INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

JORDAN STRUCTURES WITH APPLICATIONS. I

by

R. IORDANESCU

PREPRINT SERIES IN MATHEMATICS
No. 28/1990

BUCURESTI

JORDAN STRUCTURES WITH APPLICATIONS. I

by

R. IORDANESCU*)

April 1990

*) *Institute of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania.*

P R E F A C E

These notes are an extended version of the author's survey "Jordan algebras with applications" (Preprint INCREST, Bucharest, No.79/1979, ISSN 0250-3638). Suggestions for further development of results related to Jordan structures are given in the form of open problems and comments.

The author is indebted to the late Professors Drs.H.Braun (Hamburg) and M.Koecher (Münster) for their willingness and for the interest they took in his work. Their important results as well as those due to their co-workers prompted the author to write these notes. They are the following:

I. Jordan structures.

II. Jordan algebras in projective geometry.

III. Jordan algebras in differential geometry.

IV. Jordan triple systems in differential geometry.

V. Jordan algebras in analysis.

VI. Jordan triple systems and Jordan pairs in analysis.

VII. Algebraic varieties (or manifolds) defined by Jordan pairs.

VIII. Jordan structures in classical and quantum mechanics.

IX. Jordan algebras in mathematical biology.

References to these notes are given as JSA.I,..., IX.

Many thanks are due to Prof.Dr.K.McCrimmon (Charlottesville) for his valuable suggestions, those concerning algebraic aspects in particular.

Thanks are also due to Prof.Dr.E.Størmer (Oslo) for useful comments on JSA.V.

The author is also grateful to Dr.C.Gheorghe (Bucharest-Măgurele) who suggested many improvements in JSA.VIII and to Dr.D.B.Ion (Bucharest-Măgurele) for useful discussion of the same topic.

Last but not least, the author takes this opportunity to acknowledge the technical assistance furnished by the Institute of Mathematics, Bucharest.

Radu IORDANESCU.

Bucharest, April, 1990

JORDAN STRUCTURES WITH APPLICATIONS. I.

Radu IORDANESCU

The early thirties witnessed the pubblication of Pascual JORDAN's papers from which Jordan algebras emerged. For fifty five years since the creation of Jordan algebras, various kinds of Jordan structures (algebras, triple systems, pairs) have been intensively studied, and a large number of important results have been obtained. At the same time an impressive variety of applications have been explored with several surprising connections. The study of Jordan structures and their applications is at present a wide-ranging field of mathematical research.

Broadly speeking, the aim of these notes is to present the most important applications of Jordan structures (algebras, triple systems, and pairs), with due emphasis on geometrical applications. Many open problems and comments are given.

The close connection between Jordan and Lie structures is not examined in detail, as it may find its due place in a proper treatment of Jordan structures. Let us mention in this respect McCrimmon's remark that "... if you open up a Lie algebra and look inside, 9 times out of 10 there is a Jordan algebra (or pair) which makes it work" (see McCRIMMON [71 e]) and Zel'manov's contention that "... Lie algebras with finite grading may be righteously included into the Jordan theory" (see ZEL'MANOV [110 h]).

We shall present in the four sections below the definitions and basic properties which are needed to understand the applications. For an <u>intrinsic</u> treatement of Jordan structures the reader is referred to the excellent books by BRAUN and

KOECHER [18], (see also KOECHER [59 b]), JACOBSON [47 a,d], LOOS [65 a,c], MEYBERG [77 e], ZHEVLAKOV, SLIN'KO, SHESTAKOV, SHIRSHOV [115]. For a comprehensive surveys on this topic, see [11, 59 h, 71 e]. In Section 5 some of the connections between Jordan and Lie structures that have become classical references together with some recent developments are briefly recalled. In Section 6 some of the most important results of Russian school at Novosibirsk are presented.

Included is a recent rather extensive bibliography giving additional references to work not covered by the text.

§ 1. Jordan algebras

Jordan algebras emerged in the early thirties with JORDAN's papers [48 a,b,c] on the algebraic formulation of quantum mechanics. The name "Jordan algebras" was given by Albert in 1946.

Definition. Let J be a vector space over a field $\mathbb{F}$ with characteristic different from two. Let $\varphi: J \times J \longrightarrow J$ be an $\mathbb{F}$-bilinear map, denoted by $\varphi: (x,y) \longrightarrow xy$, satisfying the following conditions:

$$xy = yx \text{ and } x^2(yx) = (x^2y)x \text{ for all } x,y \in J.$$

Then J together with the product defined by φ is called a Jordan algebra over $\mathbb{F}$.

Example. If A is an associative algebra over $\mathbb{F}$, and we define a new product $\varphi(x,y) := \frac{1}{2}(x \cdot y + y \cdot x)$, where the dot denotes the associative product of A, we obtain a Jordan algebra. It is denoted by $A^{(+)}$.

Remark 1. Jordan was the first who studied the properties of the product xy from the above example in the case when $\mathbb{F}$ is the field of reals. He proved a number of properties of this product, and showed that these were all consequences of the two identities above.

Remark 2. Recently, Zel'manov obtained interesting results for infinite dimensional Jordan algebras. A good account of Zel'manov's work as well as of McCrimmon's extension to quadratic Jordan algebras, can be found in [47 d] and [71 i] (see also [110 h]) and § 6.

Comments. In [59 g], KOECHER considered commutative nonassociative algebras over unital (commutative and associative) rings of scalars and defined the Jordanator $[a,b,c;d] := (ab)(cd) + (ac)(bd) + (bc)(ad) - a((bc)d) - b((ac)d) - c((ab)d)$ to measure how far these algebras are from being Jordan. Such an algebra is Jordan if and only if all Jordanators are zero.

Remark. A Jordan algebra J is power-associative, i.e. $x^m x^n = x^{m+n}$ for any $x \in J$ and all $m,n \in \mathbb{Z}$, $m,n \geqslant 1$ (the powers are defined inductively by $x^1 := x$, $x^{n+1} := xx^n$.)

Notation. On a Jordan algebra J consider the left multiplication L given by $L(x)y := xy$, $x,y \in J$.

Remark. In general $L(xy) \neq L(x)L(y)$ for x,y from a Jordan algebra (which is not associative). This holds for, e.g., $J = A^+$ for noncommutative A.

Definition. The map P defined by $P(x) := 2L^2(x) - L(x^2)$, $x \in J$, is called the quadratic representation of J. When $J = A^{(+)}$ it assumes the form $P(x)y = x \cdot y \cdot x$.

Proposition 1.1. For any $x,y \in J$ the following fundamental formula hold: $P(P(x)y) = P(x)P(y)P(x)$.

Remark 1. For P(x,y) given by $P(x,y) := 2(L(x)L(y) + L(y)L(x) - L(xy))$ we have $P(x+y) = P(x) + P(x,y) + P(y)$, $x,y \in J$.

Remark 2. In general, $P(xy) \neq P(x)P(y)$, $x,y \in J$ (as can easily be seen for $J = A^{(+)}$).

Proposition 1.2. Suppose that J has a unit element e and let x be an element of J. Then $P(x)$ is an automorphism of J if and only if $x^2 = e$. If $P(x)$ is an automorphism of J, then it is involutive.

Definition. An element $x \in J$ is called invertible if the map $P(x)$ is bijective. In this case the inverse of x is given by $x^{-1} := (P(x))^{-1}x$. (In case $J = A^{(+)}$, this is the usual inverse in the associative algebra A).

Remark 1. We have $(P(x))^{-1} = P(x^{-1})$, $x \in J$.

Remark 2. An element x is invertible with inverse y if and only if $xy = e$, $x^2 y = x$.

Definition. Let f be an element of J. Dafine a new product on the vector space J by

$$x \perp y := x(yf) + y(xf) - (xy)f.$$

The vector space J together with this product is called the mutation (homotope) of J with respect to f and is denoted by J_f. If $J = A^{(+)}$, then $J_f = (A_f)^{(+)}$ where A_f is the associative mutation ($x \underset{f}{\circ} y := x \cdot f \cdot y$).

Proposition 1.3. Any mutation J_f of J, $f \in J$, is a Jordan algebra and its quadratic representation P_f is given by $P_f(x) = P(x)P(f)$.

Proposition 1.4. The algebra J_f, $f \in J$, has a unit element if and only if f is invertible in J; in this case the unit element of J_f is f^{-1}. In this situation we call J_f the f-isotope of J.

Remark. If f is invertible in J, then the set of invertible elements of J coincides with the set of invertible elements of J_f.

Note. From this point on many of the results require that J be finite-dimensional.

Notation. Denote by

$$\text{Invol}(J) := \{w \mid w \in J,\ w^2 = e\},$$
$$\text{Idemp}(J) := \{c \mid c \in J,\ c^2 = c\},$$

the set of involutive, respectively idempotent (zero included), elements of J. Here J is supposed to contain a unit element e.

Remark. The map Idemp (J) $\rightarrow$ Invol (J) given by $c \rightarrow 2c-e$ is a bijection.

For an element c of Idemp (J) we have:

$$L(c)(L(c)-Id)(2L(x)-Id) = 0.$$

This leads to the Peirce decomposition of J with respect to the idempotent c :

$$J = J_0(c) \oplus J_{1/2}(c) \oplus J_1(c),$$

where

$$J_\alpha(c) := \{x \mid x \in J,\ cx = \alpha x\},\ \text{for } \alpha = 0,1/2,1.$$

Theorem 1.5. $J_0(c)$ and $J_1(c)$ are subalgebras of J, and we have

$$J_0(c)J_1(c) = \{0\},\ J_\nu(c)J_{1/2}(c) \subset J_{1/2}(c),\ \text{for } \nu = 0,1,$$

and

$$J_{1/2}(c)\ J_{1/2}(c) \subset J_0(c) \oplus J_1(c).$$

Definition. Let c be an idempotent of J, $c \neq e$. Then the map $P(2c-e)$, which by virtue of Proposition 1.1, is an automorphism of J, is called the Peirce reflection with respect to the idempotent c of J.

Notation. $\text{Idemp}_1(J) := \{c \mid c \in \text{Idemp}(J),\ \dim J_1(c) = 1\}$.

Definition. The dimension of $J_1(c)$ is called the rank of the idempotent c.

Definition. An idempotent c of J is called primitive if it cannot be decomposed as sum c_1+c_2 of two orthogonal (i.e. $c_1c_2 = 0$) idempotents c_1 and c_2, $c_i \neq 0$ $(i=1,2)$.

Remark. Every element of $\text{Idemp}_1(J)$ is primitive. The converse is not true in general.

Definition. A Jordan algebra over the reals is called formally real if, for any two of its elements x and y, $x^2+y^2 = 0$ implies that $x = y = 0$.

Proposition 1.6. A primitive idempotent of a formally real finite-dimensional Jordan algebra is of rank one.

Proposition 1.7. The set $\Gamma(J)$ of bijective linear maps W on J for which there exists a bijective linear map W^* on J such that $P(Wx) = WP(x)W^*$ for all $x \in J$ is a linear algebraic group.

Note. The notation W^* is justified by the fact that if J is real semi-simple and λ denotes the trace form on J (i.e. $\lambda(x,y) := \operatorname{Tr} L(xy)$, $x,y \in J$), then for $W \in \Gamma(J)$, W^* coincides with the adjoint of W with respect to λ .

Definition (KOECHER [59 a]). The (linear algebraic) group $\Gamma(J)$ from Proposition 1.7 is called the structure group of J.

Remark. The fundamental formula (see Proposition 1.1) implies that $P(x) \in \Gamma(J)$ whenever x is an invertible element of J. Also, every automorphism of J belongs to $\Gamma(J)$. Indeed, the automorphism group Aut(J) is just the set of elements $W \in \Gamma(J)$ fixing the unit element e of J, $We = e$.

Comments. It would be interesting to reconsider GHIKA's results [32 a,b] on so-called "field-like" algebras (i.e. algebras generated by their groups of invertible elements) in this Jordan algebra setting.

Formally real Jordan algebras have been classified (in the finite-dimensional case) by JORDAN, von NEUMANN and WIGNER [49]:

Theorem 1.8. Every formally real finite-dimensional Jordan algebra is a direct sum of the following algebras

(1.1) $$H_p(\mathbb{R})^{(+)},\ H_p(\mathbb{C})^{(+)},\ H_p(\mathbb{H})^{(+)},\ H_3(\mathbb{O})^{(+)},\ J(Q).$$

Here $H_p(\mathbb{F})^{(+)}$ denotes the algebra of Hermitian (pxp)-matrices with entries in $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$)[1], the multiplication in $H_p(\mathbb{F})^{(+)}$ is given by $xy := \frac{1}{2}(x\cdot y + y\cdot x)$ ($x\cdot y$ denotes the usual matrix product), and $J(Q) = \mathbb{F}\,1 \oplus X$ where Q is a real-valued quadratic form defined on X and the multiplication is given by

$$xy := Q(x,y)\,1,\quad 1\cdot x = x\cdot 1 = x,\quad 1\cdot 1 = 1 \text{ for } x,y \in X.$$

Remark. An associative Jordan algebra is formally real if and only if it is isomorphic to a product of copies of $\mathbb{R}$ with componentwise multiplication.

Definition. A Jordan algebra J is called special if it can be embedded in an associative algebra so that $xy = \frac{1}{2}(x\cdot y + y\cdot x)$, where the dot denotes the multiplication in the associative algebra (i.e. if J is isomorphic to a (Jordan) subalgebra of some $A^{(+)}$ for A associative).

Remark. The first three algebras in (1.1) clearly lie in associative matrix algebras, and the fifth algebra can be embedded in the Clifford algebra of Q. The fourth algebra $H_3(\mathbb{O})^{(+)}$ is not special (it cannot be embedded in any associative algebra); that is why it is called exceptional. Recently, an exciting work of Zel'manov has shown that this is essentially the only well-behaved exceptional algebra.

Proposition 1.9. A Jordan algebra is formally real if and only if its trace form is positive definite.

Notation. Suppose that J has unit element e. Then we set $x^0 := e$, $\exp x := \sum_{n \geq 0} \frac{x^n}{n!}$, and $\exp J := \{\exp x \mid x \in J\}$.

Proposition 1.10. If J is a formally real Jordan algebra, then it possesses a unit element and $\exp J = \{x^2 \mid x \text{ invertible in } J\}$.

---

1) Throughout these notes $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ denote the set of reals, complex numbers, quaternions, and octonions (Cayley numbers), respectively.

Theorem 1.11. Suppose that J is a formally real Jordan algebra endowed with the natural topology of $\mathbb{R}^n$. Then the identity component of the set of invertible elements of J coincides with exp J and the map $x \longrightarrow \exp x$ is bijective.

Recall that an algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called central simple if it is central and simple, i.e. it has a unit element e, its centre $\{c \mid c \in \mathcal{A}, (ca)b = c(ab), (ac)b = a(cb)$ for all $a,b \in \mathcal{A}\}$ coincides with $\mathbb{F}e$, and it has no proper ideals.

The determination of simple Jordan algebra can be reduced to that of central simple algebras, and for these ALBERT [1] proved the existence of a finite-dimensional field $\mathcal{F}$ of the underlying field (Jacobson extended this result to any field of characteristic $\neq 2$) such that the scalar extensions $J_{\mathcal{F}}$ are cointained in the following list of split algebras:

A. the algebra $M_n(\mathcal{F})^{(+)}$ of $(n \times n)$ - matrices over $\mathcal{F}$ relative to the Jordan multiplication $xy := \frac{1}{2}(x \cdot y + y \cdot x)$, where $x \cdot y$ denotes the usual matrix product;

B. the subalgebra of $M_n(\mathcal{F})^{(+)}$ of simmetric matrices;

C. the subalgebra of $M_n(\mathcal{F})^{(+)}$, $n = 2m$, of symplectic matrices, i.e. of matrices that are symmetric relative to the involution $x \longrightarrow q^{-1} x' q$, where x' denotes the transposed of x,

$$q = \begin{pmatrix} 0 & Id_m \\ -Id_m & 0 \end{pmatrix}$$, and Id_m is the (mxm)-identity matrix;

D. the algebra with basis $\{e_0, e_1, \ldots, e_n\}$ and multiplication table $e_0 e_i = e_i$, $e_i^2 = e_0$, $e_i e_j = 0$, $(i \neq j)$;

E. the algebra of Hermitian (3×3)-matrices with entries in an octonion algebra relative to the multiplication $xy := \frac{1}{2}(x \cdot y + y \cdot x)$.

Remark. KANTOR [54 b] has obtained a classification of the real simple Jordan algebras by means of transitive differential groups.

Comments 1. IORDANESCU and POPOVICI [46] constructed and studied all (matrix) representations of Jordan algebras of the types A-D, including the new class of quasi-irreducible representations, which were previously defined and studied in some particular cases by IORDANESCU [45 a,b]. It would be interesting to find the "quasi-irreducible" representations for other classes of Jordan structures.

Comments 2. Recently, JACOBSON [47 g] determined the orbit under the orthogonal group of Jordan algebras of real symmetric matrices. His results are an outgrowth of an affirmative answer that he gave to the following question of Malley: Can every Jordan algebra $\mathbb{R}1 \oplus X$ of a positive definite symmetric bilinear form be realized as a Jordan algebra of real matrices? Malley was interested in this question for application to statistics (namely, optimal unbased estimation of variance components). Melley also showed by a method different from JACOBSON's that the Jordan algebra $\mathbb{R}1 \oplus X$ can be embedded in a Jordan algebra of real symmetric matrices.

