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by

A. BUIUM

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A. BUIUM*)

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*) Institute of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania.

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A.Buium

Introduction.

This paper is a direct continuation of [B$_1$][B$_2$] from where we borrow terminology and notations.

In section 1 we discuss some further properties of Δ-polynomial functions and apply these properties to make a rough discussion of Δ-polynomial functions on algebraic surfaces.

In section 2 we prove a theorem which implies in particular that the order of any Δ-polynomial character of the Jacobian of a curve does not decrease by restriction to the curve.

In section 3 we introduce a new concept (having no analogue in the "non-differential" algebraic geometry): a Δ-closed subset Σ of $\mathcal{U}^N$ will be called strongly Δ-closed if it remains Δ-closed in $\mathbb{P}^N$ (where we view $\mathcal{U}^N$ embedded in $\mathbb{P}^N$ via $(x_1,\ldots,x_N)\mapsto(1:x_1:\ldots:x_N)$). We shall prove that for X a smooth complete curve of genus $g \geq 3$ which is Δ-generic in the moduli space $\mathcal{M}_g$, the image of the Δ-pluricanonical map $\varphi_3 : X \longrightarrow \mathcal{U}^{8g-8}$ which is known from [B$_2$] to be Δ-closed is in fact strongly Δ-closed.

1. Further properties of Δ-polynomial functions

(1.1) Start by recalling a basic fact from [J] p.183. Assume $(f,\delta):X\to Y$ is a smooth kernel and $(\tilde{f},\tilde{\delta}):P\to X$ its canonical prolongation. Then we have a natural isomorphism:

$$\Phi=\Phi_{X/Y}:\tilde{f}^{*}\Omega_{X/Y}\longrightarrow\Omega_{P/X}$$

Here naturality of $\Phi_{X/Y}$ means compatibility in the obvious sense with morphisms between kernels. Recall for convenience how Φ is defined (cf. [J]). The composition

$$\mathcal{O}_X\xrightarrow{\tilde{\delta}}\tilde{f}_{*}\mathcal{O}_P\xrightarrow{\tilde{f}_{*}d_{P/X}}\tilde{f}_{*}\Omega_{P/X}$$

is an $\mathcal{O}_Y$ - derivation hence provides an $\mathcal{O}_X$-linear map $\Omega_{X/Y}\to\tilde{f}_{*}\Omega_{P/X}$ hence an $\mathcal{O}_P$-linear map Φ as above. It is easy to see (using for instance the fact that P has a natural structure of principal homogeneous space for $\mathbb{V}(T_{X/Y})$ or simply using [J] p.183) that Φ is an isomorphism.

In particular if X is a smooth $\mathcal{U}$-variety and $(X^n)_n$ its associated infinite prolongation sequence $[B_1]$(3.1) then we have natural isomorphisms $\mathbb{V}(T_{X^n/X^{n-1}})\simeq X^n\times_X TX$ where TX denotes as usual the tangent bundle of X. Now let $(f,\delta):X_o\to X_{-1}$ and $(g,\delta):Y_o\to Y_{-1}$ be two smooth kernels and $(u_o,u_{-1}):(f,\delta)\to(g,\delta)$ be a morphism of kernels. Let moreover $X_1=P(f,\delta)\to X_o$ and $Y_1=P(g,\delta)\to Y_o$ be the canonical prolongations. Then using the functors that X_1 and Y_1 represent one immediately checks that the natural morphism $u_1:X_1\to Y_1$ induced by (u_o,u_{-1}) is equivariant (in the obvious sense) with respect to the natural map $\mathbb{V}(T_{X_o/X_{-1}})\to\mathbb{V}(T_{Y_o/Y_{-1}})$.

In particular, if we start with a morphism of smooth $\mathcal{U}$-varieties $u:X\to Y$ and if $u^n:X^n\to Y^n$ is the associated morphism of infinite prolongation sequences then $u^{n+1}:X^{n+1}\to Y^{n+1}$ is equivariant with respect to the map

$$u^n\times Tu:X^n\times_X TX\longrightarrow Y^n\times_Y TY$$

where $Tu:TX \to TY$ is the "tangent" map of u.

Here is a quick but remarkable application of (1.1) above. Let's introduce a notation. For X a smooth projective $\mathcal{U}$-variety and $I=(i_1,\ldots,i_k)$, $i_j \geq 0$, $k \geq 1$ any multiindex we put

$$q_I(X)=\dim_{\mathcal{U}} H^0(X, S^{i_1}\Omega_{X/\mathcal{U}}\otimes\ldots\otimes S^{i_k}\Omega_{X/\mathcal{U}})$$

These are birational invariants of X. We put $|I|=k$. Note that the invariants q_I for $|I|=1$ have been studied by Sakai [Sak]. We have the following:

(1.2) PROPOSITION. Assume $q_I(X)=0$ for $I\neq 0$, $|I|\leq n$. Then $\mathcal{O}^{(n)}(X)=\mathcal{U}$. In particular, if $q_I(X)=0$ for all $I\neq 0$ then $\mathcal{O}^{\Delta}(X)=\mathcal{U}$.

To prove (1.2) let's make one more definition. A morphism of schemes $f:X\to Y$ will said to have property P_n if the map $H^0(\mathcal{O}_Y)\to H^0(\mathcal{O}_X)$ is an isomorphism and for any multiindex $I\neq 0$ with $|I|\leq n$ we have $H^0(X,S^{i_1}\Omega_{X/Y}\otimes\ldots\otimes S^{i_{|I|}}\Omega_{X/Y})=0$. Then (1.2) follows by induction applying the Lemma below to the infinite prolongation sequence $(f^n,\delta^n):X^n\to X^{n-1}$ associated to X.

