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ON THE MESH - INDEPENDENCE PRINCIPLE FOR GALERKIN DISCRETIZATIONS

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Abstract. The mesh - independence principle is obtained for Galerkin discretizations, in the norm induced by Gramm matrix. This result follows the line of Allgower, Bohmer, Potra and Rheinboldt ([3]), using the convergence of the Newton's method for the initial equation on a Hilbert space and the approximation properties on subspaces as in finite element method. This paper is a revised version.

Key words: Newton's method, Galerkin discretization, mesh independence
AMS (MOS): 65F30, 65F35, 65N30

1. INTRODUCTION

In [3] the authors proves the mesh-independence principle (M.I.P.) in the hypotheses of the Lipschitz uniform, bounded, stable and consistent discretizations. Their model, characteristic for finite difference discretizations and covering another type discretizations, is not applicable for Galerkin schemas because the stability is not valid. MIP was studied in many papers, see the refferences contained in [2] (there we reffere only at [2] and [6], that was used in [3]. MIP consist in: the above hypotheses ensure the existence of the solution for discrete equation, the convergence of this at the solution of the initial equation, the existence and convergence of the Newton's sequence for discrete equation and the "parallelism" of this convergence with the convergence of the Newton's method for the initial equation.

We use the framework of [1] for Galerkin schemas, sepparing the analysis on approximation subspaces of the analysis on the real Euclidean spaces. In this framework that is shortly explained in the following, excepting the consistence, the hypotheses of [3] are natural on approximation subspaces.

Let on the real separable Hilbert space $\mathcal{H}$, the following equation

(1.1) $$\mathcal{T}u = f$$

One of the methods for to obtain approximation equations for (1.1) is the projection method. This consist in to project (1.1) on a finite dimension subspace $S_h \subset \mathcal{H}$, i.e. to find $u_h \in S_h$ such that

(1.2) $$P_h \mathcal{T} u_h = P_h f$$

Let P_h be the orthogonal projection of $\mathcal{H}$ onto S_h. Then (1.2) is equivalent with the following equation on S_h

(1.3) $$\mathcal{T}_h u_h = P_h f \quad , \quad \mathcal{T}_h := P_h \mathcal{T} P_h$$

We restrict our considerations at the linear operators, $\mathcal{T} \in [\mathcal{H}]$. Let S_h be spanned by the linear independent family $\{\phi_h^j,\ j = 1,n_h\}$ in $\mathcal{H}$ and $\mathbf{R}^{n_h}$ the real Euclidean space with same dimension, whose inner product is indexed by h. Let $J^h \in \mathcal{L}(\mathbf{R}^{n_h}, S_h)$ wich maps the canonical basis $\{e_h^j,\ j = 1,n_h\}$ in $\mathbf{R}^{n_h}$ onto the family $\{\phi_h^j\}$: $J^h e_h^j = \phi_h^j$, $j = 1,n_h$. Denoting by J_h the adjoint of J^h related at the both inner products, the linear operator on $\mathbf{R}^{n_h}$

(1.4) $$T_h := J_h \mathcal{T}_h J^h$$

has as matrix representation in the canonical basis, the Galerkin matrix, associated at $\mathcal{T}$ for the family $\{\phi_h^j\}$: $(T_h)_{ij} = \langle \mathcal{T}\phi_h^j, \phi_h^i \rangle$. This is the natural connextion between the projection method and Galerkin method.

Identifying the operators on $\mathbf{R}^{n_h}$ with their matrix reprezentation in the canonical basis, let $\hat{J}_h := L_h^{-1} J_h$, where L_h, L_h^* are the Choleski factors of the Gram matrix of $\{\phi_h^j\}$, $G_h = J_h J^h \in [\mathbf{R}^{n_h}]$. The following result (Theorem 1 in [1]) permits the transfer of the properties on approximation subspaces at the matrix representations on $\mathbf{R}^{n_h}$: the application $\Lambda_h : [S_h] \to [\mathbf{R}^{n_h}]$,

$$\Lambda_h(\mathcal{T}_h) = \hat{J}_h \mathcal{T}_h \hat{J}^h \tag{1.5}$$

preserves the spectrum, norm and condition number, i.e. with the notation $\hat{T}_h = \Lambda_h(\mathcal{T}_h)$, we have

$$\begin{aligned} \sigma(\hat{T}_h) &= \sigma(\mathcal{T}_h) \\ \|\hat{T}_h\|_h &= \|\mathcal{T}_h\| \\ \mathcal{K}(\hat{T}_h) &= \mathcal{K}(\mathcal{T}_h) \end{aligned} \tag{1.6}$$

where the operator norms are the induced norms, and the condition number is defined by $\mathcal{K}(\mathcal{T}_h) = \|\mathcal{T}_h\| \cdot \|\mathcal{T}_h^{-1}\|$. So,

$$\hat{T}_h = L_h^{-1} T_h L_h^{-*} \tag{1.7}$$

is the preconditioned Galerkin matrix by the Choleski factors of the Gramian.

