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JORDAN ALGEBRAS IN DIFFERENTIAL GEOMETRY.
JORDAN TRIPLE SYSTEMS IN DIFFERENTIAL GEOMETRY

by

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JORDAN STRUCTURES WITH APPLICATIONS. III
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This paper gives the description of almost all important symmetric spaces (Riemannian, Hermitian, etc.) in terms of Jordan algebras, and illustrates the relations of Jordan algebras to Vagner spaces, Minkowski space, and quasi-symmetric domains.

§ 1. Formally real Jordan algebras and compact symmetric Riemannian spaces of rank one.

Let A be a formally real Jordan algebra of dimension n. Theorem 3.4. of BRAUN and KOECHER's book [24, Chapter XI] implies that A has a unit element, which we shall denote by e.

In this case, by Proposition 1.6 from JSA I , we have

$$\mathrm{Idemp}_1(\mathcal{A}) = \{ c \mid c \in \mathrm{Idemp}\,(\mathcal{A}),\ c \text{ primitive} \}.$$

<u>Definition.</u> A system of idempotents $c_1 \dots , c_s \in \mathcal{A}$ is called a <u>complete orthogonal system of idempotents</u> of $\mathcal{A}$ if $\sum_{i=1}^{s} c_i = e$ and $c_i c_j = \delta_{ij} c_i$, $(i,j = 1, \dots , s)$.

<u>Proposition 1.1.</u> A formally real Jordan algebra contains a complete orthogonal system of primitive idempotents.

TILLIER [186 a] gave a geometric characterization of primitive idempotents in a formally real Jordan algebra, namely:

every primitive idempotent belongs to an extremal ray of the domain of positivity of the algebra, and, conversely, such a ray always contains a primitive idempotent.

Proposition 1.2. All complete orthogonal systems of primitive idempotents of a formally real Jordan algebra have the same number of elements.

Definition. The number of elements of a complete orthogonal system of primitive idempotents of a formally real Jordan algebra $\mathcal{A}$ is called the degree of $\mathcal{A}$.

We shall now recall some of the results established by HIRZEBRUCH [82 b].

Suppose that $\mathcal{A}$ is simple and denote its degree by s. Then the form

$$\mu(u) := \frac{s}{n} \operatorname{Tr} L(u), \qquad u \in \mathcal{A},$$

is an associative (i.e. $\mu(x(yz)) = \mu((xy)z)$ for any $x, y, z \in \mathcal{A}$) linear form on $\mathcal{A}$ with $\mu(c) = 1$ for every $c \in \operatorname{Idemp}_1(\mathcal{A})$.

Remark. Suppose that a formally real Jordan algebra is not simple. Then it is semisimple (therefore it is a sum of simple ideals), and the associative linear form μ with value 1 on the primitive idempotents is constructed by means on the forms μ_i on the components.

Notation. For every $c \in \operatorname{Idemp}_1(\mathcal{A})$ define S_c by

$$S_c := \left\{ x \mid x \in \mathcal{A}_{1/2}(c),\ \mu(x^2) = 2 \right\}.$$

Theorem 1.3. Let $\mathcal{A}$ be a simple formally real Jordan algebra and let $c \in \operatorname{Idemp}_1(\mathcal{A})$. For every $d \in \operatorname{Idemp}_1(\mathcal{A})$ there exists a real number t, $0 \leq t \leq \pi/2$, and an element x in S_c such that $d = d(t)$, where

$$d(t) = (\cos 2t)c + (\tfrac{1}{2} \sin 2t)x + \tfrac{1}{2}(1 - \cos 2t)x^2.$$

Conversely, for each such t, d(t) is an element of $\mathrm{Idemp}_1(\mathcal{A})$. The primitive idempotents which are orthogonal to c are exactly those of the form $x^2 - c$ with $x \in S_c$. For $x \in S_c$, $x^2 = c + d$ if and only if $x \in \mathcal{A}_{1/2}(c) \cap \mathcal{A}_{1/2}(d)$.

Corollary . A formally real Jordan algebra is simple if and only if the set of its primitive idempotents is connected.