(1.3) LEMMA. Assume $(f,\delta):X\to Y$ is a smooth kernel and $(\tilde{f},\tilde{\delta}):P\to X$ its canonical prolongation. If f has property P_{n+1} then $\tilde{f}$ has property P_n.

Proof. For any multiindex $I=(i_1,\ldots,i_k)$, $i_j\geq 0$, $k\leq n$ we have by (2.1):

$$H^0(P,S^{i_1}\Omega_{P/X}\otimes\ldots\otimes S^{i_k}\Omega_{P/K})=H^0(P,\tilde{f}^*(S^{i_1}\Omega_{X/Y}\otimes\ldots\otimes S^{i_k}\Omega_{X/Y})=$$
$$=H^0(X, S^{i_1}\Omega_{X/Y}\otimes\ldots\otimes S^{i_k}\Omega_{X/Y}\otimes\tilde{f}_*\mathcal{O}_P)$$

But by [B_2] (1.2) $\tilde{f}_*\mathcal{O}_P$ has a filtration whose associated graded algebra is $S(\Omega_{X/Y})$. This immediately implies that $\tilde{f}$ has property P_n.

(1.4) Remarks. 1) One immediately checks that $q_I((\mathbb{P}^1)^N)=0$ for $I\neq 0$ and $N\geqslant 1$. So $q_I(X)=0$ for $I\neq 0$ and any unirational smooth projective X; this reproves the fact that for any such X, $\mathcal{O}^\Delta(X)=\mathcal{U}$ (cf. [B_1] (6.3)).

2) By [Sak] it follows that for any smooth projective surface X which is either K3 or Enriques or a hypersurface in $\mathbb{P}^3$ we have $q_I(X)=0$ for $I\neq 0$, $|I|=1$; so for any such X, $\mathcal{O}^{(1)}(X)=\mathcal{U}$. One might conjecture in fact that for any such X, $q_I(X)=0$ for all $I\neq 0$ hence in particular that $\mathcal{O}^\Delta(X)=\mathcal{U}$.

(1.5) Let's investigate in what follows the behaviour of Δ-polynomial functions with respect to Galois coverings (i.e. finite Galois morphisms). Assume $X\to Y$ is a Galois covering with group G, X and Y being smooth (non necessarily projective) $\mathcal{U}$-varieties. As well known $\mathcal{O}(Y)=\mathcal{O}(X)^G$. But in general we have the curious fact that $\mathcal{O}^\Delta(Y)\neq\mathcal{O}^\Delta(X)^G$. Indeed, let $f:X\to Y$ be a Galois covering of smooth projective curves with group G such that $g(X)\geqslant 1$ and $g(Y)=0$, so $Y=\mathbb{P}^1$. Then by [B_2] $\mathcal{O}^\Delta(X)\neq\mathcal{U}$ and $\mathcal{O}^\Delta(Y)=\mathcal{U}$ hence $\mathcal{O}^\Delta(X)^G$ cannot coincide with $\mathcal{O}^\Delta(Y)$. Nevertheless we have the following

(1.6) PROPOSITION. Let $u:X\to Y$ be an étale covering of smooth $\mathcal{U}$-varieties. Then:

1) If u is Galois with group G we have $\mathcal{O}^{(n)}(Y)=\mathcal{O}^{(n)}(X)^G$ hence $\mathcal{O}^\Delta(Y)=\mathcal{O}^\Delta(X)^G$.

2) u is a closed map for the Δ-topologies (we will say simply that u is Δ-closed).

Proof. Let $u^n : X^n \longrightarrow Y^n$ be the natural morphisms induced by u between the infinite prolongation sequences associated to X and Y. We prove by induction on n that u^n is an étale covering which is Galois if u is so and this will close the proof of 1). Now by (1.1) u^{n+1} is equivariant with respect to the map $u^n \times Tu : X^n \times_X TX \to Y^n \times_Y TY$. Since u is étale we have a natural isomorphism $TX \simeq X \times_Y TY$ hence $u^n \times Tu$ identifies with the map $u^n \times id : X^n \times_Y TY \longrightarrow$ $\to Y^n \times_Y TY$. Since $u^n \times id$ is an étale covering so will be u^{n+1}. If u^n is Galois then so is u^{n+1}.

To prove 2) we may assume X and Y affine. Let $u^\infty : X^\infty \longrightarrow Y^\infty$ be the morphism of D-scheme induced by u, $u^\infty = \varprojlim u^n$. Since $\mathcal{O}(X^n)$ is finite over $\mathcal{O}(Y^n)$, we get that $\mathcal{O}(X^\infty)$ is integral over $\mathcal{O}(Y^\infty)$ hence u^∞ is closed for the Zariski topology. This (and the "differential Nullstellensatz") immediately imply that u is Δ-closed.

(1.7) Remarks. 1) The Δ-closedness of étale coverings implies for instance that if $(\mathcal{A}_g)_{reg}$ is the smooth locus of the moduli space of principally polarized abelian $\mathcal{U}$-varieties of dimension g then the locus of all points in $(\mathcal{A}_g)_{reg}$ corresponding to abelian $\mathcal{U}$-varieties of Δ-rank r is Δ-closed for all r (cf. [B_1] (6.8)).

2) Any Δ-closed immersion of smooth $\mathcal{U}$-varieties (in the sense of [B_2] (1.6)) hence in particular any closed immersion of smooth $\mathcal{U}$-varieties is Δ-closed in the sense of (1.6).