If $\mathcal{T}$ is a bounded linear operator, $\|\mathcal{T}\| \leq M$ and positive definite,

$$\langle \mathcal{T}u, u \rangle \geq m \|u\|^2 \quad , u \in \mathcal{H}$$

then we have:

REMARK 1. The Galerkin matrices of $\mathcal{T}$ are spectral equivalent with the Gram matrices, with the equivalence constants independent of h:

$$m \leq \frac{\langle T_h \tilde{\xi}_h, \tilde{\xi}_h \rangle_h}{\langle G_h \tilde{\xi}_h, \tilde{\xi}_h \rangle_h} \leq M \quad , (\forall)\ \tilde{\xi}_h \in \mathbb{R}^{n_h}, \ \tilde{\xi}_h \neq \tilde{0}_h \tag{1.8}$$

Proof. For $\tilde{\xi}_h \in \mathbf{R}^{n_h}$, let $\xi_h = J^h \tilde{\xi}_h \in S_h$. We have,

$$\|\xi_h\|^2 = \|J^h \tilde{\xi}_h\|_h^2 = \langle G_h \tilde{\xi}_h, \tilde{\xi}_h \rangle_h$$

Now,

$$\langle T_h \tilde{\xi}_h, \tilde{\xi}_h \rangle_h = \langle \mathcal{T}_h \xi_h, \xi_h \rangle = \langle \mathcal{T} \xi_h, \xi_h \rangle$$

and

$$m \|\xi_h\|^2 \leq \langle \mathcal{T} \xi_h, \xi_h \rangle \leq M \|\xi_h\|^2$$

what proves the affirmation.

2. STABILITY PROPERTY

Let $\mathcal{T}_h$ the approximation operator of the nonsingular operator $\mathcal{T} \in [H]$. I do not known if the both hypotheses: $\mathcal{T}^{-1}$ is bounded and there exists $\mathcal{T}_h^{-1}$, are sufficient for stability, i.e. there exists a constant c independent of h such that $\|\mathcal{T}_h^{-1}\| < c$. But, equivalent conditions with the stability can be given. For this, let u and u_h be solutions of the equation (1.1) respectively for approximation equation obtained by projection method (1.3), corresponding at $f \in \mathcal{H}$.

LEMMA 1. If $\mathcal{T}_h^{-1}$ there exists, then the following affirmations are equivalent:

i) (Cea's lemma type property) There exist c independent of h, such that for any $u \in \mathcal{H}$,

(2.1) $$\|u - u_h\| \leq c \, \|u - P_h u\|$$

ii) (Approximation property of the inverse) There exist c independent of h, such that,

(2.2) $$\|\mathcal{T}^{-1} - \mathcal{T}_h^{-1} P_h\| \leq c$$

iii) (Stability property) There exists c independent of h, such that,

(2.3) $$\|\mathcal{T}_h^{-1}\| < c$$

Proof. **i) $\Rightarrow$ ii)** From (2.1) for any $u \in \mathcal{H}$,

$$\|(I - \mathcal{T}_h^{-1} P_h \mathcal{T}) u\| = \|u - \mathcal{T}_h^{-1} P_h f\| = \|u - u_h\| \leq c \, \|u - P_h u\| \leq c \, \|u\|$$

Now $$\|\mathcal{T}^{-1} - \mathcal{T}_h^{-1} P_h\| \leq \|\mathcal{T}^{-1}\| \cdot \|I - \mathcal{T}_h^{-1} P_h \mathcal{T}\| \leq c$$

ii) $\Rightarrow$ iii). Because

$$\|\mathcal{T}_h^{-1}\| = \|\mathcal{T}_h^{-1} P_h\| \leq \|\mathcal{T}^{-1}\| + \|\mathcal{T}^{-1} - \mathcal{T}_h^{-1} P_h\| ;$$

this implication is immediate.

iii) $\Rightarrow$ i). We have

$$\|u-u_h\| = \|u-\mathcal{T}_h^{-1}f_h\| \leq \|u-P_h u\| + \|P_h u - \mathcal{T}_h^{-1}P_h\mathcal{T}u\|$$
$$\leq \|u-P_h u\|\,(1+\|\mathcal{T}_h^{-1}\|\cdot\|\mathcal{T}\|) \leq c\,\|u-P_h u\| \qquad \blacksquare$$

A sufficient condition for stability is that $\mathcal{T}$ be a positive definite operator. In this case, $C = 1/m$ in (2.3).

By Remark 1, the Galerkin matrix corresponding at the approximation operator $\mathcal{T}_h$, is spectral equivalent with Gram matrix. A simplest example shows that the Galerkin matrices are not stable. Let $\mathcal{H}$ be the Sobolev space $H_0^1(\Omega)$, $\Omega=(0,1)^d$, $d = 1,2$, equipped with the inner product involving only the first derivatives and S_h a finite element subspace with linear functions on a uniform grid of mesh h. Then the Gram matrix coincides with the discretization of the Laplace operator, having the first eigenvalue $\lambda_{min} \cong ch^d$. So, $\|T_h^{-1}\| \leq \frac{1}{mc}h^{-d}$, i.e. the stability of Galerkin discretizations is not valid in this case, in oposition with the stability of their approximation operators.

3. STANDARD HYPOTHESES

Let on $\mathcal{H}$ the following nonlinear equation

(3.1) $$\mathcal{F}(u) = 0$$

which has an unique solution u^*. We suppose that $\mathcal{F}$ is Frechet differentiable on $\mathcal{H}$, with Frechet derivative $\mathcal{F}'$ satisfying: $\mathcal{F}'$ is Lipschitz continuous on $\mathcal{H}$

(3.2) $$\|\mathcal{F}'(u) - \mathcal{F}'(v)\| \leq \gamma\,\|u-v\| \quad , u, v \in \mathcal{H} \;;$$

$\mathcal{F}'(u^*)$ is a linear positive definite operator on $\mathcal{H}$

(3.3) $$\langle \mathcal{F}'(u^*)v, v\rangle \geq (1/\beta^*)\,\|v\|^2 \quad , v\in\mathcal{H}$$

