INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

JORDAN STRUCTURES WITH APPLICATIONS. II
JORDAN ALGEBRAS IN PROJECTIVE GEOMETRY

by

R. IORDANESCU

PREPRINT SERIES IN MATHEMATICS
No. 29/1990

BUCURESTI

JORDAN STRUCTURES WITH APPLICATIONS. II

JORDAN ALGEBRAS IN PROJECTIVE GEOMETRY

by

R. IORDANESCU*)

April 1990

*) Institute of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania.

JORDAN STRUCTURES WITH APPLICATIONS - II

JORDAN ALGEBRAS IN PROJECTIVE GEOMETRY

Radu IORDANESCU

§ 1. Octonion planes

The first investigation of octonion planes dates from 1933 and is due to MOUFANG [47]. It consisted in the construction of a projective plane coordinatized with an octonion division algebra. In this Moufang plane, Desargues'Theorem fails but the Harmonic Point Theorem is valid.

Another approach to octonion planes was given in 1945 by JORDAN [33] via the Jordan algebra $H_3^{(+)}(\mathcal{O})$. Recall the definition of the exceptional Jordan algebra $H_3^{(+)}(\mathcal{O})$: Let $H_3(\mathcal{O})$ be the set of all (3×3)-matrices with entries in an octonion algebra $\mathcal{O}$ and which are symmetric with respect to the involution $x \to \bar{x}'$. The characteristic of the underlying field is supposed to differ from two. On $H_3(\mathcal{O})$ we can define a Jordan algebra structure by means of the product $xy := \frac{1}{2}(x\cdot y+y\cdot x)$, where the dot means the usual matrix product. The resulting Jordan algebra is denoted by $H_3^{(+)}(\mathcal{O})$. Jordan focussed on a real octonion division algebra $\mathcal{O}$ and used the primitive idempotents in $H_3^{+}(\mathcal{O})$ to represent the points and lines of a projective plane. (Two years later, FREUDENTHAL [22] obtained essentially the same construction)[1)]. ATSUYAMA used the embedding defined by YOKOTA [71] to obtain [4] new results in this direction.

The construction was extended in 1960 by SPRINGER [59a], who considered $\mathcal{O}$ as an octonion division algebra over a field of charac

1) For a recent systematic treatise on applications of the real algebra of Cayley numbers, we refer the reader to BRADA's Ph.D. Thesis [11].

teristic different from two or three. In this more general setting, elements of rank one (which are either non-zero multiples of primitive idempotents or nilpotents of index two) are used to represent the points and lines of a projective plane. Springer proved the fundamental theorem relating collineations of the plane and norm semisimilarities of the Jordan algebra. JACOBSON [31] showed that the little projective group, i.e. the group generated by elations (transvections) of these planes, is simple and isomorphic to the norm-preserving group of the Jordan algebra modulo its centre. SUH [63] showed that any isomorphism between the little projective groups of two planes is induced by a collineation or correlation of the planes. SPRINGER and VELDKAMP [66a] undertook a study of Hermitian polarities of a projective octonion plane and the related hyperbolic and eliptic planes. The unitary group of collineations commuting with a hyperbolic polarity was studied by VELDKAMP [66c].

SPRINGER and VELDKAMP [60b] considered planes associated with split (i.e. not division) octonion algebras over a field of characteristic different from two or three. These planes are not projective. (For the study of these planes, see VELDKAMP [66b,d]).

In 1970, FAULKNER [18a] extended the notion of octonion planes in another direction by removing the restriction that the characteristic of the underlying field be different from two or three. After McCRIMMON [46a] had introduced the notion of quadratic Jordan algebra and verified that $H_3^{(+)}(\mathcal{O})$ possesses such a structure for any characteristic, Jacobson suggested to Faulkner (see [66 a, p.3]) that characteristic-two octonion planes could be approached in this way. As it turned out, in the setting of quadratic Jordan algebras, most of the results on octonion planes can be derived in a uniform manner, without referring to the characteristic or the type of an octonion algebra.

For collineation groups of projective planes over degenerate octaves or antioctaves, see PERSITS [48 a,b].

DAVIES [16] studied bi-axial actions on projective planes (including octonion planes), making use of their Jordan algebra description.

BIX [9a] has defined and studied octonion planes over local rings. He generalized Faulkner's result on the simplicity of PS (see for definition on page 6) of an octonion plane over a field to octonion planes over local rings. Those subgroups of the collineation group of an octonion plane over a local ring which are normalized by the little projective group have been classified. This parallels results of KLINGENBERG and BASS, who classified those subgroups of the general linear group over a local ring which are normalized by the special linear group.

In [9b] BIX proved two main theorems about octonion planes over local rings (see Theorems 2.16 and 2.17 below).

Following FAULKNER [18a] and BIX [9 a,b,c], we shall present here the octonion plane for an arbitrary octonion algebra over a field of arbitrary characteristic (or over a local ring, or over an Euclidean domain), as well as its basic geometrical structure.

<u>The (quadratic) Jordan algebra</u> $H_3(\mathcal{O},\Upsilon)$. Let $\mathcal{O}$ be an octonion algebra over an arbitrary field $\mathbb{F}$ (see SCHAFER [53, Ch. III, § 4]). We have an involution $a \to \bar{a}$, $a \in \mathcal{O}$; a trace $t(a) \in \mathbb{F}$, $a + \bar{a} = t(a)\, 1$; a norm $n(a) \in \mathbb{F}$, $a\bar{a} = \bar{a}a = n(a)\, 1$; and a symmetric nondegenerate bilinear form $n(a,b) := n(a+b) - n(a) - n(b)$. Let $\gamma_1, \gamma_2, \gamma_3$ be nonzero elements of $\mathbb{F}$ and let $\Upsilon = \operatorname{diag}\{\gamma_1, \gamma_2, \gamma_3\}$. Let $H_3(\mathcal{O},\Upsilon)$ be the subspace of matrices that are symmetric under the involution J_γ of the (3×3)-matrix algebra with entries in $\mathcal{O}$

given by $J_\gamma : x \to \gamma^{-1}\bar{x}'\gamma$. One can see that an element x of $H_3(\mathcal{O},\gamma)$ has the form:

$$x = \sum_{i=1}^{3} \alpha_i e_i + \sum_{i=1}^{3} a_i\,[jk] \quad \text{with } \alpha_i \in \mathbb{F},\ a_i \in \mathcal{O}.$$

Here (i,j,k) is a cyclic permutation of $(1,2,3)$;

$$a[ij] := \gamma_j a\, e_{ij} + \gamma_i \bar{a}\, e_{ji}\,, \qquad i \neq j,$$

in terms of the matrix units e_{ij}, and

$$1\,[ii] := e_{ii} = e_i.$$

$H_3(\mathcal{O},\gamma)$ can be viewed as a unital quadratic Jordan algebra $\mathcal{J}(N, \#, e)$ <u>via</u> the following definitions (see McCRIMMON [46 b]). If x is expressed as above and $y = \sum \beta_i e_i + \sum b_i[jk]$ with $\beta_i \in \mathbb{F}$, and $b_i \in \mathcal{O}$ then set

$$N(x) := \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_2\gamma_3\, n(a_1) - \gamma_1\alpha_2\gamma_3\, n(a_2) - \\ - \gamma_1\gamma_2\alpha_3\, n(a_3) + \gamma_1\gamma_2\gamma_3\, t(a_1a_2a_3)\,,$$

$$T(x,y) := \sum \alpha_i\beta_i + \sum \gamma_j\gamma_k\, n(a_i,b_i),$$

$$x^{\#} := \sum (\alpha_j\alpha_k - \gamma_j\gamma_k\, n(a_i))e_i + \\ + \sum (\gamma_i \overline{(a_j a_k)} - \alpha_i a_i)\,[jk],$$

$$e := e_1 + e_2 + e_3.$$

<u>Definition</u>. An element $x \in H_3(\mathcal{O},\gamma)$ such that $x \neq 0$ and $x^{\#} = 0$ is said to be of <u>rank one</u>.

<u>Notation</u>. Let $x,y \in H_3(\mathcal{O},\gamma)$ and set $T_{y,x} := \mathrm{Id} + V_{y,x} + U_yU_x$ (see KOECHER [37, p.142]). Let x,y,z be elements of rank one in $H_3(\mathcal{O},\gamma)$ such that $T(x,z) \neq 0 \neq T(y,z)$. Set $w := T^{-1}(y,z)y - T^{-1}(x,z)x$. Note that $T(w,z) = 0$ and that $T_{z,w}$ belongs to $S(H_3(\mathcal{O},\gamma))$. Put $T_{z;x,y} := T_{z,w}$.

<u>Definition</u>. $T_{z;x,y}$ is called <u>algebraic transvection</u> on

$H_3(\mathcal{O},\gamma)$ (see SPRINGER [59 b], and also FAUKNER [18 a]).

Notation. The quadratic Jordan algebra $H_3(\mathcal{O},\gamma)$ will be denoted by $\mathfrak{J}$. Denote by Π the set of elements of rank one in $\mathfrak{J}$. For $x \in \Pi$, let x_* add x^* be two copies of the set $\{\alpha x \mid \alpha \in \mathbb{F}-\{0\}\}$.

Definition. The octonion plane $\mathcal{P}(\mathfrak{J})$ consists of points $x_*, x \in \Pi$, and lines y^*, $y \in \Pi$, under the following relations

a) $x_* | y^*$, x_* incident to y^*, if $V_{y,x} = 0$;

b) $x_* \simeq y^*, x_*$ connected to y^*, if $T(x,y) = 0$;

c) $x_* \simeq y_*$, x_* connected to y_*, if $x \times y = 0$;

d) $x^* \simeq y^*$, x^* connected to y^*, if $x \times y = 0$.

Remark. If the characteristic of $\mathbb{F}$ is different from two then the above definition is equivalent to that of SPRINGER and VELDKAMP [60 b].

Notation. Let W be a semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}' = H_3(\mathcal{O}',\gamma')$ and define $\ulcorner W \urcorner : \mathcal{P}(\mathfrak{J}) \to \mathcal{P}(\mathfrak{J}')$ by $\ulcorner W \urcorner (x^*) := (\hat{W}(x))^*$, $x \in \Pi$.

Remark. $\ulcorner W \urcorner$ is a collineation of $\mathcal{P}(\mathfrak{J})$ onto $\mathcal{P}(\mathfrak{J}')$ (see the following definition).

Definition. A bijective map of the points of $\mathcal{P}(\mathfrak{J})$ onto the points (lines) of $\mathcal{P}(\mathfrak{J}')$ and of the lines of $\mathcal{P}(\mathfrak{J})$ onto the lines (points) of $\mathcal{P}(\mathfrak{J}')$ preserving the incidence and connectness relations is called a colloneation (correlation) of $\mathcal{P}(\mathfrak{J})$ onto $\mathcal{P}(\mathfrak{J}')$.

Definition. A correlation of $\mathcal{P}(\mathfrak{J})$ onto itself is called a duality.

Definition. A duality of order two is called a polarity.

Definition. Let π_0 be as follows:

$$\pi_0 : x_* \longrightarrow x^*, \; x^* \longrightarrow x_*, \qquad x \in \Pi.$$

Then π_0 is a polarity of $\mathcal{P}(\mathfrak{J})$, called the standard polarity.

Remark. If we consider $H_3(\mathcal{O},\gamma)$ and $H_3(\mathcal{O}',\gamma')$, then it can be seen that there exists a collineation of $\mathcal{P}(H_3(\mathcal{O},\gamma))$ onto $\mathcal{P}(H_3(\mathcal{O},\gamma'))$.

Thus, the structure of the plane $\mathcal{P}(H_3(\mathcal{O},\gamma))$ depends only on the octonion algebra $\mathcal{O}$, hence $\mathcal{P}(\mathfrak{J})$ will be denoted by $\mathcal{P}(\mathcal{O})$.

Definition. Three points u_{1*}, u_{2*}, u_{3*} form a three-point if $T(u_1, u_2 \times u_3) \neq 0$.

Definition. Four points form a four-point if each subset of three points is a three-point.

Remark. The elements e_i, $e_i + a[ij] + \gamma_i\gamma_j n(a) e_i$ are of rank one and span $\mathfrak{J}$. For every point there exists a point not connected to it. Moreover, since T is nondegenerate, every pair of unconnected points can be embedded in a three-point.

Notations. Denote by $P\Gamma$, PG, PS the image of Γ, G, S, respectively, under the homomorphism $W \rightarrow \ulcorner W \urcorner$ in the collineation group. (For Γ, G, S, see § 2 of JSA.I).

Proposition 1.1. PS is transitive on points, pairs of points that are not connected, and three-points, respectively. PG is transitive on four-points.

Proposition 1.2. If $a, b \in \Pi$, then there exists a $c \in \Pi$ such that $T(a,c)\,T(b,c) \neq 0$.

Corollary. If a_* and b_* are points, then there is an algebraic transvection T such that $\ulcorner T \urcorner (a_*) = b_*$.

Proposition 1.3. If a_* and b_* are not connected, then $(a \times b)^*$ is the unique line incident to both a_* and b_*. We write $a_* \times b_* := (a \times b)^*$.

