GLOBAL EXISTENCE AND UNIQUENESS FOR A CLASS OF QUASILINEAR HYPERBOLIC EQUATIONS

by

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1. INTRODUCTION

Let us consider a separable Hilbert space X, a dense subspace $D(A)\subset X$ and a linear selfadjoint operator $A:D(A)\to X$ which satisfies the condition:

(1.1) $$\langle Au,u\rangle \geq 0,$$

where $\langle .,.\rangle$ is the scalar product in X and $\|.\|$ is the corresponding norm.

We shall study the following initial value problem:

(1.2) $$\ddot{u}(t) + B(l(u(t))u(t) = 0,$$

(1.3) $$u(0) = u_0,\ \dot{u}(0) = u_1,$$

where $l:D(A^{1/2})\to R$ and $B(l(u)):D(A)\to X$ is a selfadjoint operator for each $u\in D(A^{1/2})$. This abstract framework is motivated by the study of the nonlinear mathematical models for the small vibrations of a string or of a Timoshenko beam.

If in equation (1.2) we choose:

$$B(l(u)) = g(\|A^{1/2}u\|^2)A,$$

we obtain the initial value problem:

(1.4) $$\ddot{u}(t) + g(\|A^{1/2}u(t)\|^2)Au = 0,$$

(1.5) $$u(0) = u_0\ ,\ \dot{u}(0) = u_1,$$

which contains as a particular case the equations modelling the nonlinear vibrations of a string (cf.[4],[7] ch 7,[9]).

For $g \in C^1(R_+)$ and $g>0$, a local existence and uniqueness result for the problem (1.4),(1.5) was given in [9], provided that $\langle Au,u\rangle \geq c\|u\|^2$, $u_0 \in D(A)$ and $u_1 \in D(A^{1/2})$. Global existence for (1.4), (1.5) was proved in [1],[8] for initial data much more regular.

The free vibrations of a nonlinear Timoshenko beam with the ends hinged and held a fixed distance apart were modeled in the appendix of [5] by the following initial and boundary value problem:

$$\text{(1.6)} \qquad \frac{\partial^2 w}{\partial t^2}(x,t)=[cd-a+b\int_0^1(\frac{\partial w}{\partial x}(y,t))^2dy]\frac{\partial^2 w}{\partial x^2}(x,t)-cd\frac{\partial \psi}{\partial x}(x,t),$$

for $0<x<1$, $t \geq 0$

$$\text{(1.7)} \qquad \frac{\partial^2 \psi}{\partial t^2}(x,t)=c\frac{\partial^2 \psi}{\partial x^2}(x,t)-c^2d\psi(x,t)+c^2d\frac{\partial w}{\partial x}(x,t),$$

for $0<x<1$, $t \geq 0$

$$\text{(1.8)} \quad w(x,0)=w_0(x);\frac{\partial w}{\partial t}(x,0)=w_1(x);\ \psi(x,0)=\psi_0(x);\frac{\partial \psi}{\partial t}(x,0)=\psi_1(x),$$

$$\text{(1.9)} \quad w(0,t)=w(1,t)=\frac{\partial \psi}{\partial x}(0,t)=\frac{\partial \psi}{\partial x}(1,t)=0,$$

where w represents the transverse deflection of the centerline of the beam and ψ is the rotary displacement of the cross section. The constants a, b, c, d are supposed to be strictly positive, with $cd>a$. For the physical meaning of these constants; see[4], [10].

Let us notice that the equations (1.4), (1.5) and (1.6)-(1.9) have at least two common proprieties:

they are consequences of a conservation law (see also relation (4.1)) and they act "nicely" on the eigenfunctions of a certain operator (this is the reason we introduce the assumptions (4.6)-(4.7)). This similarity lead us considering the problem (1.2),(1.3) which contains as particular cases the problems (1.4),(1.5) and (1.6)-(1.9). A local existence result for the problem (1.6)-(1.9) similar with the one given in [8] for (1.4),(1.5) was given in [10]. As far as we know no global existence result for (1.6)-(1.9) was obtained till now.