3) If $u : X \longrightarrow Y$ is a Galois covering of smooth $\mathcal{U}$-varieties with group G ramified along the divisor $D \subset Y$ and if $\varphi \in \mathcal{O}^\Delta(X)^G$ then the function $\bar{\varphi} : Y \to \mathcal{U}$ induced by φ (which is well defined set - theoretically but which is not in general Δ-polynomial on Y cf. (1.5)) is nevertheless Δ-polynomial on $Y \setminus D$. Here is a typical example. Let $u : X = \mathbb{A}^1 \longrightarrow Y = \mathbb{A}^1$ be the Galois covering defined by $u(y) = y^a$, $a \geq 2$ an interger and consider the Δ-polynomial function $\varphi : X \to \mathcal{U}$, $\varphi(y) = (y')^a$. Clearly φ is invariant with respect to the Galois group of u and we have $\varphi = \psi \circ u$ where $\psi : Y \setminus \{0\} \to \mathcal{U}$ is defined by $\psi(y) = a^{-a} y^{1-a} (y')^a$.

We see that ψ is Δ-polynomial on $Y \setminus \{0\}$ (embed $Y \setminus \{0\}$ into $\mathcal{U}^2$ via $y \mapsto (y,$

y^{-1}) and then ψ is given by the Δ-polynomial $a^{-a} w_2^{a-1}(w_1')^{a-1}$ where (w_1, w_2) are coordinate on $\mathcal{U}^2$) and not Δ-polynomial on Y. So ψ may be viewed as having a "singularity" at $0 \in Y$ which can be "resolved" by composing with a finite covering of Y ramified at 0. The study of such "singularities" of Δ-polynomial functions deserves further investigation.

(1.8) PROPOSITION. Let X be a smooth $\mathcal{U}$-variety and $U \subset X$ a Zariski open subset whose complement has codimension ≥ 2. Then the restriction maps $\mathcal{O}^{(n)}(X) \to \mathcal{O}^{(n)}(U)$, $\mathcal{O}^{\Delta}(X) \to \mathcal{O}^{\Delta}(U)$ are isomorphisms. In particular $\mathcal{O}^{(n)}(X)$, $\mathcal{O}^{\Delta}(X)$ are birational invariants of the smooth projective variety X.

Proof. If (U^n), (X^n) are the infinite prolongation sequences associated to U and X then U^n is the preimage of U via the projection $X^n \to X$ hence $X^n \setminus U^n$ has codimension ≥ 2 in X^n and we are done.

(1.9) LEMMA. Let X and Y be smooth $\mathcal{U}$-varieties and (X^n), (Y^n), $((X\times Y)^n)$ be the infinite prolongation sequences associated to X, Y and $X\times Y$. Then we have natural isomorphisms

$$(X\times Y)^n \simeq X^n \times Y^n$$

compatible with the corresponding structure maps of the two prolongation sequences $((X\times Y)^n)$ and $((X^n)\times(Y^n))$.

Proof. It is a formal consequence of the fact that a functor admitting a left adjoint commutes with products.

(1.10) PROPOSITION. Let X and Y be smooth $\mathcal{U}$-varieties. Then we have natural isomorphisms $\mathcal{O}^{(n)}(X\times Y) \simeq \mathcal{O}^{(n)}(X) \otimes \mathcal{O}^{(n)}(Y)$, $\mathcal{O}^{\Delta}(X\times Y) \simeq$ $\simeq \mathcal{O}^{\Delta}(X) \otimes \mathcal{O}^{\Delta}(Y)$.

Proof. By (1.9) plus Künneth's formula we get

$$\mathcal{O}^{(n)}(X\times Y)\simeq\mathcal{O}((X\times Y)^n)\simeq\mathcal{O}(X^n\times Y^n)\simeq\mathcal{O}(X^n)\otimes\mathcal{O}(Y^n)\simeq\mathcal{O}^{(n)}(X)\otimes\mathcal{O}^{(n)}(Y).$$

(1.11) COROLLARY. Let X be a smooth $\mathcal{U}$-variety and E a vector bundle on X. Then we have a natural isomorphish

$$\mathcal{O}^{\Delta}(\mathbb{P}(E))\simeq\mathcal{O}^{\Delta}(X)$$

Proof. The question is local on X (for the Zariski topology). So we may assume E is trivial. Then we conclude by (1.9) and (1.4).

(1.12) PROPOSITION. Let X be a smooth $\mathcal{U}$-variety with $\mathcal{O}(X)=\mathcal{U}$, assume we are given a structure of D-scheme on X and let X_D be the subset of X consisting of all points $x\in X$ such that the maximal ideal of $\mathcal{O}_{X,x}$ is a Δ-ideal. Then any Δ-polynomial function on X is constant on X_D.