Albert, Jacobson and others extended the above mentioned classification to nonsplit algebras: Any finite-dimensional simple Jordan algebra over a field of characteristic $\neq 2$ is isomorphic to either

(A) $\mathfrak{A}^{(+)}$ for a simple associative algebra $\mathfrak{A}$;

(B-C) $H(\mathfrak{A}, *)$, the subalgebra of $\mathfrak{A}$ consisting of all $*$ - symmetric elements $x^* = x$, where $\mathfrak{A}$ is a simple associative algebra with involution $*$;

(D) $J(Q)$ for a nondegenerate quadratic form Q; orto

(E) an exceptional simple algebra of dimension 27 over its centre (which becomes $H_3(\mathbb{O})^{(+)}$ over a suitable extension field).

Remark. These Albert algebras can be constructed using a cubic norm form N as J(N) by the Freudenthal-Springer-Tits constructions.

Comments. For a surprising connection of Moore-Penrose inverse with Jordan algebras see KOECHER [59h].

2. Quadratic Jordan algebras

In 1966 McCRIMMON [71 a] extended the theory of Jordan algebras to the case of an arbitrary commutative unital underlying ring. So, instead of considering a Jordan algebra as a vector space with a nonassociative bilinear composition satisfying certain identities, McCrimmon considered it as a module over a ring together with a quadratic representation satisfying a number of conditions. In this way one gets unital quadratic Jordan algebras, which for the case that the underlying ring is a field of characteristic different from two (or any underlying ring containing 1/2) turn out to be the well-known Jordan algebras considered from a different point of view.

In 1971 McCRIMMON [71 b] defined the concept of (not necessarily unital) quadratic Jordan algebra J as a module over a (commutative unital) ring together with two quadratic maps, $U: J \longrightarrow \mathrm{End}(J)$ and $^2 : J \longrightarrow J$ (squaring), satisfying certain identities.

Let Φ be a commutative associative ring with a unit element and let X be a unital Φ-module.

Definition. A map $U : X \rightarrow \mathrm{End}(X)$, denoted by $U : x \rightarrow U_x$, such that $U_{\alpha x} = \alpha^2 U_x$ and $U_{x,y} := U_{x+y} - U_x - U_y$ is bilinear for all $\alpha \in \Phi$ and $x, y \in X$, is called a quadratic representation on X.

Notation. If U is a quadratic representation on X, then there exists a trilinear composition $\{xyz\} := U_{x,z}y$ which is symmetric in x and z. We define $V_{x,y}$ by $V_{x,y}z := \{xyz\}$.

Definition. A unital quadratic Jordan algebra $\mathcal{J} = (X,U,e)$ consists of a unital Φ-module X together with a quadratic representation U and a distinguished element e such that the following identities hold under arbitrary scalar extension :

$$U_e = Id,$$

$$U_{U_x y} = U_x U_y U_x,$$

$$U_x V_{y,x} = V_{x,y} U_x = U_{U_x y,x}.$$

Example. Let $\mathcal{A}$ be an associative unital algebra over Φ. Then, denoting by 1 the unit element of $\mathcal{A}$ and defining $U : \mathcal{A} \longrightarrow \mathrm{End}(\mathcal{A})$ by $U_a b := a.b.a$, $a,b \in \mathcal{A}$, it can be easily seen that $(\mathcal{A}, U, 1)$ - denoted by $\mathcal{A}^{(q)}$ for the sake of brevity - is a unital quadratic Jordan algebra.

Definition. A unital quadratic Jordan algebra $\mathcal{J}$ is called special if it is a subalgebra of $\mathcal{A}^{(q)}$ for some associative unital algebra $\mathcal{A}$, otherwise $\mathcal{J}$ is said to be exceptional.

Notations. Given a unital quadratic Jordan algebra $\mathcal{J} = (X,U,e)$ we can introduce a quadratic composition $x^2 := U_x e$, which permits us to define the powers inductively $x^0 := e$, $x^1 := x$, $x^{n+2} := U_x x^n$, and a symmetric bilinear composition $x \circ y := U_{x,y}e = \{xey\}$. We set $V_x := V_{x,e}$, hence $V_x y = y \circ x$.

Definition. An element c of $\mathcal{J} = (X,U,e)$ is called idempotent if $c^2 = c$. Two idempotents $c,d \in \mathcal{J}$ are called orthogonal if $U_d c = U_c d = c \circ d = 0$.

Remark. Let $\mathcal{J}$ be special. It immediately follows that $cd = dc = 0$, and thus c and d are orthogonal in the usual associative sense.

Theorem 2.1. (Peirce decomposition). Let $\mathcal{J} = (X,U,e)$ be a unital quadratic Jordan algebra and let c be one of its idempotents. Then

$$\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1 \oplus \mathcal{J}_{1/2}$$

where $\mathcal{J}_0 := (\mathrm{Id} + U_c - V_c)\mathcal{J} = U_{e-c}\mathcal{J}$, $\mathcal{J}_1 := U_c\mathcal{J}$, and $\mathcal{J}_{1/2} := = (V_0 - 2\,U_c)\mathcal{J} = U_{c,e-c}\mathcal{J}$. If $e = \sum c_i$ is a sum of pairwise orthogonal idempotents, then $\mathcal{J} = \oplus \mathcal{J}_{ij}$, where $\mathcal{J}_{ij} := U_{c_i,c_j}\mathcal{J}$, $i \neq j$, $\mathcal{J}_{ii} := U_{c_i}\mathcal{J}$.

Definition. An element x of $\mathcal{J} = (X,U,e)$ is called invertible if U_x is invertible. In this case the element $(U_x)^{-1}x$ is called the inverse of x and is denoted by x^{-1}.

Remark 1. One can show that $(U_x)^{-1} = U_{x^{-1}}$.

Remark 2. The element x is invertible with inverse y if and only if $U_x y = x$ and $U_x y^2 = e$.

Definition. Let a be an invertible element of $\mathcal{J} = (X,U,e)$. Then $(X^{(a)}, U^{(a)}, e^{(a)})$, where $X^{(a)} := X$, $U_x^{(a)} := U_x U_a$, and $e^{(a)} := a^{-1}$, can be shown to be a unital quadratic Jordan algebra. It is called the a-isotope of $\mathcal{J}$.

Definition. A subspace of a quadratic Jordan algebra which is closed under the map U is called an inner ideal of that algebra.

Remark. The role of inner ideals in the theory of quadratic Jordan algebras is analogous to that played by the one-sided ideals in the associative theory.

We now give the construction of two types of unital quadratic Jordan algebras, Φ and X being as above. (See, for instance, FAULKNER [28a]).

Let Q be a quadratic form on X and let $e \in X$ with $Q(e) = 1$. Set $T(y) := Q(y,e)$, $\bar{y} := T(y) - y$, and $U_x y := Q(x,\bar{y})x - Q(x)\bar{y}$, where $Q(x,y) := Q(x+y) - Q(x) - Q(y)$. Then (X,U,e) is a unital quadratic

Jordan algebra, called the (unital) quadratic Jordan algebra of the quadratic form Q with base point e (denoted by $\mathcal{J}(Q,e)$).

Suppose that Φ and X are as above. Recall that a map $Q : X \to X$ is called quadratic if $Q(\alpha x) = \alpha^2 Q(x)$, for $\alpha \in \Phi$, $x \in X$, and if $Q(x,y) := Q(x+y) - Q(x) - Q(y)$ is bilinear in $x,y \in X$. If $Q : X \to \Phi$ then Q is called a quadratic form and $Q(x,y)$ is called the associated bilinear form. A map $N : X \to \Phi$ together with a map $\partial N : X \times X \to \Phi$ such that $\partial N(x,y)$ is linear in x and quadratic in y

$$N(\alpha x) = \alpha^3 N(x)$$

$$\partial N(x,x) = 3\, N(x)$$

$$N(x+y) = N(x) + \partial N(x,y) + \partial N(y,x) + N(y)$$

$\partial N(x,y)$ is linear in x and quadratic in y,

for $\alpha \in \Phi$ and $x,y \in X$ is called a cubic form on X. Consider a cubic form N on X, a symmetric bilinear form T on X, a quadratic map # on X, and $e \in X$ such that

$$x^{\#\#} = N(x)x,$$

$$N(e) = 1,$$

$$T(x^{\#},y) = \partial N(y,x),$$

$$e^{\#} = e.$$

Define $e \times y := T(y)e - y$, where $T(y) := T(y,e)$ and

$$x \times y := (x+y)^{\#} - x^{\#} - y^{\#},$$

and suppose that the above relations hold unde arbitrary scalar extension of Φ. Set

$$U_x y := T(x,y)x - x^{\#} \times y.$$

Then (X,U,e) is a unital quadratic Jordan algebra; it is denoted by $\mathcal{J}(N, \#, e)$.

Definition. Let $\mathcal{J} = \mathcal{J}(N, \#, e)$ and $\mathcal{J}' = \mathcal{J}(N', \#', e')$ be defined over fields $\mathbb{F}$ and $\mathbb{F}'$, respectively. Assume that

$2 < \dim \mathcal{J} = \dim \mathcal{J}' < \infty$ and that T and T' are nondegenerate. Let s be an isomorphism of $\mathbb{F}$ onto $\mathbb{F}'$. Let W be a bijective s-semilinear map of $\mathcal{J}$ onto $\mathcal{J}'$ satisfying

$$(2.2) \qquad N'(Wx) = \rho\, s(N(x)) \text{ for all } x \in \mathcal{J},$$

where $\rho \neq 0$ is fixed in $\mathbb{F}'$, and satisfying (2.2) for all field extensions $\mathcal{F}$ of $\mathbb{F}$ and $\mathcal{F}'$ of $\mathbb{F}'$ such that s can be extended to $\mathcal{F}$. Then W is called an <u>s-semisimilarity of $\mathcal{J}$ onto $\mathcal{J}'$</u> and ρ is called the <u>multiplier</u> of W.

<u>Notation</u>. Let W be an s-semisimilarity of $\mathcal{J}$ onto $\mathcal{J}'$. Denote by W^* the s^{-1}-semilinear map of $\mathcal{J}'$ onto $\mathcal{J}$ defined by

$$s(T(W^*x', y)) = T'(x', Wy), \quad x' \in \mathcal{J}', \; y \in \mathcal{J}.$$

Write $\hat{W} := W^{*-1}$.

<u>Definition</u>. If W is an s-semisimilarity for s = Id, then W is called <u>similarity</u>.

<u>Definition</u>. If W is a similarity and $\rho = 1$, then W is called a <u>norm-preserving transformation</u>.

<u>Notation</u>. The group of all semisimilarities of $\mathcal{J}$ onto itself will be denoted by $\Gamma = \Gamma(\mathcal{J})$, the group of similarities of $\mathcal{J}$ onto itself will be denoted by $G = G(\mathcal{J})$, and the group of norm-preserving transformations of $\mathcal{J}$ onto itself will be denoted by $S = S(\mathcal{J})$.

<u>Remark</u>. $\Gamma(\mathcal{J})$ coincides with the structure group as defined by Koecher, consisting of those bijective W for which a W^* exists satisfying the relation $U_{Wx} = WU_x W^*$.

<u>Theorem 2.2</u>. Every semisimple quadratic Jordan algebra which is finite-dimensional over a field $\mathbb{F}$ (of arbitrary characteristic) is a direct sum of simple ideals, and every simple ideal is isomorphic to either

(A) $\mathfrak{A}^{(q)}$ for a simple associative algebra $\mathfrak{A}$;

(B-C) an ample outer ideal in $H(\mathfrak{A},*)$, i.e. consisting of the symmetric elements of a simple associative algebra $\mathfrak{A}$ under an involution * ;

(D) an ample outer ideal in the algebra $\mathcal{J}(Q,e)$ determined by a nondegenerate quadratic form Q with base point e;

or to

(E) $\mathcal{J}(N,\#, e)$ determined by a nondegenerate cubic form N.

Note. In Theorem 2.2, a subspace $K \subset \mathcal{J}$ is an outer ideal if it is invariant under multiplications, $U_{\mathcal{J}}K \subset K$, and is ample if it contains $\mathbb{1}$; when $1/2 \in \mathbb{F}$, the only ample outer ideal is $K = \mathcal{J}$.

Remark. The condition of finite dimensionality can be weakened to having the descending chain condition (d.c.c.) for inner ideals.

§ 3. Jordan triple systems

In the study of the Koecher-Tits construction of Lie algebras from Jordan algebras in a general setting (see KOECHER [59c], TITS [105 a,b] and § 5), MEYBERG [77 b,c] defined Jordan triple systems as modules with a trilinear composition $\{xyz\}$ satisfying the following identities:

$$\{xyz\} = \{zyx\} ,$$

$$\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\}.$$

Remark. As was noticed by KOECHER [59 h], a glimps of Jordan triple systems was given by GIBBS (1839-1903) as early as 1881 (Collected Works, vol.II, p.18) in a different setting.

A connection between Jordan triple systems and Jordan algebras was proved by MEYBERG [77 b] :

Theorem 3.1. If T is a Jordan triple system and a an element of T, then T together with the product $(x,y) \to \frac{1}{2}\{xay\}$ becomes a Jordan algebra, denoted by T_a. Conversely, a Jordan algebra induces a Jordan triple system in the same vector space by setting $\{xyz\} := P(x,z)y$.

Comments. It would be interesting to reconsider the results due to BRANZEI [17], and CREANGA and TAMAŞ [21 a,b,22] on ternary structures in this Jordan triple system setting.

In 1972, MEYBERG [77 e] defined quadratic Jordan triple systems over an arbitrary (commutative unital) ring ϕ of scalars by analogy with McCrimmon's concept of quadratic Jordan algebras based on the quadratic operator P(x) (see the definition below).

Definition. Let T be a ϕ-module with a quadratic map $P : T \to End(T)$ such that the following identities hold in all scalar extensions

$$L(x,y)P(x) = P(x)L(y,x),$$
$$L(P(x)y,y) = L(x,P(y)x),$$
$$P(P(x)y) = P(x)P(y)P(x),$$

where $L(x,y)z := P(x,z)y := P(x+z)y - P(x)y - P(z)y$. Then T is called a quadratic Jordan triple system over ϕ.

Remark. Every quadratic Jordan algebra can be considered as a quadratic Jordan triple system by setting $P(x) := U_x$.

There exists a hosts of results concerning the classification theory of Jordan triple systems. The classification of simple finite-dimensional Jordan triple systems over an algebraically closed field of characteristic different from two can be found in [65 a]. For the classification of various types of Jordan triple systems over $\mathbb{R}$, see NEHER [78 a,b,d]. ZEL'MANOV classified Jordan triple systems in general (char $\phi \neq 3$, but with no finite conditions) [110 m]. NEHER showed [78 e] (see also [78 i]) that for large subclasses of Jordan and Lie triple systems, there exists a bijection between the forms of the simple objects. This bijection was used to classify forms of certain exceptional Jordan triple systems. SCHWARZ [94 a] used the known classification of Jordan pairs to obtain a classification of Jordan triple systems by classifying involutions of Jordan pairs.

As is well-known, the coordinatization theorem of Jacobson for Jordan algebras asserts that a Jordan algebra with supplementary family of $(n \times n)$-Jordan matrix units for $n \geqslant 3$ is isomorphic to a Jordan matrix algebra $H_n(D, D_o)$ of Hermitian $(n \times n)$-matrices with coordinates in an alternative algebra D, with nuclear involution, which is associative if $n \geqslant 4$. This theorem not only is fundamental in classifying the simple Jordan algebras but also immediately describes their unital bimodules. McCRIMMON and MEYBERG [72] developed a similar coordinatization for Jordan triple systems: there are three distinct cases, the rectangular $(p \times q)$-matrices $M_{p,q}(D)$, the symplectic $(n \times n)$-matrices $S_n(D)$, and the $(n \times n)$-Hermitian matrices $H_n(D,D_o,j)$ over D. They showed that a Jordan triple system with $(p \times q)$-rectangular grid $(p+q \geqslant 3)$, $(n \times n)$-symplectic grid $(n \geqslant 4)$, or $(n \times n)$-Hermitian grid $(n \geqslant 3)$ is a rectangular, symplectic, or Hermitian matrix system whose coordinate algebra is associative if $p+q \geqslant 4$ in the rectangular case or $n \geqslant 4$ in the Hermitian case, and commutative associative in the symplectic case if $n \geqslant 4$. (Grids are special families on tripotents in Jordan triple systems). The key is the fact that any two collinear tripotents are coordinatized by an alternative algebra with involution. McCrimmon and Meyberg applied their coordinatization results to the classification of rigidly unital bimodules for matrix systems.

The theory of grids, including their classification and coordinatization of their cover, is presented by NEHER in [78 k].

Among the applications given by him, there are: 1°) classification of simple Jordan triple systems covered by a grid, reproving and extending most of the known classification theorems for Jordan algebras and Jordan pairs, and 2°) a Jordan-theoretic interpretation of the geometry of the 27 lines on a cubic surface.

Finally, we mention the result of NEHER [78 f] by which a central nondegenerate Jordan triple system over a field of characteristic different from two has a nonzero centre if and only if it is scalar isomorphic to a Jordan algebra. As an application, Neher classified Jordan triple forms of Jordan algebras.

§ 4. Jordan pairs

In 1969, MEYBERG introduced [77 b] the "verbundene Paare" (connected pairs), which correspond in Loos' terminology to the linear Jordan pairs (see the definition below). Such connected pairs first arose in KOECHER's work on Lie algebras. In 1974, LOOS introduced the notion of quadratic Jordan pair (see the definition below) and gave [65 b] the main results of a structure theory of quadratic Jordan pairs, which is analogous to Jacobson's and McCrimmon's structure theory of quadratic Jordan algebras with chain conditions. A detailed development of Loos' results [65 b] was given by LOOS [65 c].