Theorem 1.4. Let $\mathcal{A}$ be a simple formally real Jordan algebra and let c be an element of $\mathrm{Idemp}_1(\mathcal{A})$. For $x, y \in S_c$ there exists a product of Peirce reflections with respect to idempotents of $\mathrm{Idemp}_1(\mathcal{A})$ that fixes c and maps x to y .

The proofs for Theorems 1.3 and 1.4 are based on three other results concerning $\mathrm{Idemp}_1(\mathcal{A})$ - which are too involved to be reported here - established by HIRZEBRUCH in [82b, pp.342-343].

Theorem 1.5. Let $\mathcal{A}$ be a simple formally real Jordan algebra and let $c_1, c_2, d_1, d_2 \in \mathrm{Idemp}_1(\mathcal{A})$ such that $\mu(c_1 c_2) = \mu(d_1 d_2)$. Then there exists a product of Peirce reflections with respect to idempotents of $\mathrm{Idemp}_1(\mathcal{A})$ which maps c_1 to d_1 and c_2 to d_2.

The proof follows from Theorems 1.3 and 1.4.

Consider now on $\mathcal{A}$ the symmetric bilinear form $\mu(x,y) :=$ $:= \mu(xy)$ and the Euclidean metric $\varrho_E(x,y) := (\mu(x-y)^2))^{1/2}$ it determines.

Remark. The automorphisms of $\mathcal{A}$ are isometries of the metric space $(\mathrm{Idemp}_1(\mathcal{A}), \varrho_E)$.

Definition. A metric space (M, ϱ) is called two-point homogeneous if there exists an isometry ι of M such that $\iota(c_1) = d_1$ and $\iota(c_2) = d_2$ for any $c_1, c_2, d_1, d_2 \in M$ with $\varrho(c_1, c_2) = \varrho(d_1, d_2)$.

Corollary of Theorem 1.5. If $\mathcal{A}$ is a simple formally real Jordan algebra, then $(\mathrm{Idemp}_1(\mathcal{A}), \varrho_E)$ is a connected, compact and two-point homogeneous metric space.

<u>Theorem 1.6.</u> Let $\mathcal{A}$ be a simple formally real Jordan algebra and let T be a one-to-one map of $\mathrm{Idemp}_1(\mathcal{A})$ onto itself such that $\mu(T(c), T(d)) = \mu(c,d)$ for all $c,d \in \mathrm{Idemp}_1(\mathcal{A})$. Then T can be extended to an automorphism of $\mathcal{A}$.

<u>Sketch of the proof.</u> If $c_1, \ldots, c_r$ is a complete orthogonal system of idempotents from $\mathrm{Idemp}_1(\mathcal{A})$, then $T(c_1),\ldots,T(c_r)$ is also such a system. For every $x \in \mathcal{A}$ we have $x = \sum_{i=1}^{\ell} \alpha_i c_i$, where c_i are pairwise orthogonel idempotents. T(x) is defined by

$$T(x) := \sum_{i=1}^{\ell} \alpha_i T(c_i).$$

The set $\mathrm{Idemp}_1(\mathcal{A})$ is a subset of the sphere $\{x \mid x \in \mathcal{A},\ \mu(x^2) = 1\}$ because, for every $c \in \mathrm{Idemp}_1(\mathcal{A})$ we have $\mu(c^2) = \mu(c) = 1$. Making use of Proposition 4.4. and Theorem 2.10 from [77, Chapter II], it follows that $\mathrm{Idemp}_1(\mathcal{A})$ is a submanifold of the sphere $\{x \mid x \in \mathcal{A},\ \mu(x^2) = 1\}$ and also a topological subspace of it.

<u>Notation.</u> Consider the Riemannian structure induced on $\mathrm{Idemp}_1(\mathcal{A})$ by $\mu(xy)$. The Riemannian manifold thus obtained will be denoted by $(\mathrm{Idemp}_1(\mathcal{A}), R)$.

<u>Remark 1.</u> The automorphisms of $\mathcal{A}$ are isometries of the Riemannian manifold $(\mathrm{Idemp}_1(\mathcal{A}), R)$.