From (3.3), we have

(3.4) $$\| \mathcal{F}'(u^*)^{-1} \| \leq \beta^*$$

i.e. u^* is a simple zeros of $\mathcal{F}$. Let $r^* = 2/3\,\beta^*\gamma$ and $B^* = \{u;\ \| u - u^* \| \leq r^*\}$, the Rheinboldt ball. In the above hypotheses, holds the local like result of Rheinboldt ([7]) what ensures the convergence of the Newton's sequence:

(3.5) $$\mathcal{F}'(u^k)(u^{k+1} - u^k) = -\mathcal{F}(u^k)$$

for any $u^0 \in B^*$, that is quadratic

(3.6) $$\| u^{k+1} - u^* \| \leq \frac{\beta^*\gamma \| u^k - u^* \|^2}{1 - \beta^*\gamma \| u^k - u^* \|}$$

Now, let $S_h \subset \mathcal{H}$ a finite dimension space having the following approximation property: there exists the subspace $W \subset \mathcal{H}$, equipped with the norm $\| \cdot \|_W := ||| \cdot |||$, such that, for any $u \in W$,

(3.7) $$\inf_{v \in S_h} \| u - v \| \leq C_0 h^\alpha |||u||| \quad , \alpha > 0 \, , \, h \leq h_0$$

For example this holds with $\mathcal{H} := H^1_0(\Omega)$ the Sobolev space when Ω is sufficient of smooth, S_h the finite element subspace with linear finite element function, $W = H^1_0(\Omega) \cap H^2(\Omega)$, $\| \cdot \|_W := \| \cdot \|_{H^2(\Omega)}$, and $\alpha = 1$.

As in § 1, for $\mathcal{F}'(u) \in [H]$, the approximation operator on S_h is given by $\mathcal{F}'_h(u) := P_h \mathcal{F}'(u) P_h$. Then, (3.2) holds for it with same constant γ. Notting that $\mathcal{F}_h := P_h \mathcal{F} P_h$, we obtain that for $\xi_h, \eta_h \in S_h$,

$$\| \mathcal{F}_h(\xi_h + \eta_h) - \mathcal{F}_h(\xi_h) - \mathcal{F}'_h(\xi_h)\eta_h \| =$$
$$\| P_h (\mathcal{F}(\xi_h + \eta_h) - \mathcal{F}(\xi_h) - \mathcal{F}'(\xi_h)\eta_h \|$$

i.e. $\mathcal{F}'_h$ is the Frechet derivative of the approximation operator $\mathcal{F}_h$. Now, because (3.3) holds for any $v \in S_h$,

$$\| \mathcal{F}'_h(u^*)^{-1} \| \leq \beta^*$$

Let $u^* \in W$. Then, $\| u^* - P_h u^* \| := \inf_{v \in S_h} \| u^* - v \| \leq c_o h^\alpha ||| u^* |||$. By

$$\mathcal{F}_h'(P_h u^*) = \mathcal{F}_h'(u^*)\left[i_h - \mathcal{F}_h'(u^*)^{-1}(\mathcal{F}_h'(u^*) - \mathcal{F}_h'(P_h u^*))\right]$$

for h such that $\beta^* \gamma c_o ||| u^* ||| h^\alpha < 1$, we obtain

$$\| \mathcal{F}_h'(P_h u^*)^{-1} \| \leq \beta^* / [1 - \beta^* \gamma ||| u^* ||| h^\alpha]$$

If $\xi_h \in S_h \cap B^*$, i.e. $\| \xi_h - u^* \| \leq r^*$, then

$$\| \mathcal{F}_h'(\xi_h) - \mathcal{F}_h'(P_h u^*) \| \leq \gamma \| \xi_h - P_h u^* \| \leq \gamma \| \xi_h - u^* \| \leq 2/3\beta^*$$

So, for h such that $2 \| \mathcal{F}_h'(P_h u^*)^{-1} \| /3\beta^* < 1$, there exists $\mathcal{F}_h'(\xi_h)^{-1}$, for any $\xi_h \in S_h \cap B^*$.

Let h_1 such that $\beta^* \gamma C_o ||| u^* ||| h_1^\alpha = 1/12$; then holds:

REMARK 2. If $h < \tilde{h} := \min \{ h_i, \ i = 0,1 \}$, $\mathcal{F}_h'(\xi_h)^{-1}$ there exists for any $\xi_h \in S_h \cap B^*$ and

(3.8) $$\| \mathcal{F}_h'(\xi_h)^{-1} \| \leq \sigma := 4\beta^*$$

This is immediate, from above considerations, proving the stability property of the approximations on $S_h \cap B^*$, in the sense of [3], for Frechet derivative.

A classical theorem ([5]) gives for $u \in W \cap B^*$,

(3.9) $$\| \mathcal{F}(u) - \mathcal{F}(P_h u) - \mathcal{F}'(u)(P_h u - u) \| \leq \frac{\gamma}{2} \| u - P_h u \|^2 \leq \frac{\gamma}{2}[C_o ||| u ||| h^\alpha]^2$$

With this, we obtain for $u, v \in W \cap B^*$:

(3.10) $$\| P_h(\mathcal{F}(u)) - \mathcal{F}_h(P_h u) \| \leq C_o [\tfrac{\gamma}{2} ||| u ||| C_o h^\alpha + M] ||| u ||| h^\alpha$$

(3.11) $$\| P_h(\mathcal{F}'(u)v) - \mathcal{F}_h'(P_h u) P_h v \| \leq C_o [\gamma ||| u ||| \cdot \| v \| + M \cdot ||| v |||] h^\alpha$$

where M is the constant of boundness for Frechet derivative.