Quadratic Jordan pairs are preferable to quadratic Jordan algebras or triple systems because they admit a natural way of defining inner automorphisms and, on the other hand, they always contain sufficiently many idempotents. Let us mention also that quadratic Jordan pairs arise naturally in the Koecher-Tits construction of Lie algebras and the associated algebraic groups.

As was shown by Loos, the quadratic Jordan pairs provide a unifying framework for both the theory of quadratic Jordan algebras and Jordan triple systems: the category of Jordan pairs with involution is equivalent to the category of Jordan triple systems, and the category of Jordan pairs with invertible elements is equivalent to the category of Jordan algebras up to isotopy.

KÜHN [62 a] showed how the theory of Jordan pairs can be derived from quasi-inversion as a basic algebraic operation (as SPRINGER [101] and McCRIMMON [71 d] derived Jordan algebras from inversion). We shall return to this on more detail in § 5.

McCRIMMON [71 e, p. 621] remarked that "a Jordan triple system is just a Jordan algebra with the unit thrown away" and "a Jordan pair is just a pair of spaces acting on each other like Jordan triple systems". On the other hand, "the study of graded Lie algebras leads naturally to Jordan pairs (including triple systems and algebras)". So, McCRIMMON [71 e, p. 622] concluded that "from several points of view, Jordan pairs are the most natural Jordan structures".

Definition. Let $\mathbb{K}$ be a unital commutative ring such that 2 is invertible in $\mathbb{K}$. Assume all $\mathbb{K}$-modules to be unital and to possess no 3-torsion (i.e. no nonzero elements x such that $3x = 0$). A pair $V = (V^+, V^-)$ of $\mathbb{K}$-modules endowed with two trilinear maps $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, written as $(x,y,z) \to \{xyz\}_\sigma$, $\sigma = \pm$, satisfying the identities

$$\{xyz\}_\sigma = \{zyx\}_\sigma ,$$

$$\{xy\,\{uvz\}_\sigma\}_\sigma - \{uv\,\{xyz\}_\sigma\}_\sigma = \{\{xyu\}_\sigma\, vz\}_\sigma - \{u\{yxv\}_{-\sigma}\, z\}_\sigma$$

for $\sigma = \pm$, is called a linear Jordan pair over $\mathbb{K}$.

Remark. Jordan algebras can be regarded as a generalization of symmetric matrices, while the linear Jordan triple structires (systems or pairs) can be regarded as a generalization of rectangular matrices.

We shall give now examples and essential properties of linear Jordan pairs needed in the theory of Jordan manifolds presented in §§ 3,4 from JSA VII. Hence, WATSON [108,I] was used.

Convention. For the sake of simplicity we shall omit the word "linear".

Notations. Define, for $\sigma = \pm$, the bilinear map $D_\sigma : V^\sigma \times V^{-\sigma} \to \mathrm{End}\,(V^\sigma)$ by

$$D_\sigma(x,y)z := \{xyz\}_\sigma .$$

The map $D_\sigma(x,y)$ is called the differential determined by the pair $(x,y) \in V^\sigma \times V^{-\sigma}$. Define, for $\sigma = \pm$, the quadratic map $Q_\sigma : V^\sigma \to \mathrm{Hom}_{\mathbb{K}\text{-mod}}\,(V^{-\sigma}, V^\sigma)$ by

$$Q_\sigma(x)y := \frac{1}{2}\{xyx\}_\sigma ,$$

so that the associated bilinear map is given by $Q_\sigma(x,z)y = \{xyz\}_\sigma$. The map $Q_\sigma(x)$ is called quadratic representation of x.

Definition. Let $V = (V^+, V^-)$ and $W = (W^+, W^-)$ be two Jordan pairs over $\mathbb{K}$. A homomorphism $h : V \to W$ is a pair $h = (h_+, h_-)$ of $\mathbb{K}$-linear maps $h_\sigma : V^\sigma \to W^\sigma$ such that

$$h_\sigma(\{xyz\}_\sigma) = \{h_\sigma(x), h_{-\sigma}(y), h_\sigma(z)\}_\sigma ,$$

for all $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$.

Notation. The set of homomorphisms $h : V \to W$ is denoted by $\Gamma(V,W)$, and the set of σ-components $\{h_\sigma | h \in \Gamma(V,W)\}$ is denoted by $\Gamma_\sigma(V,W)$.

Remark. Isomorphisms and automorphisms are defined in the abvious way.

Definition. A pair of $\mathbb{K}$-modules $U = (U^+, U^-)$ is called a subpair of a Jordan pair V if U^σ is a submodule of V^σ and $\{U^\sigma U^{-\sigma} U^\sigma\}_\sigma \subset U^\sigma$ for $\sigma = \pm$.

Definition. A subpair U of a Jordan pair V is called an ideal of V if $\{V^{\sigma} U^{-\sigma} V^{\sigma}\}_{\sigma} \subset U^{\sigma}$ and $\{V^{\sigma} V^{-\sigma} U^{\sigma}\}_{\sigma} \subset U^{\sigma}$, $\sigma = \pm$.

Definition. A Jordan pair V is called simple if it has only the trivial ideals V and 0 and if $\{V^{\sigma} V^{-\sigma} V^{\sigma}\} \neq 0$.

Remark. The direct sum of two Jordan pairs V and W over $\mathbb{K}$ is defined as $V \oplus W := (V^+ \oplus W^+, V^- \oplus W^-)$, with componentwise operations.

Definition. If V is a Jordan pair over $\mathbb{K}$, then the opposite V^{op} of V is the Jordan pair (V^-, V^+) with the trilinear maps $\{\ ,\ ,\ \}_-$ and $\{\ ,\ ,\ \}_+$.

Remark. The symmetry between V^+ and V^- is one of the essential features of the theory of Jordan pairs.

Convention. The indices σ and τ will always assume the value + and - . They will be omitted whenever no ambiguities are possible. $\mathbb{F}$ will denote a field of characteristic different from two or three, and all Jordan pairs will be assumed to be finite-dimensional.

Definition. For $(x,y) \in V^{\sigma} \times V^{-\sigma}$, define $B_{\sigma}(x,y) \in End(V^{\sigma})$ by $B_{\sigma}(x,y) := Id_{V^{\sigma}} - D_{\sigma}(x,y) + Q_{\sigma}(x)\ Q_{-\sigma}(y)$. The map $B(x,y)$ is called the Bergmann trnsformation determined by the pair $(x,y) \in V^{\sigma} \times V^{-\sigma}$.

Let $V = (V^+, V^-)$ be a Jordan pair over $\mathbb{F}$ and let y be an element of V^-. Define the bilinear product $(a,b) \to \frac{1}{2}\{ayb\}$ on the $\mathbb{F}$-module V^+. With this product, V^+ become a Jordan algebra, denoted by V^+_y. The quadratic operator in V^+_y is $U_x = Q_+(x)\ Q_-(y)$.

Definition. A pair $(x,y) \in V = (V^+, V^-)$ is called quasi-invertible if x is quasi-invertible in V^+_y (i.e., 1-x is invertible in the unital Jordan algebra $\mathbb{F}\,1 \oplus V^+_y$). In this case there exists a unique $z \in V^+$ such that $(1-x)^{-1} = 1 + z$; the element z is called the quasi-inverse of (x,y) and is denoted by x^y.

Remark. A pair $(x,y) \in V$ is quasi-invertible if and only if $B(x,y)$ is invertible; in this case $x^y = B^{-1}(x,y)\ (x - Q(x)y)$.

Examples of Jordan pairs. a) Let $M_{p,q}(\mathbb{F})$ be the vector space of $(p \times q)$-matrices over the field $\mathbb{F}$. Then $\underline{M}_{p,q}(\mathbb{F}) := (M_{p,q}(\mathbb{F}), M_{p,q}(\mathbb{F}))$ is a Jordan pair over $\mathbb{F}$ with $Q(x)\ y := xy'x$, where y' denotes the transpose of y. The quasi-inverse of (x,y) in $\underline{M}_{p,q}(\mathbb{F})$ is given by

$$x^y = x(e_q - y'x)^{-1} = (e_p - xy')^{-1}x,$$

where e_q and e_p are the identity $(q \times q)$- and $(p \times p)$- matrices, respectively.

The next three examples are subpairs of $\underline{M}_n(\mathbb{F}) := \underline{M}_{n,n}(\mathbb{F})$.

a_1) $\underline{A}_n(\mathbb{F}) := (A_n(\mathbb{F}), A_n(\mathbb{F}))$, where $A_n(\mathbb{F})$ is the space of skewsymmetric $(n \times n)$-matrices over $\mathbb{F}$;

a_2) $\underline{S}_n(\mathbb{F}) := (S_n(\mathbb{F}), S_n(\mathbb{F}))$, where $S_n(\mathbb{F})$ is the space of symmetric $(n \times n)$-matrices over $\mathbb{F}$;

a_3) (U_n, L_n) where U_n (L_n) is the space of upper (lower) triangular $(n \times n)$-matrices over $\mathbb{F}$.

b) Let X be an n-dimensional vector space over $\mathbb{F}$, equipped with a symmetric bilnear form $\langle x,y \rangle$. Then $\underline{X} := (X,X)$ is a Jordan pair over $\mathbb{F}$ with $Q(x)y := 2\langle x,y \rangle x - \langle x,x \rangle y$.

The quasi-inverse of (x,y) in $\underline{X}$ is given by

$$x^y = (x - \langle x,x \rangle y)(1 - 2\langle x,y \rangle + \langle x,x \rangle \langle y,y \rangle)^{-1}$$

c) Consider now $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\mathbb{R}_{p,q}(\mathbb{F})$ denote the set of $(p \times p)$-matrices over $\mathbb{F}$, considered as a real vector space. Then $\underline{\mathbb{R}}_{p,q}(\mathbb{F}) := (\mathbb{R}_{p,q}(\mathbb{F}), \mathbb{R}_{p,q}(\mathbb{F}))$ is a Jordan pair over $\mathbb{R}$ with $Q(x)y := xy'x$. The quasi-inverse of (x,y) is as in a).

Set $\underline{\mathbb{R}}_n(\mathbb{F}) := \underline{\mathbb{R}}_{n,n}(\mathbb{F})$, $\mathbb{R}_n(\mathbb{F}) := \mathbb{R}_{n,n}(\mathbb{F})$, and let x^* denote the conjugate transpose of $x \in \mathbb{R}_n(\mathbb{F})$. For $\mathbb{F} = \mathbb{R}$, $x^* = x'$. The next two examples are subpair of $\underline{\mathbb{R}}_n(\mathbb{F})$.

c_1) $\underline{SH}_n(\mathbb{F}) := (SH_n(\mathbb{F}), SH_n(\mathbb{F}))$, where $SH_n(\mathbb{F})$ is the space of skew-Hermitian matrices ($x^* = -x$) in $\mathbb{R}_n(\mathbb{F})$.

c_2) $\underline{H}_n(\mathbb{F}) := (H_n(\mathbb{F}), H_n(\mathbb{F}))$, where $H_n(\mathbb{F})$ is the space of Hermitian matrices ($x^* = x$) in $\mathbb{R}_n(\mathbb{F})$.

d) Let J be a Jordan algebra over $\mathbb{F}$ and let $S : J \to End(X)$ be an associative specialization of J into the endomorphisms of the $\mathbb{F}$-vector space X. That is, we have a linear map S with the property $S(ab) = \frac{1}{2}(S(a)\,S(b) + S(b)\,S(a))$ for all $a,b \in J$. Hence, $J \oplus X$ becomes a Jordan algebra with $(a+x)\,(b+y) := ab + \frac{1}{2}(S(a)y + S(b)x)$ and $P(a)\,x = 0$, for all $a,b \in J$ and $x,y \in X$. Following the Remark on page 21, we have a Jordan pair $(J \oplus X, J \oplus X)$ with $(J \oplus X, J)$ as subpair.

Comments. As was noticed by TILGNER [1o4 e] the formula for the quasi-inverse x^y in the above example b) appears also in electrodynamics as the formula for special conformal transformations, $\langle\, ,\rangle$ being replaced by $-\langle\, ,\rangle$ (see, for instance, TILGNER [1o4 d]). Given the large number of results already obtained in the theory of Jordan pairs, TILGNER's remark could be fruitfully used in modelling some facts in electrodanamics.

Convention. If X and Y are finite-dimensional vector spaces over an infinite field $\mathbb{F}$, then the vector space of rational maps $X \to Y$ is denoted by $Rat(X,Y)$ (see LOOS [65c, 18.7]).

Remark 1. Every $f \in Rat(X,Y)$ has a reduced expression $f = gh^{-1}$, where g is a polynomial map $X \to Y$, h is a polynomial function $X \to \mathbb{F}$, and the components of g (with respect to a basis of Y) and h have no nonconstant common divisor.

Remark 2. Since $\mathbb{F}$ is infinite, one can consider f to be a partial map $X \to Y$ from its domain of definition Dom f = $= \{x \in X \mid h(x) \neq 0\}$ into Y. One says that f is defined at x if $x \in$ Dom f.

Definition. If X,Y,Z are finite-dimensional vector spaces over $\mathbb{F}$ and $f_1 \in \mathrm{Rat}(X,Y)$, $f_2 \in \mathrm{Rat}(Y,Z)$ with reduced expressions $f_i = g_i h_i^{-1}$, then the rational maps f_1 and f_2 are called composable if there exists an $x \in \mathrm{Dom}\, f_1$ such that $h_2(f_1(x)) \neq 0$.

Notation. The composite of f_1 and f_2 is an element of Rat (X,Z) and is denoted by $f_2 \circ f_1$.

Let V be a Jordan pair over $\mathbb{F}$. For $y \in V^{\sigma}$, define $T^{\sigma}(y) := (T_+^{\sigma}(y), T_-^{\sigma}(y))$ by

$$T_{\sigma}^{\sigma}(y)z := z+y, \quad T_{-\sigma}^{\sigma}(y)x := x^{-y}, \quad \text{for } z \in V^{\sigma},\ x \in V^{-\sigma}.$$

Then $T^{\sigma}(y) \in \mathrm{Rat}\,(V,V)$ and $T_{\tau}^{\sigma}(y) \in \mathrm{Rat}\,(V^{\tau}, V^{\tau})$. Note that $T_{-\sigma}^{\sigma}(y)0 = 0$ for all $y \in V^{\sigma}$.

Definition. The transformations $T^{\sigma}(y)$, $y \in V^{\sigma}$, $\sigma = \pm$, and the automorphisms of V generate a group, $\sum(V)$, of birational transformations of V, called the group of linear fractional transformations of V.

Notation. $\sum_{\sigma}(V)$ denotes the group of σ-components f_{σ} where $f = (f_+, f_-)$ is an element of $\sum(V)$.

Definition. Let V and W be Jordan pairs over $\mathbb{F}$. Then the set $\sum(V,W)$ of linear functional maps from V to W is the set of $f = g \circ h \in \mathrm{Rat}\,(V,W)$, where $g \in \sum(W)$ and $h \in \Gamma(V,W)$ are composable.

Definition. The set $LF_{\sigma}(V,W)$ of linear fractional maps from V^{σ} to W^{σ} is the set of $f = g \circ h \in \mathrm{Rat}(V^{\sigma}, W^{\sigma})$, where $g \in \sum_{\sigma}(W)$ and $h \in \Gamma_{\sigma}(V,W)$ are composable.

Remark 1. The Jordan pairs over $\mathbb{F}$ form a quasi-category $\sum$, with $\mathrm{Mor}(V,W) = \sum(V,W)$ and with $\circ$ for composition of morphisms.

Remark 2. The σ-spaces V^{σ} of Jordan pairs V over $\mathbb{F}$ also form a quasi-category LF_{σ}, with $\mathrm{Mor}(V^{\sigma}, W^{\sigma}) = LF_{\sigma}(V,W)$. In LF_{σ}, the objects are the vector spaces V^σ equipped with the structure of a Jordan pair V, and as such they should be denoted by (V^σ, V); whenever confusion is impossible, we shall simply write V^σ, however.

Notation. If V is a Jordan pair, then we denote by G(V) the group of automorphisms in L F_+ of V^+.

Definition. Let V be a Jordan pair over $\mathbb{F}$ and put
Rad $V^\sigma := \{ x \in V^\sigma \mid (x,y)$ is quasi-invertible for all $y \in V^{-\sigma} \}$.
Then Rad V := (Rad V^+, Rad V^-) is called the radical of V.

Remark. Rad V is an ideal of V.

Definition. A Jordan pair $V \neq 0$ is called semi-simple if Rad V = 0.

Remark. A Jordan pair is semi-simple if and only if it is a direct sum of simple Jordan pairs.

Definition. A Jordan pair V is called radical if V= Rad V.

Definition. Let $V = (V^+, V^-)$ be a Jordan pair. Then an element $u \in V^\sigma$ is called invertible if $Q_\sigma(u) : V^{-\sigma} \to V^\sigma$ is invertible, and, in this case, the inverse of u is the element $u^{-1} \in V^{-\sigma}$ defined by

$$i_\sigma(u) := u^{-1} := Q_\sigma(u)^{-1}\, u .$$

Remark 1. Note that $i_\sigma \in \mathrm{Rat}(V^\sigma, V^{-\sigma})$.

Remark 2. For invertible $u \in V^\sigma$ we have $Q_\sigma(u)^{-1} = Q_{-\sigma}(u^{-1})$ and $(u^{-1})^{-1} = u$; hence $i_{-\sigma} \circ i_\sigma = \mathrm{Id}_{V^\sigma}$.

Definition. A Jordan pair V is called unital if V^- (or V^+) contains an invertible element.

Recently, POPUTA was stimulated by [45c] to define [88 ,a] a special kind of Jordan pairs called Jordan duals. These objects are Jordan pairs (L^+, L^-) of a linear space and its dual over a field of characteristic different from two or three, the operations $\{\ \ \}_\sigma$, $\sigma = \pm$, being defined as follows

$$\{xyz\}_+ := y(x)z + y(z)x, \text{ and } \{xyz\}_- := x(y)z + z(y)x,$$

where $L^+ := L$, $L^- := L^*$.