The paper is organised as follows. In section 2 we prove a local existence result for the problem (1.2), (1.3). Section 3 contains some energetic estimates for the solutions of a linear evolution equation. Section 4 contains the global existence theorem and in section 5 the applications for the nonlinear string and Timoshenko beam models are given.

2. THE LOCAL EXISTENCE RESULT

In this section we shall use notations and assumptions from [3] in order to apply an existence and uniqueness result given there.

We begin by noticing that due to the assumption (1.1) and to the selfadjointness of the operator A we can define the fractional powers of A (cf.[6],ch.1). Let us make the notations:

(2.1) $$H_\gamma = D(A^{\gamma/2}),\ \gamma\geq 0,\ H_0=X$$

Under these circumstances it is well known that the family of Hilbert spaces $\{H_\gamma\}_{\gamma\in[0,m]}$, with $m\geq 2$ represents an interpolated scale of real, separable Hilbert spaces.(see [6], ch.1 and [10] for the background in interpolation theory). We denote by H_{-1} the dual space of H_1 constructed by means of the inner product of H_0 (cf.[6]) and by $\langle .,. \rangle$ the corresponding duality. To simplify our notations we set:

(2.2) $$\mathcal{L} = \bigcap_{j=-1}^{m-2} \mathcal{L}(H_{j+2};H_j),$$

equiped with the obvious operator norm, where by $\mathcal{L}(Z,Y)$ we denoted the space of bounded linear operators from Z to Y. We denote by $||.||_k$ the norm in H_k.

We consider a family of linear operators $B(\xi)$, $\xi\geq 0$, satisfying the following conditions:

(B1) $B\in W^{m,\infty}([0,D],\mathcal{L})$ for each $D>0$

(B2) For each $\xi\geq 0$ and $k=0,\dots,m-2$ the conditions $v\in H_k$ and $B(\xi)v\in H_k$ imply that $v\in H_{k+2}$. Moreover there is a constant $C_0>0$ such that:

$$||v||_{k+2} \leq C_0(||v||_k+||B(\xi)v||_k),\quad \xi\in[0,D]$$

(B3) There is a constant $C_1>0$ such that

$\langle B(\xi)v,v\rangle \geq C_1 ||v||_1^2$, $\langle B(\xi)v,z\rangle = \langle B(\xi)z,v\rangle$

for any $v,z \in H_1$, $\xi \in [0,D]$.

Let us consider now a functional $l: D(A^{1/2}) \to R_+$.

As a straightforward consequence of the Theorem 4.1 from [3] we obtain the following result:

THEOREM 2.1 Assume that (B1) through (B3) hold and and that the functional l is localy bounded and m-1 times continously differentiable, with $m \geq 2$. Then for sufficiently small $T>0$ the initial value problem (1.2),(1.3) has a unique solution:

$$u \in \bigcap_{k=0}^{m} C^k([0,T), H_{m-k}),$$

for each $u_0 \in H_m$, $u_1 \in H_{m-1}$.

REMARK 2.1 Due to the special choice of the interpolated scale of Hilbert spaces and to the proprieties of $B(l(u))$, the higher order compatibility conditions necessary in [3] are satisfied in our situation without any other asumptions.

REMARK 2.2 In the following we shall suppose that the solution of (1.2), (1.3) defined on $[0,T)$ is a maximal one.

3. ENERGY ESTIMATES FOR A LINEAR EQUATION

In this section we shall make use of the fact that, due to the assumptions made on the operator A, the space X admits an orthonormal basis $\{v\}_{k \geq 1}$ such that for each k=1,2,...

(3.1) $$Av_k = \lambda_k^2 v_k, \quad \lambda_k > 0,$$

provided that the imbedding $D(A) \subset X$ is compact.

In order to establish the global existence for (1.2),(1.3) we shall need stronger assumptions on the initial data u_0, u_1. More precisely we shall consider them A-analytic. According to [1],[8], a vector $v \in D(A^{1/2})$ is A-analytic if there exists constants K, Λ such that:

$$A^j v \in D(A^{1/2}), \quad |\langle A^j v, v\rangle|^2 \leq K\Lambda^j j! \ , \quad j=0,1,\ldots.$$

In [1] Arosio ans Spagnolo proved the following characterisation of A-analicity.