DEFINITION. We name the standard hypotheses (S.H.) the following:

Lipschitz continuity of $\mathcal{F}'$, (3.2);

positivity of the derivative in the solution u^*, (3.3);

approximation property of the approximation subspaces, (3.7).

Summarising, we have

THEOREM 1. Suppose that SH hold. If $u^* \in W$, the Newton's sequence $\{u^k\} \subset W \cap B^*$, and there exists C_1 such that

(3.12) $$|||u^k||| < C_1 \quad , \quad k \geq 0$$

Then, there exists $\tilde{h}$, such that for $h < \tilde{h}$, the approximation schemas for operator equation (3.1) is:

Lipschitz uniform on S_h: for any $\xi_h, \eta_h \in S_h$,

(3.13) $$\|\mathcal{F}_h'(\xi_h) - \mathcal{F}_h'(\eta_h)\| \leq \gamma \, \|\xi_h - \eta_h\|$$

bounded on $\mathcal{H}$: for any $u \in \mathcal{H}$,

(3.14) $$\|P_h u\| \leq \|u\|$$

stable on $S_h \cap B^*$: for any $\xi_h \in S_h \cap B^*$, there exists σ independent of h,

(3.15) $$\|\mathcal{F}_h'(\xi_h)^{-1}\| \leq \sigma := 4\beta^*$$

consistent of order α on the Newton's sequence: there exists the constants C_2 and C_3 independent of h, such that:

(3.16) $$\|P_h \mathcal{F}(u^k) - \mathcal{F}_h(P_h u^k)\| \leq C_2 h^\alpha \quad , \quad k \geq 0 \quad , \quad \text{and}$$

(3.17) $$\|P_h(\mathcal{F}'(u^k)u^j) - \mathcal{F}_h'(P_h u^k) P_h u^j\| \leq C_3 h^\alpha, \quad k \geq 0, \; j \geq 0$$

Proof. The consistence property is a consequence of (3.10)-(3.12). The stability was obtained in remark 2; (3.14) is evident because P_h is the orthogonal projection and (3.13) is immediate. ∎

We point out that the above properties are the work hypotheses in [3] and here in the SH these are natural. Now, we are in the same position as in [3] for proving MIP for approximations.

4. MIP FOR APPROXIMATIONS

We follow the line of [3] in the proof of MIP for approximations and transfer the estimations at the Galerkin discretizations. For first part, we work in B^*. By projection method, the approximation equation on S_h for (3.1) is $P_h\mathcal{F}(\xi_h) = 0$, or in equivalent form,

$$\mathcal{F}_h(\xi_h) = 0 \tag{4.1}$$

LEMMA 2. Let $u^* \in W$. If SH hold, then there exists $\tilde{h}$ such that for $h < \tilde{h}$ the approximation equation (4.1) has an unique solution ξ_h^* in the Kantorowich ball $B_h(P_hu^*, r_h)$, where $r_h \leq C_4 h^\alpha$. Moreover, there exists $\tilde{h}$ such that, for $h < \tilde{h}$, $\xi_h^* \in B^*$.

Proof. For the last $\tilde{h}$, we have $\| u^* - P_hu^* \| \leq C_0 ||| u^* ||| h^\alpha < r^*$, i.e. $P_hu^* \in B^*$. So, by remark 2, there exists $\mathcal{F}_h'(P_hu^*)^{-1}$ and the estimation (3.8). Let $a_h := 2\gamma \| \mathcal{F}_h'(P_hu^*)^{-1} \| \cdot \| \mathcal{F}_h'(P_hu^*)^{-1}\mathcal{F}_h(P_hu^*) \|$. Then, by (3.10),

$$a_h \leq 2\sigma^2\gamma \| \mathcal{F}_h(P_hu^*) \| = 2\sigma^2\gamma \| \mathcal{F}_h(P_hu^*) - \mathcal{F}(u^*) \|$$
$$\leq 2\sigma^2\gamma C_0 [\tfrac{\gamma}{2} ||| u^* ||| C_0 h^\alpha + M] ||| u^* ||| h^\alpha := C_5 h^\alpha$$

with h_2 such that $C_5h_2^\alpha < 1$, redifine $\tilde{h} := \min\{h_i, i = 0,2\}$. By a classical theorem of Kantorowich, for $h < \tilde{h}$, there exists ξ_h^* in the ball $B_h(P_hu^*, r_h)$, where

$$r_h := (1 - \sqrt{1 - a_h}) \| \mathcal{F}_h'(P_hu^*)^{-1}\mathcal{F}_h(P_hu^*) \| / a_h$$
$$\leq \sigma C_0 [1/6\, \beta^* + M] ||| u^* ||| h^\alpha := C_4 h^\alpha$$

Now, by

$$\| \xi_h^* - u^* \| \leq r_h + \| u^* - P_hu^* \| \leq [C_4 + C_0 ||| u^* |||] h^\alpha := C_6 h^\alpha$$

for $h < \tilde{h} := \min\{h_i, i = 0,3\}$, where h_3 verifies $C_6 h_3^\alpha < r^*$, we have $\xi_h^* \in B^*$. ∎

Because $\xi_h^* \in S_h \cap B^*$ we have

$$\| \mathcal{F}_h'(\xi_h^*)^{-1} \| := \beta_h^* \leq \sigma$$

So, we can consider the Rheinboldt ball in S_h, $B_h^*(\xi_h^*, r_h^*)$, where

$$r_h^* = 2/3\beta_h^*\gamma$$

what includes the following ball with constant radius $\tilde{r}^* := 2/3\sigma\gamma$, $\tilde{B}_h^*(\xi_h^*, \tilde{r}^*)$. The following lemma is in fact the theorem of Rheinboldt applied in our context.