Then he studied [88 ,b] the derivations of Jordan duals and defined [88 ,c] Lie algebras associated with Jordan duals.

Definition. A quadratic Jordan pair over a unital commutative ring $\mathbb{K}$ is a pair $V = (V^+, V^-)$ of $\mathbb{K}$-modules endowed with two quadratic maps $Q_\sigma : V^\sigma \longrightarrow \mathrm{Hom}(V^{-\sigma}, V^\sigma)$, $\sigma = \pm$, satisfying the identities

$$L_\sigma(x,y)Q_\sigma(x) = Q_\sigma(x)L_{-\sigma}(y,x),$$
$$L_\sigma(Q_\sigma(x)y,y) = L_\sigma(x,Q_{-\sigma}(y)x),$$
$$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x),$$

in all scalar extensions, where $L_\sigma : V^\sigma \times V^{-\sigma} \longrightarrow \mathrm{End}(V^\sigma)$ are the bilinear maps defined by $L_\sigma(x,y)z := Q_\sigma(x,z)y$, where $Q_\sigma(x,z) := Q_\sigma(x+z) - Q_\sigma(x) - Q_\sigma(z)$.

A pair $(x,y) \in V$ is called an idempotent if $x = Q_+(x)y$ and $y = Q_-(y)x$.

By analogy with the theory of quadratic Jordan algebras (see § 2) one can define the Peirce decomposition of a Jordan pair with respect to an idempotent (x,y). Then, we have

$$V^\sigma = V_0^\sigma \oplus V_1^\sigma \oplus V_{1/2}^\sigma,$$

where $V_0^\sigma := (\mathrm{Id} - L_\sigma(x,y) + Q_\sigma(x)Q_{-\sigma}(y))V^\sigma$, $V_1^\sigma := Q_\sigma(x)Q_{-\sigma}(y)V^\sigma$, and $V_{1/2}^\sigma := (L_\sigma(x,y) - 2Q_\sigma(x)Q_{-\sigma}(y))V^\sigma$.

Let us note here LOOS'basic result [65 c] concerning the classification of quadratic Jordan pairs: a semi-simple Jordan pair with d.c.c. on inner ideals is a direct sum of simple pairs which are either

(O) Jordan division pairs;

(I) sets of rectangular matrices $(M_{n,m}(D), M_{n,m}(D))$ for an associative division algebra D;

(II) sets of alternating matrices $(A_n(k), A_n(k))$ for an extension field k;

(III) sets of Hermitian matrices $(H_n(D,D_o), H_n(D,D_o))$;

(IV) sets of ample outer ideals in the Jordan algebra $\mathcal{J}(Q,e)$ of a nondegenerate quadratic form Q;

(V) sets of (1×2)-Cayley matrices $(M_{1,2}(k), M_{1,2}(k))$;

or

(VI) sets of Hermitian 3×3 Cayley matrices $(H_3(K), H_3(K))$.

The last two pairs, of dimension 16 and 27 over their centres, are the only pairs which are exceptional in the sense that they cannot be embedded in associative systems.

Concerning Jordan superalgebras, let us mention that KATS [56], generalized KANTOR's method [54 a,b] and applied it to the classification of the simple Jordan superalgebras (i.e., $\mathbb{Z}_2$-graded algebras $J = J_{\bar 0} \oplus J_{\bar 1}$ with an operation $\circ$ which satisfies $a \circ b = (-1)^{\alpha\beta}\, b \circ a$ and $(-1)^{\alpha\gamma}[L(a\circ b),L(c)] + (-1)^{\beta\alpha}[L(b\circ c),L(a)] + (-1)^{\gamma\beta}[L(c\circ a),L(b)] = 0$ for $a\in J_\alpha$, $b\in J_\beta$, $c\in J_\gamma$ and where $[\,,]$ are the brackets in the Lie superalgebra End J [1]). KÜHN [62 b] gave a partial classification of central simple Jordan superalgebras. BARS and GÜNAYDIN [13] introduced superternary algebras, involving Bose and Fermi variables. Following the classification of Jordan superalgebras given by KATS [56], Bars and Günaydin constructed Jordan superternary algebras corresponding to certain Jordan superalgebras, and speculated on their possible physical

1) By setting $\mathrm{End}_\alpha J := \{a \mid a\in \mathrm{End}\, J,\ a(J_s)\subset J_{s+\alpha}\}$ we obtain an associative superalgebra $\mathrm{End}\, J = \mathrm{End}\, J_{\bar 0} \oplus \mathrm{End}\, J_{\bar 1}$; the brackets $[a,b] := ab - (-1)^{\alpha\beta}\, ba$ make End J into a Lie superal-

applications (see [13, pp.1990-1991]). In [36], GÜNAYDIN used the ternary algebraic technique to define the derivation, structure and Tits-Koecher algebras of Jordan superalgebras. Explicit forms of these algebras were listed for the simple Jordan superalgebras and then a suggestion made to extend Koecher's theory of linnear fractional groups defined by Jordan algebras to the case of Jordan superalgebras.

Comments. Some identities which are now used in the study of Lie and Jordan superalgebras were given by ION as early as 1965 in a different setting. He established [44 a-d] essential properties of the structure constants of Jordan algebras of the types A-E and obtained, for their roots and weights, properties analogous to those of roots and weights of semisimple Lie algebras. Such results may be useful in quantum physics.

Finally, let us comment on the work of NEHER [78 c] about involutive gradings of Jordan structures.

Involutive automorphisms of Jordan pairs (i.e. automorphisms of order two) have different properties in characteristic different from two and in characteristic two. This mainly comes from the fact that in characteristic different from two the pair can be decomposed into the eigenspaces of the eigenvalues 1 and -1 which is impossible in characteristic two. In the latter case there are on the other hand many examples of decompositions which behave like the eigenspace decompositions in characteristic different from two. This suggests the following procedure: throw away the involutive automorphism, keep only their eigenspace decomposition and study these decompositions, called involutive gradings (see definition below), for Jordan pairs aver an arbitrary ring of scalars. This includes a study of involutive automorphisms for characteristic different from two.

Definition. An involutive grading of a Jordan pair $V = (V^+, V^-)$ is a decomposition $V = V_+ \oplus V_-$ into subpairs $V_+ := (V_+^+, V_+^-)$ and $V_- := (V_-^+, V_-^-)$ such that

$$Q(V_\varepsilon^\sigma)V_\mu^{-\sigma} \subset V_\mu^\sigma \quad \text{and} \quad \{V_\varepsilon^\sigma\, V_\varepsilon^{-\sigma} V_\mu^\sigma\} \subset V_\mu^\sigma$$

for $\sigma, \varepsilon, \mu = \pm$.

Important examples of involutive gradings of a Jordan pair V can be constructed by means of idempotents. If $V = V_0(c) \oplus V_1(c) \oplus V_2(c)$ is a Peirce decomposition relative to an idempotent c of V, then $V = V_+ \oplus V_-$ with $V_+ := V_0(c) \oplus V_2(c)$ and $V_- := V_1(c)$ is an involutive grading of V. In fact, every abstract decomposition $V = V_0 \oplus V_1 \oplus V_2$ which satisfies the same multiplication rules as a Peirce decomposition induces an involutive grading of V. These decompositions are called Peirce gradings. In general not every Peirce grading comes from a Peirce decomposition relative to an idempotent. However NEHER [78 c] showed that this is the case for a special class of Jordan pairs including the Jordan pairs of symmetric matrices over a field.

Another result due to NEHER [78 c] is the following structure theorem for involutive gradings: if V is a simple and semisimple Jordan pair which has d.c.c. and a.c.c. on principal inner ideals, then there are two possibilities for an involutive grading $V = V_+ \oplus V_-$ with $V_+ \neq 0$, either

1) V_+ is simple,

or

2) V_+ is the direct sum of two simple ideals V_0 and V_2 and $V = V_0 \oplus V_1 \oplus V_2$ $(V_1 = V_-)$ is a Peirce grading.

NEHER [78 c] proved that the structure theorem for involutive gradings is also true for Jordan triple systems and Jordan algebras.

§ 5. J-structures and Q-structures

In order to give an exposition of a part of the theory of finite-dimensional Jordan algebras using linear algebraic groups SPRINGER [101] introduced the concept of J-structure, based on the notion of inverse. So, stress is laid on the role of algebraic groups in the theory of Jordan algebras. The structure groups, first introduced for a Jordan algebra by KOECHER [59a] (see also KOECHER [59 d] and BRAUN and KOECHER [18, p.79]), is now incorporated in the definition of J-structure. In this algebraic group approach, the classification of simple Jordan algebras, for instance, is derived from the Cartan-Chevalley theory of semisimple linear algebraic groups and their representations (see [101, §§ 12,13]). We shall give here the notion of J-structure, together with some comments (see SPRINGER [101]).

Let V be a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$.

<u>Notation</u>. Denote by $\mathbb{F}[V]$ the symmetric algebra on the dual V^* of V. (Recall that $\mathbb{F}[V]$ can be defined as the quotient of the tensor algebra $T(V^*)$ on V^* by the two-sided ideal generated by elements $x \otimes y - y \otimes x$, $x,y \in V^*$.)

Consider a basis $\{e_i\}$, $i=1,\ldots,n$, of V and let $\{x_i\}$, $i=1,\ldots,n$, be the dual baisis of V^*. The x_i are identified with their canonical images in $\mathbb{F}[V]$. Then we have $\mathbb{F}[V] = \mathbb{F}[x_1,\ldots,n_n]$. The n elements x_i are algebraically independent over $\mathbb{F}$, hence $x_i \to X_i$ define an isomorphism of $\mathbb{F}[V]$ onto the polynomial algebra $\mathbb{F}[X_1,\ldots,X_n]$ in n indeterminates $\{X_i\}$, $i=1,\ldots,n$. Let

$$f = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$$

be an element of $\mathbb{F}[V]$. Identify f with the function on V as follows : if $x = \sum_{i=1}^{n} a_i e_i \in \mathbb{F}$, then

(5.1) $$f(x) := \sum_{i_1,\ldots,i_n \geq 0} a_{i_1\ldots i_n} a_1^{i_1}\ldots a_n^{i_n}.$$

Since by infiniteness of $\mathbb{F}$, $f(x) = 0$ for all $x \in \mathbb{F}$ if and only if $f = 0$, $\mathbb{F}[V]$ can be identified with an algebra of functions on $\mathbb{F}$ (viz., the functions given by an expression (5.1)). The elements of $\mathbb{F}[V]$ are called polynomial functions on V.

Remark. If $\mathbb{F}[V]^n$ denotes the subspace of $\mathbb{F}[V]$ consisting of the polynomial functions which are homogeneous of degree n, then $\mathbb{F}[V] = \bigoplus_{n\geq 0} \mathbb{F}[V]^n$ is a grading.

Definition. If V and W are two finite-dimensional vector spaces over $\mathbb{F}$, then a map $\varphi : V \to W$ is called a polynomial map if the coordinates of $\varphi(x)$ are polynomial functions of $x \in V$ with respect to some basis of W.

Remark. The polynomial maps form a vector space $\mathbb{F}[V,W]$ which is a free $\mathbb{F}[V]$-module, isomorphic to $\mathbb{F}[V] \otimes_{\mathbb{F}} W$.

Definition. Let $\mathbb{F}(V)$ denote the quotient field of $\mathbb{F}[V]$. The elements of $\mathbb{F}(V)$ are called rational functions on V.

For an $f \in \mathbb{F}(V)$, there exist $g,h \in \mathbb{F}[V]$ such that $h \neq 0$ and

(5.2) $$f = gh^{-1}.$$

Because of the isomorphisms $\mathbb{F}[V] \simeq \mathbb{F}[X_1,\ldots,X_n]$; there is unique factorization in $\mathbb{F}[V]$. It follows that there exists an expression (5.2) such that g and h have no common factor of strictly positive degree. This expression is called a reduced expression of f, g is called a numerator of f, and h is called a denominator of f. Note that these g,h are unique up to a nonzero scalar factor: a denominator of f is a polynomial function h of minimal degree such that (5.2) holds.

Consider now a nonempty open subset U of V and let $\mathbb{F}[U]$ be the ring of functions f on U such that there exist $g,h \in \mathbb{F}[V]$ with

$h(x) \neq 0$ for all $x \in U$ and $f(x) = g(x)h(x)^{-1}$, $x \in U$. $\mathbb{F}(V)$ can be identified with the inductive lim ind $\mathbb{F}[U]$ (see BOREL [16, § 8, p.35]). Hence rational functions on V can be viewed as ordinary functions defined in open subsets of V. One says that $f \in \mathbb{F}(V)$ is <u>defined</u> at x or <u>regular</u> at x if there exists an expression (5.2) with $h(x) \neq 0$, and one writes $f(x) = g(x)h(x)^{-1}$.

<u>Definition</u>. If V and W are two finite-dimensional vector spaces over $\mathbb{F}$, then the elements of $\mathbb{F}(V,W) := \mathbb{F}(V) \otimes_{\mathbb{F}[V]} \mathbb{F}[V,W]$ are called <u>rational maps</u> of V into W.

<u>Remark</u>. $\mathbb{F}(V,W)$ is a finite-dimensional vector space over $\mathbb{F}(V)$, isomorphic to $\mathbb{F}(V) \otimes_{\mathbb{F}} W$.

<u>Definition</u>. If $\varphi \in \mathbb{F}(V,W)$, then there exists a polynomial map $\psi \in \mathbb{F}[V,W]$ and an $h \in \mathbb{F}[V]$ such that $\varphi = h^{-1}\psi$. An h of minimal degree is called a <u>denominator</u> of φ, and ψ is called a <u>numerator</u> of φ. One says that φ is <u>defined</u> at x or <u>regular</u> at x if $h(x) \neq 0$, and one writes $\varphi(x) = h(x)^{-1} \psi(x)$.

If V,W,Z are three finite-dimensional vector spaces over $\mathbb{F}$, then there exists a composition map $(\varphi,\psi) \rightarrow \varphi \circ \psi$ of $\mathbb{F}(W,Z) \times \mathbb{F}(V,W)$ to $\mathbb{F}(V,Z)$, defined in the obvious way. In particular, one can compose rational maps of V into V. A map $\varphi \in \mathbb{F}(V,V)$ is called <u>birational</u> if there exists a $\psi \in \mathbb{F}(V,V)$ such that $\psi \circ \varphi = \varphi \circ \psi = Id$.

Let $j : V \rightarrow V$ be a rational map, and denote by H the subset of $GL(V) \times GL(V)$ consisting of the pairs (g,h) such that $g \circ j = j \circ h$. Denote by $\pi : GL(V) \times GL(V) \rightarrow GL(V)$ the projection on the first factor. Then $\pi(H)$ is a closed subgroup of GL(V), called the <u>structure group</u> of j.

<u>Definition</u>. A <u>J-structure</u> is a triple (V,j,e), where V is a finite-dimensional vector space over $\mathbb{F}$, j a birational map of V, and e a nonzero element of V, satisfying the following axioms:

1) (i) j is a homogeneous birational map of V of degree -1, and $j = j^{-1}$;

(ii) j is regular at e, and $j(e) = e$;

2) if $x \in V$ is such that j is regular at x, $e+x$, and $e + j(x)$, then $j(e+x) + j(e+j(x)) = e$;

3) the orbit of e under the structure group of j is Zariski open in V. 1)

Remark 1. The notion of a J-structure contains an axiomatization of the notion of inverse. Axioms 1) and 2) are then obvious requirements; the importance of axiom 3) was first realized by BRAUN and KOECHER [18,p.152], who showed that properties of this nature can be used to characterize Jordan algebras in the case of characteristic not two.

Remark 2. The algebraic group which is central in the theory, the structure group (first defined for a Jordan algebra in a somewhat different manner by KOECHER [59 a,p.70], [59 d]; see also BRAUN and KOECHER [18, p.79] and § 1), is already included in the definition of J-structure.

Remark 3. If char $\mathbb{F} \neq 2$, then a J-structure is essentially the same as a Jordan algebra.

Remark 4. The notion of J-structure offers the advantage that the case char $\mathbb{F} = 2$, at least in the elementary theory, needs no special treatment. This does not hold in Jordan algebra theory, where the case char $\mathbb{F} = 2$ needs quadratic Jordan algebras.

KÜHN [62 a] generalized SPRINGER's results [101] for finite-dimensional (quadratic) Jordan pairs over a field. For this,

1) Recall that the Zariski topology on V is the topology whose closed sets are the algebraic subsets of V, i.e., sets such that there exists a set $f_1, \ldots, f_d$ of polynomial functions on V such that $S = \{ x \mid x \in V, f_1(x) = \ldots = f_d(x) = 0 \}$.

Kühn introduced the concept of Q-structure based on the notion of quasi-inverse. McCRIMMON's results [71 d] on H-structures can be also deduced from Kühn's results on Q-structures.

Consider a finite-dimensional (quadratic) Jordan pair $V = (V^+, V^-)$ over a field $\mathbb{F}$. Let $x^y \in V^\sigma$ denote the quasi-inverse of a quasi-invertible pair $(x,y) \in V^\sigma \times V^{-\sigma}$. The rational maps $q_\sigma : (x,y) \longrightarrow x^y$ satisfy the relations:

$$
\begin{aligned}
&q_\sigma(x,o) = x,\\
&q_\sigma(x,y+u) = q_\sigma(q_\sigma(x,y),u),\\
(5.3)\qquad &q_\sigma(x+z,y) = q_\sigma(x,y) + H_\sigma(x,y)\, q_\sigma(z, q_{-\sigma}(x,y)),\\
&q_\sigma(x,y) = \alpha\, q_\sigma(x,\alpha y), \qquad \alpha \text{ scalar,}
\end{aligned}
$$

where $H_\sigma(x,y)$ is the inverse of the Bergmann transformation $B_\sigma(x,y)$ defined in the foregoing section.