<u>Remark 2.</u> $(\mathrm{Idemp}_1(\mathcal{A}), R)$ is a compact symmetric Riemannian space.

<u>Notation.</u> The Riemannian distance between two elements c and d of $(\mathrm{Idemp}_1(\mathcal{A}), R)$ will be denoted by $\rho_R(c,d)$.

Since the relations

$$0 \leq \rho_R(c,d) \leq \pi/\sqrt{2} \quad \text{and} \quad \rho_E(c,d) = \sqrt{2}\,\sin\left(\frac{1}{\sqrt{2}}\ \rho_R(c,d)\right)$$

hold, it follows that $\rho_R(c_1,c_2) = \rho_R(d_1,d_2)$ is equivalent to $\mu(c_1,c_2) = \mu(d_1,d_2)$ for $c_1,c_2,d_1,d_2 \in \mathrm{Idemp}_1(\mathcal{A})$.

Remark. Consequently, $(\mathrm{Idemp}_1(\mathcal{A}), R)$ is a two-point homogeneous symmetric Riemannian space and hence (see HELGASON [77, p.355]) of rank one.

Let c be an element of $\mathrm{Idemp}_1(\mathcal{A})$. Clearly, $\rho_R(c,d)$, $d \in \mathrm{Idemp}_1(\mathcal{A})$, is maximal only when $\rho_E(c,d)$ is maximal. Because $\mu(c) = \mu(d) = 1$, we have $\rho_E(c,d) = \sqrt{2}\ \sqrt{1 - \mu(cd)}$, which is maximal only when $\mu(cd) = 0$, i.e., when $cd = 0$.

Notation. For every $c \in \mathrm{Idemp}_1(\mathcal{A})$ we define

$$A_c := \left\{ d \mid d \in (\mathrm{Idemp}_1(\mathcal{A}), R),\ cd = 0 \right\}.$$

Remark 1. By Theorem 1.3 we have $A_c = \left\{ x^2 - c \mid x \in S_c \right\}$.

Remark 2. A_c is a submanifold of $(\mathrm{Idemp}_1(\mathcal{A}), R)$ and is called the antipodal manifold of c.

Notation. Forany simple formally real Jordan algebra $\mathcal{A}$ of dimension $n > 1$ there exists a natural number $q(\mathcal{A})$ such that, for every pair of orthogonal primitive idempotents $c_1, c_2 \in \mathcal{A}$, the relation $\dim(\mathcal{A}_{1/2}(c_1) \cap \mathcal{A}_{1/2}(c_2)) = q(\mathcal{A})$ holds. If $s = s(\mathcal{A})$ denotes the degree of $\mathcal{A}$, then $\mathcal{A}$ is said to be of type $(s, q(\mathcal{A}))$.

Remark. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are simple formally real Jordan algebras with $s(\mathcal{A}_1) = s(\mathcal{A}_2)$ and $q(\mathcal{A}_1) = q(\mathcal{A}_2)$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic.

Comments. It would be interesting to extend HIRZEBRUCH's results [82b, pp.350-351] on Betti numbers of $\mathrm{Idemp}_1(\mathcal{A})$, critical points of differentiable functions on $\mathrm{Idemp}_1(\mathcal{A})$, etc., to other kinds of Jordan algebras.

Using the well-known classification of compact symmetric Riemannian spaces of rank one, HIRZEBRUCH [82 b] proved that each of these spaces can be described in terms of a suitable formally real Jordan algebra, namely:

a) Type (1,0). $\mathcal{A} = \mathbb{R}$, and $\mathrm{Idemp}_1(\mathcal{A})$ consists of point alone.

b) Type (2,q); $q \geq 1$. Let V' be a $(q+1)$-dimensional vector space over $\mathbb{R}$ and let σ be a positive definite bilinear form on V'. Define on $V := \mathbb{R}e \oplus V'$ a bilinear product by $uv := \sigma(u,v)e$ for $u,v \in V'$, e being the unit element. It is immediate that V endowed with this product is a Jordan algebra J(Q) (as in Theorem 1.8 from JSAI). $\mathrm{Idemp}_1(V)$ is homeomorphic to the q-dimensional sphere S^q.