PROPOSITION 3.1 Assume that conditions (3.1) holds. Then a vector $v=\sum_{k\geq 1} y_k v_k \in H_1$ is A-analytic if and only if

there exists some $\delta>0$ such that:

$$\sum_{k\geq 1} |y_k|^2 \exp(2\delta\lambda_k)<\infty. \tag{3.2}$$

In this section we shall consider a family of linear operators $C(t)$: $D(A) \to X$, $t\in[0,T)$, satisfying certain assumptions in connection with the bases $(v_i)_{i\geq 1}$. We suppose that the space X is decomposable in direct sum of orthogonal subspaces $X_1,\dots,X_p$ to which it corresponds a partition $N_1,\dots,N_p$ of the set of the natural numbers such that:

(3.3) $(P_j v_i)_{i\in N_j}$ is an orthonormal basis in X_j, $j=1,p$;

(3.4) $P_j(C(t)v_i) = c_{ij}(t)\lambda_i^2 P_j v_i$, $i\in N_j$, $j=1,p$, $t\in[0,T)$

where P_j is the orthogonal projector on X_j, $c_{ij}: (0,\infty)\to R$ are continously differentiable and $c_{ij}(t) \geq c_0 > 0$.

(3.5) $P_j(C(t)v_i) = 0$ if $i\notin N_j$, $t\in[0,T)$.

Let us make the notations:

(3.6) $e_k(u,t)=(1/2)\lambda^2_k|y_k(t)|^2+|y'_k(t)|^2$,

where $u(t)=\sum_{k\geq 1} y_k(t)v_k$.

Consider the initial value problem:

(3.7) $\ddot{u}(t)+C(t)u(t)=0$

(3.8). $u(0)=u_0$, $\dot{u}(0)=u_1$.

Concerning the solutions of (3.7),(3.8) the following lemma which is a slight variation of a result in [1] holds:

LEMMA 3.1 Let u be a solution of (3.7) on [0,T) and suppose that : $|c_{ij}(t)|< M$, $i\geq 1$, $j=1,p$, $t\in[0,T)$. Then we have for each $k=1,2,\dots$ and $0\leq t<T$

(3.9) $e_k(u,t) \leq e_k(u,0)M_1\exp(\eta\lambda_k)$, for any $\eta>0$, where M_1 is a positive constant depending on T and η.

Proof:

Consider $u(t)=\sum_{k\geq 1} y_k(t)v_k$, the Fourier expansion of the solution of (3.8).

If we take the projection of (3.7) on X_j and take into consideration (3.5) we get:

$$\sum_{k\in N_j} [\ddot{y}_k(t)P_jv_k+y_k(t)P_j(C(t)v_k)]=0.$$

From (3.3) and (3.4) it follows that:

$$\ddot{y}_k(t)+\lambda_k^2 c_{kj}(t)y_k=0, \quad k\in N_j. \tag{3.10}$$

From (3.10), following exactly the proof of Lemma 2 from [1] we obtain that (3.9) holds for all $k\in N_j$. As the sets $(N_j)_{j=1,p}$ cover the set of natural numbers the conclusion of the lemma follows.

We can prove now that, in certain conditions, the solution of (3.8) will not blow up in finite time.

THEOREM 3.1 Let u be a solution of (3.7) on $[0,T)$ such that $u(0)$ and $\dot{u}(0)$ are A-analytic. Then, if $c_{kj}(t)$ is bounded on $[0,T)$, $u(t)$, $\dot{u}(t)$ and $Au(t)$ admit a limit in H_0 for $t\to T^-$. Moreover $u(T^-)$ and $\dot{u}(T^-)$ are A-analytic vectors.