Proof. We shall denote by X^* the D-scheme X so $X=X^{*!}$. Let $\pi:X^{\infty!}\to X$ be the canonical projection. By adjunction the identity $1_X\in$ $\in \mathrm{Hom}_{\mathcal{U}\text{-sch}}(X^{*!},X)$ corresponds to a morphism $s\in \mathrm{Hom}_{D\text{-sch}}(X^*,X^{\infty})$ so $\pi\circ s^!=1_X$. For any $\varphi\in\mathcal{O}^{\Delta}(X)$ let $\tilde{\varphi}\in\mathcal{O}(X^{\infty})=\mathrm{Hom}_{\mathcal{U}\text{-sch}}(X^{\infty},\mathbb{A}^1)$ correspond to φ. Since $\mathcal{O}(X)=\mathcal{U}$ the composition $\tilde{\varphi}\circ s^!:X\to\mathbb{A}^1$ is constant i.e. factors through some $a\in\mathrm{Hom}_{\mathcal{U}\text{-sch}}(\mathrm{Spec}\,\mathcal{U},\mathbb{A}^1)$. Now let $\alpha\in\mathrm{Hom}_{\mathcal{U}\text{-sch}}(\mathrm{Spec}\,\mathcal{U},X)$ correspond to a point in X_D. Then α provides a morphism $\alpha^*\in\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{U},X^*)$ hence $s\circ\alpha^*:\mathrm{Spec}\,\mathcal{U}\to X^{\infty}$ is a morphism of D-schemes such that $\pi\circ(s\circ\alpha^*)^!=\alpha$. So $s\circ\alpha^*\in\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{U},X^{\infty})$ corresponds via adjunction to $\alpha\in\mathrm{Hom}_{\mathcal{U}\text{-sch}}(\mathrm{Spec}\,\mathcal{U},X)$. Consequently $\varphi(\alpha)=\tilde{\varphi}\circ s\circ\alpha^*=a$ and we are done.
The following corollary completes [B_2] Theorem 2:

(1.13) COROLLARY. Let X be a smooth $\mathcal{U}$-variety with $\mathcal{O}(X)=\mathcal{U}$ and assume X descends to $\mathcal{K}$. Then any Δ-polynomial function on X is constant on $X_{\mathcal{K}}$.

Proof. If $X\simeq X_0\otimes_{\mathcal{K}}\mathcal{U}$ (X_0 a $\mathcal{K}$-variety) then X has the natural "split" D-scheme structure obtained by lifting δ from $\mathcal{U}$ to $X_o \otimes_{\mathcal{K}} \mathcal{U}$; in this case $X_D = X_{\mathcal{K}}$ and we are done.

(1.14) Remarks. 1) Proposition (1.12) can be useful also in the case when X does not descent to $\mathcal{K}$. For instance if X is the universal extension of an abelian $\mathcal{U}$-variety A then $\mathcal{O}(X)=\mathcal{U}$ and δ lifts from $\mathcal{U}$ to X (cf. [B_3], Chapter 3 section 2; the latter is also a consequence of Grothendieck's theorem that X is "crystalline in nature"). On the other hand if A dose not descend to $\mathcal{K}$ the same will hold for X!.

2) Assume X is a smooth $\mathcal{U}$-variety which descends to $\mathcal{K}$ with $\mathcal{O}(X)=\mathcal{U}$ and assume that for any $x \in X$ there exists $\sigma \in$ Aut X such that $x \in \sigma(X_{\mathcal{K}})$ and $\sigma(X_{\mathcal{K}}) \cap X_{\mathcal{K}} \neq \emptyset$.

Then any Δ-polynomial function on X is constant (indeed any $\varphi \in \mathcal{O}^{\Delta}(X)$ is constant on any set of the form $\sigma(X_{\mathcal{K}})$ with $\sigma \in$ Aut X). This gives in particular one more proof of the that $\mathcal{O}^{\Delta}(\mathbb{P}^N)=\mathcal{U}$ (cf. [B_1] (6.3) or (1.4) of the present paper);

(1.16) The above remarks can be used to make a rough discussion of Δ-polynomial functions on algebraic surfaces. We expect a deeper discussion could be done if more results along the work of Sakai [Sak] could be obtained. Let X be a smooth projective surface over $\mathcal{U}$. By (1.8) we have $\mathcal{O}^{\Delta}(X) \simeq \mathcal{O}^{\Delta}(X_1)$ where X_1 is a minimal model. So we may assume X itself is a minimal model. Then according to the classification of surfaces X is in one of the following situations:

1) X is rational. Then $\mathcal{O}^{\Delta}(X)=\mathcal{U}$ cf.(1.4)

2) X is ruled irrational. If $X\to B$ is the rulling then $\mathcal{O}^{\Delta}(X)\simeq\mathcal{O}^{\Delta}(B)$ cf.(1.11)

3) X is a K3 surface. Then $\mathcal{O}^{(1)}(X)=\mathcal{U}$ cf.(1.4).

4) X is an Enriques surface. Then $\mathcal{O}^{(1)}(X)=\mathcal{U}$ cf.(1.4).

5) X is an abelian surface. Then $\mathcal{O}^{\Delta}(X)$ is a Hopf Δ-$\mathcal{U}$-algebra generated as an algebra by its space of primitives, which is a finitely generated D-module (cf. $[B_1]$ (6.1) for additional information).

6) X is a bielliptic surface, $X=(E_1 \times E_2)/G$ where E_i are elliptic curves, $G\subset E_1$ is a finite subgroup acting on E_1 by translations and on E_2 by algebraic group automorphisms. Then $\mathcal{O}^{\Delta}(X)=(\mathcal{O}^{\Delta}(E_1)\otimes\mathcal{O}^{\Delta}(E_2))^G$ cf.(1.10) and (1.6). Now $\mathcal{O}^{\Delta}(E)=\mathcal{U}\{y\}$ (= Δ-polynomial algebra in one variable) for any elliptic curve, cf. $[B_1]$ (1.6); if $\mathrm{rank}_{\Delta} E=0$, y corresponds to an element of order 1 while if $\mathrm{rank}_{\Delta} E=1$, y corresponds to an element of order 2. So $\mathcal{O}^{\Delta}(X)=\mathcal{U}\{y_1,y_2\}^G$ with y_1,y_2 Δ-indeterminates and y_1 is fixed by G.