LEMMA 3. Let $u^* \in W$. If SH hold, then there exists $\tilde{h}$ such that for $h < \tilde{h}$, the Newton's sequence defined by

(4.2) $$F_h'(\xi_h^K)(\xi_h^{K+1} - \xi_h^K) = -F_h(\xi_h^K) \quad , K \geq 0$$

converges at the solution of (4.1), ξ_h^*, for any ξ_h^o that is the projection on S_h of the ball $B_q^*(u^*, qr^*)$, $q < 1/4$, and this convergence is quadratic:

(4.3) $$\| \xi_h^{K+1} - \xi_h^* \| \leq \beta_h^* \gamma \| \xi_h^K - \xi_h^* \|^2 / (1 - \beta_h^* \gamma \| \xi_h^K - \xi_h^* \|)$$

Proof. Consequently to work in B^*, we wish to have $\tilde{B}_h^* \subset B^*$. For $\xi_h \in \tilde{B}_h^*$,

$$\| \xi_h - u^* \| \leq \| \xi_h - \xi_h^* \| + \| \xi_h^* - u^* \| \leq 2/3\sigma\gamma + c_6 h^\alpha$$
$$\leq r^*/4 + c_6 h^\alpha$$

For h_4 that verifies $c_6 h_4^\alpha \leq \frac{3}{4} r^*$, $\xi_h \in B^*$. So redefining $\tilde{h} := \min \{ h_i, i = 0,4 \}$, with $h < \tilde{h}$, $\tilde{B}_h^* \subset B^*$.

Now, let $u^o \in B_q^*$, $q < 1/4$. Then,

$$\| P_h u^o - \xi_h^* \| \leq \| P_h (u^o - u^*) \| + \| P_h u^* - \xi_h^* \| \leq q r^* + r_h$$

Let h_5 such that $r_{h_5} < (\frac{1}{4} - q) r^*$. Then for $h < \tilde{h} := \min \{ h_i, i = 0,5 \}$ we have $\xi_h^o := P_h u^o \in \tilde{B}_h^* \subset B^*$. Applying the theorem of Rheinboldt, the Newton's sequence for approximation equation with starting point ξ_h^o, we have the quadratic convergence, as in the lemma. ∎

Let $u^o \in B_q^*$. Then $\xi_h^o := P_h u^o \in B^*$ and the linear equation

$$F'(\xi_h^o)(\xi^1 - \xi_h^o) = - F(\xi_h^o)$$

is well defined with an unique solution in B^*. By projection method, we obtain the approximation equation

$$\mathcal{F}_h'(\xi_h^0)(\xi_h^1 - \xi_h^0) = -\mathcal{F}_h(\xi_h^0)$$

what by above considerations has an unique solution $\xi_h^1 \in B^* \cap S_h$. Repeating, the iterates in Newton's sequence $\{\xi_h^k\}$ are the solutions of linear approximation equations obtained by projection method. The "parallelism" of the convergence of this Newton's sequence with the Newton's sequence on initial equation is the object of the following theorem, that is the variant of the main result of [3] in our hypotheses.

THEOREM 2 (Mesh-Independence Principle). If SH hold, $u^* \in W$, $\{u^k, k \geq 0\} \subset W$, with $u^0 \in B_q^*$, $q < 3/16$ and there exists c_1 such that $|||u^k||| < c_1$, then MIP holds, i.e. there exists $\tilde{h}$ such that for $h < \tilde{h}$, and for $\xi_h^0 = P_h u^0$, we have:

(4.4) $$\xi_h^k = P_h u^k + O(h^\alpha)$$

(4.5) $$\mathcal{F}_h(\xi_h^k) = P_h \mathcal{F}(u^k) + O(h^\alpha)$$

(4.6) $$\xi_h^k - \xi_h^* = P_h(u^k - u^*) + O(h^\alpha)$$

and in the stronger form, for any $\varepsilon > 0$, there exists $\tilde{h}_\varepsilon$ such that for $h < \tilde{h}_\varepsilon$:

(4.7) $$\left| \min\{k;\ \|u^k - u^*\| < \varepsilon\} - \min\{k;\ \|\xi_h^k - \xi_h^*\| < \varepsilon\} \right| \leq 1.$$

Proof. Following [3], (4.4)-(4.6) are proved for a approximation schemas that is Lipschitz uniform, bounded, stable and consistent of order α, what are ensured here by the theorem 1. We schetch this proof in our context.

If $C_0 |||u^*||| h_6^\alpha < (1-q)r^*$, then for $h < \tilde{h} := \min\{h_i, i = 0,6\}$, $P_h u^k \in B^*$, $k \geq 0$.