Proposition 1.4. If $a_* \simeq b_*$, then there is a line c^* incident to both a_* and b_*.

Remark. For $a, b \in \Pi$, $a_* | b^*$ implies that $a_* \simeq b^*$.

Proposition 1.5. The following statements hold in $\mathcal{P}(\mathcal{O})$:

a) $a_* \simeq b^*$ if and only if there exists a $c_* | b^*$ such that $a_* \simeq c_*$;

b) if $a_* \not\simeq b^*$, then for $c_* | b^*$ either $a_* \simeq c_*$ or $(a \times c)^* \simeq b^*$.

Proposition 1.6. Given $a_* | b^*$, there exists a $c_* | b^*$ such that $a_* \not\simeq c_*$.

Corollary 1. PS is transitive on pairs $a_* | b^*$.

Corollary 2. If $x_*, y_* \not| z^*$ and $x_* \not\simeq y_*$, then $\ulcorner T_{z;x,y} \urcorner$ has z^* as its unique line of fixed points. This line is called the axis of $\ulcorner T_{z;x,y} \urcorner$.

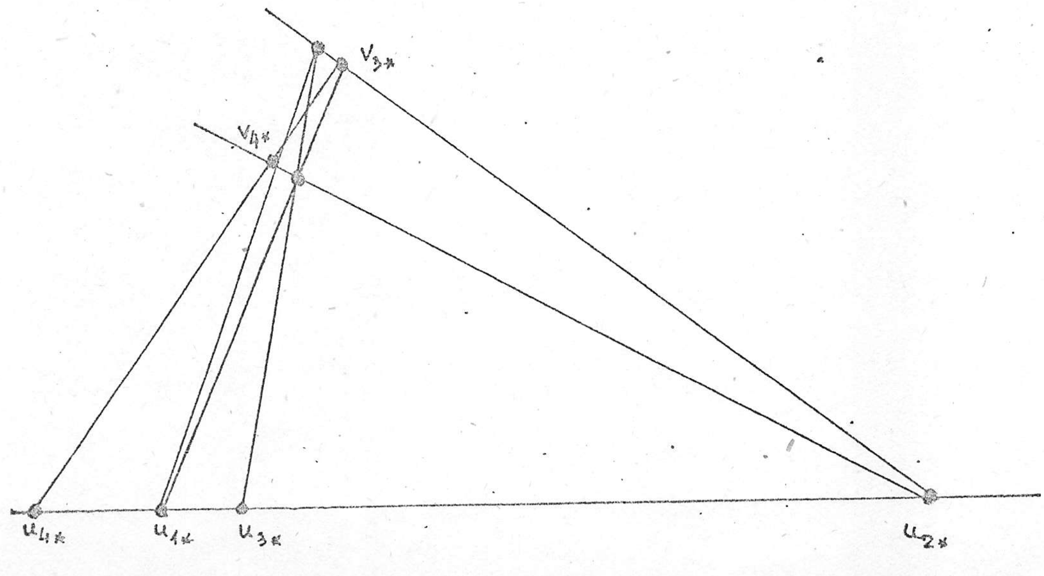
Proposition 1.7. PS is transitive on pairs $a_* \simeq b_*$.

Proposition 1.8. If σ is a collineation of $\mathcal{P}(\mathfrak{V})$ fixing a_* , b_* , and all points on c^* where $a_* \not\simeq b_*$, $a_* \not| c^*$ and $b_* \not| c^*$, then σ is the identity.

Proposition 1.9. If $a_* \not\simeq b_*$ and if σ is a collineation of $\mathcal{P}(\mathfrak{V})$ fixing all points $c_* | (a \times b)^*$ such that either $c_* \not\simeq a_*$ or $c_* \not\simeq b_*$, then σ fixes all points incident to $(a \times b)^*$.

Theorem 1.10. (The Fundamental Theorem for octonion planes). If $\mathfrak{J}_i = H_3(\mathfrak{V}^{(i)}, \gamma^{(i)})$, $i=1,2$, and if σ is a collineation of $\mathcal{P}(\mathfrak{V}^{(1)})$ onto $\mathcal{P}(\mathfrak{V}^{(2)})$, then σ is induced by a semisimilarity W of $\tilde{\mathfrak{J}}_1$ onto $\tilde{\mathfrak{J}}_2$; i.e. $\sigma = \ulcorner W \urcorner$.

Definition. Let u_{i*}, $i=1,2,3,4$, be four points of $\mathcal{P}(\mathfrak{V})$ with $u_{1*} \not\simeq u_{2*}$, $u_{3*} | u_{1*} \times u_{2*}$, and $u_{4*} | u_{1*} \times u_{2*}$. If there exist $v_3{}^*$, $v_4{}^*$ such that $u_1{}^*$, $u_2{}^*$, $v_3{}^*$, $v_4{}^*$ form a four-point and the incidence relations indicated in the figure below are valid, then one says that the $u_i{}^*$, $i=1,2,3,4$, are in harmonic position.



Proposition 1.11. If $u_{1*} \not\sim u_{2*}$ and $u_{3*} | u_{1*} \times u_{2*}$, then there exists a unique point $u_{4*} | u_{1*} \times u_{2*}$ such that u_{1*}, u_{2*}, u_{3*}, u_{4*} are in harmonic position. Moreover, $u_{3*} = u_{4*}$ if and only if the characteristic of $\mathbb{F}$ is 2.

Remark. Note that in a projective plane, the uniqueness of the fourth harmonic point in Proposition 1.11 is a consequence of Desargues'theorem, more exactly, his weaker Little Theorem (see, for instance, PICKERT [50, pp.190-191]). Conversely, although Desargues'Theorem is not valid for projective octonion planes (see, for instance, HALL [28, p.374]), the Little Theorem of Desargues holds if the fourth harmonic point is unique and distinct from the third harmonic point in a projective plane (see PICKERT [50, p.191]). Therefore, Proposition 1.11 implies that Desargues'Little Theorem holds in a projective octonion plane if the characteristic of $\mathbb{F}$ is 2. For arbitrary characteristic FAULKNER [18 a] has given a direct proof of a result true for an arbitrary octonion plane, which reduces to Desargues'Little Theorem in case the plane is projective (see Proposition 1.12 below).

Proposition 1.12. If c_*, p_*, r_*, q_* form a four-point, if $u^* | c_*$, $u^* \not\sim p_*, q_*$ $\longleftarrow$ - and if $v_* | c_* \times q_*$, $v_* \not\sim c_* \times r_*$, then the trree points $x_* := (p_* \times r_*) \times u^*$, $s_* := (((r_* \times q_*) \times u^*) \times v_*) \times (c_* \; r_*)$ and $t_* : = (((p_* \times q_*) \times u^*) \times v_*) \times (c_* \times p_*)$ are uniquely determined and incident to a unique line.

Now consider (quadratic) Jordan algebra $H_3(\mathfrak{C},\gamma) =: \mathfrak{J}$; $\mathfrak{C}$ is an octonion algebra which is a free module over a local ring R with maximal ideal m. Let Γ be the group of semilinear R-module automorphisms φ of $\mathfrak{J}$ satisfying the condition

$$N(\varphi x) = \rho s(N(x))$$

for some $\rho \in R$-m, and consider its subgroup

$$S = \{\varphi \mid \varphi \in G,\ s = \mathrm{Id},\ \rho = 1\}.$$

Notation. For any subgroup H of Γ and any ideal I of R put

$$H_I := \{\varphi \mid \varphi \in H,\ \varphi(x) \equiv x \pmod{I \mathfrak{J}},\ x \in \mathfrak{J}\}.$$

BIX [9 a] has proved

Theorem 1.13. A subgroup H of Γ is normalized by S if and only if $S_I \subseteq H \subseteq (R\text{-m})\ \Gamma_I$ for some ideal I of R.

Remark. The collineation group of $\mathcal{P}(\mathfrak{J})$ is isomophic to $\Gamma/(R\text{-m})$, hence Theorem 1.13 classifies the subgroups of the collineation group which are normalized by the little projective group.

Recently, BIX [9c] defined octonion planes over Euclidean domains as follows: Let $\mathcal{O}$ be an octonion algebra over an Euclidean domain D containing 1/2 and suppose that $\mathcal{O}$ is split (i.e., contains a subalgebra that is a hyperbolic plane with respect to the norm of $\mathcal{O}$). Denote by S the group of norm-preserving transformations of the Jordan algebra $H_3^{(+)}(\mathcal{O})$. For any ideal I of D, let S_I be the subgroup of S consisting of elements that induce the identity map on $H_3^{(+)}(\mathcal{O})/\ I\ H_3^{(+)}(\mathcal{O})$, and let S_I^+ be the subgroup of S consisting of elements that induce scalar multiplication on $H_3^{(+)}(\mathcal{O})/I\ H_3^{(+)}(\mathcal{O})$.

BIX [9c] proved the following:

Theorem 1.14. If N is a subgroup of S, then N equals S if and only if N is normalized by S and N is not contained in S_m^+ for any maximal ideal m of D. If D consists of all rational numbers whose denominators are powers of 2, then N is normalized by S if and only if $S_I \subset N \subset S_I^+$ for some ideal I of D.

§ 2. Transvections and polarities

Since the structures of $\mathcal{P}(\mathfrak{J})$, PG, P$\Gamma$ and PS depend only on the octonion algebra $\mathcal{O}$, throughout this section γ is assumed to be diag$\{1,1,1\}$.

Theorem 2.1. S is generated by algebraic transvections and PS is a simple group.

Proposition 2.2. The following statements are equivalent for $\mathcal{P}(\mathfrak{O})$:

a) $a_* \simeq b_*$ implies that $a_* = b_*$;

b) $a^* \simeq b^*$ implies that $a^* = b^*$;

c) $a_* \simeq b^*$ implies that $a_* | b^*$;

d) $\mathfrak{O}$ is a division algebra;

e) $\mathcal{P}(\mathfrak{O})$ is a projective plane.

Definition. Let $\mathcal{P}(\mathfrak{O})$ be a projective plane. A collineation σ of $\mathcal{P}(\mathfrak{O})$ which fixes all points incident to a line y^* and all lines incident to a point $x_* | y^*$ is called a transvection with centre x_* and axis y^*.

Remark 1. A line y^* and two points w_* and z_* not on y^* determine a unique transvection with axis y^* and mapping w_* to z_*.

Remark 2. If $T_{y;w,z}$ is an algebraic transvection, then $\ulcorner T_{y;w,z} \urcorner$ is a transvection with axis y^* and mapping w_* to z_*.

FAULKNER [18 a, pp.51-57] investigated involutions (i.e. collineations of order two) in the group PG and proved that any isomorphism between the groups PS (or PG) of two projective planes $\mathcal{P}(\mathfrak{O}^{(1)})$ and $\mathcal{P}(\mathfrak{O}^{(2)})$ is deduced by a collineation or correlation. These results were established in case the characteristic of $\mathbb{F} \neq 2$ or 3 by SUH [63] and VELDKAMP [66a]. The methods used therein also apply for characteristic 3, but for characteristic 2 the structure is essentially different. FAULKNER [18 a] gave proofs only for characteristic 2 (see Proposition 2.3, 2.4 and 2.5 below).

Proposition 2.3. If the field $\mathbb{F}$ has characteristic two, then every involution of a projective plane $\mathcal{P}(\mathfrak{O})$ is either a transvection or an involution with a fixed four-point.

<u>Proposition 2.4</u>. If σ and τ are transvections in a projective plane $\mathcal{P}(\mathcal{V})$ over a field of characteristic two, then the following conditions are equivalent:

(i) $\sigma\tau = \tau\sigma$;

(ii) the centre of one lies on the axis of the other, viceversa;

(iii) $\sigma\tau$ is the identity or a transvection.

<u>Proposition 2.5</u>. If α is an isomorphism of the group PS (or PG, respectively) of the projective plane $\mathcal{P}(\mathcal{V}^{(1)})$ to the corresponding group of the projective plane $\mathcal{P}(\mathcal{V}^{(2)})$, both defined over a field of characteristic two, i=1,2, and if σ is a transvection of $\mathcal{P}(\mathcal{V}^{(1)})$, then $\alpha(\sigma)$ is a transvection of $\mathcal{P}(\mathcal{V}^{(2)})$.

<u>Theorem 2.6</u>. If α is an isomorphism of the group PS (or PG, respectively) of the projective plane $\mathcal{P}(\mathcal{V}^{(1)})$ to the corresponding group of the projective plane $\mathcal{P}(\mathcal{V}^{(2)})$, both defined over a field of characteristic two, then α is given by $\alpha(\sigma) = \rho^{-1}\sigma\rho$ for $\sigma \in PS(\mathcal{V}^{(1)})$ ($\sigma \in PS(\mathcal{V}^{(2)})$ respectively), where ρ is a collineation or a correlation of $\mathcal{P}(\mathcal{V}^{(1)})$ onto $\mathcal{P}(\mathcal{V}^{(2)})$.

<u>Note</u>. Now assume that $\mathbb{F}$ is of arbitrary characteristic.

<u>Notation</u>. If π is a polarity of $\mathcal{P}(\mathfrak{V})$, then $PU(\pi)$ denotes the subgroup of elements of PG which commute with π.