Proof: Since $u(0)$ and $\dot{u}(0)$ are A-analytic, by Proposition 3.1 there exists a constant $\delta=\delta(u_0,u_1)>0$ such that:

$$\sum_{k\geq 1} e_k(u,0)\exp(2\delta\lambda_k)<+\infty \quad .$$

Using the estimate (3.10) with $\eta=\delta$ we have

$$\sum_{k\geq 1} e_k(u,t)\exp(\delta\lambda_k) \leq M_2(T,\delta)\sum_{k\geq 1} e_k(u,0)\exp(2\delta\lambda_k),$$

hence

$$\sum_{k\geq 1} e_k(u,t)\exp(\delta\lambda_k) \leq M_3, \text{ where } M_3 \text{ depends on } T,\ u_0,\ u_1. \tag{3.12}$$

Particularly:

$$\sup_{0\leq t\leq T} ||AC(t)u(t)||^2 = \sup_{0\leq t\leq 1} \sum_{k\geq 1} \lambda_k^8 e_k(u,t)<\infty ,$$

and hence by equations (3.8)

$$\int_0^T ||A\ddot{u}(t)|| <\infty .$$

This implies the existence of $A\dot{u}(T^-)$, hence of $(Au)(T^-)$, $u(T^-)$ and $\dot{u}(T^-)$. By the use (3.12) and of Proposition 3.1 we obtain that the vectors $u(T^-)$ and $\dot{u}(T^-)$ are A-analytic.

4. THE GLOBAL EXISTENCE RESULT

In order to prove the global existence result we have to assume that (1.2) is the expression of a conservation law. More precisely we shall suppose that the following condition holds:

(4.1) $\langle Dl(u),h_1\rangle\langle B'(l(u))u,h_2\rangle=\langle Dl(u),h_2\rangle\langle B'(l(u))u,h_1\rangle$,

for any $u,h_1,h_2 \in H_1$, where by $Dl(u):H_1\to H_{-1}$ we denoted the Frechet differential of l.

According to the Kerner-Vainberg potentiality theorem (cf.[11], ch.2) we obtain that the nonlinear operator

$$P:H_1\to H_{-1},\ Pu=B(l(u))u$$

is the gradient of the functional:

(4.2) $$F:H_1\to R\ ;\ F(u)=\int_0^1 \langle B(l(tu))tu,u\rangle$$

In these conditions the following result holds:

PROPOSITION 4.1 Let $u:[0,T)\to X$ be a solution of (1.2),(1.3), and suppose that (4.1) and (B3) hold. Then there is a constant $M>0$ such that:

(4.3) $$\|\dot{u}(t)\|_0^2+\|u(t)\|_1^2 \le M, \text{ for any } t\in[0,T).$$

PROOF If we take the inner product of (1.2) with u and integrate on $[0,t]$ we obtain:

(4.4) $(1/2)\|\dot{u}(t)\|_0^2+F(u(t))=(1/2)u_1^2+F(u_0)$, $t\in[0,T)$.

From (4.2) and (4.3) we obtain that:

(4.5) $$F(u)\ge C_3\|u\|_1^2,\ C_3>0.$$

Relations (4.4),(4.5) imply (4.3).

We shall also make the following assumptions, related with (3.3)-(3.5):

(4.6) $P_j(B(\xi)v_i) = b_{ij}(\xi)\lambda_i^2 P_j v_i$, $i \in N_j, j=1,p$,

$b_{ij}: (0,\infty) \to (C_0,\infty)$ continously differentiable

(4.7) $P_j(B(\xi)v_i) = 0$, if $i \notin N_j$, $\xi \geq 0$.

We can state now the main result.

THEOREM 4.1 Assume that conditions (B1)-(B3), (3.3), (4.1), (4.6), (4.7) hold true. Then the problem (1.1),(1.2) has a solution u in $C^2([0,\infty),X)$ for any choice of the A-analytic initial data u_0 and u_1. Moreover the vectors $u(t)$, $\dot{u}(t)$ and $\ddot{u}(t)$ are A-analytic for each $t \geq 0$

PROOF By Theorem 2.1 we know that the problem (1.2)-(1.3) has a maximal solution u in $C([0,T), H_2) \cap C^1([0,T),H_1) \cap C^2([0,T),H_0)$.