Moreover there is a character $\chi: G\to\mathcal{U}$ such that $gy_2=\chi(g)y_2$; indeed G acts on $\mathcal{O}^{\Delta}(E_2)$ via its action on $Ch_{\Delta}(E_2)$ and this action is compatible with the filtration by orders so G invariates the subspace $\mathcal{U}y_2$ of $\mathcal{U}\{y_2\}$.

Consequently $\mathcal{O}^{\Delta}(X)=\mathcal{U}\{y_2\}^G\{y_1\}$ while $\mathcal{U}\{y_2\}^G$ can be computed in an obvious way.

7) X is an elliptic surface with Kodaira dimension 1.

8) X has Kodaira dimension 2.

In cases 7) and 8) very different things can happen. For instance in case 8) if X is a hypersurface in $\mathbb{P}^3$, $\mathcal{O}^{(1)}(X)=\mathcal{U}$ cf.(1.4) while if $X=C_1\times C_2$ where C_i are curves of genus $\geqslant 2$ then by (1.10) $\mathcal{O}^{\Delta}(X)=\mathcal{O}^{\Delta}(C_1)\otimes\mathcal{O}^{\Delta}(C_2)$. In the latter case if C_i do not descend to $\mathcal{K}$ then there exist $\varphi_1,\dots,\varphi_N\in\mathcal{O}^{(1)}(X)$ providing a Δ-closed embedding $X\to\mathcal{U}^N$ (because by (1.9) $X^1=C_1^1\times C_2^1$ hence X^1 is affine cf [B$_1$] (2.1)).

2. Non-degeneracy of characters

Let $u:X\to Y$ be a morphism of smooth $\mathcal{U}$-varieties. For any Δ-polynomial function φ on Y we have

$$\mathrm{ord}(\varphi\circ u)\leq\mathrm{ord}(\varphi)$$

where "ord(φ)" means "order of φ" cf. [B$_1$] (3.3). We shall say that φ is non-degenerate with respect to u if equality $\mathrm{ord}(\varphi\circ u)=\mathrm{ord}(\varphi)$ holds. The aim of this section is to prove the following:

(2.1) THEOREM. Let $u:X\to G$ be a morphism from a smooth $\mathcal{U}$-variety to a commutative algebraic $\mathcal{U}$-group such that $0\in u(X)$ and $u(X)$ generates G. Then all Δ-polynomial characters of G are non-degenerate with respect to u.

We need a preparation

(2.2) Let $u:X\to G$ be as in (2.1), dim X=r. Then the "Gauss map" $\gamma:X\to \mathrm{Grass}(r,L(G))$ sending each $x\in X$ to $(T_x(L_{u(x)^{-1}}\circ u))(T_xX)\subset T_oG\simeq L(G)$ (where $L_y:G\to G$ denotes the translation with y) is non-degenerate in the sense that there is no hyperplane in L(G) containing $\gamma(x)$ for all $x\in X$. Indeed there exists an integer N such that the induced map $u_N:X\times\ldots\times X\to G$,

$u_N(x_1,\dots,x_N)=\sum_{i=1}^{N} u(x_i)$ is dominant. Hence there exists a point $x=(x_1,\dots$ $\dots,x_N)$ such that the tangent map $T_x u_N : T_x(X\times\dots\times X)\longrightarrow T_{u_N(x)}G$ is surjective hence the map $\bigoplus_{i=1}^{N} T_{x_i}X \longrightarrow L(G)$ defined by $(t_1,\dots,t_N)\longmapsto \sum (T_{x_i}(L_{u(x_i)}-u))(t_i)$ is surjective and we are done.

(2.3) Let G be as in (2.1), let B its linear part i.e. its maximum connected subgroup and $A=G/B$. Then we claim that there is an exact sequence

$$0\longrightarrow X_a(G)\xrightarrow{\ res\ } X_a(B)\xrightarrow{\ a\ } H^1(\mathcal{O}_A)$$

where a is the "defining map" of the extension $0\to B\to G\to A\to 0$ according to Serre's description of Ext(A,B) (cf. $[B_1]$(2.3) for notitions). To check this, recall that we have $G\simeq B_1\times G_1$ where B_1 is an algebraic vector group and $\mathcal{O}(G_1)=$ $=\mathcal{U}$ (cf $[B_3]$, Chapter 3, section 2) so if B_2 is the linear part of G_1 we have induced isomorphisms $B\simeq B_1\times B_2$, $X_a(G)\simeq X_a(B_1)$, $X_a(B)\simeq X_a(B_1)\oplus X_a(B_2)$, the induced map $X_a(B_2)\to H^1(\mathcal{O}_A)$ is injective and $X_a(B_1)\to H^1(\mathcal{O}_A)$ is the zero map. This checks our claim.