Let $\delta_h^{k+1} := \|\xi_h^{k+1} - P_h u^{k+1}\|$. Then, ([3])

$$\delta_h^{k+1} \leq \|\mathcal{F}_h'(\xi_h^k)^{-1}\| \cdot \{$$

(4.8$_i$) $$\|\mathcal{F}_h'(\xi_h^k)(\xi_h^k - P_h u^k) - \mathcal{F}_h(\xi_h^k) + \mathcal{F}_h(P_h u^k)\| +$$

(4.8$_{ii}$) $$\|(\mathcal{F}_h'(\xi_h^k) - \mathcal{F}_h'(P_h u^k)) P_h(\mathcal{F}'(u^k)^{-1}\mathcal{F}(u^k))\| +$$

(4.8$_{iii}$) $$\| \mathcal{F}_h'(P_h u^k) P_h(\mathcal{F}'(u^k)^{-1}\mathcal{F}(u^k)) - P_h\mathcal{F}(u^k)\| +$$

(4.8$_{iv}$) $$\| P_h\mathcal{F}(u^k) - \mathcal{F}_h(P_h u^k)\| \}$$

Now, (4.8$_{iv}$) is bounded by $c_2 h^\alpha$ because holds (3.16); from (3.17) with $j = k, k+1$, (4.8$_{iii}$) is bounded by $2c_3 h^\alpha$; (4.8$_{ii}$) is bounded $\gamma \delta_h^k \cdot \| u^{k+1} - u^k \| \leq$ $\leq 2\gamma \delta_h^k \cdot \| u^o - u^* \| \leq 2\gamma q r^* \cdot \delta_h^k$. Using a standard estimation for (4.8$_i$), we obtain for it the bound $\frac{\gamma}{2}(\delta_h^k)^2$. The last estimation,

$$\| \mathcal{F}_h'(\xi_h^k)^{-1} \| \leq \| \mathcal{F}_h'(P_h u^k)^{-1} \| / [1 - \gamma \| \mathcal{F}_h'(P_h u^k)^{-1}\| \cdot \| \xi_h^k - P_h u^k \|]$$
$$\leq \sigma / (1 - \sigma\gamma\delta_h^k)$$

With $c = \max\{c_2, 2c_3\}$, we have

(4.9) $$\delta_h^{k+1} \leq \frac{\sigma}{1 - \gamma\sigma\delta_h^k}\left\{ \frac{\gamma}{2}(\delta_h^k)^2 + 2\sigma q r^* \delta_h^k + 2c h^\alpha \right\}$$

Because $q < 3/16$, the cantity $b := 1 - 2\sigma\gamma q r^* > 0$. Let h_7 such that $h_7^\alpha <$ $< b^2/12c\sigma^2\gamma$ and redifine $\tilde{h} := \min\{h_i, i = 0,7\}$. Then the equation $3\sigma\gamma x^2 - 2bx +$ $+ 4c\sigma^2 h^\alpha = 0$ has the smallest solution $\tau := 4c\sigma h^\alpha / [b + \sqrt{b^2 - 12c\sigma^2\gamma h^\alpha}]$, solution that verifies:

(4.10) $$\frac{\sigma}{1 - \sigma\gamma\tau}\left\{ \frac{1}{2}\gamma\tau^2 + 2\sigma q r^* \tau + 2c h^\alpha \right\} = \tau$$

Now, $\delta_h^o = 0 < \tau$ and $\delta_h^1 = 2c\sigma h^\alpha < \tau$. By induction, from (4.9) and (4.10), we obtain that

$$\delta_h^k \leq \tau \leq c_7 h^\alpha$$

and (4.4) is proved. By

$$\| \mathcal{F}_h(\xi_h^k) - P_h\mathcal{F}(u^k) \| \leq \| \mathcal{F}_h(\xi_h^k) - \mathcal{F}_h(P_h u^k) \| + \| \mathcal{F}_h(P_h u^k) -$$
$$- P_h\mathcal{F}(u^k) \| \leq \frac{\gamma}{2}(\delta_h^k)^2 + M\delta_h^k + c_2 h^\alpha \leq c_8 h^\alpha$$

(4.5) is proved. For (4.6), we have

$$\| (\xi_h^k - \xi_h^*) - P_h(u^k - u^*) \| \leq \delta_h^k + r_h \leq c_9 h^\alpha$$

The last relation (4.7) is the corollary 1 in [3], for wich is verified the condition

$\lim_{h\to 0} \| P_h u \| = \| u \|$, $u \in \mathcal{W}$. ∎

In the situation when $\mathcal{F}$ is defined only on the domain $\mathcal{D} \subset \mathcal{H}$, and $B^* \subset \mathcal{D}$, the above results remain valid. So, $\mathcal{F}_h$ is defined on $\mathcal{D}_h := S_h \cap \mathcal{D}$. The our restriction to work in B^* for approximations, makes that the balls of Kantorowich and Rheinboldt B_h respectively $\tilde{B}_h^*$ lies in $\mathcal{D}_h$.

5. MIP FOR GALERKIN DISCRETIZATIONS

We are pointed that the iterates of the Newton's sequence (4.2) are the solutions in $B^* \cap S_h$ of the sequence of the linear approximation equations obtained by projection method.

For to pass at the matrix formulation of the MIP, we turn out at the § 1.

We note that the entries of the vector $\tilde{\xi}_h \in R^{n_h}$ are the coefficients of the $\xi_h := J^h \tilde{\xi}_h \in S_h$, in the basis $\{\phi_h^j, j = 1, n_h\}$. The following notations are used in the following for Galerkin matrix representations:

$$F_h := J_h \mathcal{F}_h J^h \quad , \quad F_h'(\tilde{\xi}_h) := J_h \mathcal{F}'(\xi_h) J^h \quad : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$$

where the last has the Galerkin matrix representation of $\mathcal{F}'(\xi_h)$ in canonical basis. By (1.5),

$$\hat{F}_h'(\hat{\xi}_h) := \Lambda_h(F_h'(\tilde{\xi}_h)) = L_h^{-1} F_h'(\tilde{\xi}_h) L_h^{-*}$$

where $\hat{\xi}_h := L_h^* \tilde{\xi}_h \in R^{n_h}$.