Let V^+, V^- be finite-dimensional vector spaces over a field $\mathbb{F}$.

<u>Notation</u>. If M is a countably infinite set of algebraically independent elements of $\mathbb{F}$, then we put $V^\sigma(M) := V^\sigma \otimes \mathbb{F}(M)$, where $\mathbb{F}(M)$ is the quotient field of the polynomial ring in M.

Assuming that x is a generic element of V^+ and y is a generic element of V^-, let $q_+(x,y)$, $q_-(y,x)$ be rational maps with values in $V^+(M)$, $V^-(M)$, respectively, and let $H_+(x,y)$, $H_-(y,x)$ be rational maps with values in End $(V^+(M))$, End $(V^-(M))$ respectively. We put $V := (V^+, V^-)$, $q := (q_+ , q_-)$, and $H := (H_+ , H_-)$.

<u>Definition</u>. A triple (V,q,H) with V,q,H as defined above is called a <u>Q-structure</u> if, for $x,z \in V^\sigma(M)$, $y,u \in V^{-\sigma}(M)$ and $\alpha \in M$, the identities (5.3) hold.

<u>Remark</u>. As was shown by KÜHN [62 a], there exists a unique quadratic Jordan pair structure on (V^+, V^-) whose quasi-inverse is given by q_σ.

§ 6. Connections with Lie structures

As was already mentioned, the close connections between Jordan and Lie structure have their due place in an <u>intrinsic</u> treatment. However, taking into account that some of these connections have become classical references in the field, we shall briefly recall them here together with some recent developments.

<u>Note</u>. The close relations between Jordan algebras, Lie algebras, Jordan triple systems and the enveloping algebras of Jordan and Lie algebras are clearly presented and the historical roots of these theories are disclosed by KOECHER [59 h].

It is well-known that the exceptional Lie algebras G_2 may be obtained as the derivation algebras of Cayley-Dickson algebras, the Lie algebras F_4 may be obtained as the derivation algebras of Jordan algebras H_3 ($\mathbb{C}$) and their forms, $\mathbb{C}$ being Cayley-Dickson algebras over a field of characteristic different from two. TITS [105 a,b] used Jordan algebras to construct models for exceptional Lie algebras E_6, E_7, E_8. KOECHER [59 c] and KANTOR [54 c] imbedded an arbitrary linear Jordan algebra J into the $\mathbb{Z}$-graded Lie algebra

$$K(J) = \mathcal{L} = \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1, \; \mathcal{L}_i = 0 \quad \text{for } |i| > 1.$$

Conversely, in any $\mathbb{Z}$-graded Lie algebra $\mathcal{L} = \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$, the operations

$$(\mathcal{L}_\sigma, \mathcal{L}_{-\sigma}, \mathcal{L}_\sigma) \ni (a_\sigma, b_{-\sigma}, c_\sigma) \rightarrow [[a_\sigma, b_{-\sigma}], c] \quad , \sigma = \pm 1$$

are very close to a Jordan triple product.

<u>Freudenthal-Tits magic square</u> (see [105 b], and also [47 b], [29 a]). Let $\mathcal{A}$, $\mathcal{B}$ be composition algerbas over a commutative field $\mathbb{K}$ of characteristic zero, and let J be the Jordan algebra $H_3(\mathcal{B})$. Denote by T the trace bilinear form on J, by t the analogous form on $\mathcal{A}$, and by $\mathcal{A}_0$ (resp. J_0) the space of elements of

trace zero in $\mathcal{A}$ (resp. J). Define a product $*$ on $\mathcal{A}_o$ (resp. J_o) by projecting the usual product with respect to the decomposition $\mathcal{A} = \mathbb{K}1 \oplus \mathcal{A}_o$ (resp. $J = \mathbb{K}1 \oplus J_o$). Define $D_{a,b} \in \mathrm{Der}\,\mathcal{A}$ and $D_{x,y} \in \mathrm{Der}\,J$ by

$$D_{a,b} := [L_a, L_b] + [L_a, R_b] + [R_a, R_b], \text{ and } D_{x,y} := [R_x, R_y],$$

where R_a (resp. L_a) denotes the right (resp. left) multiplication by a. Consider the $\mathbb{K}$-vector space

$$\mathcal{L}(\mathcal{A}, J) := \mathrm{Der}\,\mathcal{A} \oplus (\mathcal{A}_o \otimes J_o) \oplus \mathrm{Der}\,J$$

endowed with the product

(i) $[X,Y]$ the usual Lie product in $\mathrm{Der}\,\mathcal{A} \oplus \mathrm{Der}\,J$;

(ii) $[a \otimes x, D+E] := aD \otimes x + a \otimes xE$, for $a \in \mathcal{A}_o$, $x \in J_o$, $D \in \mathrm{Der}\,\mathcal{A}$, $E \in \mathrm{Der}\,J$;

(iii) $[a \otimes x, b \otimes y] := \frac{1}{2} T(x,y) D_{a,b} + a * b \otimes x * y + \frac{1}{2} t(a,b) D_{x,y}$ for $a,b \in \mathcal{A}_o$, $x,y \in J_o$.

Thus, $\mathcal{L}(\mathcal{A}, J)$ becomes a Lie algebra. As $\mathcal{A}$ and $\mathcal{B}$ vary over the possible composition algebras, the resulting algebras are those displayed in the following "magic square" :

(6.1)

dim $\mathcal{A}$ \ dim $\mathcal{B}$	K	$K^{(3)}$	1	2	4	8
1	0	0	A_1	A_2	C_3	F_4
2	0	$\mathcal{A}$	A_2	$A_2 \oplus A_2$	A_5	E_6
4	A_1	$A_1 \oplus A_1 \oplus A_1$	C_3	A_5	D_6	E_7
8	G_2	D_4	F_4	E_6	E_7	E_8

Remark (see [29 a]). A number of other well-known contructions for exceptional Lie algebras can be found in the above table (6.1), as follows:

a) Der J occurs in row 1, column 6, while the derivation algebra of the octonion algebra occurs in row 4, column 1;

b) the earlier construction of E_7, due to TITS [1o5 a],

$$\mathcal{L} = (\mathcal{L}_1 \otimes J) \oplus \text{Der } J,$$

where $\mathcal{L}_1$ is a simple three-dimensional Lie algebra, occurs in row 3, column 6 (identifying Der$\mathcal{A}$ with $\mathcal{L}_1 \otimes 1$);

c) the KOECHER construction [59 c] for E_7,

$$\mathcal{L} = J \oplus \bar{J} \oplus \widehat{\mathcal{L}}(J),$$

where J is exceptional simple Jordan and

$$\widehat{\mathcal{L}(J)} := \{ R_x + D \mid x \in J, \ D \in \text{Der } J \},$$

occurs in row 3, column 6 if $\mathcal{A}$ is split;

d) the E_8 construction (see FAULKNER [28 d]),

$$\mathcal{L} = \mathcal{L}(J) \oplus (J \otimes \mathcal{B}) \oplus (\bar{J} \otimes \bar{\mathcal{B}}) \oplus \mathbb{K}_3';$$

for J exceptional simple Jordan, $\mathcal{B}$ a three-dimensional vector space over $\mathbb{K}$, $\mathbb{K}_3'$ the algebra of transformations of trace zero in $\mathcal{B}$, $\bar{\mathcal{B}}$ the contragradient module to $\mathcal{B}$; this appears in row 4, column 6.

ATSUYAMA [9] gave a construction of all compact real simple Lie algebras. In the case of the exceptional algebras, he described an isomorphism from his models onto those constructed by TITS [1o5 b].

<u>Definition</u> (see HEIN [39 a], ALLISON [2]). A vector space $\mathcal{M}$ with trilinear composition satisfying

(i) $L_{u,v} - L_{v,u} = R_{u,v} - R_{v,u}$,

(ii) $[R_{v,w}, R_{x,y}] = R_{vR_{x,y},w} + R_{v,wR_{y,x}}$,

for all $x,y,z,u,v,w \in \mathcal{M}$, where

$$L_{u,v} : x \to uvx \quad \text{and} \quad R_{u,v} : x \to xuv,$$

is called a J-ternary algebra.

Remark 1. The name J-ternary arises since by (i) and (ii) combined, the transformation $\langle u,v\rangle := R_{u,v} - R_{v,u}$ span a Jordan subalgebra J of $(\mathrm{End}\,\mathcal{M})^{(+)}$ so that $\mathcal{M}$ is a special J-module.

Note. For the structure of reduced J-ternary algebras, see HEIN [39 b,c].

Remark 2. A kind of structure closely linked to J-ternary algebras are the so-called "Freudenthal triple systems", studied and classified by MEYBERG [77 a]. In 1978, KANTOR and SKOPETS [55] established a one-to-one correspondence between Freudenthal triple systems (as defined by Meyberg except that the associated quartic form is allowed to be zero) over an algebraically closed field of characteristic zero and simple Lie algebras, in a way very similar to the construction of FAULKNER [28 b] (see also FERRAR [30]).

J-ternary algebras have been used in another construction of exceptional Lie algebras similar to that of KOECHER [59 c] and MEYBERG [77 c]. If $\mathcal{M}$ is a J-ternary algebra and J spanned by $\{\langle u,v\rangle \mid u,v\in\mathcal{M}\}$ is a simple Jordan algebra,

$$R_a : (u,b) \to (\tfrac{1}{2}\,us,\ .b),\quad A_{v,w} : (u,b) \to (uvw, \langle v,bw\rangle)$$

for $a,b\in J$, $v,w,u\in\mathcal{M}$, then one forms

$$\mathcal{L}(J,\mathcal{M}) := \bar{J}\oplus\bar{\mathcal{M}}\oplus\mathcal{L}_0\oplus\mathcal{M}\oplus J$$

($\mathcal{L}_0$ = span of $\{A_{v,w}\in\mathcal{M}\}$, $\bar{J}, \bar{\mathcal{M}}$ copies of $J,\mathcal{M}$) with product defined by

$[A,B]$ usual Lie product in $\mathcal{L}_0$

$$\left.\begin{array}{l}[a+v,b+w] := \langle v,w\rangle,\ [\overline{a+v},\ \overline{b+w}] := \overline{\langle v,w\rangle}\\ [a+v,A] := (a+v)A,\ [\overline{a+v},A] := \overline{(a+v)A^{\varepsilon}}\end{array}\right\}\begin{array}{l}a,b\in J\\ v,w\in\mathcal{M}\\ A,B\in\mathcal{L}_0\end{array}$$

$$[a+v,\ \overline{b+w}] := 2R_{a\cdot b} - 2[R_a,R_b] + A_{v,w} + aw + \overline{bv}\ ,$$

where $\varepsilon : A \rightarrow A - 2 R_{eA}$ and a is the identity of J. The algebra $\mathcal{L}(J, \mathcal{M})$ is a Lie algebra which in most cases is simple (see ALLISON [2]).

For a particular $\mathcal{M}$ (see FAULKNER [28 c], and also FAULKNER and FERRAR [29 a]) with J exceptional simple one obtains the algebra E_8 (see FAULKNER [28 b]) as it was first introduced by FREUDENTHAL [31].

Note. Let us mention here the constructions of Lie algebras from J-ternary algebras given by KANTOR [54 d], YAMAGUTI [109 a], HEIN [39 a], and ALLISON [2], which have as common origin that given by FREUDENTHAL for exceptional Lie algebra from Jordan algebras [31]. KAKICHI [50 a] proved that all simple Lie algebras (except for those of type A_1) over an algebraically closed field can be constructed from a J-ternary algebra. For "superternary" algebras, see Comments 1, § 4, as well as BARS [12].

MEYBERG [77 d] generalized the Koecher-Tits construction and studied certain Lie algebras which are constructed from the (-1)-eigenspaces of an involution of a Jordan algebra. He gave necessary conditions in terms of the Jordan algebras for the Lie algebras to be simple. If the (-1)-spaces are Peirce - $\frac{1}{2}$ - components, then as was noted by Meyberg there exists a close relation between the Lie algebras under consideration and the structure algebras of Jordan algebras. Finally, he gave a list of those types of simple Lie algebras which ca be formed by this construction and among these are the Lie algebras of type E_6 and E_7.

KÜHN and ROSENDAHL [63] generalized the Koecher-Tits construction to Jordan pairs (and, in a similar manner, to Jordan triples). The functor obtained goes from Jordan pairs (or Jordan triples) to Lie algebras. Koecher's remark that Levi's theorem for Lie algebras of characteristic zero implies, via the functor, the Wedderburn principal theorem for Jordan algebras is extended to Jordan pairs (and Jordan triples) over a field of characteristic zero.

Each simple Koecher-Tits construction of a Lie algebra from semisimple complex Jordan triple systems may be characterized by the existence of nontrivial elements u with $(\mathrm{ad}u)^3 = \mathrm{ad}u$. However, real forms of such algebras are not in each case Koecher-Tits constructions as this is well-known from compact real forms. So, RINOW [91] asked for a new method for constructing real Lie algebras characterized by the existence of elements $u \neq 0$ with $(\mathrm{ad}u)^3 = \alpha\, \mathrm{ad}u$ ($\alpha \in \mathbb{R}$). Let J be a finite-dimensional real Jordan triple and let W be a two-dimensional simple Jordan triple over $\mathbb{R}$ which is given by

$$\{xyz\} := \langle x,y\rangle z + \langle y,x\rangle x - \langle x,x\rangle y,$$

$\langle\ ,\ \rangle$ being a nondegenerate symmetric bilinear form on W. By direct calculation, one can see that $W \otimes J$ together with the componentwise product is again a Jordan triple, from which Rinow got a Lie triple by alternation of the first two arguments. The subject became the standard embedding Lie algebra L(W,J) of the Lie triple $W \otimes J$ and it turned out that a real finite-dimensional simple Lie algebra is a construction L(W,J) if and only if can find an element $u \neq 0$ with $(\mathrm{ad}u)^3 = \alpha \cdot \mathrm{ad}u$.

HIRZEBRUCH [40 a] described a generalization of TITS' construction of Lie algebras from Jordan algebras [105 a] to a construction of Lie algebras from Jordan triple systems. This generalization incorporates MEYBERG's construction of Lie algebras by Jordan triple systems [77 e] in the same way as Tits construction incorporates KANTOR's [54 a] and KOECHER's [59 c], and as a Tits' construction it has the advantage that it allows us to obtain different forms of a Lie algebra starting with the same Jordan triple system.

ASANO and YAMAGUTI [8] showed that HIRZEBRUCH's construction [40 a] is valid also for a more general structure, namely, generalized Jordan triples of second order in the sense of KANTOR [54 d]. This construction involves a two-dimensional space with bilinear form $\langle , \rangle$; in the case when the field of definition is algebraically closed and $\langle , \rangle$ is non-degenerate, the construction is essentially that given by KANTOR [54 d]. A Lie algebra construction analogous to that of Asano-Yamaguti was given by KAKIICHI [50 b], and generalized by himself to graded structures [50 c].

KOECHER [59 e] pointed out how algebraic constructions of Lie algebras by means of Jordan algebras of Jordan triple systems, the study of holomorphic vector fields, and other methods and results in the domain of nonassociative algebras, can be linked by a general concept. Koecher's fundamental ideas are the following:

Let X be a left $\mathbb{K}$-module, $\mathbb{K}$ being an associative and commutative ring with unity. Consider the $\mathbb{K}$-module

$$\mathrm{Alg}\ X := \mathrm{Hom}_{\mathbb{K}} (X, \mathrm{End}_{\mathbb{K}} X).$$

To each element A of Alg X there belongs an algebra X_A on X defined by $(u,v) \to uAv := A_u v$ with $A_u \in \mathrm{End}\ K$ and $u,v \in X$. In Alg X a product which depends on $u \in X$ is defined as follows: if A, B are elements of X, then

$$x(AuB)y := (uAx)By + xB(uAy) - uA(xBy), \qquad (6.2)$$

and one gets an algebra denoted by $\mathrm{Alg}_u\ X$.

Alg X together with the product (6.2) can be considered as an algebra of algebras Alg_u X. If $T \in$ End X, $A \in Alg$ X, then an element $T \cdot A$ of Alg X is defined by

$$u(T \cdot A)v := T(uAv) - (Tu)\, Av - uA\, (Tv).$$

For a submodule M of Alg X, the statements "M is a left ideal in $Alg_u X$" and "$T \cdot M \subset M$ for all T of End X" are equivalent.

If $\mathbb{K}$ is a field and $u \neq 0$, then Alg_u X is simple and all algebras are mutually isomorphic.

The principal idea is to consider the so-called <u>standard algebra</u>

$$\text{Stand } X := X \oplus \text{End } X \oplus \text{Alg } X,$$

in which the product of the elements $\phi = u \oplus T \oplus A$ and $\psi = v \oplus S \oplus B$ is defined by $(\phi, \psi) \rightarrow [\phi, \psi]$ with

$$[T,S] := TS-ST, \; [T,u] := Tu, \; [T,A] := T \cdot A,$$
$$[A,u] := A_u, \; [u,v] = 0, \; [A,B] = 0,$$

for $u,v \in X$, $T,S \in$ End X, $A,B \in$ Alg X.

Stand X is an anticommutative algebra in which the Lie subalgebras are of special interest. Various new results were derived and well-known ones were put in a new context. KOECHER [59 e] also gave a study of the connection between the Lie subalgebras of Stand X (in which X is finite-dimensional) and a group of birational mappings of X.