(2.4) Let G be as in (2.1) and (G^n) its in finite prolongation sequence. Let $Ch_n(G)=Ch_\Delta(G)\cap\mathcal{O}^{(n)}(G)$. By $[B_1]$ section 3, $Ch_n(G)$ identifies with $X_a(G^n)$. We claim there is an exact sequence

$$0\to X_a(G^{n-1})\xrightarrow{\ \alpha_n\ } X_a(G^n)\xrightarrow{\ \rho_n\ } L(G)^o$$

where α_n is induced by the natural projection $\pi_n : G^n\to G^{n-1}$ while ρ_n is induced by restriction (we identify $L(G)$ with $\ker\pi_n$ cf $[B_1]$ (2.2)). Indeed if B^n is the linear part of G^n and $A=G^n/B^n$ then we have an exact sequence of algebraic groups $0\to L(G)\to B^n\to B^{n-1}\to 0$ hence an induced exact sequence $0\to X_a(B^{n-1})\xrightarrow{\beta_n} X_a(B^n)\to L(G)^o$. Now looking at the diagram with exact rows (cf. (2.3) above):

$$\begin{array}{ccccccc}
0 \longrightarrow X_a(G^{n-1}) & \xrightarrow{\text{res}} & X_a(B^{n-1}) & \xrightarrow{a_{n-1}} & H^1(\mathcal{O}_A) \\
\downarrow \alpha_n & & \downarrow \beta_n & & \Vert \\
0 \longrightarrow X_a(G^n) & \xrightarrow[\text{res}]{} & X_a(B^n) & \xrightarrow[a_n]{} & H^1(\mathcal{O}_A)
\end{array}$$

we get that coker α_n is contained in coker β_n and our claim is proved.

(2.3) Proof of (2.1). Let $\varphi \in Ch_n(G) \setminus Ch_{n-1}(G)$. We must prove that $\varphi \circ u \notin \mathcal{O}^{(n-1)}(X)$. By (2.4) φ corresponds to an element (still denoted by) $\varphi \in X_a(G^n)$ whose restriction to $L(G)$ is a non-zero functional on $L(G)$. Let G_o^{n-1} be an open Zariski neighbourhood of 0 in G^{n-1} on which the $L(G)$-torsor $G^n \longrightarrow G^{n-1}$ has a section $s: G_o^{n-1} \longrightarrow G^n$ and let G_o^n be the preimage of G_o^{n-1} in G^n; let moreover X_o^{n-1} be any open subset of $u^{-1}(G_o^{n-1})$ over which the $\mathbb{V}(T_{X^{n-1}/X^{n-2}})$-torsor $X^n \longrightarrow X^{n-1}$ has a section $t: X_o^{n-1} \longrightarrow X^n$ and let X_o^n be the preimage of X_o^{n-1} in X^n. Consider now the diagram

$$\begin{array}{ccccc}
X_o^n & \xrightarrow{u^n} & G_o^n \subset G^n & \xrightarrow{\varphi} & G_a \\
\tau \uparrow & & \uparrow \sigma & \nearrow \psi & \\
X_o^{n-1} \times_X TX & \xrightarrow[v^n]{} & G_o^{n-1} \times L(G) & &
\end{array}$$

where $\psi = \varphi \circ \sigma$, $\sigma(g_{n-1}, \theta) = s(g_{n-1}) + \theta$, $g_{n-1} \in G_o^{n-1}$, $\theta \in L(G)$ (the sum "+" is taken in the group G^n) and where $v^n := \sigma^{-1} \circ u^n \circ \tau$, $\tau(x_{n-1}, \theta) = t(x_{n-1}) + \theta$, $(x_{n-1}, \theta) \in X_o^{n-1} \times_X TX$ (the sum "+" here indicates the torsor operation). With notations above we have

$$(*) \quad \psi(g_{n-1}, \theta) = \varphi(s(g_{n-1}) + \theta) = \varphi(s(g_{n-1})) + \varphi(\theta)$$

By (1.1) u^n is equivariant with respect to the map

$$u^{n-1} \times Tu : X_o^{n-1} \times_X TX \to G_o^{n-1} \times_G TG = G_o^{n-1} \times L(G)$$

hence we have

$$v^n(x_{n-1},\theta) = (u^{n-1}(x_{n-1}), \bar{v}^n(x_{n-1}) + (T_x(L_{u(x)^{-1}} \circ u))(\theta)$$

where $x \in X$ is the image of $x_{n-1} \in X_o^{n-1}$ and $\bar{v}^n : X_o^{n-1} \longrightarrow L(G)$ is the composition

$$X_o^{n-1} \xrightarrow{\ t\ } X_o^n \xrightarrow{\ u^n\ } G_o^n \xrightarrow{\sigma^{-1}} G_o^{n-1} \times L(G)^{\bullet} \xrightarrow{\ proj\ } L(G)$$

Let $\pi : X_o^{n-1} \times_X TX \to X_o^{n-1}$ be the first projection. Then for $x_{n-1} \in X_o^{n-1}$ the image of $\pi^{-1}(x_{n-1})$ via v^n equals

$$\{u^{n-1}(x_{n-1})\} \times (\bar{v}^n(x_{n-1}) + \gamma(x))$$

where $\gamma(x) = (T_x(L_{u(x)^{-1}} \circ u))(T_x X) \subset L(G)$. Consequently by equality (*) above $\varphi(v_n(\pi^{-1}(x_{n-1})))$ equals

$$(**)\ \varphi(s(u^{n-1}(x_{n-1}))) + \varphi(\bar{v}^n(x_{n-1})) + \varphi(\gamma(x))$$

Let $\pi_n : X^n \longrightarrow X^{n-1}$ be the canonical projection. Then the image of $\pi_n^{-1}(x_{n-1})$ via $\varphi \circ u^n$ equals (**) too. Since $\varphi(L(G)) \neq 0$, by (2.2) we have $\gamma(x) \not\subset \operatorname{Ker} \varphi$ for x_{n-1} generic in X_o^{n-1}. So for such an x_{n-1} the set (**) does not reduce to a point; consequently $\varphi \circ u^n$ does not factor through π_n and our Theorem is proved.