<u>Proposition 2.7</u>. Let π be a polarity of $\mathcal{P}(\mathfrak{V})$. Then

a) $\pi = \ulcorner T \urcorner \pi_0$, where T is a t-semisimilarity of $\mathfrak{V}$ such that $t^2 = Id$ and $T = T^*$;

b) $\ulcorner S \urcorner \in PU(\pi)$ if and only if there exists $\lambda \in \mathbb{F}$ such that $STS^* = \lambda T$; such a λ satisfies $\lambda t(\lambda) = 1$;

c) every element of $PU(\pi)$ can be written as $\ulcorner S \urcorner$ where $STS^* = T$; for such S, $\rho t(\rho) = 1$, where S is a similarity with multiplier ρ ;

d) if $t = Id$, then every element of $PU(\pi)$ can be written as $\ulcorner S \urcorner$ with $STS^* = T$, where S has multiplier $\rho = 1$; such an S is uniquely determined by $\ulcorner S \urcorner$.

Definition. Two polarities π and π' are called equivalent if $\pi' = \sigma^{-1}\pi\sigma$ with σ a collineation.

Remark 1. π' is equivalent to π if and only if $\pi' = \ulcorner S^{-1}TS^{*-1} \urcorner \pi_0$ for some $S \in \Gamma$, where $\pi = \ulcorner T \urcorner \pi_0$.

Remark 2. If π and π' are equivalent, then $PU(\pi') \cong PU(\pi)$.

Definition. A three-point u_{1*}, u_{2*}, u_{3*} is called a polar three-point if $\pi(u_{i*}) = (u_j \times u_k)^*$ for i,j,k distinct.

Notation. Suppose that e_{i*}, $i=1,2,3$, form a polar three-point with respect to the polarity $\pi = \ulcorner T \urcorner \pi_0$, with T as in a) of Proposition 2.7. We then have $T(e_i) = \lambda_i e_i$ for some $\lambda_i \in \mathbb{F}-\{0\}$. Let $T_i := T|_{\mathfrak{J}_{jk}}$ and put $T := [\lambda_1, \lambda_2, \lambda_3; T_1, T_2, T_3]$.

Definition. If π is a polarity and x_* is a point of $\mathcal{P}(\mathfrak{J})$, then x_* is called weakly isotropic if $x_* \simeq \pi(x_*)$, and x_* is called isotropic if $x_* \mid \pi(x_*)$.

Definition. If $\mathcal{P}(\mathfrak{J})$ contains a (weakly) isotropic point, then $\mathcal{P}(\mathfrak{J})$ is said to be (weakly) hyperbolic.

Proposition 2.8. If x_*, y_*, and z_* are collinear isotropic points relative to the polarity, if $\pi = \ulcorner T \urcorner \pi_0$, $T = T^*$, and if $y_* \not\simeq \pi(x_*)$, $z_* \not\simeq \pi(x_*)$, then $\ulcorner T_{T(x);y,z} \urcorner$ is an element of $PU(\pi)$.

Notation. The group generated by all transvections $\ulcorner T_{T(x);y,z} \urcorner$ from Proposition 2.8 is denoted by $PT(\pi)$.

Proposition 2.9. If π is a hyperbolic polarity with two isotropic points v_* and w_* with $v_* \not\simeq \pi(w_*)$, then π is equivalent to a polarity of the form $\ulcorner T \urcorner \pi_0$, where $T = [1,1,1; -J,-J,J]$ and J is a t-semiautomorphism of Γ such that $J^2 = id$.

Definition. A polarity π is called linear if $\pi = \ulcorner T \urcorner \pi_0$ with T a linear map.

Remark. A polarity equivalent to a linear polarity is linear.

Note. Throughout the remainder of this section, $\mathbb{F}$ will be assumed to have characteristic two.

If π is a linear polarity satisfying the conditions of Proposition 2.9, then either $\pi = \pi_0$ or $\pi = \ulcorner T \urcorner \pi_0$, $T = [1,1,1;J,J,J]$, where J is an automorphism of Γ of order two.

First consider the case $\pi = \pi_0$.

Proposition 2.10. If π_0 is the standard polarity, then the group $PT(\pi_0) \leq PU(\pi_0)$ is transitive on isotropic points. Moreover, if x_* is an isotropic point and y^* is any line through x_* with $y^* \not\simeq x^* = \pi_0(x_*)$, then there exists an isotropic point $z_* | y^*$ with $z_* \not\simeq x_*$.

Proposition 2.11. $PU(\pi_0)$ is a faithful permutation group of the set of isotropic points.

Consider now polarities π other than π_0.

Proposition 2.12. If $\pi = \ulcorner T \urcorner \pi_0$, where $T = [1,1,1;J,J,J]$ and J is an automorphism of Γ of order two, then there exists an isotropic point w_* and a line $y^* | w_*$ with $y^* \not\simeq \pi(w_*)$, such that every isotropic point $z_* | y^*$ satisfies $z_* \simeq w_*$.

Corollary 1. If π is as in Proposition 2.12, then π is not equivalent to π_0.

Corollary 2. $PU(\pi_0)$ is transitive on pairs of isotropic points v_*, w_* such that $v_* \not\simeq w^*$.

Corollary 3. $PU(\pi_0)$ is transitive on pairs of points u_{1*}, x_* such that $u_{1*}, x_* | v^*$ for some isotropic point v_* and $u_{1*} \not\simeq x_* \not\simeq v_* \not\simeq u_{1*}$.

Recalling that an idempotent $c \in \mathfrak{J}$, $c \neq 0$, is primitive if the Peirce 1-space with respect to c has no idempotents other than c and 0, we give

Proposition 2.13. Let v_*, w_*, z_* be collinear isotropic points with respect to Π_0 such that $v_* \not\sim z^*$, $w_* \not\sim z^*$, then

a) there exists a primitive idempotent c, such that $c_* | z^*$;

b) if c is a primitive idempotent such that $c_* | z^*$, then $T_{z;v,w} \in \mathrm{Aut}\, \mathfrak{J}/\mathbb{F}c$, the set of automorphisms of $\mathfrak{J}$ fixing c.

Definition. If u_* is a point such that $\Pi(u_*)$ contains an isotropic point v_* for which $u_* \not\sim v_*$, then u_* is called an outer point.

Theorem 2.14. $PT(\Pi_0)$ is a simple group and is transitive on outer points.

Theorem 2.15. If $\mathbb{F}$ is a field of characteristic two, then $PT(\Pi_0) = PU(\Pi_0)$. Hence, Aut $\mathfrak{J}$ is simple.

Remark. The simplicity of Aut $\mathfrak{J}$ was first proved by JONKER [32] for fields with at most two elements.

Consider now two (quadratic) Jordan algebras $H_3(\mathcal{O}^{(i)}, \gamma) =: \mathfrak{J}^{(i)}$, i=1,2, where $\mathcal{O}^{(i)}$ are octonion algebras over local rings BIX [9b] proved the following:

Theorem 2.16. There is a collineation between $\mathcal{P}(\mathfrak{J}^{(1)})$ and $\mathcal{P}(\mathfrak{J}^{(2)})$ if and only if there is a ring isomorphism between $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$.

Faulkner's result, here given as Theorem 2.6, was generalized by BIX [9b] as follows:

Theorem 2.17. Let $PH^{(i)}$, i=1,2, be subgroups of the collineation groups of $\mathcal{P}(\mathfrak{J}^{(i)})$ containing $PS(\mathfrak{J}^{(i)})$, where $\mathfrak{J}^{(i)} = H_3(\mathcal{O}^{(i)}, \Gamma)$ with $\mathcal{O}^{(i)}$ defined over local rings which contain 1/2. Let β be an isomorphism of $PH^{(1)}$ onto $PH^{(2)}$ such that $\beta(PS(\mathfrak{J}^{(1)})) = PS(\mathfrak{J}^{(2)})$. Then there exists a collineation or a correlation $\rho: \mathcal{P}(\mathfrak{J}^{(1)}) \to \mathcal{P}(\mathfrak{J}^{(2)})$ such that $\beta(\varphi) = \rho^{-1}\varphi\rho$ for all $\varphi \in PH^{(1)}$.

Remark 1. The assumption that $\beta(PS(\mathfrak{J}^{(1)})) = PS(\mathfrak{J}^{(2)})$ in Theorem 2.17. can be removed if either:

1) $PH^{(i)}$, $i=1,2$, are contained in $P\Gamma(\mathfrak{J}^{(i)})$,

or

2) $\Gamma^{(i)}$, $i=1,2$, are fields.

Remark 2. The octonion plane over the real local ring algebra of dual numbers $\mathbb{R}(\varepsilon)$ (with basis $(1,\varepsilon)$, $\varepsilon^2 = 0$) has been considered by KUZNETSOVA [39].

Using concepts from valuation theory, CARTER and VOGT [15] have given a characterization of all collinearity-preserving functions from one affine or projective Desarguesian plane into another. Lineations (i.e. point functions f from one plane into another with the property that whenever x,y and z are collinear points, f(x), f(y) and f(z) are collinear points) whose ranges contain a quadrangle, called in [15] full lineations, have been algebraically characterized in various setting by KLINGENBERG ([34], lineations from a Desarguesian plane onto another), SKORNYAKOV ([57], lineations from an arbitrary plane onto another), RADÓ ([51b], full lineations from a Desarguesian plane into another, the infinite-dimensional case being considered in [51a]), and GARNER ([24], lineations from a Pappian coordinate plane into another taking the reference quadrangle of one plane to that of the other).

Carter's and Vogt's results allow one or both planes to be affine and include cases where the range contains a triangle but no quadrangle. A key theorem is that, with the exception of certain embeddings defined on planes of order 2 and 3, every collinearity-preserving function from an affine Desarguesian plane into another can be extended to a collinearity-preserving function between the enveloping projective planes. Full lineations defined on finite-dimnensional affine spaces can also be extended to the enveloping projective space (BREZULEANU and RADULESCU [13]).

FAULKNER and FERRAR [20a] has shown that, up to conjugation by collineations, there exists at most one surjective homomorphism from an octonion plane to a Moufang plane. They also established the existence of proper homomophisms between octonion planes and of homomorphisms from octonion planes onto Desarguesian planes.

FERRAR and VELDKAMP [21] studied neighbour-preserving homomophisms between projective ring planes (i.e., mappings preserving incidence and the neighbour relation between points and lines). These are generalization of the homomophisms between ordinary Desarguesian projective planes which have first been studied by KLINGENBERG [34] . On the other hand, in the context of projective planes over rings of stable rank 2 as studied by VELDKAMP [66e], an obvious question to ask was, what mappings between such planes are induced by homomorphisms between coordinatizing rings. If one requires that the ring homomorphisms carry 1 to 1, then they induce distant-preserving homomorphisms which are mappings incidence and the negation of the neighbour relation between a point and a line. VELDKAMP [66f] proved that any distant-preserving homomorphism ψ is induced by a ring homomorphism carrying 1 to 1, provided the two planes are coordinatezed with respect to basic quadrangles which correspond under ψ . VELDKAMP [66f] also studied homomorphisms between projective ring planes which only preserve incidence. They turn out to be products of a bijective neighbour-preserving homomophism followed by an arbitrary distant-preserving homomorphisms.

In 1983, FAULKNER and FERRAR [19b], utilizing methods similar to those of SKORNYAKOV [57], extended the results of KLINGENBERG [34] from Desarguesian planes to Moufang planes. Let us note in this respect that a systematic study of places of octonion algebras over discrete valuation rings has been carried out by PETERSSON [49]. Recently, BROŽIKOVA [14], making use of previous results of HAVEL [30] and FAULKNER and FERRAR [19b], provided a Jordan-theoretic description of all homomorphisms between Moufang planes having the property that the points identified via Springer's isomorphism (see SPRINGER [59a]) with (0,0), 0, (∞) and (1,1) are mapped to their analogues in the image plane.

Open problem. It would be interesting to deal with topics similar to those mentioned above in the case of octonion planes.

§ 3. Barbilian structures

As is well known, BARBILIAN [5a,b] was the first who made an axiomatic study of projective planes over arbitrary associative rings.

He has proved [5b] that the rings that can be underlying rings for projective geometries are (with a few exceptions) rings with a unit element in which any one-sided inverse is a two-sided inverse. BARBILIAN [5b,I] called these rings "Z-rings" (from "Zweiseitigsinguläre Ringe") and gave a set of 11 axioms of projective geometry over a certain type of Z-ring (see [5b, II]). LEISSNER [41a] has developed a plane geometry over an arbitrary Z-ring R, in which a point is an element of $R \times R$ and a line is a set of the form

$$\left\{(x+ra,\ y+rb) \mid (x,y) \in R \times R,\ \ r \in R,\ (a,b) \in B\right\},$$

where B is a "Barbilian domain", i.e. a set of unimodular pairs from $R \times R$ satisfying certain axioms.[1] RADÓ [51c] extended LEISSNER's results [41a] to affine Barbilian planes over an arbitrary ring with a unit element and investigated the corresponding affine Barbilian structures and translation Barbilian planes. Corresponding to the algebraic representation of affine Barbilian spaces as affine geometries over unitary free modules, LEISSNER [41b] has recently characterized algebraic properties of the underlying ring R, respectively module M_R, respectively Barbilian domain $B \subset M_R$ by geometric properties of the affine Barbilian space and viceversa.