Let us mention now the following KOECHER construction [59 f] of Jordan algebras given a few years later: For a unitary commutative ring $\mathbb{K}$, a $\mathbb{K}$-module X and a Lie algebras T of endomorphisms of X, let M(T) denote the submodule of End X consisting of all $\mathbb{K}$-linear mappings B : K $\rightarrow$ End X, u $\rightarrow$ Bu, such that $Bu \in T$ for $u \in X$. Suppose that there exists a commutative (nonassociative) algebra A on X such that the algebras defined on X <u>via</u> $(u,v) \rightarrow$ $\rightarrow B_u v$, $B \in M(T)$, are mutations of A. Then one can construct a

Jordan algebras of A. Let $\eta : X \to \mathbb{K}$ be a nondegenerate rational map of the finite-dimensional vector space X over a field $\mathbb{K}$. Denote by T_η the Lie algebra of the "invariance group of η" and by A_η the commutative algebra associated with η and let $e \in X$. Then the assumptions are satisfied and one obtains a Jordan sub algebra of A_η.

As is well known, the notion of the structure group introduced by KOECHER [59 a] (see § 1) provides a useful tool for the theory of Jordan algebras. JACOBSON's paper [47 c] aims at determining this group as explicitly as possible for the Jordan algebra of symmetric elements in an associative algebra with involution. After defining the basic concepts associated with quadratic Jordan algebras, Jacobson presents the main idea of his paper: given a special quadratic Jordan algebra J, the elements of the structure group being just the isomorphisms of J onto its various isotopes induce certain automorphisms of the special universal envelope of J, and the group of these automorphisms can be explicitly described. The structure group of any Jordan algebra J gives rise, by the standard pattern <u>via</u> dual numbers, to its associated Lie algebra, called the <u>structure algebra</u> of J. In the final section of the paper [47 c], JACOBSON showed how his results about the structure group imply corresponding results about the structure algebra.

GORDON [34 a] determined the components of the automorphisms group and of the structure group of a semisimple Jordan algebra J over an algebraically closed field of characteristic zero, as well as the action of these on J. The results thus obtained are highly elaborated but are modeled on the corresponding results for Lie algebras; indeed, the main procedure is to work with the Lie algebras of derivations, the structure Lie algebra, and the Koecher-Tits algebra. In [34 b] GORDON proceeded from a complex, semisimple, finite-dimensional Jordan algebra J with the structure Lie algebra $\mathcal{L}$ and Koecher-Tits algebra $\mathcal{R} \geqq \mathcal{L}$ to prove that a suitable selected Chevalley basis for $\mathcal{R}$ yields simultaneous integral bases for J and $\mathcal{L}$. Then, GORDON [34 c] undertook a thorough-going study of the structure group of a split semi-simple Jordan algebra.

FAULKNER and FERRAR [29 b] defined the concept of an anti-Jordan pair which differs from the concept of a Jordan pair only by the sign in the second identity. However, this difference induces a connection between anti-Jordan pairs and graded consistent Lie superalgebras instead of the known connection between Jordan pairs and graded Lie algebras.

ZEL'MANOV [110 h] remarked that "...under certain restrictions on the characteristic of the ground field, the theory of Lie algebras with finite grading turned out to be «parallel» to the Jordan theory. This parallelism is not formal: the most important notions and methods of the theory of Jordan algebras admit the natural analogoues for Lie algebras with finite grading. That is why the Lie algebras with finite grading may be rightfully included into the Jordan theory as its most general (up today) object. This ideology made it possible to classify the simple (infinite-dimensional) Lie algebras with finite grading" (see Theorem 6.1 below).

Theorem 6.1 (ZEL'MANOV [110 f]; see also [110 h]). Let $\mathcal{L} = \sum_{i=-n}^{n} \mathcal{L}_i$ be a simple $\mathbb{Z}$-graded Lie algebra over a field of characteristic $p \geqslant 4n+1$ (or zero), $\sum_{i \neq 0} \mathcal{L}_i \neq 0$. Then one of the following assertions is valid :

1) there exists a simple $\mathbb{Z}$-graded associative algebra $R = \sum_{i=-n}^{n} R_i$ such that $\mathcal{L} \simeq [R,R] / Z \cap R_o$, where Z is a cent... of the commutant $[R,R]$;

2) there exists a simple $\mathbb{Z}$-graded associative algebra $R = \sum_{i=-n}^{n} R_i$ with involution $* : R \rightarrow R$, $R_i^* = R_i$, such that $\mathcal{L} \simeq [S,S] / Z([S,S]) \cap R_o$, where $S=S(R,*) := \{ a \in R \mid a^* = -a \}$ is a Lie algebra $*$-skew-symmetric elements;

3) $\mathcal{L}$ is isomorphic to a Koecher-Kantor-Tits construction of a Jordan algebra of symmetric bilinear form over some extension of the ground field;

4) $\mathcal{L}$ is of one of the types G_2, F_4, E_6, E_7, E_8, D_4.

Comments. As ZEL'MANOV [110 h] observed, one may consider an even more general situation, namely, when Λ is a torsion-free abelian group, $\mathcal{L} = \sum_{\lambda \in \Lambda} \mathcal{L}_\lambda$ is a Λ-graded Lie algebra, and the set $\{ \lambda \in \Lambda \mid \mathcal{L}_\lambda \neq 0 \}$ is finite. Then, under certain restrictions on the characteristic of the ground field, an analogue of the Theorem 6.1 above can be proved.

Let us finally mention some of the advances made in view of applications in physics.

If we take the totally symmetrized monomials in q and p as a basis in the Weyl algebra, which is the associative algebra generated by p and q modulo the canonical commutation relations, the polynomials of the first and second degree can be given Lie and Jordan algebra structures that are isomorphic to well-known matrix algebras. As an application, the relation between formally real Jordan algebras, domain of positivity, and symmetric spaces was used by TILGNER [104 c] to give a classification of the second-degree Hamiltonians that are invariant under invertible linear transformations of q and p and has some influence on the representation theory of the solvable spectrum generating groups of these Hamiltonians which were earlier described by TILGNER [104 a,b]. In the final part of his paper [104 c], TILGNER gave the relation of the Weyl algebra to the Clifford algebra over an orthogonal vector space, and discussed the minimal imbedding of an arbitrary Lie algebra into the Weyl algebra.

BARS [12] described the role of ternary (super) algebras as building blocks for all Lie (super) algebras. This mathematical construction is tentatively applied to the physical gauge theory in Lagrangian formulation.

Every involution α defines, by $\alpha(X) = -X$, the elements of one of the classical Lie algebras $\sigma(n, \mathbb{C})$ and sp $(2n,\mathbb{C})$ over $\mathbb{C}$, and $\sigma(p,q)$, sp$(2n, \mathbb{R})$, u(p,q), $\sigma^*(2n)$, and sp$(2p,2q)$ over $\mathbb{R}$, also defines α-symmetric matrices $Y = \alpha(Y)$ spanning the classical Jordan algebras. PATERA nad ROUSSEAU [83] gave an explicit description of all perturbations of α-symmetric matrices, considered up to equivalence, under the action of the corresponding classical Lie group. Such exhaustive perturbations are called versal deformations.

GÜNAYDIN, SIERRA and TOWNSEND [37] derived the magic square from the geometry of a special class of N = 2 Maxwell-Einstein supergravity theories. They also showed that each of these theories is obtainable by truncation of N = 8 supergravity theories in various spacetime dimensions d, except for an "exceptional" subclass, unique for a given d, which is associated with the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$.

Note. Much information on the so-called Lie-admissible and Jordan-admissible algebras can be found in "Mathematical studies on Lie-admissible algebras" edited by H.C.Myung and published by

Hadronic Press, Inc. Nonautum, Mass. vols.1-4, 1984-1986.

§ 7. Russian school in Jordan structures

In this section we shall briefly recall some of the most important result obtained by the Novosibirsk school of the late Professor Shirshov, particularly Zel'manov's results concerning the infinite-dimensional case. These advances essentially complete the general structure theory for linear Jordan algebras. These results are expected to bear largely on the applications of Jordan structures as they become better known. For a detailed description of the recent breakthrough of the Russian school in Novosibirsk, the reader is referred to ZEL'MANOV [110 h] and McCRIMMON [71 i].

Note. For Russian contributions to geometrical applications of Jordan structures, see §§ 2,5 in JSA.III, and § 1 in JSA.IV, while to analytical applications, see § 3 in JSA.V.

SHIRSHOV [96 a] proved by combinatorial methods that any special Jordan algebra which satisfies the identity $x^n = 0$ is locally nilpotent. More generally, he proved that the Burnside-like problem (or the Kurosh problem cf. [64]) has positive solution in the class of special Jordan algebras which satisfies an essential polynomial identity (see SHIRSHOV [96 a], and also [115]). From his result raised the following

Question (SHRSHOV [96 a] , see also [115]). Is there any Jordan nil-algebra of bounded degree that is locally nilpotent ?

The above-mentioned question is equivalent to the following: Does the McCrimmon radical of a Jordan algebra always lie in its locally nilpotent radical ?

Suppose that a linear Jordan algebra J is generated by its absolute zero divisors (an element a of J, a ≠ 0, is called an absolute zero divisor of J if $\{a,b,a\} := (ab)a+a(ba)-ba^2 = 0$ for all b from J). Let us consider its Koecher-Tits construction $K(J) = J(J)_{-1} \oplus K(J)_0 \oplus K(J)_1$. ZEL'MANOV [110 b,d] used KOSTRIKIN's techniques [61] and the Jordan origin of the Lie algebra K(J) to prove its nilpotence which is equivalent to the local nilpotence of J (see Theorems 7.1 and 7.2 below). This solved in the affirmative Shirshov's question.

Theorem 7.1. McCrimmon's radical of a Jordan algebra lies in its locally nilpotent radical.

Theorem 7.2. Any Jordan nil algebra of bounded degree is locally nilpotent.

The Burnside-like problem in the class of Jordan PI-algebras has also positive solution as is shown in

Theorem 7.3 (ZEL'MANOV [110 c]). Any algebraic Jordan PI-algebra is locally finite-dimensional.

Comments. SLIN'KO [100 b] showed that a special Jordan algebra with minimum condition on inner ideals inside a quasi-invertible ideal I has I nilpotent, and if a special algebra was generated by a finite number of absolute zero-divisors, then its special universal envelope is nilpotent. (For Jordan algebras over an arbitrary ring of scalars, see SKOSYRSKII [98] .) SLIN'KO and ZEL'MANOV [115] extended this result to arbitrary algebras. By introducing the important notion of annihilator 1), ZEL'MANOV [110 a] was able to handle simultaneously both the minimum and the maximum condition.

Remark 1. From ZEL'MANOV's [110 c] the conclusions of one statement due to OSBORN and RACINE [82 a,b] follow easily.

Remark 2. In order to develop a module theory more pro-

1) Extended to quadratic Jordan algebras by McCRIMMON [71 g].

perly suited to the study of the structure theory of nonassociative algebras (primarily alternative, flexible and Jordan algebras), OSBORN [81 b] presented two module theories and developed the corresponding structure theory. These theories were also independently discovered by ZHEVLAKOV [113].

Note. For results on radicals of Jordan algebras, the reader is referred to SLIN'KO [100 a,c], ZHEVLAKOV and SHESTAKOV [114], NIKITIN [80 a,b], ZHELYABIN [112] and ZEL'MANOV [110 h].

Let us recall now three other results due to ZEL'MANOV [110 e] (see also [110 h]) concerning the classification of simple or prime (i.e. have no orthogonal ideals) Jordan algebras (see Theorem 7.4, 7.5 and 7.6 below).

Theorem 7.4. Any simple Jordan algebra is isomophic to one of the following algebras:

1) $R^{(+)}$, there R is a simple associative algebra;
2) $H(R,*) := \{x \in R^{(+)} \mid x^* = x\}$, where R is a simple associative algebra with the involution $* : R \to R$;
3) a Jordan algebra J(Q) of a nondegenerate symmetric bilinear form Q in a vector space V over some extension of the basic field, $\dim_{\Gamma} V > 1$;
4) the simple exceptional Jordan algebra which is 27-dimensional over its centre.

Remark. This important result completely clasifies simple algebras without any finiteness conditions.

Comments. Using a special case of Theorem 7.4 one can classify the Jordan division algebras (done by ZEL'MANOV [110 c] in 1979 by more complicated methods).

Theorem 7.5. A prime nondegenerate Jordan algebra is either spectral or an Albert ring (i.e. a prime Jordan algebra J which has its centre Z(J) different from zero and its central closure

$Z(J)^{-1}J$ is a simple exceptional algebra of dimension 27 over its center $Z(J)^{-1}Z(J)$).

Theorem 7.6. Let J be a special nondegenerate Jordan algebra. Then one of the following statements is valid:

I) the centre $Z(J)$ is nonzero and the central closure $Z(J)^{-1}J$ is a Jordan algebra of a nondegenerate symmetric bilinear form over the field $Z(J)^{-1}Z(J)$;

II) J contains the nonzero ideal I which is invariant under all automorphisms and derivations of J and either;

II_1) $I \simeq R^{(+)}$, R is a prime associative algebra $R^{(+)} \trianglelefteq J \subseteq$ $\subseteq Q(R)^{(+)}$, where Q(R) is a Martindale quotient ring of R (cf.[64]), or

II_2) $I \simeq H(R,*)$, R is a prime associative algebra with involution and $H(R,*) \trianglelefteq J \subseteq H(Q(R),*)$.

Remark. Theorems 7.5 and 7.6 above (see ZEL'MANOV [110 g, II]) are improved versions of the earlier prime theorem given by ZEL'MANOV [110 g,I] in 1979.

In ref. [71 i] McCRIMMON listed several fundamental questions raised in [18], [71 c,e]. Two on these questions are: Can one develop a theory of Jordan algebras satisfying polynomial identities? Are the J(Q)'s the only simple PI-algebras which are infinite-dimensional over their centres? The following theorem due to ZEL'MANOV [110 g, II] answers these two questions.

Theorem 7.7. Each nonzero ideal of a nondegenerate Jordan PI-algebra has nonzero intersection with the centre (so if the centre is a field the algebra is simple). The central closure of a prime nondegenrate PI-algebra is central simple. Any primitive PI-algebra is simple. Each simple PI-algebra is either finite-dimensional or an algebra J(Q) over its centre.

Two other questions broached up by McCRIMMON in ref. [71 i], namely: Is the special universal envelope of a finitely generated Jordan PI-algebra an associative PI-algebra? Are the J(Q)'s essentially the only examples of special Jordan PI-algebras whose envelope is not PI ?, are given an answer in SHESTAKOV's [95 b] (see Theorem 7.8 below).

Theorem 7.8. If J is a special Jordan PI-algebra, then its special universal envelope su (J) is locally finite: if J is finitely generated, su (J) is an associative PI-algebra.

Theorem 7.9 (see ZEL'MANOV [110 g,I]). A semi-primitive algebra will be i-special (satisfies all s-identities) so soon as it satisfies the s-identity $\mathcal{G}_8$. Hence, the ideal s(X) of all s-identities (for universal X) is quasi-invertible modulo the ideal generated by Glennie's identity $\mathcal{G}_8$.

Note. McCRIMMON extended Zel'manov's results to quadratic Jordan algebras: for the nilpotence theorem, see [71 f], while for the prime theorem, see [71 h]. ZEL'MANOV [110 g,II] showed that any ideal in a strongly prime (i.e. prime and with no trivial elements) Jordan algebra is again strongly prime. McCRIMMON [71 j] extended this result to quadratic Jordan structures (algebras, triple systems, and pairs).

Worth mentioning here is also SHESTAKOV's comprehensive paper [95 a] on some classes of noncommutative Jordan algebras, as well as those due to MEDVED'EV [75 a], SLIN'KO [100 d], PCHELINTSEV [84 a], and SVERCHKOV [102 b,d] for results on varieties of Jordan algebras. For thorough studies on the varieties of algebras, see ARTAMANOV [6] and OSBORN [81 a].

Results on solvable Jordan algebras (see [76], [99]), and bimodule representations of Jordan algebras (see [20], [110 m]) were also obtained. A recent work of ZEL'MANOV [110 n] deals with Goldie's theorems for Jordan algebras.

References

1. ALBERT,A.A., A structure theory for Jordan algebras, Ann.of Math. 48 (1947), 546-567.

2. ALLISON,B.N., A construction of Lie algebras from J-ternary algebras, Amer.J.Math. 98 (1976), 2, 285-294.

3. ALLISON,B.N., FAULKNER,J.R., A Cayley-Dickson process for a class of structurable algebras, Trans.Amer.Math.Soc. 283 (1984), 1, 185-210.

4. ANH,P.N., Simple Jordan algebras with minimal inner ideals, Comm.Algebra 14 (1986), 3, 489-492.

5. ANH,P.N.,WIEGANDT,R., Semisimple classes of nonassociative rings and Jordan algebras, Comm.Algebra, 13 (1985), 12, 2669-2690.

6. ARTAMANOV,V.A., Structures of varieties of linear algebras (in Russian), Uspehi Mat.Nauk, 33 (1978), 2, 135-167.

7. ASANO,H., KAMIYA,N., YAMAGUTI,K., On Freudenthal-Kantor supertriple systems and Lie supertriple systems,

8. ASANO,H., YAMAGUTI,K., A construction of Lie algebras by generalized Jordan triple systems of second order, Nederl. Akad. Wetensch. Indag. Math. 42 (1980), 3, 249-253.

9. ATSUYAMA,K., Another construction of real, simple Lie algebras, Kodai Math.J. 6, (1983), 1, 122-133.

10. AYUPOV,SH.A., ZHELYABIN,V.N., Compatibility of elements in Jordan algebras (in Russian), Mat.Zametki 37 (1985), 3, 305-312.