c) Type (s,1); $s \geq 3$. Let V be the vector space of symmetric $(s \times s)$ - matrices over $\mathbb{R}$. For $A,B \in V$, let $AB := \frac{1}{2}(A.B+B.A)$, where A.B denotes the usual matrix product in V. We have $\mu(A) = \mathrm{Tr}\, A$ and $\mathrm{Idemp}_1(V) = \{ A \mid A \in V, A^2 = A, \mathrm{Tr}\, A = 1 \}$. It follows that $\mathrm{Idemp}_1(V)$ is homeomorphic with the real $(s-1)$-dimensional projective space $P_{s-1}(\mathbb{R})$. For $c \in \mathrm{Idemp}_1(V)$, the antipodal manifold A_c is $P_{s-2}(\mathbb{R})$.

d) Type (s,2); $s \geq 3$. Let V be the ordinary real vector space of complex Hermitian $(s \times s)$-matrices. Define the product as in c). Then $\mathrm{Idemp}_1(V)$ is homeomorphic with $P_{s-1}(\mathbb{C})$. For $c \in \mathrm{Idemp}_1(V)$, A_c is $P_{s-2}(\mathbb{C})$.

e) Type (s,4); $s \geq 3$. Let V be the real vector space of Hermitian $(s \times s)$-matrices over $\mathbb{H}$. Define the product as in c). Then $\mathrm{Idemp}_1(V)$ is homeomorphic to $P_{s-1}(\mathbb{H})$. For $c \in \mathrm{Idemp}_1(V)$, A_c is $P_{s-2}(\mathbb{H})$.

f) Type (3,8). Let V be the real vector space of Hermitian (3×3)-matrices over $\mathbb{O}$. Define the product as in c). Then Idemp_1 is homeomorphic with the projective octonion plane. For $c \in \mathrm{Idemp}_1(V)$, A_c is an eight-dimensional sphere.

Remark. The geodesics through a point $c \in \mathrm{Idemp}_1(\mathcal{A})$, as a set of points, are $\mathrm{Idemp}_1(\mathcal{A}) \cap \mathcal{A}(c,x)$, $x \in \mathcal{A}_{1/2}(c)$, $\mathcal{A}(c,x)$ being a simple three-dimensional subalgebra of $\mathcal{A}$ containing c (see HIRZEBRUCH [82 b, p.348]). For a detailed discussion on geodesics in a more general case see NEHER [125] and § 3 in this paper.

Comments 1. CRAIOVEANU and PUTA [37] proved that, under certain hypotheses, linear continuous operators acting on the space $A^p(M)$ of smooth p-forms of a compact symmetric Riemannian space M of rank one are functions of the Laplace-Beltrami operator acting on $A^p(M)$. This is a p-version of some results established by BEN-ABDALLAH [13] and LEMOINE [104] in the case p=0. In contrast to the smooth function-space case, the result of Craioveanu and Puta holds under stronger assumptions.

Open problem. To reconsider the above mentioned of study Craioveanu and Puta as well as ROGOV's results [152] in the Jordan algebra setting.

Comments 2. Taking into account, on the one hand, the above description of complex projective space as $\mathrm{Idemp}_1(V)$ (see case d)) and, on the other, the results of ATIYAH-PENROSE on the role of $P_3(\mathbb{C})$ in physical problems (involving Minkowski space) - see, for instance ATIYAH and WARD [6] and the references threin- it would be interesting to use the Jordan structure of V for obtaining new properties for the entities under consideration. It would be also useful to employ the above Jordan algebra description of complex projective spaces (particularly of $P_3(\mathbb{C})$) in the study of instantons as given by DRINFELD and MANIN in [48], who basically regarded the algebraic approach.

Comments 3. SAKAMOTO [160] defined a helical geodesic immersion of order d as follows: let $\varphi : M \to M'$ be an isometric immersion of a connected complete Riemannian manifold M into a Riemannian manifold M'. If, for each geodesic γ of M, the curve $\varphi \circ \gamma$ in M' has constant curvatures of osculating order d which are independent of γ , then φ is called a helical geodesic immersion of order d. The above Jordan algebra description of compact symmetric Riemann spaces of rank one could be used in solving TSUKADA's conjecture [188]: If φ is a helical geodesic minimal immersion of a compact Riemann manifold M into a unit sphere, then M is isometric to a compact symmetric space of rank one and φ is equivalent to a standard minimal immersion.