VELDKAMP [66e], gave an axiomatic description of plane geometries of the kind considered by BINGEN [8]. A most satisfactory situation is reached by extending the class of rings used for coordinatization from semiprimary rings, which Bingen used, to rings of stable rank 2. These rings have played a role in algebraic K-theory, and seem to form a natural framework for many geometric problems.

Note. For simplicity VELDKAMP [66e] confines himself to the case of planes (called projective Barbilain planes), but a generalization to higher dimensions is straighforward (see below projective Barbilian spaces defined also by Veldkamp).

Veldkamp has chosen an approach somewhat along the lines of ARTIN [3] rather than to follow Barbilian.

The basic relations in the plane are incidence and the neighbour relation. The axioms consist of a number of axioms expressing elementary relations between points and lines such as, e.g., the existence of a unique line joining any two non-neigh-

---

1) Let us mention in this context that LANTZ extended [40] BENZ' results [6a] by showing that large classes of commutative rings admit only one Barbilian domain.

bouring points, and a couple of axioms ensuring the existence of transvections and dilatations.

In 1987, VELDKAMP [66h] extended all this above mentioned results to arbitrary finite dimension. Basic objects in the axioms are points and hyperplanes, by analogy with the self-dual set-up for clasical projective spaces over skew fields given by ESSER [17]. As basic relations again serve incidence and the neighbour relation. The self-dual approach is quite natural since incidence and the neighbour relation between points and hyperplanes have a simple algebraic description in coordinates. Homomorphisms are more or less the same as in the plane case, things becoming a bit more complicated because Veldkamp included homomophisms between spaces of unequal dimension.

Note. Veldkamp confine himself to full homomorphisms, which can only increase the dimension or leave it the same. Thus he excluded homomorphisms which lower the dimension, an example of which was given by FRITSCH and PRESTEL [23].

Recently, FAULKNER [18e] defined and studied the so-called F-planes which generalize the projective planes. Planes considered by BARBILIAN in the Zusatz to [5b] are connected F-planes in Faulkner's setting. Besides extending the class of coordinate ring, FAULKNER's work [18e] introduces some new concepts, techniques, and connections with other areas. These include a theory of covering planes and homotopy although there is no topology, a theory of tangent bundle planes and their sections although there is no differential or algebraic geometry, a purely geometric and coordinate-free construction of the Lie ring of the group generated by transvections, and connections to the K-theory of the coordinate ring.

Let us now recall, making use of ref. [66h] by VELDKAMP, the definition and the fundamental properties of projective Barbilian spaces.

Definition. A (projective) Barbilian space of dimension $n \geq 1$, or Barbilian n-space, $P = (P_*, P^*, |, \approx)$ consist of non-empty sets P_* and P^*, whose elements are called points and hyperplanes, respectively, together with two relations $x|h$, incidence, and $x \approx h$, neighbour, between P_* and P^* which satisfy the following axioms: [1)]

1. If $x|h$, then $x \approx h$;
2. If $a_1, \ldots, a_k$ are independent points ($1 \leq k \leq n$), then there exists a hyperplane $h|a_1, \ldots, a_k$. This h is unique if k=n, and is then denoted by $h := a_1 \vee \ldots \vee a_n$, called the join of these points.

2'. If $h_1, \ldots, h_k$ are independent hyperplanes ($1 \leq k \leq n$), then there exists a point $a|h_1, \ldots, h_k$. This a is unique if k=n, and is then denoted by $a := h_1 \wedge \ldots \wedge h_n$, calles the meet of these hyperplanes.

3. If a is a point and $h_1, \ldots, h_n$ are independent hyperplanes with $a|h_1, \ldots, h_{n-1}$ and $h_1 \wedge \ldots \wedge h_n \not\approx a$, then $a \not\approx h_n$.
4. For any two points x, x' there exists a hyperplane h with $h \not\approx x$, $h \not\approx x'$.

---

1) For $x|h$ and $x \approx h$ we shall also write $h|x$ and $h \approx x$, respectively. Characters $a, b, c, x, y, z, \ldots$ denote elements of P_*, while $h, k, l, m, n, \ldots$ denote elements of P^*. Dualizing a statement means interchanging "point" and "hyperplane". We shall call x and h distant if $x \not\approx h$. Points $a_1, a_2, \ldots, a_k$ ($1 \leq k \leq n+1$) are called independent provided there exist hyperplanes $h_1, h_2, \ldots, h_{k-1}$ such that $a_i \not\approx h_i \mid a_{i+1}, \ldots, a_k$ for $1 \leq i \leq k-1$. Independent hyperplanes are defined in the dual manner. Points a_1, a_2 are distant, $a_1 \not\approx a_2$, if they are independent, and independent hyperplanes h_1, h_2 are called dually distant : $h_1 \overset{*}{\not\approx} h_2$.

5. If a_1, a_{n-1} are independent points and h_n, h_{n+1} independent hyperplanes such that $a_1, \ldots, a_{n-1} \mid h_n, h_{n+1}$, then $x \mid h_n, h_{n+1}$ and $h \mid a_1, \ldots, a_{n-1}$ imply $x \mid h$.

Examples. Projective spaces $P_n(R)$ iver a ring R of stable rank 2 are, for $n \geq 3$, Barbilian n-spaces. For n=2, one has to add axioms on the existence of transvections, dilatations, and affine and dual affine dilatations to characterize the ring planes $P_2(R)$ (see VELDKAMP [66e]).

Proposition 3.1. Assume $a_1, \ldots, a_k$ are independent points in a Barbilian n-space. Then:

i) Any subrow of $a_1, \ldots, a_k$ is independent too;

ii) $a_{\pi(1)}, \ldots, a_{\pi(k)}$ are independent for any permutation π of $1, \ldots, k$;

iii) For any other row $b_1, \ldots, b_\ell$ of independent points with $\ell \leq k$ there exist point $b_{\ell+1}, \ldots, b_k$, $c_1, \ldots, c_{n+1-k}$ such that both $c_1, \ldots, c_{n+1-k}, a_1, \ldots, a_k$ and $c_1, \ldots, c_{n+1-k}, b_1, \ldots, b_\ell, \ldots, b_k$ are independent rows;

iv) If $a_1, \ldots, a_{n+1}$ are independent points, and $h_i = a_1 \vee \ldots \vee a_{i-1} \vee a_{i+1} \vee \ldots \vee a_{n+1}$, then $h_1, \ldots, h_{n+1}$ are independent hyperplanes.

Lemma 3.2. For any three hyperplanes h, m_1, m_2 with $h \neq m_1, m_2$ there exists a point $x \mid h$ such that $x \nmid m_1, m_2$.

Convention. The notation "$a_1, \ldots, a_k$ indep. $\mid$ indep. $h_{k+1}, \ldots, h_\ell$" means: $a_1, \ldots, a_k$ are independent points, $h_{k+1}, \ldots, h_\ell$ independent hyperplanes and $a_i \mid h_j$ for all $i=1, \ldots, k$ and $j=k+1, \ldots, \ell$.

Proposition 3.3. i) If $a_1, \ldots, a_k$ indep. $\mid$ indep. $h_{k+1}, \ldots, h_\ell$, then $\ell \leq n+1$. Moreover, if $\ell \leq n$, there exist hyperplanes $h_{\ell+1}, \ldots \ldots h_{n+1}$ such that $a_1, \ldots, a_k$ indep. $\mid$ indep. $h_{k+1}, \ldots, h_{n+1}$;

ii) If $a_1, \ldots, a_k$ indep. $\mid$ indep. $h_{k+1}, \ldots, h_{n+1}$, then for any point $x \mid h_{k+1}, \ldots, h_{n+1}$ and hyperplane $h \mid a_1, \ldots, a_k$ we have $x \mid h$.

Proposition 3.4. The duals of axioms 1 to 5 are valid for any Barbilian space.

VELDKAMP [66h, § 2] introduced "flats" in Barbilian spaces as the analogs of subspaces in projective ring spaces, and considered relations between them as follows.

Definition. Let $a_o, a_1, \dots, a_d$ be independent points in a Barbilian space $P = (P_*, P^*, |, \approx)$. The flat spanned be $a_o, a_1, \dots$ $\dots, a_d$ is the set

$F := F(a_o, a_1, \dots, a_d) := \{x \in P_* | x/h \text{ for all } h \in P^* \text{ with } h | a_o, \dots, a_d\}$.

The points $a_o, a_1, \dots, a_d$ are said to form a basis of F. Further, $F(\emptyset) = \emptyset$, the empty flat.

Proposition 3.5. For any set of independent points $a_o, a_1, \dots, a_d$ in a Barbilian n-space there exist n-d independent hyperplanes $h_{d+1}, \dots, h_n$ (which is meant to be the empty set if d=n) such that

$$F = F(a_o, \dots, a_d) = \{x \in P_* | x/h_{d+1}, \dots, h_n\}.$$

Convention. The hyperplanes $h_{d+1}, \dots, h_n$ are called a dual basis of F, and F is alco denoted by $F := F^*(h_{d+1}, \dots, h_n)$.

Remark. Similarly for $F(\emptyset)$ with n+1 independent hyperplanes.

Convention. Proposition 3.5 and its dual allow to identify flats and "dual flats", which one always does.

From Proposition 3.3 it follows that any two bases for a flat F have the same number of elements, say d+1 with $-1 \leq d \leq n$. We call d the dimension of F, and F a d-flat.

Convention. Whenever the notation $F(a_o, \dots, a_d)$ is used, it is tacitly understood that $a_o, \dots, a_d$ are independent, and similarly for $F^*(h_{d+1}, \dots, h_n)$. Any point x will be identified with the flat $F(x)$, and any hyperplane h with the flat $F^*(h)$.

Proposition 3.6. i). $F(a_o, \dots, a_k) \leq F(b_o, \dots, b_l)$ if and only if $a_0, \ldots, a_k \in F(b_0, \ldots, b_\ell)$;

ii) Any set of independent points in a flat F can be extended to a basis of F ;

iii) $A \subseteq B$ for flats A and B if and only if B has a basis which contains a basis of A.

<u>Definition</u>. Two flats A and B will be called <u>transversal</u>, $A \tau B$, if there exists a set of independent points such that each of A and B is spanned by a subset thereof.

<u>Remark</u>. The relation τ is symmetric since a permutation of independent points is also independent.

<u>Proposition 3.7</u>. i) Transversality is a self-dual notion, i.e., $A \tau B$ if and only if each of them has a dual basis taken from one row of independent hyperplanes;

ii) If $A \tau B$, then $A \cap B$ is a flat, viz., $F(a_0, \ldots, a_k)$ if $A = F(a_0, \ldots, a_k, a_{k+1}, \ldots, a_\ell)$ and $B = F(a_0, \ldots, a_k, a_{\ell+1}, \ldots, a_m)$ with $a_0, \ldots, a_m$ independent;

iii) If $A \tau B$, there exists a unique minimal flat $A + B$ containing both A and B, viz., $F(a_0, \ldots, a_m)$ in the notation of ii);

iv) For $A \tau B$, we have $\dim (A + B) + \dim (A \cap B) = \dim A + \dim B$;

v) If $A \tau B$, then for every basis $b_0, \ldots, b_\ell$ of A which contains a basis $b_0, \ldots, b_k$ of $A \cap B$ and for every basis $b_0, \ldots, b_k, b_{\ell+1}, \ldots, b_m$ of B, the points $b_0, \ldots, b_k, b_{k+1}, \ldots, b_\ell, b_{\ell+1}, \ldots, b_m$ are independent.

<u>Definition</u>. Two flats A and B are called <u>distant</u>, $A \not\approx B$, if $A \tau B$ and $A \cap B = \emptyset$, and <u>dually distant</u>, $A \overset{*}{\not\approx} B$, if $A \tau B$ and $A + B = P_*$. We say that A and B are

complementary flats if both $A \not\tau B$ and $A \overset{*}{\not\tau} B$. The notation $A \oplus B$ is used for $A + B$ provided $A \not\tau B$.

Remark. $A \tau B$ if and only if there exist flats A_1, B_1, C such that $A = A_1 \oplus C$, $B = B_1 \oplus C$ and $B_1 \not\tau A$. Then $C = A \cap B$, and $A + B = A_1 \oplus B_1 \oplus C$.

Convention. If x is a point and h a hyperplane, then we shall use notations like $x \tau A$, $x + A$, $h \cap A$, etc., instead of $F(x) \tau A$, $F(x) + A$, $F^*(h) \cap A$, respectively. Similarly we shall write $x_1 + \ldots + x_k$ for $F(x_1) + \ldots + F(x_k)$ if $x_1, \ldots, x_k$ are independent, etc.

In lattice theory and projective geometry is well known the so-called "Modular Law". This holds also for Barbilian spaces, as follows:

Proposition 3.8. If A, B and C are flats satisfying $A \tau B$, $(A+B) \tau C$ and $A \subseteq C$, then $B \tau C$, $A \tau B \cap C$ and $(A+B) \cap C = A + B \cap C$.

Remark. One can easily prove that, if A is a d-flat, $d \geqslant 1$, then the points of A and the intersections of A with hypperplanes which do not contain all of A form a Barbilian space of dimension d with as incidence relation the inclusion and the neighbour relation defined by the negation of the distant relation between flats defined as above.