(2.6) The most interesting case in our Theorem above is when X is a smooth projective curve and $u : X \to G$ is its canonical map into its Jacobian $G = J(X)$. It happens in this case that there are plenty of Δ-polynomial functions on G which are degenerate with respect to u. So the non-degeneracy property in (2.1) is specific to characters! For instance if X descends to $\mathbb{K}$

the map $u^1:X^1 \to G^1$ identifies with the tangent map $Tu:TX \to TG=G\times L(G)$ and as well known the image of the composition

$$TX \xrightarrow{Tu} G\times L(G) \xrightarrow{proj} L(G)$$

is the cone over the canonical image of X into $\mathbb{P}(H^0(\omega_X))$. So any hypersurface in $\mathbb{P}(H^0(\omega_X))$ containing this image will provide a non-zero form in Δ-polynomial characters of G which vanishes on X hence is degenerate with respect to $u:X \to G$.

(2.7) Let us see what Theorem (2.1) gives in case $G=G_a^N=\mathcal{U}^N$ and X a closed subvariety of $\mathcal{U}^N$ containing $0=(0,\dots,0)$ and not contained in any hyperplane passing through 0. If J is the ideal of X in $\mathcal{U}[y_1,\dots,y_N]$ then (2.1) says that the ideal $[J]\subset \mathcal{U}\{y_1,\dots,y_N\}$ does not contain Δ-polynomials of the form L+F with L a homogeneous non-zero linear Δ-polynomial of order n and F a Δ-polynomial of order $\leq n-1$. We expect that the latter statement can be proved "directly" using the usual theory Δ-ideals in the ring of Δ-polynomials (characteristic sets, a.s.o).

3. Strongly Δ-closed sets

(3.1) Let Σ be a Δ-closed subset of $\mathcal{U}^N$. We will say that Σ is strongly Δ-closed in $\mathcal{U}^N$ if, upon embedding $\mathcal{U}^N$ into $\mathbb{P}^N$ via $(x_1,\dots,x_N)\mapsto(1:x_1:\dots:x_N)$, Σ remains Δ-closed in $\mathbb{P}^N$. Note that $\mathcal{U}^N$ and $\mathcal{K}^N$ are not strongly Δ-closed in $\mathcal{U}^N$ (although they are of course Δ-closed). It is easy to check the following invariance property of the notion defined above. Let $\alpha:\mathcal{U}^N \to \mathcal{U}^M$ be an affine morphisms (i.e. the composition of a translation with a linear map); then a subset Σ of $\mathcal{U}^N$ is strongly Δ-closed in $\mathcal{U}^N$ if and only if $\alpha(\Sigma)$ is strongly Δ-closed in $\mathcal{U}^M$. The aim of this section is to prove the following:

(3.2) THEOREM. Assume X is a smooth projective curve over $\mathcal{U}$ of genus $g \geqslant 2$ which does not descend to $\mathcal{K}$ and let $\varphi_d : X \to \mathcal{U}^{N_d}$ be the Δ-pluricanonical map of degree d cf. $[B_1]$. Then for $d \gg 0$, $\varphi_d(X)$ is strongly Δ-closed in $\mathcal{U}^{N_d}$. Moreover if X is non-hyperellintic and $\mathrm{rank}_\Delta(X) = g$ then $\varphi_d(X)$ is strongly Δ-closed for $d \geqslant 3$.

Proof. We may assume the first component of φ_d is 1 and denote by $\overline{\varphi}_d : X \to \mathcal{U}^{N_d - 1}$ the composition of φ_d with the projection $\mathcal{U}^{N_d} \to \mathcal{U}^{N_d - 1}$ onto the last $N_d - 1$ components. By (3.1) it is sufficient to prove that $\overline{\varphi}_d(X)$ is strongly Δ-closed in $\mathcal{U}^{N_d - 1}$. With notations from $[B_2]$, section 2 we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{\ \varphi_{dD}\ } & \mathbb{P}^{N_d - 1} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(dD))) \\ \uparrow & & \uparrow \\ X^1 & \xrightarrow[\ \overline{\varphi}_d\ ]{} & \mathcal{U}^{N_d - 1} \end{array}$$

where $D = \mathbb{P}(\omega_X) \subset P(\mathcal{E})$, we still denoted by $\overline{\varphi}_d$ the morphism of schemes induced by the Δ-polynomial map $\overline{\varphi}_d : X \to \mathcal{U}^{N_d - 1}$, φ_{dD} is the natural rational map defined by the linear system $|dD|$ and $\mathcal{U}^{N_d - 1} \to \mathbb{P}^{N_d - 1}$ is defined by $(u_1, \dots, u_{N_d - 1}) \mapsto$ $\mapsto (1 : u_1 : \dots : u_{N_d - 1})$. By $[B_2]$ (2.1), φ_{dD} is a closed immersion for $d \gg 0$ so to check that $\overline{\varphi}_d(X)$ is strongly Δ-closed for $d \gg 0$ it is sufficient to check that the composed map $h : X \xrightarrow{\nabla_1} X^1 \to \mathbb{P}(\mathcal{E})$ has a Δ-closed image. But this map is a section of the projection $\pi : \mathbb{P}(\mathcal{E}) \to X$. Now h corresponds to a morphism of D-schemes $h^\infty : X^\infty \to \mathbb{P}(\mathcal{E})^\infty$ which is a section of $\pi^\infty : \mathbb{P}(\mathcal{E})^\infty \to X^\infty$. Since π^∞ is separated (indeed $\mathbb{P}(\mathcal{E})^{\infty !} \to \mathbb{P}(\mathcal{E})$ is affine hence separated, $\mathbb{P}(\mathcal{E}) \to$ $\to \mathrm{Spec}\, \mathcal{U}$ is separated hence $\mathbb{P}(\mathcal{E})^{\infty !} \to \mathrm{Spec}\, \mathcal{U}$ is separated hence so is π^∞) and since an $\mathcal{U}$-point of $\mathbb{P}(\mathcal{E})^{\infty !}$ belongs to the image of $h^\infty_{\mathcal{U}}$ if and only if it lies in the pull-back of the diagonal of $(\mathbb{P}(\mathcal{E})^\infty \times_{X^\infty} \mathbb{P}(\mathcal{E})^\infty)_{\mathcal{U}}$ via the morphism $\mathrm{id} \times h^\infty_{\mathcal{U}} \pi^\infty_{\mathcal{U}} : \mathbb{P}(\mathcal{E})^\infty_{\mathcal{U}} \to (\mathbb{P}(\mathcal{E})^\infty \times_{X^\infty} \mathbb{P}(\mathcal{E})^\infty)_{\mathcal{U}}$ one immediately gets that the image of h is Δ-closed.