Comments 4. ZILLER [213] has classified the homogeneous Einstein metrics on compact symmetric spaces of rank one and has studied some of their properties.

Open problem. To obtain new properties of the above-mentione metrics by using the Jordan algebra description of the spaces of interest.

§ 2. Jordan algebras and symmetric spaces

A symmetric Riemannian space, defined as usual, is a Riemannian manifold such that the geodesic symmetry S_x around every point x is an isometry. By writting $x.y := S_x(y)$, Loos has obtained a (nonassociative) multiplication on the manifold satisfying certai algebraic identities, which in turn suffice to characterize symmetric spaces. In this way, one obtains an elementary "algebraic" definition of a symmetric space not involving the manifold structure of the underlying topological space. This defi nition was first given by LOOS [106 a].

Definition. A manifold $\mathcal{M}$ with a differentiable multiplication $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, denoted by $(x,y) \longrightarrow x.y$, and having the properties

1. $x.x = x$,

2. $x.(x.y) = y$,

3. $x.(y.z) = (x.y) . (x.z)$,

4. every x has a neighborhood U such that $x.y = y$ implies $y = x$ for all y in U,

is called a symmetric space.

Note. In Loos'terminology, a manifold is a differentiable manifold of class C^{∞} which is Hausdorff and paracompact as a topological space. It may have several connected components, which may be of different (yet finite) dimensions.

Remark. Spaces satisfying only (1), (2), and (3) ("relection spaces") have been studied by LOOS in [106 b]. They turn out to be fibre bundles over symmetric spaces (see, for instance, NEHER [125] and § 3 in this paper).

Definition. Left multiplication by x in $\mathcal{M}$ is denoted by S_x, i.e. $S_x y = x.y$ for all $x, y \in \mathcal{M}$, and is called symmetry around x.

Remark. The following properties are immediate:

(i) x is an isolated fixed point of S_x;

(ii) S_x is an involutive automorphism of $\mathcal{M}$.

Examples. 1°. Lie groups. Let L be a Lie group and put $x.y := xy^{-1}x$, where xy denotes the product in L. In particular, a vector space becomes a symmetric spaces with the product $x.y := 2x-y$.

2°. Spheres. Let (x,y) be a nonsingular symmetric bilinear form an $\mathbb{R}^n$ and let $M_{\alpha} := \{ x \in \mathbb{R}^n \mid (x,x) = \alpha \}$ where $\alpha \neq 0$, be the "sphere" with radius $\sqrt{\alpha}$. Define

$$x.y := 2 \frac{(x,y)}{(x,x)} x - y.$$

3°. Grassmann manifolds. Let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $\mathbb{K}^n$ have the usual Hermitian scalar product $(x,y) = \sum \bar{x}_i y_i$. Let $M := M(n, \mathbb{K})$ be the set of all linear subspaces of $\mathbb{K}^n$ (with the usual topology), and $M_q := M(q, n; \mathbb{K})$ the set of subspaces of dimension q. We have $M = M_o \cup M_1 \cup \dots \cup M_n$. For $V \in M$ let S_V be the reflection in V, i.e., if $x = x_1 + x_2$ is the decomposition of a vector with respect to $\mathbb{K}^n = V \oplus V^{\perp}$, then $S_V(x) = x_1 - x_2$. Clearly S_V is an orthogonal transformation of $\mathbb{K}^n$ having V as fixed point set and is uniquely determined by this property. Define

$$V.W := S_V(W).$$

4°. Jordan algebras. The set of invertible elements of a real or complex Jordan algebra becomes a symmetric space with the product $x.y := P(x)y^{-1}$.