11. BAHTURIN,A.Yu., SLIN'KO, A.M.,SHESTAKOV,I.P., Nonassociative rings (in Russian), Itogi nauki i tehniki, Algebra, Topologia, Geometria, 18 (1981), 3-72. Akad.Nauk SSSR, VINITI

Moscow, 1981; English translation: J.Sov.Math. 18 (1982), 169-211.

12. BARS,I., Ternary algebras as building blocks of Lie algebras and Lie superalgebras, in "Group theoretical methods in physics", pp.58-72, Ann.Israel Phys.Soc.3, Hilger, Bristol,1980.

13. BARS,I., GÜNAYDIN,M., Construction of Lie algebras and Lie superalgebras from ternary algebras, J.Math.Phys. 20 (1979), 9, 1977-1993.

14. BEIDAR,K.I., MIKHALEV,A.V., SLIN'KO,A.M., Primeness criteria for nondegenerate alternative and Jordan algebras (in Russian) Trudy Moskov.Mat.Obshch. 50 (1987), 13o-137.

15. BIX,R., Separable Jordan algebras commutative rings III, J. Algebra 86 (1984), 1, 35-59 (for part I see the same journal 57 (1979), 1, 111-143, and for part II: ibid. 79 (1982), 2, 341-374).

16. BOREL,A., Linear algebraic groups, Benjamin, New York, 1969.

17. BRANZEI,D., Structures affines et opérations ternaires, An. Ştiinţ. Univ."Al.I.Cuza" Iaşi, 23 (1977), 33-38.

18. BRAUN,H., KOECHER,M., Jordan-Algebren, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

19. CHEHONADSKII,A.V., Algebraic Jordan algebras with the maximal condition for associative subalgebras, Siberian Math. J. 25 (1984), 5, 823-831.

20. CHEHONADSKII,A.V., ZEL'MANOV,E.I., On the bimodules representation of the simple exceptional Jordan algebra (in Russian), Sibirsk. Mat. Zh.

21. CREANGA,I., a) Sur les algèbres ternaires, An.Stiinţ. Univ. "Al.I.Cuza" Iaşi 16 (1970), 249-255.

b) Ternary formations, An.Stiinţ. Univ. "Al.I.Cuza", Iaşi, 31 (1985), 46-50.

22. CREANGA,I., TAMAŞ,V., Contributions to the study of the ternary structures, An. Stiinţ.Univ. "Al.I.Cuza" Iaşi, 28 (1982), 7-12.

23. DIXMIER,J., Certaines algebres non associatives simples définie par la transvection des formes binaires, J.Reine Angew. Math. 346 (1984), 110-128.

24. DRENSKY,V.S., a) Polynomial identities in simple Jordan algebras, C.R.Acad.Bulgare Sci.35 (1982), No.10, 1327-1330.
b) On the Hilbert series of relativity free algebras, Comm. Algebra 12 (1984), 2335-2347.
c) Polynomial identities for the Jordan algebra of a symmetric bilinear form, J.Algebra 1o8 (1987), No.1, 66-87.

25. DRENSKY,V.S., KOSHLUKOV,P.E., Polynomial identities for quadratic Jordan algebras, Dokl.Bolgarsk.Akad.Nauk 40 (1987), No.9, 5-8.

26. EL-AGAWANY,M., Algèbre mésonique d'une forme bilineaire symétrique II, Arch.Math. (Basel) 45 (1985), 5, 442-450, (for part I see EL-AGAWANY,M. and MICALI,A., Real.Sem.Mat.Univ. Politec. Torino 38 (1980), 1-12).

27. EL-AGAWANY,M., MICALI,A., Algèbre mésonique d'un module, Comm. Algebra 13 (1985), 399-418.

28. FAULKNER,J.R., a) Octonion planes defined by quadratic Jordan algebras, Memoirs Amer.Math.Soc.,No.1o4 (1970).
b) A construction of Lie algebras from a class of ternary algebras, Trans.Amer.Math.Soc. 155 (1971), 397-4o8.
c) A geometry of E_7, Trans.Amer.Math.Soc. 167 (1972), 49-58.
d) Groups with Steinberg relations and coordinatization of polygonal geometries, Memoirs Amer.Math.Soc., 10 (1977), Nr.185.
e) Identity classification in triple systems, J.Algebra, 94 (1985), 2, 352-362

29. FAULKNER,J.R., FERRAR, J.C., a) Exceptional Lie algebras and related algebraic and geometric structures, Bull.London Math.Soc. 9 (1977), 1, 1-35.

b) Simple anti-Jordan pairs, Comm. Algebra, 8 (1980), 11, 993-1o13.

30. FERRAR,J.C., On the classification of Freudenthal triple systems and Lie algebras of type E_7, J.Algebra, 62 (1980), 2, 276-282.

31. FREUDENTHAL,H., Beziehungen der E_7 und E_8 zur Oktavenebene I, Nederl.Akad.Weten.Proc.Ser.A, 57 (1954), 218-230.

32. GHIKA,Al., a) "Paranormed" modules (in Romanian), Comm. Acad. RPR, 5 (1955), No.2, 317-324.

b) Algebras of field type (in Romanian), Comm.Acad. RPR, 6 (1956), No.2, 239-244.

c) Ternary-linear algebrico-topological structure (in Romanian) Comunicările Academiei RPR, 8 (1958), 447-450.

d) Ternary algebraic structures (in Romanian), Comunicările Academiei RPR 8 (1958), 457-461.

33. GONZÁLEZ,S., MARTINEZ, C., a) Order relation in Jordan rings and a structure theorem, Proc.Amer.Math.Soc. 98 (1986), No.3, 379-388.

b) Order relation in quadratic Jordan rings and a structure theorem, Proc.Amer.Math.Soc. 1o4 (1988), No.1, 51-54.

34. GORDON,S.R., a) The components of the automorphism group of a Jordan Algebra,Trans.Amer.Math.Soc.,153 (1971), 1-52.

b) An integral basis theorem for a Jordan algebra, J.Algebra, 24 (1973), 258-282.

c) On the structure group of a split semisimple Jordan algebra I, II, Comm.Algebra 5 (1971), 10, 1009-1023, 1025-1055.

35. GUANGYU,S., On Lie algerbas associated with nodal noncommentative Jordan algebras, Acta Math.Sinica N.S. 2 (1986), No.1, 14-24.

36. GÜNAYDIN,M., Quadratic Jordan formulation of quantum mechanics and construction of Lie (super) algebras from Jordan (super) algebras, in "Group theoretical methods in physics", pp.279-296, Ann.Israel Phys.Soc. 3, Hilger, Bristol, 1980.

37. GÜNAYDIN,M., SIERRA,G., TOWNSEND,P.K., Exceptional supergravity theories and the magic square, Phys.Lett. 133 B (1983), 1/2, 72-76.

38. HEEG,R.A., a) A theorem on central fixed Jordan rings, Houston J.Math. 10 (1984), 495-5o1.

b) Jordan automorphisms on direct sums of simple rings, J. Korean Math. Soc. 21 (1984), 31-39.

39. HEIN,W., a) A construction of Lie algebras by triple systems, Trans.Amer.Math.Soc. 2o5 (1975), 79-95.

b) On the structure of reduced J-ternary algebras of degree >2, Math.Z. 176 (1981), 4, 521-539.

c) On the structure of reduced J-ternary algebras of degree two. J.Algebra 82 (1983), 1, 157-184.

d) Durch fraduierte Lie-Algebren definierte Paar-Algebren, Manuscripta Math. 52 (1985), No.1-3, 97-130.

e) Imbedding of pair algebras into Lie algebras, Proc. Koninkl Acad. Wetensch. A 89 (1986), No.3, 319-336.

40. HIRZEBRUCH,U., a) A generalization of Tits'construction of Lie algebras by Jordan algebras to Jordan triple systems, Nederl.Akad.Wetensch.Indag.Math.,40 (1978), 4, 456-459.

b) Invariant polynomial functions on Jordan algebras, Algebras Groups Geom. 1 (1984), No.4, 442-445.

c) Ein Einfachheitskriterium für nicht-assoziative Algebren, Proc.Koninkl.Nederl.Akad. Wetensh.A 90 (1987), No.4,393-396.

41. HOGBEN,L., McCRIMMON,K., Maximal modular inner ideals and the Jacobson radical of a Jordan algebra, J.Algebra 68 (1981), No.1, 155-169.

42. HOPKINS,N.C., Decompositions of Lie module triple systems, Tagunsbericht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

43. IL'TYAKOV,A.V., Radical of a finitely generated Jordan PI-algebra, Siberian Math.J. 26 (1985), 3, 342-344.

44. ION,D.B., a) On Jordan algebras of type A (in Romanian), Stud.Cerc.Mat. 17 (1965), 3o1-31o.

b) On some properties of the roots of Jordan algebras of type A (in Romanian). Stud.Cerc.Mat. 18 (1966), 3o9-313.

c) On some properties of the structure constants of Jordan algebras of type A (in Romanian), Stud.Cerc.Mat. 19 (1967), 1o31-1o37.

d) On Jordan algebras of types B-E (unpublished).

45. IORDANESCU,R., a) Sur des représentations des algèbres $A^{p}_{m,n}$, Rev.Roumaine Math.Pures Appl.,10 (1965), 14o3-1421.

b) Les représentations quasi-irréductibles des algèbres $A^{2}_{m,n}$ dans des algèbres $A^{q}_{m',n'}$, Rev.Roumaine Math.Pures Appl. 10 (1965), 1583-1591.

c) Jordan structures and applications, Lectures given at the University of Timişoara, December 1983.

46. IORDANESCU,R., POPOVICI,I., On representations of special Jordan algebras, Rev.Roumaine Math.Pures Appl., 13 (1968), 1o89-11oo.

47. JACOBSON,N., a) Structure and representations of Jordan algebras, Amer.Math.Soc.Colloq.Publ. 39 Amer.Math.Soc., Providence, 1968.

b) Exceptional Lie algebras, Marcel Dekker, NewYork, 1971.

c) Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution, Advances in Math., 20 (1976), 2, 106-150.

d) Structure theory of Jordan algebras, Univ.of Arkansas, Lecture notes in math. vol.5, 1981.

e) Forms of the generic norm of a separable Jordan Algebra, J.Algebra 86 (1984), 1, 76-84.

f) Generic splitting fields revisited, Tagungabericht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

g) Jordan algebras of real symmetric matrices, Algebras Groups Geom. 4 (1987), 291-304.

48. JORDAN,P., a) Über eine Klasse nicht assoziatieverhyperkomplexen Algebren, Gött. Nachr. (1932), 569-575.

b) Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik, Gött, Nachr. (1933), 209-217.

c) Über die Multiplikation quantenmechanischen Grössen, Z. Phys. 80 (1933), 285-291.

49. JORDAN,P., von NEUMANN,J., WIGNER,E., On an algebraic generalization of the quantum mechanical formalism, Ann. Math., 35 (1934), 29-64.

50. KAKIICHI,Y., a) The relation between Hein's construction of Lie algebras and the construction of Lie algebras from symplectic triple systems. Comment,Math.Univ.St.Paul. 26 (1977/78), 2, 195-199.

b) Another construction of Lie algebras by generalized Jordan triple systems of second order, Proc.Japan Acad.Ser. A Math.Sci. 57 (1981), 3, 194-198.

c) A construction of Lie-graded algebras by grade generalized Jordan triple o second order, Proc.Japan Acad. Ser. A Math.Sci. 57 (1981), 5, 276-280.

d) A construction of Lie algebras and Lie superalgebras by Freudenthal-Kantor triple system I, Proc.Japan Acad.Ser. Math. 61 (1985), 7, 232-234.

51. KAMIYA,N., A structure theory of Freudenthal-Kantor triple systems, J.Algebra 110 (1987), No.1, 1o8-123.

52. KANEYUKI,S., The Sylvester's law of inertia for Jordan algebras, Proc.Japan Acad.Ser.A 64 (1988), No.8, 311-313.

53. KANEYUKI,S., ASANO,H., Graded Lie algebras and generalized Jordan triple systems, Nagoya Math. J. 112 (1988), 81-115.

54. KANTOR I.L., a) Classification of irreducible transitively differential groups (in Russian), Dokl.Akad.Nauk SSSR, 158 (1964), 1271-1274.

b) Transitive differential groups and invariant connections in homogeneous spaces (in Russian), Trudy Sem.Vektor. Tenzor. Anal. 13 (1966), 31o-398.

c) Some generalizations of Jordan algebras (in Russian), Trudy Sem.Vektor.Tenzor.Anal. 16 (1972), 4o7-499.

d) Models of exceptional Lie algebras (in Russian). Dokl.Akad. Nauk SSSR, 2o8 (1973), 1276-1279.

55. KANTOR,I.L., SKOPETS,I.M., Freudenthal trilinear systems (in Russian), Trudy Sem.Vektor.Tenzor.Anal.18 (1978), 25o-263.

56. KATS,V.G., Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras, Comm.Algebra, 5 (1971), 13, 1375-1400. (Erratum: Hogben,L., Kats, V.G., Comm. Algebra 11 (1983), 10, 1155-1156).

57. KHAN,M.R., a) On quasicommutative Jordan algebras, Tamkang J.Math. 9 (1978), No.2, 171-188.

b) The Wedderburn principal theorem for quasicommutative Jordan algebras, Tamkang J.Math. 11 (1980), No.2, 237-248.

58. KHAN,M.R., KAZIM,M.A., On quasicommutative Jordan algebras, Math.Japon. 24 (1979/80), No.5, 479-487.

59. KOECHER,M., a) Analysis in reellen Jordan-Algebren, Nachr. Akad. Wiss. Göttingen (1958), 67-74.

b) Jordan algebras and their applications, Lecture Notes of the University of Minneapolis, 1962.

c) Imbedding of Jordan algebras into Lie algebras, I,Amer. J.Math. 89 (1967), 787-816.

d) On Lie algebras defined by Jordan algebras, Aarhus Univ. Lecture Notes, Aarhus, 1967.

e) Über Standard-Konstructionen von nicht-assoziativen Algrbren, Bayer, Akad.Wiss.Math.-Natur.Kl.S.-B. 1974, 35-57 (1975).

f) Eine Konstruktion von Jordan-Algebren, Manustripta Math. 23 (1977/78), 4, 387-425.

g) On commutative nonassociative algebras, J.Algebra 62 (1980), 2, 479-493.

h) Von Matrizen zu Jordan-Tripelsystemen, Lectures at the Rhine-Westphalia Academy of Sciences, No.307, pp.53-68. Westdeutscher Verlag, Opladen, 1982.

i) The generalized inverse of integral matrices, Linear Algebra Appl. 71 (1985), 187-198.

60. KOSHLUKOV,P., Polynomial identities for a family of simple Jordan algebras, Comm. Algebra 16 (1988), No.7, 1325-1372.

61. KOSTRIKIN,A.I., On the Burnside problem (in Russian), Izv.Akad. Nauk SSSR, Ser.Mat., 23 (1959), 3-34.

62. KÜHN,O., a) Construction of Jordan pairs by the quasi-inverse, J.Algebra, 50 (1978), 265-273.

b) Classification of central simple Jordan superalgebras, Tagungsbericht 33/1982, Jordan-Algebren (8.8-14.8,1982), Ober-

63. KÜHN,O., ROSENDAHL,A., Wedderburnzerlegung für Jordan-Paare, Manustripta Math. 24 (1978), 4, 4o3-435.

64. KUROSH,A.G., Problems in ring theory which are related to Burnside problem on periodic groups (in Russian), Iz.Akad. Nauk SSSR, Ser.Mat. 5 (1941), 233-240.

65. LOOS,O., a) Lectures on Jordan triples, Lecture Notes Univ. of British Columbia, Vancouver, 1971.

b) A structure theory of Jordan pairs, Bull.Amer.Math.Soc. 80 (1974), 67-71.

c) Jordan pairs, Lecture notes in math. 460, Springer, Berlin-Heidelberg-New York, 1975.

66. LOPEZ,A.F., a) Modular annihilator Jordan algebras, Comm. Algebra 13 (1985), No.12, 2597-2614.

b) Ideals in nondegenrate noncommutative Jordan algebras, Comm.Algebra 14 (1986), No.6, 1111-1116.

67. LOPEZ,A.F., PALACIOS,A.R., a) Primitive noncommutative Jordan algebras with nonzero socle, Proc.Amer.Math.Soc. 96 (1986), No.2, 199-2o6.

b) On the socle of a noncommutative Jordan algebra, Manuscripta Math. 56 (1986), No.3, 269-278.

68. LOUSTAU,J.A., The Arens closure of a nonassociative algebra, Comm. Algebra, 13 (1985), 12, 2523-2535.

69. MARTINDALE III, W.S., A note on nilpotent Jordan rings, Canadian Math.Bull. 30 (1987), No.4, 399-4ol.

70. MARTINDALE III, W.S., McCRIMMON,K., Imbedding nondegenerate Jordan algebras in semi-primitive algebras, Proc.Amer.Math. Soc. 1o3 (1988), No.4, 1o31-1o36.