With this, we observe that the Newton's sequence (3.5) has the following "preconditioned" matrix representation

(5.1) $$\hat{F}_h'(\hat{\xi}_h^k)(\hat{\xi}_h^{k+1} - \hat{\xi}_h^k) = -\hat{F}_h(\hat{\xi}_h^k) \quad , \; k \geq 0 \; ; \; \hat{\xi}_h^0 = J_h P_h u^0$$

of the Galerkin matrix representation

(5.2) $$F_h'(\tilde{\xi}_h^k)(\tilde{\xi}_h^{k+1} - \tilde{\xi}_h^k) = -F_h(\tilde{\xi}_h^k) \quad , \; k \geq 0 \; ; \; \tilde{\xi}_h^0 = \bar{J}^h P_h u^0$$

We wish to show that the sequences defined by (5.1) and (5.2) are Newton's sequences on R^{n_h}, converging at the $\hat{\xi}_h^* = \hat{J}_h \xi_h^*$ and $\tilde{\xi}_h^* = J^{-h} \xi_h^*$ respectively. Because

$$\| L_h^{-1} \{ F_h(\tilde{\xi}_h + \tilde{\eta}_h) - F_h(\tilde{\xi}_h) - F_h'(\tilde{\xi}_h)\tilde{\eta}_h \|_h =$$

$$\| \hat{F}_h(\hat{\xi}_h + \hat{\eta}_h) - \hat{F}_h(\hat{\xi}_h) - \hat{F}_h'(\hat{\xi}_h)\hat{\eta}_h \|_h =$$

$$\| \mathcal{F}_h(\xi_h + \eta_h) - \mathcal{F}_h(\xi_h) - \mathcal{F}_h'(\xi_h)\eta_h \|$$

F' respectively $\hat{F}'$ are the Frechet derivatives for F respectively $\hat{F}$. The second equality, holds because, for any $\xi_h := \hat{J}^h \hat{\xi}_h \in S_h$, we have

(5.3) $$\| \xi_h \| = \| \hat{\xi}_h \|_h = \| \tilde{\xi}_h \|_{G_h}$$

where $\| \tilde{\xi}_h \|_{G_h} := \langle G_h \tilde{\xi}_h, \tilde{\xi}_h \rangle_h^{1/2} = \| L_h^* \tilde{\xi}_h \|_h$, is the norm induced by the Gram matrix of $\{\phi_h^j\}$.

THEOREM 3. In the same hypotheses as in theorem 2, the MIP for Galerkin discretizations, holds in the norm induced by Gram matrix, i.e. the Newton's sequence (5.2) for discretized equation $F_h(\tilde{\xi}_h) = 0$ in R^{n_h}, converges with the starting point $\tilde{\xi}_h^o := (P_h u^o)^\sim$, and the iterates verifies:

$$\| \tilde{\xi}_h^k - (P_h u^k)^\sim \|_{G_h} \leq c h^\alpha$$

$$\| \mathcal{F}_h(\tilde{\xi}_h^k) - (P_h \mathcal{F}(u^k))^\sim \|_{G_h} \leq c h^\alpha$$

$$\| (\tilde{\xi}_h^k - \tilde{\xi}_h^*) - (P_h(u^k - u^*))^\sim \| \leq c h^\alpha$$

and in the stronger formulation: for any $\varepsilon > 0$, there exists $\tilde{h}_\varepsilon$ such that if $h < \tilde{h}_\varepsilon$,

$$| \min\{k;\ \|u^k - u^*\| < \varepsilon\} - \min\{k;\ \|\tilde{\xi}_h^k - \tilde{\xi}_h^*\|_{G_h} < \varepsilon\} | \leq 1.$$

Proof. We remark that $\{\hat{\xi}_h^*\}$ is the Newton sequence for equation $\hat{F}_h(\hat{\xi}_h) = 0$ if and only if $\{\xi_h^k\}$ is the Newton's sequence of equation $\mathcal{F}_h(\xi_h) = 0$. Because

$$\| \xi_h^k - \xi_h^* \| = \| \hat{\xi}_h^k - \hat{\xi}_h^* \|_h$$

we have the quadratic convergence for $\{\hat{\xi}_h^k\}$, as in Rheinboldt theorem. Moreover MIP holds for it with same estimations in the Euclidean norm as the estimations for approximations. This implies same estimations with respect at the Newton sequence for Galerkin discretizations $\{\tilde{\xi}_h^*\}$ because holds the second equality in (5.3) for $(\tilde{\xi}_h^k - \tilde{\xi}_h^*)$, and MIP holds in this case in the norm induced by Gram matrix. $\blacksquare$

We point out that the sequence (5.2) is really the Newton's sequence for the Galerkin discretization. Using the framework of § 1, with the starting point $\tilde{\xi}_h^o := (P_h u^o)^{\sim} \in R^{n_h}$, we have for $k = 0$ for example, in (5.2) an system of linear equations, whose lines are

$$\left(\mathcal{F}_h'(\tilde{\xi}_h^o)(\tilde{\xi}_h^1 - \tilde{\xi}_h^o) + \mathcal{F}_h(\tilde{\xi}_h^o) \right)_i = \langle \mathcal{F}_h'(\tilde{\xi}_h^o)(\tilde{\xi}_h^1 - \tilde{\xi}_h^o) + \mathcal{F}_h(\tilde{\xi}_h^o), e_h^i \rangle_h =$$

$$\langle \mathcal{F}_h'(\xi_h^o)(\xi_h^1 - \xi_h^o) + \mathcal{F}_h(\xi_h^o), \phi_h^i \rangle =$$

$$\langle \mathcal{F}'(\xi_h^o)(\xi_h^1 - \xi_h^o) + \mathcal{F}(\xi_h^o), \phi_h^i \rangle$$

So, (5.2) is obtained by projection of the residuum calculated in ξ_h^{k+1} of k-th linear Newton's equation onto the family $\{\phi_h^j, j = n_h\}$, what span S_h; but this is Galerkin method.