5°. Homogeneous spaces and spaces of symmetric elements. Let L be a connected Lie group with an involutive automorphism σ. Let L^{σ} be the fixed point set of σ and K a subgroup such that $(L^{\sigma})_o \subset K \subset L^{\sigma}$, $(L^{\sigma})_o$ being the connected unit component. Then K is closed. Let $M = L/K$, and for $x \in L$, let $\tau(x)$ be the translation $\tau(x)(yK) = xyK$ of M. Also put $L_{\sigma} := \{ x\,\sigma(x)^{-1} \mid x \in L \}$. We call L_{σ} the space of symmetric elements of L. M is a symmetric space with the product $xK.yK := x\,\sigma(x)^{-1}\sigma(y)K$ and L_{σ} is a symmetric space with the product $x.y := xy^{-1}x$.

<u>Comments 1.</u> As ATSUYAMA remarked in [7] the two known models of real projective plane, namely, as the set of all lines through the origin in 3-dimensional Euclidean space or as the set of all subalgebras isomorphic to the field of complex

numbers in the quaternion field, are slightly different. More precisely, in the first case the lines appear to have no algebraic structure, whereas in the latter case, the subalgebras are endowed with a structure that can be used to obtain the symmetric space. Starting from this remark, ATSUYAMA presumed that a model similar to the latter is suitable for an explicit construction of symmetric spaces from various algebras, and in this connection he raised the question of whether symmetric spaces (in the sense of the above definition) can be constructed from the set of all subalgebras satisfying suitable conditions in a given algebra. In the paper [7] referred to above, ATSUYAMA gave an affirmative answer to this question.

Comments 2. KOWALSKI [97c , Chapter II] characterized in the Loos' spirit a wider class of spaces which are "s-regular affine manifolds" in the sense of LEDGER [67] (see also LEDGER and OBATA [102]). Modifying the above axioms of Loos, Kowalski [97a] defined the so-called tangentially regular s-manifolds. Next (see [97b]) he gave a new definition of generalized affine symmetric space. For some theory that is important for the classification of these latter spaces see WEGRZYNOWSKI [202]. SÁNCHEZ [164a] extended the Ferus' definition to the Kowalski's regular s-manifolds.

1. Let $\mathcal{A}$ be a real Jordan algebra of dimension n and with unit element e.

Notation. The set of all invertib e elements of $\mathcal{A}$ will be denoted by Inv ($\mathcal{A}$).

Remark 1. The vector space $\mathcal{A}$ carries a natural topology, and the set Inv ($\mathcal{A}$) is open in $\mathcal{A}$.

Remark 2. The trace form λ of $\mathcal{A}$, given by $\lambda(a,b) := \operatorname{Tr} L(ab)$ for all $a,b \in \mathcal{A}$, is an associative form, i.e. $\lambda(a,bc) = \lambda(ab,c)$ for all $a,b,c \in \mathcal{A}$.

Suppose that λ is nondegenerate. Then the (not necessarily positive definite) line element $ds^2 := \lambda(\dot{x}, P(x^{-1})\dot{x})dt^2$, where $x = x(t)$ is a curve in Inv $(\mathcal{A})$, is invariant under the maps $x \longrightarrow Wx$, $W \in \Gamma(\mathcal{A})$, and $x \longrightarrow x^{-1}$. In order to discuss the induced (pseudo-) Riemannian structure, let C be a component of Inv $(\mathcal{A})$. Then there exists an $f \in C$ such that $f^2 = e$.

Notation. Denote by $\operatorname{Inv}_0(\mathcal{A})$ the component of Inv $(\mathcal{A})$ containing e.

Remark 1. $C = \operatorname{Inv}_0(\mathcal{A}_f)$ whenever $f \in C$, $f^2 = e$.

Remark 2. Since $\mathcal{A}_f$ is again a Jordan algebra (with unit element f^{-1}), for the discussion of the induced (pseudo-) Riemannian structure it suffices to consider $\operatorname{Inv}_0(\mathcal{A})$.

Comments. TILLIER [186b,c] gave a definition of the connected components of Inv $(\mathcal{A})$ in the formally real case, and established relations between their groups of transvections and the group generated by the quadratic representation of $\mathcal{A}$.