Let us make the notation:

$$C_u(t)=B(l(u(t))),\ 0 \leq t < T.$$

By the use of Proposition 4.1 and of the assumptions (4.6),(4.7) we see thet $C_u(t)$ satisfies the hypotesis of Lemma 3.1 so that u belongs to $C^1([0,T],H_1)$, Au to $C^0([0,T],H_0)$, $u(T^-)$ and $\dot{u}(T^-)$ are A-analytic. Using again Theorem 2.1 with initial instant T and initial data $u(T^-)$ and $\dot{u}(T^-)$ we deduce that u may be extended to a solution u in $C^1([0,T+\sigma],H_1)$ with Au in $C([0,T+\sigma],H_0)$ for some $\sigma>0$. This contradicts the maximality of u, so we obtain that $T=+\infty$. The vectors $u(t)$ and $\dot{u}(t)$ are A-analytic for each $t \geq 0$ by Theorem 3.1; u is in $C([0,\infty),H_0)$ and $\ddot{u}(t)$ is A-analytic because u is a solution of (4.6).

5. APPLICATIONS

A straighforward application of Theorems 2.1 and 4.1 can be obtained by choosing:

$$B(l(u))=g(\|A^{1/2}u\|^2)A,$$

with g a C^1 function, g strictly positive. In this particular case from Theorem 2.1 we obtain the local result given in Theorem 1 from [9], already stated in the introduction. For initial data u_0 and u_1 A-analytic we obtain the global existence result proved in [8]. The equation modelling the free vibrations of a nonlinear string can be obtained from (1.4) by choosing:

$$X=L^2(0,1),\ D(A)=H_0^1(0,1) \cap H^2(0,1),\ Au= -\frac{d^2u}{dx^2},$$

$$g(v)=a+bv^2,\ a>0,\ b \geq 0.$$

So we obtain the initial value problem:

$$(5.1)\quad \frac{\partial^2 u}{\partial t^2}(x,t)-[(a+b\int_0^1(\frac{\partial u}{\partial x}(y,t))^2dy)]\frac{\partial^2 u}{\partial x^2}=0,\quad 0<x<1,\ t>0.$$

$$(5.2)\quad \mu(0,x)=u_0(x),\quad \frac{\partial u}{\partial t}(0,x)=u_1(x)\ ,\quad 0<x<1.$$

$$(5.3)\quad u(0,t)=u(1,t)=0.$$

A local existence result for the problem (5.1)-(5.3) was proved for the first time by Bernstein in [2] and then by Dickey in [4] by the use of Galerkin's method. As a consequence of Theorem 2.1 we obtain :

PROPOSITION 5.1 Let $u_0\in H_0^1(0,1)\cap H^2(0,1)$, $u_1\in H_0^1(0,1)$. Then for T small enough the initial and boundary value problem (5.1)-(5.3) has a unique solution $u\in C([0,T),H_0^1(0,1)\cap H^2(0,1))\cap C^1([0,T),H_0^1(0,1))\cap C^2([0,T),L^2(0,1))$.

Let us consider now the linear operator:

$$A:H^1{}_0(0,1)\cap H^2(0,1)\to L^2(0,1),\quad Au=-\frac{d^2u}{dx^2}.$$

In this particular case the function $v:[0,1]\to R$ is A-analytic if and only if (cf.[5])

(5.4) v is analytic in some neighbourhood of [0,1];

$$(5.5)\quad \frac{d^{2k}v}{dx^{2k}}(0)=\frac{d^{2k}v}{dx^{2k}}(1)=0,\ k=0,1,\dots\ .$$

By applying Theorem 4.1 we obtain the following result:

PROPOSITION 5.2 Assume that u_0 and u_1 satisfy the conditions (5.4),(5.5). Then the initial and boundary value problem (5.1)-(5.3) has a unique solution:

$$u\in C^2([0,\infty),L^2(0,1))\cap C^1([0,\infty),H_0^1(0,1))\cap C^0([0,\infty),H_0^1(0,1)\cap H^2(0,1)).$$

The above result was proved for the first time in [2] and in its abstract version for the problem (1.4),(1.5) in [8],[1].

We shall consider now another choice for A and B(l(u)) in order to obtain existence results for the problem (1.6)-(1.9).