Definition. The Barbilian space from the above remark is called a Barbilian subspace and a $(d-1)$-flat contained in A is called an A-hyperplane. If h is a hyperplane and $A = F^*(h)$, then we may similarly speak of the Barbilian subspace h, and of h-hyperplanes.

Lemma 3.9. Let $h_0, h_1, \ldots, h_d$ be hyperplanes, $1 \leq d \leq n$, satisfying the conditions: $h_i \overset{*}{\not\tau} h_0$ for $1 \leq i \leq d$ and

$h_1 \cap h_o$, ..., $h_d \cap h_o$ are independent h_o-hyperplanes in the Barbilian subspace h_o. Then h_o, h_1, ..., h_d are independent.

In order to get the coordinatization of Barbilian spaces, VELDKAMP [66e] generalized the notion of general quadrangle in a Barbilian plane.

Definition. A frame in a Barbilian n- space P is an ordered set of n+2 points such that every subset of n+1 points is independent. A d- frame in P, for $1 \leq d \leq n$, is a frame in any d- flat in P.

Proposition 3.10. In every Barbilian space there exist frames. In fact, any set of n+1 independent points in n-space can be extended to a frame.

Definition. A collineation between two Barbilian spaces consists of bijective mappings ψ_* for the points and ψ^* for the hyperplanes such that $x|h \Longleftrightarrow \psi_* x \mid \psi^* h$ and $x \approx y \Longleftrightarrow \psi_* x \approx \psi^* h$.

Note. If there is no denger of confusion we shall write just ψ for ψ_* and ψ^*.

Definition. Let c be o point and h a hypperplane. For $c \not\approx h$, we call a dilatation with centre c and axis h (or (c,h)-dilatation) any collineation leaving c and all points $|h$ fixed. For $c|h$, we call a central transvection with centre c and axis h (or (c,h)-transvection) any collineation leaving all points $|h$ and hyperplans $|c$ fixed.

Theorem 3.11. Let c,a,a' be points and h a hyperplane in a Barbilian space of dimension ≥ 3 satisfying the conditions $a, a' \not\approx h$; $a, a' \not\approx c$ and $a' | a+c$. If $c \not\approx h$, then there exists a unique (c,h)-dilatation T with Ta=a'. If, on the other hand, $c|h$, then there exists a unique (c,h)-transvection T with Ta=a'.

Theorem 3.12. Let P be a Barbilian space of dimension $n \geqslant 3$. If c, a, a' are points and h is a hyperplane such that c, a, $a' \not\sim h$; $a \not\sim c$ and $a' | a+c$, then there exists a unique affine (c,h)-dilatation T carrying a to a'.

Remark. As in the plane case, the product of central transvections with the same axis h need not have a centre.

Definition. A transvection with axis h is a product of (c_i,h)-transvections with $c_i | h$.

Proposition 3.13. The group of transvections with a given axis h in a Barbilian space is commutative, and its action on the points $\not\sim$ h is sharply transitive.

Proposition 3.14. In a Barbilian n-space, the little projective group (i.e., the group generated by all transvections) is transitive on the set of bases, and also on d-frames for any d with $1 \leq d \leq n$. Further, the full projective group (i.e., the group generated by all transvections and dilatations) is transitive on n-frame.

Theorem 3.15. Any Barbilian space of dimension $n \geqslant 3$ is isomorphic to a projective space over a ring R of stable rank 2. R is unique up to isomorphism. Collineations are induced by bijective semilinear transformations in R^{n+1}. A collineation belongs to the full projective group or to the little projective group if and only if it is induced by an element of $GL_{n+1}(R)$ or $E_{n+1}(R)$, respectively.

For some fundamental properties of full homomorphisms between Barbilian spaces as well as for their algebraic description we refer the reader to VELDKAMP [66b,I].

FAULKNER and FERRAR [19c] surveyed the development which leads from classical Desarguesian projective plane via Moufang planes to Moufang-Veldkamp planes. They first sketched inhomogeneous and homogeneous coordinates in the real and projective planes and in ring planes, the Jordan algebra construction of Moufang planes, and the representation of all these planes as homogeneous spaces for their groups of transvections. Then attention is focussed on Moufang-Veldkamp planes, i.e. projective Barbilian planes in which all possible transvections exist and which satisfy the little quadrangle section condition for quadrangles in general position. As coordinates for the affine plane one easily obtains an alternative ring of stable rank 2. Unfortunately, the Bruck and Kleinfeld theorem for alternative division rings does not carry over to alternative rings in general, i.e. such a ring need neither be associative nor be an octonion algebra. Therefore, to coordinatize the whole projective plane one cannot rely on either homogeneous coordinates (as in the associative case) or the Jordan algebra construction (as in the octonion case). In this case, one has to follow a more complicated way, namely: first to construct a certain Jordan pair from the given alternative ring, then to define a group of transformations of this Jordan pair and, finally, to represent the projective plane as a homogeneous space for that group.

FAULKNER [18c] proved that for a connected Barbilian transvection plane P (i.e, a plane with incidence and neighbouring generalizing Moufang projective planes) one can construct a connected Barbilian plane T(P) called tangent bundle plane . This construction agrees with the usual tangent bundle when such exists. If T (P) is also a transvection plane, then the set S of sections of T(P) is a Lie ring. The group G generated by all transvections of P acts on S. Since S is isomorphic to the Koecher-Tits Lie ring constructed from the Jordan pair $(M_{12}(R), M_{21}(R))$, where R is the associated alternative ring, one can determine G and therebay P from R.

In 1987, SPANICCIATI [58] introduced near Barbilian planes (NBP) and strong near Barbilian planes (SNBP) as a variation of Barbilian planes. Recently, HANSSENS and van MALDEGHEM [29] showed that a NBP is an SNBP, and classified all NBP up to the classification of linear spaces (many examples follows as a result of a universal construction). They also showed that only NBPs that are also BPs are those mentioned in [58], namely the projective planes.

In 1984, ALLISON and FAULKNER [2] have given an algebraic construction of degree 3 Jordan algebras (including the exceptional one) as trace zero elements in a degree 4 Jordan algebra. Recently, FAULKNER [18d] translated this algebraic construction to give a geometric construction of Barbilian planes coordinatized by composition algebras (including the Moufang plane) as skew polar line pairs and points on the quadratic surface determined by a polarity of projective 3 - space over a smaller composition algebra.

§ 4. Groups with Steinberg relations and parametrization theorems

In order to give (in § 5) coordinatization theorems (similar to MOUFANG's [47]) for several polygonal geometries, we present here parametrization theorems for various rank-two groups with Steinberg relations, namely A_2, G_2, B_2, and BC_2. In doing so, we shall use FAULKNER's formulations [18 e], where nonassociative division algebras which parametrize the groups are explicitly constructed.

Notation. Let E^n be the Euclidean space of dimension n with inner product denoted by (x,y) for $x,y \in E^n$. Then, for

$\alpha \in E^n$, $\alpha \neq 0$, the reflection w_α in the hyperplane through the origin and orthogonal to α is given by $w_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$.

Definition. A subset $\sum$ of E^n is called a root system if

a) $\sum$ is finite, spans E^n and $0 \notin \sum$;

b) $w_\alpha(\sum) = \sum$, for $\alpha \in \sum$;

c) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer for $\alpha, \beta \in \sum$.

Definition. A subset S of $\sum$ is called closed if $\alpha, \beta \in S$ and $\alpha + \beta \in \sum$ imply $\alpha + \beta \in S$.

Remark. $S_{\alpha,\beta} := \{ \gamma \in \sum \mid \gamma = i\alpha + j\beta , i,j > 0\}$ is closed.

Definition. A closed subset P of $\sum$ is called a set of positive roots if either α or $-\alpha$ (but not both) is an element of P for each $\alpha \in \sum$.

Recall the following facts about root systems (see STEINBERG [61, Appendix] and SERRE [56]):

a) The w_α 's, $\alpha \in \sum$, generate a finite group W, called the Weyl group;

b) if P is a set of positive roots, then the subset P_s of P of all roots α which cannot be written in the form $\alpha = \beta + \gamma$, $\beta, \gamma \in P$, has the property that each $\beta \in P$ can be written as $\beta = \sum_{\alpha \in P_s} n_\alpha \alpha$, where n_α is a positive integer; P_s is called a set of simple roots;

c) the w_α 's, $\alpha \in P_s$, generate W ;

d) if $\alpha \in P_s \subseteq P$, then w_α permutes the $\beta \in P$, $\beta \neq \alpha, 2\alpha$, while $w_\alpha(\gamma) = -\gamma$, for $\gamma = \alpha$ or 2α ;

e) W acts transitively on the sets of positive roots;

f) if P is the set of positive roots, then there exists an ordering " $<$ " of $\sum$ such that $\alpha<\beta$ implies $\beta-\alpha\in P$;

g) if S is a closed subset of $\sum$ which does not contain α and $-\alpha$ for each $\alpha\in\sum$, then $S\subseteq P$ for some set P of positive roots.

Notation. For a group G, the following notation will be used : $h^g := g^{-1}hg$ and $(g,h) := g^{-1}h^{-1}gh$, g and h being elements of G. Also, if G has subgroups X_α indexed by $\alpha\in\sum$, $\sum$ being a root system, and if $S\subseteq\sum$, by X_S we shall denote the subgroup generated by the X_α's, $\alpha\in S$ (here $X_\emptyset = \{1\}$).

Definition. A group G has Steinberg relations of type $\sum$ if G is generated by nontrivial subgroups X_α, $\alpha\in\sum$, satisfying

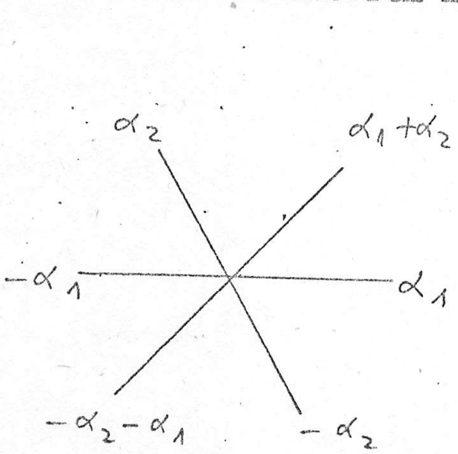
1) $X_{2\alpha}\subseteq X_\alpha$;

2) if $\alpha\neq-\beta$, then $(X_\alpha, X_\beta)\subseteq X_{S_{\alpha,\beta}}$, for $S_{\alpha,\beta}$ as in the above Remark;

3) for each $x\in X_\alpha$, $x\neq 1$, there exists a $w\in X_\alpha X_{-\alpha}$ with $X_\beta^w = X_{w_\alpha(\beta)}$ for all $\beta\in\sum$;

4) for some set P of positive roots, $X_P\cap X_{-P}=1$.

A root system of type A_2 is given in the figure below. The following parametrization theorem holds in this case.



Theorem 4.1. If G has Steinberg relations of type A_2, then there exists an alternative division ring $\mathcal{A}$ and bijections $x\ : \mathcal{A} \longrightarrow X_\alpha$, $\alpha \in A_2$, such that

1) $x_\alpha(a)x_\alpha(b) = x_\alpha(a+b)$ for $a, b \in \mathcal{A}$;

2) $(x_{\alpha_1}(a), x_{\alpha_2}(b)) = x_{\alpha_1+\alpha_2}(ab)$ for fixed simple roots α_1, α_2 and $a, b \in \mathcal{A}$;

3) if $w_\alpha(a) := x_\alpha(a)x_{-\alpha}(a^{-1})x_\alpha(a)$, $\alpha \in A_2$, then :

a) $w_\alpha(a) = w_{-\alpha}(a^{-1})$;

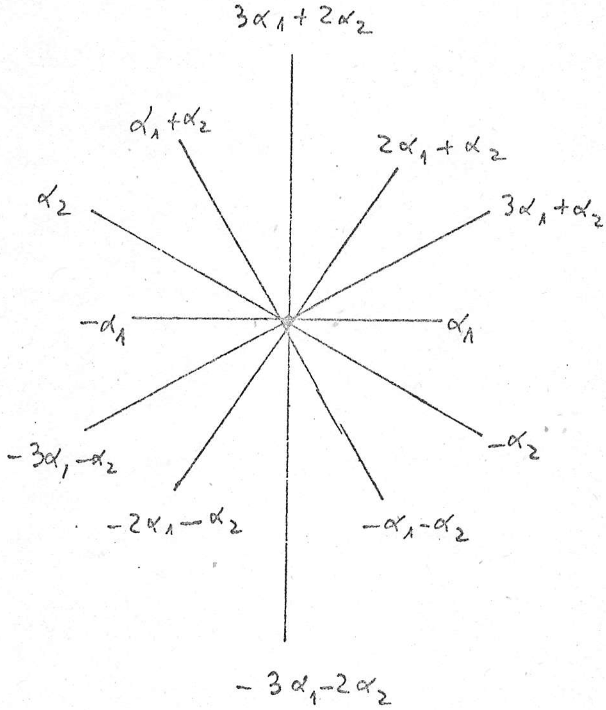
b) $x_\alpha(a)^{w_{\alpha_1}(1)\, w_{\alpha_2}(1)} = x_{w_{\alpha_2} w_{\alpha_1}(\alpha)}(a)$;

c) $x_{\alpha_1}(a)^{w_{\alpha_1}(b)} = x_{-\alpha_1}(b^{-1}ab^{-1})$;

d) $x_{\alpha_2}(a)^{w_{\alpha_1}(b)} = x_{\alpha_1+\alpha_2}(-ba)$.