6. COMMENTS

We discuss here on the hypotheses of the theorem 2. For $\mathcal{H} := H_o^1(\Omega)$ the Sobolev space, the approximation property on finite element subspaces holds with $\alpha = 1$ for $W = H_o^1(\Omega) \cap H^2(\Omega)$, dense subspace of $H_o^1(\Omega)$, equipped with the norm of the space $H^2(\Omega)$.

Let $\mathcal{T}$ be the functional from the first term of the following problem, that is linear in the second argument: to find $u \in \mathcal{D}$, such that,

(6.1) $$\mathcal{T}(u,v) := \int_\Omega \left(\frac{du}{dx} \frac{dv}{dx} + f v \right) dx = 0, \quad (\forall)\, v \in H_o^1(\Omega),\ \Omega = (0,1)$$

where $\mathcal{T} : \mathcal{D} \times H_o^1(\Omega) \to R$, $\mathcal{D} \subset H_o^1(\Omega)$. We suppose that $\mathcal{D}$ and f are such that (6.1) has an unique solution $u^* \in \mathcal{D}$, and the Newton's sequence is well defined, by: given $u^o \in \mathcal{D}$,

$$\mathcal{J}'(u^k, u^{k+1}-u^k; v) = -\mathcal{J}(u^k, v) \quad, (\forall) v \in \mathcal{H} \quad , k \geq 0 \tag{6.2}$$

Here $\mathcal{J}'$ is Frechet derivative obtained from (6.1) by derivation with respect at the first argument.

The problem (6.1) is the variational formulation of the problem: to find $u \in D \subset c^2(0,1)$ solution of:

$$-\frac{d^2u}{dx^2} + f = 0 \qquad u(0) = u(1) = 0 \quad , \ f := f(x, u, \frac{du}{dx}) \tag{6.3}$$

that is the same type as in example of [3]. By our considerations, u^* and $\{u^k\}$ are solutions of (6.3) and of the Newton's sequence of linear equations for it, in the sense of distribution. We suppose that they are classical solutions, i.e. u^*, $\{u^k\} \subset C^2(0,1)$, and u^* is the unique solution in D. Defining the norm of $C^2(0,1)$ by $\| u \|_{c^2(0,1)} = \max \{\sup |u^{(i)}(x)|, x \in \Omega \ ; i = 0,1,2 \}$, we have $\| u^k \|_{C^2(0,1)} \leq C_1$ in the ball of convergence for (6.3). Then,

$$\| u^k \|_{H^2(\Omega)} \leq C_2 \| u^k \|_{C^2(0,1)}$$

that proves the approximation property for $W = H^1_0(\Omega) \cap H^2(\Omega) \cap C^2(0,1)$.

We point out that the Newton's process and the variational formulation process are commutative for problem (6.3), i.e. both ordering conducts at the same equations on $\mathcal{D} \times H^1_0(\Omega)$, and this in a simple exercise.

Now, we wish to show that (6.1) has an equivalent operator formulation on $\mathcal{D}$. Because

$$\begin{aligned} |\mathcal{J}(u,v)| &\leq \| \frac{du}{dx} \|_{L^2} \| \frac{dv}{dx} \|_{L^2} + \| f \|_{L^2} \| v \|_{L^2} \\ &\leq \max \{ \| \frac{du}{dx} \|_{L^2} ; \| f \|_{L^2} \} \| v \|_{H^1(\Omega)} \\ &\leq C(u) \| v \|_{H^1_0(\Omega)} \end{aligned}$$

the linear functional $\mathcal{J}_u$ for $u \in \mathcal{D}$ fixed, is bounded on $H^1_0(\Omega)$. Then, by the Riesz representation, there exist $t_u \in H^1_0(\Omega)$ such that,

$$\mathcal{J}_u(v) := \mathcal{J}(u,v) = \langle t_u, v\rangle \quad ,(\forall)\, v \in H_0^1(\Omega)$$

So, $u \xrightarrow{\mathcal{F}} t_u$, for $u \in \mathcal{D}$, defines a nonlinear operator in $\mathcal{D}$. We have $\mathcal{J}(u^*,v) = 0$ for any $v \in H^1_0(\Omega)$ if and only if $\mathcal{F}(u^*) = 0$. If the derivative $\mathcal{J}'$ of $\mathcal{J}$ is a bilinear functional for any $v \in H^1_0(\Omega)$, that is bounded and elliptique, then their representation is a linear operator, bounded and positive definite, that is $\mathcal{F}'$. By our considerations, $\{u^k\}$ is the Newton's sequence for $\mathcal{F}(u) = 0$ as well as for (6.1) and (6.3).

In the finish of the work, we remark that the positive definite property in the solution u^* is not restrictive. For example in [4] the hypothese for Frechet derivative is that it is positive definite onto any compact subsets in $\mathcal{H}$. This condition implies the positivity in the solution u^*.

We remark that for a similar problem as in [3], we need no supplementary smoothing properties for the solution u^* and for Newton's sequence as in $C^2(0,1)$.

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