This choice is:

$$X=L^2(0,1)\times L^2(0,1);\ \langle (w_1,\psi_1),(w_2;\psi_2)\rangle_X = c\int_0^1 w_1 w_2 + \int_0^1 \psi_1\psi_2$$

$$D(A)=\{(w,\psi)\in X \mid w\in H_0^1(0,1)\cap H^2(0,1),\ \psi\in H^2(0,1),\ \frac{d\psi}{dx}(0)=\frac{d\psi}{dx}(1)=0\};$$

$$A(w,\psi)=\left(-\frac{d^2w}{dx^2},\ -\frac{d^2\psi}{dx^2}\right);$$

$$l(w,\psi)=\int_0^1 \left[\frac{dw}{dx}(y)\right]^2 dy;$$

$$(5.6)\qquad B(\xi)=\left(-(cd-a+b\xi)\frac{d^2w}{dx^2}+cd\frac{d\psi}{dx},\ -c\frac{d^2\psi}{dx^2}+c^2d\psi-c^2d\frac{dw}{dx}\right)$$

It is easy to check that assumptions (B1)-(B3) are satisfied so from Theorem 2.1 we get:

<u>PROPOSITION</u> <u>5.3</u> Assume that $w_0\in H_0^1(0,1)\cap H^2(0,1)$, $w_1\in H_0^1(0,1)$;

$$\psi_0\in H^2(0,1),\quad \psi_1\in H^1(0,1),\quad \frac{d\psi}{dx}(0)=\frac{d\psi}{dx}(1)=0.$$

Then the initial and boundary value problem (1.6)-(1.9) has a unique solution $(w,\psi):[0,T)\to L^2(0,1)\times L^2(0,1)$ for T small enough such that:

$$w\in C([0,T),H^2(0,1)\cap H_0^1(0,1))\cap C^1([0,T),H_0^1(0,1))\cap C^2([0,T),L^2(0,1)),$$

$$\psi\in C([0,t),H^2(0,1))\cap C^1([0,T),H^1(0,1)))\cap C^2([0,T),L^2(0,1)).$$

This result was obtained in [10] by using Galerkin's method.

Due to the independence of two components of the operator A we get that a vector (w,ψ) is A-analytic if and only if

(5.7) w,ψ are analytic in some neighbourhood of $[0,1]$;

$$(5.8)\qquad \frac{d^{2k}w}{dx^{2k}}(0)=\frac{d^{2k}w}{dx^{2k}}(1)=\frac{d^{2k+1}\psi}{dx^{2k+1}}(0)=\frac{d^{2k+1}\psi}{dx^{2k+1}}(1)=0,\quad k=0,1,\ldots .$$

As a consequence of Theorem 4.1 we obtain:

PROPOSITION 5.4 Assume that w_0, w_1, ψ_0, ψ_1 satify (5.7),(5.8). Then the initial and boundary problem (1.6)-(1.9) has a unique solution :

$$w \in C([0,\infty), H_0^1(0,1) \cap H^2(0,1)) \cap C^1([0,\infty), H_0^1(0,1)) \cap C^2([0,\infty), L^2(0,1)).$$

PROOF: We can choose the basis v_i in the following way:

$$v_{2i}(x) = (\sin i\pi x,\ 0),$$

$$v_{2i+1}(x) = (0, \cos i\pi x),\quad i \geq 0.$$

It is obvious that $(v_k)_{k\geq 0}$, the spaces $X_1 = L^2(0,1)\times\{0\}$, $X_2 = \{0\}\times L^2(0,1)$ and the family of operators $B(\xi)$ from (5.6) satisfy the assumptions (3.3)-(3.5) so we can apply Theorem 4.1.

REMARK 5.1 The local solution for (1.6)-(1.9) obtained in Proposition 5.3 are much smoother with respect to t if the initial data are smoother. In fact if

$\{H_\gamma\}_{\gamma\geq 0} = D(A^{\gamma/2})$, with A given by 5.6, we have that:

$$(w,\psi) \in \sum_{k=0}^{m} C^k([0,T], H_{m-k}).$$

In order to study the dynamic buckling of the nonlinear Timoshenko beam Hirschhorn and Reiss in [4] didn't use directly (1.6)-(1.9), but the following initial and boundary value problem for the transverse deflection $w(x,t)$:

$$(5.10)\quad \frac{\partial^4 w(x,t)}{\partial t^4} - \frac{\partial^2}{\partial t^2}\left(\mu(t)\frac{\partial^2 w}{\partial x^2}(x,t)\right) - c(d+1)\frac{\partial^4 w}{\partial t^2 x^2}(x,t) + c^2 d\frac{\partial^2 w}{\partial t^2}(x,t) =$$