Note. Recall that the notions of associative, alternative and power-associative ring can be defined in a uniform fashion as follows: a ring is asociative, alternative or power-associative if and only if any of its subring generated by 3 elements, 2 elements or one element, respectively, is associative. Note that alternativity is equivalent to the fact that, for any elements x,y, the identities $(xx)y = x(xy)$ and $(yx)x = y(xx)$ hold.

A root system of the type G_2 is given in the figure below. In this case the following parametrization theorem holds.



Theorem 4.2. If G has Steinberg relations of type G_2, then there exists a quadratic Jordan division algebra $\mathcal{J}(N, \#, e)$ over a field $\mathbb{F}$ and bijections $x_\alpha : \mathcal{J} \longrightarrow X_\alpha$, α short, $x_\beta : \mathbb{F} \longrightarrow X_\beta$, β long, $\alpha, \beta \in G_2$, such that:

1) $x_\alpha(a_1)x_\alpha(a_2) = x_\alpha(a_1+a_2)$, $x_\beta(\tau_1)x_\beta(\tau_2) = x_\beta(\tau_1+\tau_2)$ for $a_i \in \mathcal{J}$, $\tau_i \in \mathbb{F}$;

2) a) $(x_{\alpha_1}(a), x_{\alpha_2}(\tau)) =$
$= x_{3\alpha_1+2\alpha_2}(\tau^2 N(a)) x_{3\alpha_1+\alpha_2}(\tau N(a)) x_{2\alpha_1+\alpha_2} \cdot (-\tau a^\#) x_{\alpha_1+\alpha_2}(\tau a)$;

b) $(x_{\alpha_1}(a), x_{\alpha_1+\alpha_2}(b)) = x_{3\alpha_1+2\alpha_2}(T(a,b^\#)) x_{3\alpha_1+\alpha_2} \cdot (-T(a^\#,b))\, x_{2\alpha_1+\alpha_2}(a \times b)$;

c) $(x_{\alpha_1}(a), x_{2\alpha_1+\alpha_2}(b)) = x_{3\alpha_1+\alpha_2} \cdot (T(a,b))$;

d) $(x_{\alpha_2}(\tau), x_{3\alpha_1+\alpha_2}(\sigma)) = x_{3\alpha_1+2\alpha_2}(\tau\sigma)$;

3) if $w_\alpha(a) := x_\alpha(a)x_{-\alpha}(a^{-1})x_\alpha(a)$,

$w_\beta(\tau) := x_\beta(\tau)x_{-\beta}(\tau^{-1})x_\beta(\tau)$, $0 \neq a \in \mathcal{J}$, $0 \neq \tau \in \mathbb{F}$, then

a) $w_\alpha(a) = w_{-\alpha}(a^{-1})$, $w_\beta(\tau) = w_{-\beta}(\tau^{-1})$;

b) $x_\alpha(a)^{w_{\alpha_1}(e)\, w_{\alpha_2}(1)} = x_{w_{\alpha_2}w_{\alpha_1}(\alpha)}(a)$,

$x_\beta(\tau)^{w_{\alpha_1}(e)\, w_{\alpha_2}(1)} = x_{w_{\alpha_2}w_{\alpha_1}(\beta)}(\tau)$;

c) $x_{\alpha_1}(b)^{w_{\alpha_1}(a)} = x_{-\alpha_1}(U_{a^{-1}}(b))$;

d) $x_{\alpha_2}(\tau)^{w_{\alpha_1}(a)} = x_{3\alpha_1+\alpha_2}(-\tau N(a))$;

e) $x_{\alpha_1+\alpha_2}(b)^{w_{\alpha_1}(a)} = x_{2\alpha_1+\alpha_2}(N(a)U_{a^{-1}}(b))$;

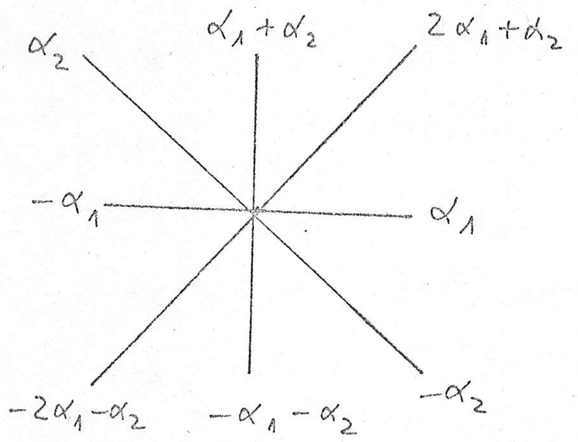
f) $x_{\alpha_1}(a)^{w_{\alpha_2}(\tau)} = x_{\alpha_1+\alpha_2}(\tau a)$;

g) $x_{\alpha_2}(\sigma)^{w_{\alpha_2}(\tau)} = x_{-\alpha_2}(\sigma\tau^{-2})$;

h) $x_{3\alpha_1+\alpha_2}(\sigma)^{w_{\alpha_2}(\tau)} = x_{3\alpha_1+2\alpha_2}(-\tau\sigma)$;

i) $i(w_\alpha(a)^2)$ (resp. $i(w_\beta(\tau)^2)$) is Id on X_γ if $\gamma = \pm\alpha$ or $\gamma \perp \alpha$ (resp., $\gamma = \pm\beta$ or $\gamma \perp \beta$) and is $-$ Id otherwise. (Here $i(g)$ denotes the inner automorphism of G given by $g \in G$). Here α_1 short, and α_2, long, are fixed simple roots for G_2, and $a, b \in \mathcal{J}$, $\sigma, \tau \in \mathbb{F}$.

A root system of type B_2 is given in the figure below. The following parametrization theorem holds.



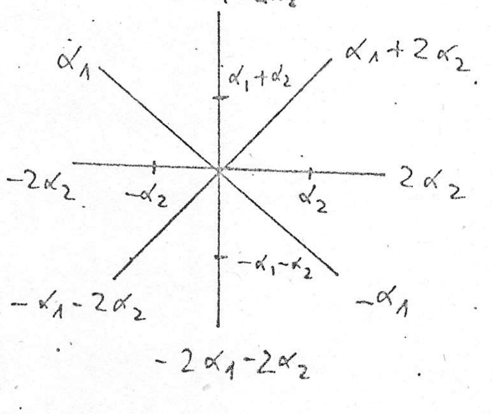
<u>Theorem 4.3.</u> If G has Steinberg relations of type B_2 and if X_γ, $\gamma \in B_2$, has no 2-torsion, then there exists a pair $(\mathcal{A}, \mathcal{H})$ in which either

(a) $\mathcal{A}$ is an associative division algebra with involution $a \longrightarrow \bar{a}$ and $\mathcal{H}$ is the Jordan algebra of symmetric elements of $\mathcal{A}$,

or (b) $\mathcal{H}$ is a field and $\mathcal{A}$ is the Jordan division algebra of a quadratic form.

<u>Remark.</u> There are bijections $x_\alpha : \mathcal{A} \longrightarrow X_\alpha$, $x_\beta : \mathcal{H} \longrightarrow X_\beta$, $\alpha, \beta \in B_2$, α short, β long, such that certain identities hold. (see FAULKNER [18b, pp.52-53]).

A root system of type B_2 is given in the figure below. The following parametrization theorem holds.



Theorem 4.4. If G has Steinberg relations of type BC_2 and if X_γ, $\gamma \in BC_2$, has no 2-torsion, then there exists a triple $(\mathcal{A}, \mathcal{H}, V)$ in which either

(a) $\mathcal{A}$ is an associative division algebra with involution $a \longrightarrow \bar{a}$, $\mathcal{H}$ is the Jordan algebra of symmetric elements of $\mathcal{A}$, and V is a vector space over $\mathcal{A}$ with a nonsingular skew-Hermitian form H,

or

(b) $\mathcal{H}$ is a field, $\mathcal{A}$ is the Jordan division algebra of a quadratic form Q, and V is a unital special $\mathcal{A}$-module with an $\mathcal{H}$-bilinear form $\mathbb{H}$ with values in $\mathcal{A}$ satisfying

(i) $\mathbb{H}(u,v) = -\overline{\mathbb{H}(v,u)}$;

(ii) $U_a(\mathbb{H}(u,v)) = \mathbb{H}(au, \bar{a}v)$;

(iii) $\mathbb{H}(av,v) = 0$ implies $a = 0$ or $v = 0$;

(iv) $Q(a, \mathbb{H}(u,v)) = \mathbb{H}(u,av) - \mathbb{H}(av,u)$,

(here $\mathcal{H}$ was identified with $\mathcal{H}_e \subseteq \mathcal{A}$).

Remark 1. There are bijections $x_\alpha : \mathcal{A} \longrightarrow X_\alpha$, $x_\beta : \mathcal{H} \longrightarrow X_\beta$, $x_\gamma : V \longrightarrow X_\gamma$, for $\alpha, \beta, \gamma \in BC_2$, with $|\gamma| < |\alpha| < |\beta|$, such that certain identities as those in Theorem 4.3 hold. (see FAULKNER [18b , pp.66-67]).

Remark 2. Faulkner's parametrization above may be compared with SELIGMAN's treatment [55] of algebraic Lie algebras.

§ 5. Polygonal geometries and coordinatization theorems

We now give coordinatization theorems similar to MOUFANG's [47] for several polygonal (projective, quadrilateral, and hexagonal, all admitting all elations- see below) geometries. These coordinatize a projective Moufang plane by an alternative division algebra. The essential information needed is contained in the group generated by all elations. These groups are groups with Steinberg relations for which parametrization theorems were presented in the foregoing section. Here again we shall use FAULKNER's [18 c] formulations.

Definition. A geometry consists of a set $\mathbb{P}$ of points, a set $\mathbb{L}$ of lines (disjoint from $\mathbb{P}$), and a subset $\mathbb{I}$ of $\mathbb{P} \times \mathbb{L}$.

Convention. For $(P, \ell) \in \mathbb{I}$ we write $P|\ell$ or $\ell|P$, and we say that P and ℓ are incident. We use the terminology common in geometry; e.g., P lies on ℓ or ℓ passes through P if $P|\ell$; P, Q are collinear if $P|\ell$ and $Q|\ell$ for some $\ell \in \mathbb{L}$; $\ell, m \in \mathbb{L}$ intersect if there is a $P \in \mathbb{P}$ with $P|\ell$ and $P|m$.

Definition. A chain of length k joining two elements a and b in $\mathbb{P} \cup \mathbb{L}$ is a sequence of distinct elements $a_i \in \mathbb{P} \cup \mathbb{L}$, $i = 0, \dots, k$, $a_0 = a$, $a_k = b$, with $a_i \,|\, a_{i+1}$, $i = 0, \dots, k-1$.

Definition. A subset N of $\mathbb{P} \cup \mathbb{L}$ is called an n-gon if $N = C_1 \cup C_2$ and $C_1 \cap C_2 = \{a, b\}$, where C_1, C_2 are chains of length n joining a and b.

Definition. A geometry in which every element of $\mathbb{P} \cup \mathbb{L}$ is incident to at least three distinct elements of $\mathbb{P} \cup \mathbb{L}$ and is such that every pair of elements of $\mathbb{P} \cup \mathbb{L}$ is contained in an n-gon, but there is not a pair in a k-gon for $2 \leq k < n$, is called an n-gonal geometry.

Remark 1. A geometry is 3-gonal or triangular if and only if it is a projective plane.

Remark 2. The above definition for an n-gonal geometry is equivalent to that given by TITS [64, p.82].

Proposition 5.1. A geometry in which every element of $\mathbb{P} \cup \mathbb{L}$ is incident to at least three other elements is an n-gonal geometry if and only if every pair of elements of $\mathbb{P} \cup \mathbb{L}$ can be joined by a chain of length $k \leq n$ and all chains joining this pair of elements have the length k.

Definition. If a and b are two elements in an n-gonal geometry, then the unique length of chains joining a with b is called the distance from a to b and is denoted by d(a,b).

Definition. A bijection σ which is defined on $\mathbb{P} \cup \mathbb{L}$, maps $\mathbb{P}$ bijectively onto $\mathbb{P}$ and $\mathbb{L}$ bijectively onto $\mathbb{L}$ such that $a \mid b$ if and only if $\sigma(a) \mid \sigma(b)$ for $a \in \mathbb{P}$, $b \in \mathbb{L}$ is called a collineation.

Proposition 5.2. If d(a,b)=n for some $a, b \in \mathbb{P} \cup \mathbb{L}$ in an n-gonal geometry, and if σ is a collineation fixing a, b and all $x \mid a$, then σ fixes all $y \mid b$.

Proposition 5.3. If σ is a collineation of an n-gonal geometry fixing all x incident to a or b, where $a \mid b$, and also fixing all a' with d(a,a') = n, then $\sigma = \mathrm{Id}$.