71. McCRIMMON,K., a) A general theory of Jordan rings, Proc.Nat. Acad.Sci. USA 56 (1966), 1o72-1o79.

b) Quadratic Jordan algebras and cubic operations, Trans. Amer.Math.Soc. 153 (1971), 265-278.

c) Quadratic methods in nonassociative algebras, Proc. Internat.Congress Mathematicians 1974, Vancouver, 325-330.

d) Axioms for inversion in Jordan algebras, J.Algebra 47 (1977), 2o1-222.

e) Jordan algebras and their applications, Bull.Amer.Math. Soc. 84 (1978), 4, 612-627.

f) The Zel'manov nilpotence theorem for quadratic Jordan algebras, J.Algebra, 63 (1980), 1, 76-97.

g) The Zel'manov annihilator and nilpotence of the nilradical in quadratic Jordan algebras with chain conditions, J. Algebra, 67 (1980), 1, 23o-253.

h) Zel'manov prime theorem for quadratic Jordan algebras, J.Algebra 76 (1982), 2, 297-326.

i) The Russian revolution in Jordan algebras, Algebras Groups Geom. 1 (1984), 1-61.

j) Strong prime inheritance in Jordan systems, Algebras Group Geom.1 (1984), 217-234.

k) Amitsur shrinkage of Jordan radicals, Comm. Algebra 12 (1984), 7/8, 777-826.

l) Invariance of Jordan radicals, Comm. Algebra 12 (1984), 7/8, 827-855.

m) Adjoints and Jordan Algebras, Comm. Algebra, 13 (1985), 12, 2567-2596.

n) Reduced elements in Jordan triple systems, J.Algebra 97 (1985), 54o-564.

o) Semiprimenese of special Jordan algebras, Proc.Amer.Math. Soc. 96 (1986), 1, 29-33.

p) Semiprimeness of special Jordan algebras, Proc. Amer. Math. Soc. 96 (1986), No.1, 29-33.

r) A not-natural s-identity, Comm. Algebra 15 (1987), No.10, 2099-2118.

72. McCRIMMON, K., MEYBERG, K., Coordinatization of Jordan triple systems, Comm. Algebra 9 (1981), No.14, 1495-1542.

73. McCRIMMON, K., NEHER, E., Coordinatization of triangulated Jordan systems, J. Algebra, 114 (1988), No.2, 411-451.

74. McCRIMMON, K., ZEL'MANOV, E.I., The structure of strongly prime quadratic Jordan algebras, Advances Math. 69 (1988), No2, 133-222.

75. MEDVED'EV, Yu.A., a) Identities of finite Jordan Φ-algebras (in Russian), Algebra i Logika, 18 (1979), 6, 723-748.

b) On nil-elements of free Jordan algebras (in Russian), Sibirk. Mat. Zh. 26 (1985), 2, 140-148.

c) An analogue of Andrunakievich's lemma for Jordan algebras Sibir. Math. J. 28 (1987), No.6, 928-935.

d) Representations of finitely generated Jordan PI-algebras (in Russian), Izv. Akad. Nauk SSSR 52 (1988), No.1, 64-78.

e) Radicals of free finite-generated Jordan Algebras (in Russian), Sibirsk. Mat. Zh. 29 (1988), No.4, 139-148.

76. MEDVED'EV, Yu.A., ZEL'MANOV, E.I., Solvable Jordan algebras, Comm. Algebra 13 (1985), No.6, 1389-1414.

77. MEYBERG, K., a) Eine Theorie der Freudenthalschen Tripelsysteme, I, II, Nederl. Akad. Weten. Proc. Ser. A 71 (1968), 162-174, 175-190.

b) Jordan-Triplesysteme und die Koecher-Konstruktion von Lie-Algebren, Habilitionsschrift, München, 1969.

c) Jordan-Tripelsystemen unde die Koecher-Konstrucktion

d) Zur Konstruktion von Lie-Algebren aus Jordan-Tripelsystemen, Manustripta Math. 3 (1970), 115-132.

e) Lecture on algebras and triple systeme, Lecture notes, The University of Virginia, Charlottesville, 1972.

f) Trace formulas and derivations in simple Jordan pairs, Comm. Algebra 12 (1980), 1311-1326.

78. NEHER, E., a) Klassification der einfachen reellen Jordan-Tripelsysteme, Dissertation, Münster, 1978.

b) Klassification der einfachen reellen speziellen Jordantripelsysteme, Manuscripta Math. 31 (1980), 197-215.

c) Involutive gradings of Jordan structures, Comm. Algebra 9 (1981), No.6, 575-599.

d) Klassifikation der einfachen reellen Ausnahme-Jordan-Tripelsysteme, J.Reine Angew.Math. 322 (1981), 145-169.

e) On the classification of Lie and Jordan triple systems, Habilitationschrift, Münster, 1983

f) Jordan riple forms of Jordan algebras, Proc.Amer.Math. Soc. 87 (1983), No.3, 386-388.

g) On triagonal elements and centralizers in Jordan triple systems, J.Algebra 90 (1984), No.1, 18-36.

h) Jordan triple systems with completely reducible derivation or structure algebras, Pacific J. Math. 113 (1984), No.1, 137-164.

i) On the classification of Lie and Jordan triple systems, Comm. Algebra, 13 (1985), 12, 2615-2667.

j) Grids in Jordan triple systems, Tagungebericht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

k) Jordan triple systems by the grid approach, Lecture Notes in Math. 1280, Springer 1987.

79. NÉMETI,I., Decidability of relation algebras with weakened associativity, Proc.Amer.Math.Soc. 100 (1987), No.2,34o-344.

80. NIKITIN,A.A., a) On the heredity of radicale of rings (in Russian), Algebra i Logika 17 (1978), 3, 3o3-315.

b) On minimal radicals of some class of rings (in Russian), Algebra i Logika 17 (1978), 5, 596-610.

81. OSBORN,J.M., a) Varieties of algebras, Advances in Math. 8 (1972), 2, 163-369.

b) Modules over nonassociative rings, Comm. Algebra, 6 (1978), 13, 1297-1358.

c) Lie admissible noncommutative Jordan loop rings, Algebras Groups Geom. 1 (1984), 453-489

82. OSBORN,J.M., RACINE,M., a) Jordan rings with nonzero socle, Trans.Amer.Soc. 251 (1979), 375-387.

b) Prime Jordan PI algebras with nonzero socle and Jordan division algebras, J.Algebra 62 (1980), 2, 249-261.

83. PATERA,J., ROUSSEAU,C., Versal deformations of elements of classical Jordan algebras, J.Math.Phys. 24 (1983), 6, 1375-1380.

84. PCHELINCHEV, S.V., a) Solvable index 2 varieties of algebras (in Russian), Mat.Sb.(N.S.) 115 (1981), 2, 179-2o3.

b) Absoluts zero divisors of special Jordan algebras (in Russian), Algebra i Logika 21 (1982), 6, 7o6-72o.

85. PETERSSON,H.P., a) On linear and quadratic Jordan division algebras, Math.Z. 177 (1981), No.4, 541-548.

b) Generic reducing fields of Jordan pairs, Trans.Amer.Math. Soc. 285 (1984), No.2, 825-843.

86. PETERSSON,H.-P., RACINE,M.L., a) Cubic subfields of exceptional simple Jordan algebras, Proc.Amer.Math.Soc.91 (1984), 31-36.

b) A norm theorem for central simple algebras of degree 3, Arch. Math. (Basel) 42 (1984), 3, 224-228.

c) Springer forms and the first Tits construction of exceptional Jordan division algebras, Manuscripta Math. 45 (1984), 3, 249-272.

d) The toral Tits process of Jordan algebras, Abh. Math. Sem. Univ. Hamburg 54 (1984), 251-256.

e) Jordan algebras of degree 3 and the Tits process, 98 (1986), 1, 211-254.

f) Classification of algebras arising from the Tits process, 98 (1986), 1, 244-279.

87. PIACENTINI-CATTANEO,G.M., A method for recognizing certain varieties of nonassociative algebras, J. Algebra 97 (1985), 2, 347-375.

88. POPUTA,V., a) Duales de Jordan, Bul.Stiinţ.Tehn.Inst. Politehn. "Traian Vuia", Timişoara, Nov.1985, 29-30.

b) Derivées géneralisées sur les paires de Jordan, Bul. Stiinţ.Techn.Inst. Politehn. "Traian Vuia", Timişoara, Nov. 1985, 27-3o.

c) Algèbres de Lie associées à une duale de Jordan, Proc. Symposium Math.and its Appl. Timişoara, Nov.1985, 127-13o.

89. RACINE,M.L., a) Minimal identities for Jordan algebras of degree 2, Comm.Algebra, 13 (1985), 12, 2493-25o6.

b) Identities of low degree algebras, Tagungsbericht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

90. RAO,S.D. RAMASWAMY,V., Invertibility in Jordan triple systems, Math. Japon. 29 (1984), 5o9-517.

91. RIMOW,G., Eine Konstruktion von Lie-Algebren aus Jordan-Tripel-Systemen, Manuscripta Math., 34 (19[illegible]1), 2/3,17[illegible]-1[illegible]7.

92. ROOS,H., A note on Jordan automorphisms, Ann.Inst. H.Poincaré, Sect. A 42 (1985), 1, 17-30.

93. SCHAFER,R.D., a) On structurable algebras, J.Algebra, 92 (1985), 400-412.

b) Structurable algebras and bimodules, Tagungsbericht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

94. SCHWARZ, T., a) Einfache Jordan Tripelsysteme, Dissertation, Hagen, 1983.

b) Automorphism group of Jordan pairs, Algebras Groups Geom. 1 (1984), 49o-5o9.

c) Special simple Jordan triple systems, Algebra Groups Geom. 2 (1985), 117-128.

d) Simple exceptional 16-dimensional Jordan triple systems, Proc.Amer.Math.Soc.100 (1987), No.4, 623-626.

e) Simple exceptional Jordan triple systems in characteristic 2, Comm. Algebra 16 (1988), No.11, 2247-2257.

95. SHESTAKOV,I.P., a) On some classes of noncommutative Jordan rings (in Russian), Algebra i Logika, 10 (1971), 4,4o7-448.

b) Finitely generated special Jordan algebras and alternative PI-algebras (in Russian), Mat.Sb. 122 (1983), 31-4o.

c) The varieties of algebras that are nearly associative, Tagungaberícht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

96. SHIRSHOV,A.I., a) On certain nonassociative nil-rings and algebraic algebras (in Russian), Mat.Sb. 41 (1957), 381-394.

b) Some problems in the theory of rings which are nearly associative (in Russian), Uspehi Mat.Nauk 13 (1958), 3-20.

c) Selected works. Rings and algebras, Nauka, Moskva, 1984.

97. SHTERN,A.S., a) Superdifferentiation of simple Jordan superalgebras (in Russian), Sibirsk.Mat.Zh. 28 (1987),No.5,223.

b) Representations of exceptional Jordan algebras (in Russian), Funktsional.Anal.i Prilozhen.21 (1987), No.3, 93-94.

98. SKOSYRSKII,V.G., Nilpotency in Jordan and right alternative algebras (in Russian), Algebra i Logika 18 (1979), 1,73-85.

99. SKOSYRSKII,V.G., ZEL'MANOV,E.I., On the solvable special Jordan algebra (in Russian), Algebra i Logika

100. SLIN'KO,A.M. a) On radicals of Jordan rings (in Russian), Algebra i Logika 11 (1972), 2, 206-215.

b) On the Jacobson radical and absolute zero divisors in special Jordan algebras (in Russian), Algebra i Logika 11 (1972), 6, 711-723.

c) Radicals of Jordan rings connected to alternative (in Russian), Mat.Zametki 16 (1974), 1, 135-140.

d) On special varieties of Jordan algebras (in Russian), Mat.Zametki 26 (1979), 3, 337-344.

e) Structure of alternative and Jordan compact rings (in Russian), Mat.Sb. 129 (1986), 3, 378-385.

f) Modules and representations of nonassociative topological compact rings (in Russian), Mat.Zametki 39 (1986), 6,791-797.

g) Nonassociative topological bimodules, Mat.Sb. 133 (1987), No.1-2, 259.

h) Locally-compact Jordan rings (in Russian), Mat.Zametki 43 (1988), No.6, 713-724.

101. SPRINGER,T.A., Jordan algebras and algebraic groups, Springer, Berlin-Heidelberg-New York, 1973.

102. SVERCHKOV,S.R., a) On solvable Jordan algebras of index two (in Russian), Mat.Sb. 121 (1983), 1, 40-47.

b) Quasivariety of special Jordan algebras (in Russian), Algebra i Logika 22 (1983), 5, 567-573.

c) On strong associative Jordan algebras (in Russian), Sibirsk.Mat.Zh. 29 (1988), No.6, 111-122.

d) Varieties of special algebras, Comm. Algebra 16 (1988), No.9, 1877-1920.

103. THEDY,A., a) Radicals of right alternative and Jordan rings, Comm. Algebra 12 (1984), 857-887.

b) Z-closed ideals of quadratic Jordan algebras, Comm. Algebra 13 (1985), 12, 2537-2566.

c) A natural s-identity of Jordan algebras, Comm. Algebra 15 (1987), No.10, 2081-2098.

104. TILGNER,H., a) A class of solvable Lie groups and their relation to the cannonical formalism, Ann.Inst. H.Poincaré Sect. A (N.S.) 13 (1970), 103-127.

b) A spectrum generating nilpotent group for the relativistic free particle, Ann.Inst. H.Poincaré, Sect.A (N.S.), 13 (1970), 129-136.

c) A class of Lie and Jordan algebras realized by means of the canonical commutation relations, Ann.Inst.H.Poincaré Sect.A (N.S.) 14 (1971), 179-188.

d) Symmetric Lie algebras on non-linear transformations of conformal type in quantum mechanics, Internat.J.Theoret. Phys. 7 (1973), No.5, 379-388.

e) Private communication via correspondênce, November 1980.

105. TITS,J., a) Un classe d'algèbres de Lie en relation avec les algèbres de Jordan, Nederl.Akad.Wetensch Proc.Ser. A 65 = Indag.Math. 24 (1962), 530-534.

b) Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionelles, Nederl.Akad.Weten.Proc.Ser. A, 69 (1966), 223-237.

106. TOLOKONNIKOV,G.K., On classification of function algebras that are nearly associative by means of the functional equations method (in Russian), Mat.Zametki 39 (1986), 6, 8o6-818.

107. WATERHOUSE,W.C., Symmetric determinants and Jordan norm similarities in characteristic 2, Proc.Amer.Math.Soc. 93 (1985), 4, 583-589.

108. WATSON,K.S., Manifolds of algebraic type over Jordan pairs, Ph.D.Thesis, Flinders Univ. of South Australia, 1978.

109. YAMAGUTI,K., a) On weak representations of a class of ternary systems, Mem.Fac.Gen.Ed. Kumamoto Univ.Natur.Sci. no.9 (1974), 1, 1-8.

b) Constructions of Lie (super) algebras from Freudenthal-Kantor triple system U(ε,δ), in Proceedings of 14th ICGTMP, Seoul, Korea, 1985. This was talked also at the meeting on Jordan algebras (4.8.-10.8.1985), Oberwolfach.

110. ZEL'MANOV,E.I., a) Jordan algebras with finiteness conditions (in Russian), Algebra i Logika 17 (1978), 693-7o4.

b) Jordan nil-algebras of bounded degree (in Russian), Dokl. Akad.Nauk SSSR 249 (1979), 1, 3o-35.

c) Jordan division algebras (in Russian), Algebra i Logika 18 (1979), 3, 286-31o.

d) Absolute zero divisor and algebraic Jordan algebras (in Russian), Sibirsk.Mat. Z 23 (1982), 1oo-116.

e) Lie algebras with finite gradings (in Russian), Materials of the 5th All-Union Ring Theory Conference, Novosibirsk, 1982, p.58.

f) Classification theorems for Jordan systems (in Russian), Sibirsk.Mat. Zh. 23 (1982), 186-187.

g) On prime Jordan algebras I, II (in Russian), Algebra i Logika 18 (1979), 2, 162-175; Sibirsk.Mat.Zh. 24 (1983), 1, 73-85.

h) On the theory of Jordan algebras, Proc.Intern.Congress Mathematicians, August, 15-24, 1983, Warzawa, P.W.N. & North-Holland, 1984, vol.I, pp.455-463.

i) On prime Jordan triple systems I,II,III (in Russian), Sibirsk.Math.Zh. 24 (1983), 23-37; 25 (1984), 5o-61; 26 (1985), 71-76.

j) Characterization of the McCrimmon radical (in Russian) Sibirsk.Mat. Zh. 25 (1984), 5, 19o-192.

k) Lie algebras with a finite grading, Math.USSR - Sb. 124 (1984), 347-386.

l) Jordan superalgebras, Lie algebras with Engel property and Jordan triple systems, Tagungaberricht Jordan-Algebren (4.8.-10.8.1985), Oberwolfach.

m) Birepresentations of infinite-dimensional Jordan algebras (in Russian), Sibirsk.Mat.Zh. 27 (1986),No.6, 849-862.

n) Goldie's theorems for Jordan algebras I, II, Sibir. Math. J. 28 (1987),No.6,899-9o5; 29 (1988), No.4, 68-74.

111. ZEL'MANOV,E.I., SKOSYRSKII,V.G., Special Jordan ni-algebras of bounded index (in Russian), Algebra i Logika 22 (1983), No.6, 626-635.

112. ZHELYABIN,V.N., The theorem on the splitting of the radical for Jordan algebras over a Hensel ring (in Russian), in "Groups and other algebraic systems with finiteness conditions" (in Russian), Novosibirsk, 1984, 5-28.

113. ZHEVLAKOV,K.A., Radicals and representations of alternative rings (in Russian),Algebra i Logika 11 (1982), 2, 162-173.

114. ZHEVLAKOV,K.A., SHESTAKOV,I.P., On local finiteness in the sense of Shirshov (in Russian),Algebra i Logika 12 (1973),1, 41-73.

115. ZHEVLAKOV,K.A., SLIN'KO,A.M., SHESTAKOV,I.P.,SHIRSHOV,A.I., Rings that are nearly associative (in Russian),Nauka, Moscow,1978 (English translation: Academic Press,New Jork,1982).