$$= -c^2 d\frac{\partial^4 w(x,t)}{\partial x^4} - c\mu(t)\frac{\partial^4 w}{\partial x^4}(x,t) + c^2 d\mu(t)\frac{\partial^2 w}{\partial x^2}(x,t),\quad 0<x<1,\ t\geq 0,$$

where $\mu(t) = -a + b\int_0^1 \left(\frac{\partial w}{\partial x}(y,t)\right)^2 dy.$

$$(5.11)\quad \frac{\partial^i w(x,0)}{\partial t^i} = w_i(x),\quad i=0,1,2,3,\ 0<x<1.$$

$$(5.12)\quad w(0,t) = w(1,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0,\ t>0.$$

We shall prove the following existence and uniqueness result for the problem (5.10)-(5.12).

PROPOSITION 5.5 Assume that w_i, $i=0,4$ satisfy (5.5). Then the problem (5.8)-(5.10) has a unique solution :

$$w\in \bigcap_{k=0}^{4} C^k([0,\infty),H_{m-k}),$$

where $H_\gamma = D(A^{\gamma/2})$ and A is defined by (5.6).

PROOF

i) Existence

Let ψ_0, ψ_1 be two functions satisfying the conditions:

$$(5.13)\qquad w_2 = (cd+\mu(0))\frac{d^2w_0}{dx^2} - cd\frac{d\psi_0}{dx};$$

$$(5.14)\qquad w_3 = (cd+\mu(0))\frac{d^2w_1}{dx^2} - cd\frac{d\psi_1}{dx} + \dot{\mu}(0)\frac{d^2w_0}{dx^2}.$$

The above relations imply that w_0, w_1, ψ_0, ψ_1 satisfy conditions (5.8),(5.9). Consider the solution w,ψ of (1.6)-(1.9) with initial data w_0, w_1, ψ_0, ψ_1.

From Remark 4.1 we obtain that:

$$w,\psi\in \bigcap_{k=0}^{4} C^k([0,\infty),H^{4-k}(0,1)).$$

This means that all the derivates necessary to eliminate ψ from (1.2),(1.3) make sense so we obtain that w satisfies (5.10) and the last two boundary conditions from (5.12).

ii) Uniqueness

Let w be a solution of (5.10)-(5.12) having the regularity

$$w\in \bigcap_{k=0}^{4} C^k([0,\ ),H^{4-k}(0,1)).$$

We define ψ (x,t) as the unique solution of the equation (1.7) with the initial and boundary conditions:

$$w(x,0)=w_0(x)\ ,\ \frac{\partial\psi}{\partial t}(x,0)=\psi_1(x)\ ,\ \frac{\partial\psi}{\partial x}(0,t)=\frac{\partial\psi}{\partial x}(1,t)=0,$$

where ψ_0, ψ_1 satisfy (5.13). It is easy to see that w_0, w_1, ψ_0, ψ_1 satisfy (5.8),(5.9) and by the use of Theorem 3.1 from [3] we obtain:

$$\psi \in \bigcap_{k=0}^{4} C^{4-k}([0,\infty), H^{4-k}(0,1)).$$

Define $G:[0,1]\times[0,\infty)\to R$ by:

$$G(x,t) = \frac{\partial^2 w}{\partial t^2}(x,t) - (\mu(t) + cd)\frac{\partial^2 w}{\partial x^2}(x,t) + cd\frac{\partial \psi}{\partial x}(x,t).$$

Taking into account (5.10),(5.13) we obtain that G is a solution of the initial-boundary value problem:

$$\frac{\partial^2 G}{\partial t^2}(x,t) - c\frac{\partial^2 G}{\partial x^2}(x,t) + G(x,t) = 0$$

$$G(0,t) = G(1,t) = 0, \quad G(x,0) = \frac{\partial G}{\partial t}(x,0) = 0.$$

From the standard uniqueness result for linear hyperbolic equations we obtain that $G = 0$, i.e. w,ψ are solutions of (1.6) - (1.9). By the use of Proposition 5.3 the uniqueness result follows.

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