COORDINATIZATION OF PROJECTIVE PLANES. Let us begin with the following

Definitions. An elation (also, transvection) of a projective plane (triangular geometry) is a collineation fixing all $x \in \mathbb{P} \cup \mathbb{L}$ incident to some elements c or a, where a is a line, called the axis, and c is a point on this line, called the centre.

Remark. Let b be a point not lying on a; i.e., d(b,a)=3. Proposition 5.3 shows that an elation σ is uniquely determined by c, a, b, and σ(b). Note that $\sigma(b) \neq c$ lies on the line ℓ joining c and b.

Given c, a, b, ℓ as above and $d \mid \ell$, $d \neq c$, an elation σ with $\sigma(b) = d$ may or may not exist. If σ does exist for all such choices, then one says that the projective plane admits all elations (or, is a Moufang plane).

Theorem 5.4. The group generated by all elations of a projective plane admitting all elations has Steinberg relations of type A_2.

Coordinatization Theorem 5.5. A projective plane admitting all elations may be coordinatized by an alternative division ring A as follows: the points are (∞), (m), (a,b), $m,a,b \in A$, the lines are $[\infty]$, $[a]$, $[m,d]$, $a,m,d \in A$, and the incidences are $(\infty)|[\infty]$, $(\infty)|[a]$, $(m)|[\infty]$, $(m)|[m,d]$, $(a,b)|[a]$, and $(a,b)|[m,d]$ if $b = am+d$.

COORDINATIZATION OF HEXAGONAL GEOMETRIES. First, we give the following.

Definition. A collineation σ of a hexagonal geometry is called an axial elation if σ fixes all lines intersecting a particular line a, called the axis, and all points on two lines b and c, called secondary axes, which intersect a but not each other.

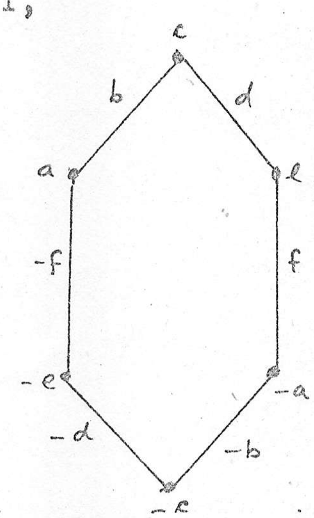
Note. If the roles of points and lines are reversed, then the notion of a central elation, with its centre and secondary centres, is obtained.

Remark. If σ is an elation (either axial or central) with a,b,c as above, and $d(d,b)=2$, $d(d,a) \neq 2$, then σ is determined by a,b,c,d, and $\sigma(d)$.

If for all choices a,b,c,d,d' with $d(a,b) = d(a,c) = d(d,b) = d(d',b) = d(d,d') = 2$, $d(d,a) \neq 2$, and $d(b,c) \neq 2$, there exists an elation σ with axis (or centre) a, with secondary axes (or centres) b and c, and with $\sigma(d)=d'$, then one says that the hexagonal geometry admits all elations.

Note. That a hexagonal geometry admits all elations is tantamount to saying that the group of elations with axis a and secondary axes b and c acts transitively on the set of all lines other than b through a point p which lies on b but not on a. <u>Only hexagonal geometries admitting all elations are considered here.</u>

Fix a hexagonal H as in the adjoining figure, so that a is a point and a,b,c,d,e,f, -a and a, -f,-e, -d, -c, -d, -a are distinct elements. Let X_a be the group of elations with centre a and secondary centres c an -e, and let X_b, etc. be defined similarly, and let X_S be the subgroup generated by the groups X_u for all $u \in S \subseteq H$.



<u>Definition.</u> A collineation σ of a hexagonal geometry is called a <u>strong elation</u> if σ fixes all x with $d(x,y) \leq 3$ for some y of the given hexagonal geometry.

<u>Remark 1.</u> If y in the above definition is a line, then this is equivalent to saying that σ is an axial elation with axis y, and each line intersecting y is a secondary axis.

<u>Remark 2.</u> The strong elations in X_u form a subgroup, denoted by $\tilde{X}_u$.

<u>Theorem 5.6.</u> Let $\mathcal{H}$ be a hexagonal geometry which admits all elations.

a) If all central elations are strong or if all axial elations are strong, then the group generated by all elations is a group with Steinberg relations of type G_2.

b) If there are central and axial elations which are not strong, then for any hexagon H of $\mathcal{H}$ and $u_o \in H$, the group generated by all $\tilde{X}_u$, X_v with $d(u,v_o)$ even and $d(v,u_o)$ odd, $u,v \in H$, is a group with Steinberg relations of type G_2.

<u>Coordinatization Theorem 5.7.</u> If $\mathcal{H}$ is a hexagonal geometry which admits all elations and if all central elations are strong, then there exists a quadratic Jordan division algebra $\mathcal{J}(N, \#, e)$ over a field $\mathbb{F}$, such that:

a) the points of $\mathcal{H}$ may be uniquely represented by (∞) and k-tuples $(\tau_1, a_2, \tau_3, a_4, \ldots)$ with $k \leq 5$, where $\tau_i \in \mathbb{F}$, $a_{2i} \in \mathcal{J}$, $i = 1, \ldots, k$;

b) the lines of $\mathcal{H}$ may be uniquely represented by $[\infty]$ and k-tuples $[a_1, \tau_2, a_3, \ldots]$ with $k \leq 5$, where $a_i \in \mathcal{J}$, $\tau_{2i} \in \mathbb{F}$, $i = 1, \ldots, k$;

c) the incidences are

(i) $(\infty) | [\infty]$, $(\tau) | [\infty]$, $(\infty) | [a]$ for $\tau \in \mathbb{F}$, $a \in$;

(ii) $(\tau_1, a_2, \tau_3, \ldots) | [a_2, \tau_3, \ldots]$ and $(\tau_2, a_3, \ldots) | [a_1, \tau_2, \ldots]$;

(iii) $(\sigma_1, b_2, \sigma_3, b_4, \sigma_5) | [a_1, \tau_2, a_3, \tau_4, a_5]$ provided that

$$-a_1 = \sigma_5 a_5 + b_4,$$

$$\tau_2 = \sigma_5^2 N(a_5) + \sigma_5 T(a_5^{\#}, b_4) + T(a_5, b_4^{\#}) - \sigma_5 \sigma_1 + T(b_2, b_4) + \sigma_3,$$

$$a_3 = \sigma_5 a_5^{\#} + a_5 \times b_4 + b_2,$$

$$-\tau_4 = \sigma_5 N(a_5) + T(a_5^{\#}, b_4) + T(a_5, b_2) + \sigma_1.$$

<u>COORDINATIZATION OF QUADRILATERAL GEOMETRIES.</u> We begin with the following

<u>Definition.</u> A collineation σ of a quadrilateral geometry $\mathcal{Q}$ is called an <u>elation</u> if σ fixes all elements incident to any one of three elements a_1, a_2, a_3 where a_1, a_2, a_3 form a chain. If a_2 is a line (a point), then σ is called an <u>axial</u> (<u>central</u>) elation with <u>axis</u> (<u>centre</u>) a_2 and <u>centres</u> (<u>axes</u>) a_1 and a_3.

<u>Definition.</u> A central elation is called <u>strong</u> if every line through the centre is an axis, and similarly for axial elations.

One says that $\mathcal{Q}$ admits all elations if for all choices of chains a_1, a_2, a_3, a_4, a_5 and $a_5' \mid a_4$, $a_5' \neq a_3$, there exists an elation with axis (or centre) a_2, with centres (or axes) a_1 and a_3, and with $\sigma(a_5) = a_5'$.

Theorem 5.8. If $\mathcal{Q}$ is a quadrilateral geometry which admits all elations and if $\mathcal{Q}$ has no elation of order two, then the group generated by all elations is a group with Steinberg relations of type B_2 (if all central or all axial elations are strong) or type BC_2 (if there exist central and axial elations which are not strong).

Coordinatization Theorem 5.9. If $\mathcal{Q}$ is a quatrilateral geometry admitting all elations and having no elation of order two, then all nontrivial strong elations are either central or axial. If all nontrivial strong elations are central, then

a) if all central elations are strong, then there exists a pair $(\mathcal{A},\mathcal{H})$ as in Theorem 4.3, §4, such that the points of $\mathcal{Q}$ may be represented by (∞), (a_1), (a_1,b_1), (a_2,b_1,a_1), the lines by $[\infty]$, $[b_1]$, $[b_1,a_1]$, $[b_1,a_2,b_2]$ with $a_i \in \mathcal{A}$, $b_i \in \mathcal{H}$, and with incidences

(i) $(\infty) \mid [\infty]$, $(a_1) \mid [\infty]$, $(\infty) \mid [b_1]$;

(ii) $(a_1,b_1) \mid [b_1]$, $(a_1,b_1,a_2) \mid [b_1,a_2]$,

$(a_1) \mid [b_1,a_1]$, $(a_1,b_2) \mid [b_1,a_1,b_2]$;

(iii) $(a_1,b_2,a_2) \mid [k_1,b_1,k_2]$ provided that

$$k_1 = \bar{a}_2 k_2 a_2 + a_1 a_2 + \bar{a}_2 \bar{a}_1 + b_2, \quad b_1 = -k_2 a_2 - \bar{a}_1 \;;$$

b) if not all central elations are strong, then there exists a triple $(\mathcal{A},\mathcal{H},V)$ as in Theorem 4.4, §4, such that the points of $\mathcal{Q}$ may be represented by (∞), (a_1), $(a_1,(v_1,b_1))$, $(a_1,(v_1,b_1),a_2)$, the lines by $[\infty]$, $[(v_1,b_1)]$, $[(v_1,b_1),a_1]$, $[(v_1,b_1), a_2, (v_2,b_2)]$ with $a_i \in \mathcal{A}$,

$h_i \in \mathcal{H}$, $v_i \in V$ and incidences

(i) $(\infty) \,|\, [\infty]$, $(a_1) \,|\, [\infty]$, $(\infty) \,|\, [(v_1,h_1)]$;

(ii) $(a_1,(v_1,h_1)) \,|\, [(v_1,h_1)]$, $(a_1,(v_1,h_1),a_2) \,|\, [(v_1,h_1),a_2]$,

$(a_1) \,|\, [(v_1,h_1),a_2]$, $(a_1,(v_2,h_2)) \,|\, [(v_1,h_1),a_1,(v_2,h_2)]$;

(iii) $(a_1,(v_2,h_2),a_2) \,|\, [(w_1,k_1),b_1,(w_2,k_2)]$ provided that

$$w_1 = \bar{a}_2 w_2 + v_2 ,$$

$$k_1 = \bar{a}_2 k_2 a_2 + a_1 a_2 + \bar{a}_2 \bar{a}_1 + h_2 ,$$

$$b_1 = \tfrac{1}{2} H(w_2, \bar{a}_2 w_2) + H(w_2, v_2) - k_2 a_2 - \bar{a}_1 ,$$

H being defined as in Theorem 4.4, § 4 . If all nontrivial strong elations are axial, then the duals of a) and b) hold.

Comments. For a brief survey on the relations between various exceptional notions in algebra and geometry (e.g., non-classical Lie algebras, nonassociative alternative algebras, non-special Jordan algebras, non-Desarguesian projective planes), the reader is referred to FAULKNER and FERRAR [19a], who proved that all these notions are related, one way or another, to the octonions.

References

1. AKIVIS, M.A., Smooth lines on projective planes over certain associative algebras, Math. Notes 41 (1987), No.1-2, 131-136.

2. ALLISON, B.N., FAULKNER, J.R., A Cayley-Dickson process for a class of structurable algebras, Trans. Amer. Math. Soc. 283 (1984), 185-210.

3. ARTIN, E., Geometric Algebra, Interscience Publishers, New York, London, 1957.

4. ATSUYAMA, M., On the embeddings of the Cayley plane into the exceptional Lie group of type F_4, Kōdai Math.Sem.Rep. 28 (1977), 129-134.

5. BARBILIAN, D., a) Grundiss einer geometrischen Algebre lehre, Bull.Math.Soc.Roumaine Sci. 40 (1939), No.2;

b) Zur Axiomatik der projektiven Ringgeometrien, I,II, Jahresber. Deutsch.Math.Verein, 50 (1940),179-229; 51 (1941), 34-76.

6. BENZ,W., a) On Barbilian domains over commutative rings, J.Geom 12 (1979), 2, 146-151.

b) On mappings preserving a single Lorentz-Minkowski-distance in Hjelmslev's classical plane, Aequationes Math.28 (1985),No.1-2, 80-87.

7. BHAMBRI, S.K., On generalized perspectivities of projective planes, Ph.D.Thesis, Westfield College, Univ.of London, 1981.

8. BINGEN, F., Géométrie projective sur un anneau semiprimaire, Acad.Roy.Belg.Bull.Cl.Sci.(5), 52 (1966), 13-24.

9. BIX,R., a) Octonion planes over local rings, Trans.Amer.Math. Soc.,261 (1980), 2, 417-438;

b)Isomorphism theorems for octonion planes over local rings, Trans.Amer.Math.Soc. 266 (1981), 2, 423-440.

c)Octonion planes over Euclidean domains,Tagungsbericht Jordan-Algebren (1985), Oberwolfach.

