)|^2 + \iint_0^t \varphi(\tau)d\tau .$$

Using Remark 2 we obtain

$$\varphi(t) \leq C \max(h^1, h^2)$$

and the proof is complete.

Proof of Theorem 2.

It follows the same scheme as in [2] for proving Theorem 1. It can be explained shortly as follows. Using Lemma 7 and Remark 3 it follows that we can divide the two-dimensional interval [0,T] into small rectangles $\Delta(h)$ with meas $\Delta(h) = h^1h^2$, $h^i = T^i/N$, and each $\Delta(h)$ is divided into L disjoint rectangles I_j, $j = 1,\ldots,L$, such that, meas $I_j = h^1h^2/L$ and for $t \in I_j$ the equation (1) is replaced by one in which the drift

$$f_o(t,x) + L v_j(t,x)h_j(t,x)$$

replaces the original one $f_o(t,x) + \sum_{j=1}^{L} v_j(t,x)h_j(t,x)$. On each fixed interval I_j we apply Theorem 1 and it is obtained an equation for which the drift has the following form

$$f_o(t,x) + u_i^h(t,x)g_i(t,x) + v^h(t)\tilde{h}(t,x), \text{ for some } i \in \{1,\ldots,m\},$$

where u_i^h, v^h are scalar functions and $\tilde{h} \in S(g_1,\ldots,g_m)$ but the length of $\tilde{h}$ is strictly smaller then the length of the original h_j and the length is the number of $g_1,\ldots,g_m$, appearing in the composition of $h_j(t,x)$.

Actually both procedures contained in Lemma 7 and Theorem 1 are performed jointly and the first approximate equation generate a solution which compared with that in (1) fulfils the same estimate as in Theorem 1. Now we have to start with the new equation as the original one but its coefficients like $u_i^h(t,x)$ and $v^h(t)$ are unbounded with respect to $h = (h^1,h^2)$.

This new equation becomes with bounded coefficients with respect to h by making a "time change" $t^i = N\tilde{t}^i$, where $0 \leq \tilde{t}^i \leq NT^i = \tilde{T}^i$, $h^i = T^i/N$, $i = 1,2$, and the previous scheme has to be repeated considering the last equation in the place of (1).

The interval $[0,\tilde{T}]$, $\tilde{T} = (\tilde{T}^1,\tilde{T}^2)$ is divided into small rectangles $\Delta(h)$ with meas $\Delta(h) = (h^1h^2)^5$ and a corresponding unbounded coefficients equation with respect to h is defined with the property that the length of the brackets decreases strictly. The estimation of the two solutions in $L_2(\Omega,P)$ is upper bounded by $C\max[(h^1)^{5/2},(h^2)^{5/2}]$ on each interval $[k_1T^1,(k_1+1)T^1]\times[k_2T^2,(k_2+1)T^2]$, $k_i = 0,1,\dots,N-1$ and on the whole interval $[0,\tilde{T}]$ the estimation will be bounded by $C\max[(h^1)^{\frac{1}{2}},(h^2)^{\frac{1}{2}}]$.

By reversing "time change" we refind the first approximation equation and the second one with bounded coefcients with respect to h, but with the corresponding solutions fulfilling in $L_2(\Omega,P)$ an estimate having the previous upper bounded

$$C\max[(h^1)^{\frac{1}{2}},(h^2)^{\frac{1}{2}}].$$

Now in the second approximate equation we perform the "time change" $t^i = N^6\tilde{t}^i$, $0 \leq \tilde{t}^i \leq N^6T^i = \tilde{T}^i$, $i = 1,2$, to obatin a bounded coefficients equation with respect to h and T^i/N^6 appears in the definition of the corresponding auxiliary functions $p(t,h)$, $q(t,h)$. The new interval $[0,\tilde{T}]$ is divided into small rectangles $\Delta(h)$ with meas $\Delta(h) = [h^1\cdot h^2]^{25}$ and a corresponding approximate equation is defined with the property that the estimate of the two solutions on each interval $[k_1T^1,(k_1+1)T^1]\times[k_2T^2,(k_2+1)T^2]$, $k_i = 0,1,\dots,N^6-1$ is upper bounded by $C\cdot\max[(h^1)^{25/2},(h^2)^{25/2}]$ and on the whole interval $[0,\tilde{T}]$ the estimate in $L_2(\Omega,P)$ of the two solutions has the upper bound $C\max[(h^1)^{\frac{1}{2}},(h^2)^{\frac{1}{2}}]$. By reversing "time change" we get the second estimate equation and the third one defined on $t\in[0,T]$ for which the corresponding solutions fulfils an $L_2(\Omega,P)$ estimate upper bounded by $C\max[(h^1)^{\frac{1}{2}},h^2)^{\frac{1}{2}}]$. By a finite number of steps we get the last estimate equation which doesn't contain any bracket but only original g_i, $i = 1,\dots,m$, as it is defined in (2) with the property stated in Theorem 2.

References

[1] C. Varsan, Continuous dependence and time change for ito's equation J.Diff. Eq. vol. 58, No.3, 1985.

[2] C. Varsan, Approximation and existence of periodic solutions for controlled disffusion equations Stochastics and Stochastics Reports vol. , No , 1990.