Let's prove the assertion about non-hyperelliptic curves of Δ-rank g.

By $[B_2]$(2.5) for any such X, $|dD|$ separates points and tangent vectors on $\mathbb{P}(\mathcal{E})\setminus D$ for $d\geq 3$. If $p\in\mathbb{P}(\mathcal{E})\setminus D$, $q\in D$ then clearly $|dD|$ separates p and q.

Claim. If $d\geq 3$ then 1) for any $p,q\in D$, $p\neq q$, $|dD|$ separates p and q and 2) for any $p\in D$ and any $t\in T_pD$ there exists $E\in|dD|$ passing through p such that $t\notin T_pE$. To check this, note first that $\mathcal{O}_D(D)\simeq\omega_D$; indeed for $d\geq 1$ applying π_* to the exact sequence $0\to\mathcal{O}((d-1)D)\xrightarrow{s}\mathcal{O}(dD)\to\mathcal{O}_D(dD)\to 0$ we get (identifying D with X via π) an exact sequence

$$0\to S^{d-1}\mathcal{E}\xrightarrow{s}S^d\mathcal{E}\to\mathcal{O}_D(dD)\to R^1\pi_*\mathcal{O}((d-1)D)=0$$

hence $\mathcal{O}_D(D)=\mathrm{Coker}\,(s:\mathcal{O}\to\mathcal{E})$; but the latter is ω_D. Consequently $\mathcal{O}_D(dD)\simeq\omega_D^{\otimes d}$ is very ample for $d\geq 1$ and to check our Claim it is sufficient to check that for $d\geq 3$ all sections of $\mathcal{O}_D(dD)$ lift to sections of $\mathcal{O}(dD)$. But this follows from the fact that $H^1(\mathcal{O}((d-1)D))=H^1(S^{d-1}\mathcal{E})=0$, cf $[B_2]$(2.3), and our Claim follows.

To conclude the proof of (3.2) we must prove that if $d\geq 3$ then for any $p\in D$ and any $t\in T_p\mathbb{P}(\mathcal{E})\setminus T_pD$ there exists $E\in|dD|$ such that $t\notin T_pE$. It is sufficient to find $F\in|(d-1)D|$ with $p\notin\mathrm{Supp}\,F$ because then we may take $E=D+F$.

If $d\geq 4$ this is clear from the Claim above. So assume d=3. By $[B_2]$(2.3) we have an exact sequence

$$H^0(\mathcal{O}(2D)) \to H^0(\omega_D^{\otimes 2}) \xrightarrow{\partial} H^1(\mathcal{E}) = \mathcal{U}$$

and $\mathrm{Ker}\,\partial = \mathrm{Ker}(\lambda_X : H^0(\omega_X^{\otimes 2}) \to \mathcal{U})$ where $\lambda_X \in H^0(\omega_X^{\otimes 2})^0$ is the element corresponding to the Kodaira-Spencer class $\rho_X(\delta) \in H^1(\omega_X^{-1})$ $(\rho_X : \mathrm{Der}\,\mathcal{U} \to H^1(\omega_X^{-1}))$ via Serre duality. Now we can consider the non-complete linear system $|\mathrm{Ker}\,\lambda_X|$ contained in the complete linear system $|\omega_X^{\otimes 2}|$. Clearly an F as above can be found if we can prove that $|\mathrm{Ker}\,\lambda_X|$ is base point free. Let $V = H^0(\omega_X^{\otimes 2})$ and let $b : V \times V \to \mathcal{U}$ be the non-degenerate bilinear map induced by $V \otimes V \to H^0(\omega_X^{\otimes 2}) \xrightarrow{\lambda_X} \mathcal{U}$. For any subspace $W \subset V$ (respectively for any element $x \in V$) we let $W^{\perp} \subset V$ (respectively $v^{\perp} \in V$) be the orthogonal with respect to b. Now for any $p \in X$ let V_p be the space of all $v \in V$ which vanish at p and choose $v_1 \in V$ such that $v_1 \notin V_p^{\perp}$ and $v_1 \notin V_p^{\perp}$. Moreover choose $v_2 \in v_1^{\perp}$ such that $v_2 \notin V_p$. Then the section $v_1 \otimes v_2 \in H^0(\omega_X^{\otimes 2})$ does not vanish at p and belongs to $\mathrm{Ker}\,\lambda_X$. This proves the fact that $|\mathrm{Ker}\,\lambda_X|$ is base point free and closes the proof of our Theorem.

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