10. BOMMIREDDY, M., NAGAMUNI REDDY, L., SITARAM,K., Planar ternary rings with zero of translation, Moufang and Desarguesian planes, Arch.Math.(Basel) 45 (1985),No.5, 476-480.

11. BRADA,C., Éléments de la géométrie des octaves de Cayley, Dissertation,Univ.d'Aix-Marseille I, Marseille,1984, Univ.Claude-Bernard,Département de Mathématdque,Lyon, 1986

12. BRADA, C., PECAUT-TISON, F., Géométrie du plan projectif des octaves de Cayley, Geom. Dedicata 23 (1987), No.2, 131-154.

13. BREZULEANU, A., RADULESCU, D.C., a) About full or injective lineations, J.Geom. 23 (1984), 45-60;
b) Characterizing lineations defined on open subsets of projective spaces over ordered division rings, Abh.Math.Sem.Univ.Hamburg 55 (1985), 171-181;
see also INCREST Preprints No.65/1981 and No.58/1982.

14. BROŽIKOVA, E., Homomorphisms of Jordan algebras and homomorphisms of projective planes, Czechoslovak Math. J. 36 (1986), No.1, 48-58.

15. CARTER, D.S., VOGT, A., Collinearity-preserving functions between Desarguesian planes, Mem.Amer.Math.Soc. 27, No.235 (1980).

16. DAVIES, M., Universal G-manifolds, Amer.J.Math. 103 (1981), 1, 103-141.

17. ESSER, M., Self-dual portulates for n-dimensional geometry, Duke Math.J. 18 (1951), 475-479.

18. FAULKNER, J.R., a) Octonion planes defined by quadratic Jordan algebras, Mem.Amer.Math Soc. No.104 (1970).
b) Groups with Steinberg relations and coordinatization of polygonal geometries, Mem.Amer.Math.Soc. 10 (1977), Nr.185.
c) Tangent bundles for Barbilian planes, Tagunsbericht Jordan-Algebren (1985), Oberwolfach.
d) A geometric construction of Moufang planes, Geom. Dedicata 29 (1989), No.2, 133-140.
e) Barbilian planes, Geom.Dedicata 30 (1989), No.2, 125-181.

19. FAULKNER, J.R., FERRAR, J.C., a) Exceptional Lie algebras and related algebraic and geometric structures, Bull.London Math.Soc. 9 (1977), 1, 1-35.
b) Homomorphisms of Moufang planes and alternative places, Geom. Dedicata 14 (1983), No.3, 215-223.
c) Generalizing the Moufang plane, in "Ring and Geometry", Reidel, Dordrecht, 1985, pp.235-288.

20. FERRAR, J.C., a) Homomorphisms of projective planes, Tagungsbericht 33/1982, Jordan-Algebren, Oberwolfach.
b) Homomorphisms of Moufang-Veldkamp planes, Tagungsbericht Jordan-Algebren (1985), Oberwolfach.

21. FERRAR, J.C., VELDKAMP, F.D., Neighbor-preserving homomorphisms between projective ring planes, Geom. Dedicata 18 (1985), 11-35.

22. FREUDENTHAL, H., Oktaven, Ausnahmengruppen und Oktavengeometrie, Math.Inst.Rijksuniversiteit Utrecht, Utrecht, 1951, (mimeographed); new revised edition, 1960; republished in Geom.Dedicata 19 (1981), No.1, 7-64.

23. FRITSCH, R., PRESTEL, A., Bewertungsfortsetzungen und nichtinjektive Kollineationen, Geom.Dedicata 13 (1982), 107-111.

24. GARNER, L.E., Fields and projective planes: a category equivalence, Rocky Mountain J.Math. 2 (1972), 605-610.

25. GRUNDHÖFER, T., a) Antomorphism groups of compact projective planes, Geom.Dedicata 21 (1986), No.3, 291-298.
b) Compact disconnected Moufang planes are Desarguesian, Arch.Math.(Basel) 49 (1987), No.2, 124-126.

26. GUMM, H.P., The Little Desarguesøian Theorem for algebras in modular varieties, Proc.Amer.Math.Soc. 80 (1980), No.3, 393-398.

27. HÄHL, H., a) Sechzehndimensionale lokalkompakte Translationsebene mit Spin (7) als Kollineationsgruppe, Arch. Math. (Basel) 48 (1987), No.3, 267-276.
b) Eine Kennzeichung der Oktavenebene, Nederl. Akad. Wetensch. Proc. A 90 = Indag. Math. (1987), No.1, 29-39.

28. HALL ,M., The theory of groups, The Macmillan Company, New York, 1959.

29. HANSSENS, G., an MALDEGHEM, H., A note on near-Barbilian planes, Geom. Dedicata 29 (1989), No.2, 233-235.

30. HAVEL, V., Ein Einbettungssatz für die Homomorphosmen von Moufang-Ebenen, Czechoslovak. Math. J. 20 (1970), 340-347.

31. JACOBSON, N., Some groups of transformations defined by Jordan algebras, III, J. Reine Angew. Math. 207 (1961) 61-85.

32. JONKER, P., Restricted Lie algebras over a field of characteristic 2, Dissertation, Math. Inst. Rijksuníversiteit Utrecht, Utrecht, 1968.

33. JORDAN, P., Über eine nicht-Desarguessche Ebene projective Geometrie, Abh. Math. Sem. Univ. Hamburg 16 (1949), 74-76.

34. KLINGENBERG, W., Projective Geometrien mit Homomorphismus, Math. Ann. 132 (1956), 180-200.

35. KNÜPPEL, F., a) Regular homomorphismus of generalized projective planes, J. Geometry 29 (1987), No.2, 170-181.
b) Projective planes over rings, Resultate Math. 12, (1987), No.3-4, 348-356.

36. KNÜPPEL, F., SALOW, E., Plane elliptic geometry over rings, Pacific J. Math. 123 (1986), No.2, 337-384.

37. KOECHER,M., Über eine Gruppe von rationalen Abbildungen, Invent.Math. 3 (1967), 136-171.

38. KULKARNI,M., Fundamental theorem of projective geometry over a commutative ring, Indian J. Pure Appl.Math.11 (1980), nO.12, 1561-1565.

39. KUZNETSOVA,T.A., Duo-octavic and duoantioctavic planes(in Russian). Geometry, No.6, pp.65-72,Leningrad,Gos. Ped.Inst.,Leningrad,1977.

40. LANTZ,D.C.,Uniqueness of Barbilian domains, J.Geom. 15 (1981), 1, 21-27.

41. LEISSNER,W., a) Affine Barbilian-Ebenen, I,II, J.Geom. 6 (1975), 1 and 2, 31-57 and 105-129.

b) On classifying affine Barbilian spaces,Resultate Math. 12 (1987), No.1-2, 157-165.

42. LEISSNER,W., SEVERIN, R., WOLF,K., Affine geometry over free unitary modules, J.Geom. 25 (1985), No.2, 101-120.

43. LORIMER, J.W., Dual numbers and topological Hjelmslev planes, Canadian Math.Bull.26 (1983), No.3, 297-302.

44. MARTINEZ,A.,PÉREZ, J.D., Real hypersurfaces in quaternionic projective space, Ann.Mat.Pura Appl.145 (1986), 355-384.

45. MATHIAK,K, Valuations of skew fields and projective Hjelmslev spaces, Lecture Notes in Math.1175, Springer,Berlin-Heidelberg-New York,1986.

46. McCRIMMON,K., a) A general theory of Jordan rings, Proc.Nat. Acad.Sci.USA 56 (1966), 1072-1079;

b) The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras, Trans.Amer.Math.Soc. 139 (1969), 495-510; 148 (1970), 293-314.

47. MOUFANG, R., Alternativekörper und der Satz von vollständigen Vierseit (D_9), Abh.Math.Sem.Univ.Hamburg 9 (1933),

48. PERSITS, D.B., a) Geometries over degenerate octaves (Russian), Dokl.Akad.Nauk SSSR 173 (1967), 1010-1013;
b) Geometries over degenerate octaves and degenerate antioctaves (Russian), Trudy Sem.Vektor.Tenzor. Anal. 15 (1970), 165-187.

49. PETERSSON,H.P., Composition algebras over a field with discrete valuation, J.Algebra 29 (1974), 414-426.

50. PICKERT,G., Projective Ebenen, Springer, Berlin, 1955.

51. RADÓ, F., a) Darstelkung nicht-injective Kollineationen eines projectiven Raumes durch verllgemeinerte semilineare Abbildungen, Math.Z. 110 (1969), 153-170;
b) Non-injective collineations on some sets in Desarguesian projective planes and extension of non-commutative valuations, Aequations Math. 4 (1970), 3, 307-321;
c) Affine Barbilian structures, J.Geom. 14 (1980), 1, 75-102.

52. SARATH,B., VARADARAJAN,K., Fundamental theorem of projective geometry, Comm.Algebra 12 (1984), No.7-8, 937-952.

53. SCHAFER,R.D., An introduction to non-associative algebras, Academic Press,New York, 1966.

54. SEIER,W., Quasi-translations of Desarguesian affine Hjelmslev planes, Math.Z. 177 (1981), No.2, 181-186.

55. SELIGMAN,G., Rational methods in Lie algebras, Marcel Dekker, New York, 1976.

56. SERRE, J.P., Algèbres de Lie semi-simple complexes, W.A. Benjamin Inc.,New York, 1966.

57. SKORNYAKOV,L.A., Homomorphisms of projective planes and T-homomorphisms of ternaries(in Russian), Mat.Sb. 43 (1957), 285-294.

58. SPANICCIATI, R., Near-Barbilian planes, Geom.Dedicata 24 (1987), No.3, 311-318.

59. SPRINGER, T.A., a) The projective octave plane, Nederl.Akad. Wetensch.Proc. A 63 = Indag.Math. 22 (1960), 74-101;

b) On the geometric algebra of the octave planes, Nederl.Akad.Wetensch.Proc.A 65 = Indag.Math. 24 (1962), 451-468.

60. SPRINGER, T.A., VELDKAMP, F.D., a) Elliptic and hyperbolic octave planes, Nederl.Akad.Wetensch.Proc. A 65 = Indag.Math. 25 (1963), 413-451;

b) On Hjelmslev-Moufang planes, Math.Z. 107 (1968), 249-263.

61. STEINBERG, R., Lectures on Chevalley groups, Yale University Notes, 1967.

62. STEINKE, C.F., Topological affine planes composed of two Desarguesian halfplanes and projective planes with trivial collineation group, Arch.Math.(Basel) 44 (1985), No.5, 472-480.

63. SUH, T., On isomorphisms of little projective groups of Cayley planes, Nederl.Akad.Wetensch.Proc. A 65 = Indag. Math. 24 (1962), 320-339.

64. TITS, J., Sur la trialité et certains groups s'en déduisant, IHES, Publ.Math. 2, (1959), 37-84.

65. VEDDER, K., Generalized elations, Bull.London Math.Soc. 18 (1986), 573-579.

66. VELDKAMP, F.D., a) Isomorphisms of little and middle projective groups of octave planes, Nederl.Akad. Wetensch.Proc. A 67 = Indag.Math. 27 (1964), 280-289;

b) Collineation groups in Hjelmslev-Moufang planes, Math.Z. 108 (1968), 37-52;

c) Unitary groups in projective octave planes, Comp.Math. 19 (1968), 213-258;

d) Unitary groups in Hjelmslev-Moufang planes, Math.Z. 108 (1969), 288-312.

e) Projective planes over rings of stable rank 2, Geom.Dedicata 11 (1981), No.5, 285-308.

f) Distant-preserving homomorphisms between projective ring planes, Nederl.Akad.Wetensch.Proc. Ser.A. 88 = Indag.Math.47 (1985), No.4, 443-454;

g) Incidence-preserving mappings between projective ring planes, Nederl.Akad.Wetensch.Proc.Ser. A 88 = Indag.Math.47 (1985), No.4, 455-459.

h) Projective Barbilian spaces I,II, Resultate Math.12 (1987), No.1-2, 222-240; 12 (1987), No. 3-4, 434-449.

67. VDOVIN,V.V., a) Simple projective planes, Arch.Math.(Basel) 47 (1986), No.5, 469-480.

b) Homomorphisms of freely generated projective planes. Comm.Algebra 16 (1988), No.11, 2209-2230.

68. WEIDA, R.A., Replaceable partial spreads and the construction of non-Desarguesian translation planes, Ph.D.Thesis Univ.of Delaware, 1986.

69. YAMAGUTI,K., An algebric system arising from the metasymplectic geometry I,II,III, Bull.Fac.School Ed.Hiroshima Univ.Part.II, 4 (1981), 57-63; 5 (1982), 65-74; 9 (1986), 65-73.

70. YAMAGUTI,K., ATSUKO,O., On representations of Freudenthal--kantor triple systems U(ε,δ), Bull.Fac. School Ed.Hiroschima Univ.Part II, 7 (1984), 43-51.

71. YOKOTA,I., Embeddings of projective spaces into elliptic projective Lie groups, Proc.Japan Acad. 35 (1959), 281-283.