Theorem 2.1 (KOECHER [93 c]). Let $\mathcal{A}$ be a real Jordan algebra such that its trace form is nondegenerate. Then $\operatorname{Inv}_0(\mathcal{A})$ is a homogeneous symmetric space.

The proof follows from Theorem 2.4 in [24, Chapter XI].

Theorem 2.2 (KOECHER [93 b]). If $\mathcal{A}$ is a formally real Jordan algebra, then $\operatorname{Inv}_0(\mathcal{A})$ is a symmetric Riemannian space and

a) at the point e the geodesic symmetry is the inversion $x \longrightarrow x^{-1}$, and the exponential map is $\exp_e(x) = \exp x$;

b) the coefficients Γ^i_{jk} of the affine connection coincide with the structure constants of the Jordan algebra $\mathcal{A}$.

Comments. Taking into account b) from Theorem 2.2., it would be interesting to reconsider in an algebraic setting some of the results obtained by VRANCEANU [198 c] and FAVA [53].

Open problem. To define a metric of Finsler type and to discuss the induced Finsler structure, analogously to the (pseudo-) Riemannian metric defined by Koecher, on a real Jordan algebra.

Comments. The Finsler metrics for the interior of a simple connected domain defined and studied by BARBILIAN [11 b] and BARBILIAN and RADU [12] may be of related interest here.

Let as mention in this respect that a strong school on Finsler spaces has developed in Romania. For detailed information on Romanian research in this field the reader is referred to the Proceedings of the National Seminars on Finsler Spaces, held each two years at the University of Braşov with the beginning in 1980. The first Romanian-Japan Seminar on Finsler spaces held at the University of Iassy in 1986. A good account of the most important Romanian contributions in the field is given also by MATSUMOTO [114].

Open problem. To identify the kind of Finsler spaces that can be descibed in terms of Jordan algebras, and to reconsider, in an algebraic setting, some of the geometrical properties of these Finsler spaces.

Remark. As was already noticed by MIRON [116], the open problems metioned above are interesting, but difficult to solve because the notion of symmetric Finsler space has not yet been properly defined.

We shall give now HELWIG's construction [80 c], which embraces many other earlier descriptions of symmetric spaces using Jordan algebras and triple systems (see, for example, BRAUN and KOECHER [24], HIRZEBRUCH [82 a,b,c], KOECHER [93 d].

Notation. The mutation of $\mathcal{A}$ with respect to q^{-1}, $q \in \mathrm{Inv}(\mathcal{A})$, will be denoted by $\mathcal{A}^q$.

Remark 1. The product of two elements $a, b \in \mathcal{A}^q$ is given by $a \perp b = a(bq^{-1}) + b(aq^{-1}) - (ab)q^{-1}$.

Remark 2. Propositions 1.3 and 1.4 from JSA I, imply that the mutation $\mathcal{A}^q$ is a Jordan algebra with unit element q, and that the quadratic representation P_q of $\mathcal{A}^q$ is given by $P_q(a) = P(a)\,P^{-1}(q)$, $a \in \mathcal{A}^q$.

Remark 3. One can also see that $\mathrm{Inv}(\mathcal{A}^q) = \mathrm{Inv}(\mathcal{A})$ and $\Gamma(\mathcal{A}^q) = \Gamma(\mathcal{A})$.

Remark 4. The set $\mathrm{Inv}_o(\mathcal{A})$ with the multiplication $q \circ p := P(q)p^{-1}$ becomes a symmetric space in the sense of the above definition (see LOOS [106c, pp.64 and 68]).

Notation. Denote by V the F-module of all vector fields of class C^∞ on $\mathrm{Inv}_o(\mathcal{A})$, where F is the ring of the real infinitely differentiable functions on $\mathrm{Inv}_o(\mathcal{A})$.

The F-module V will be identified in what follows with the module of all maps of class C^∞ X from $\mathrm{Inv}_o(\mathcal{A})$ to $\mathcal{A}$, $X : q \longrightarrow X_q$. The module V can be endowed, in a natural manner, with a Jordan structure $V(\mathcal{A})$, namely: for all $X, Y \in V$ one defines a product $X \circ Y \in V$ such that for all $q \in \mathrm{Inv}_o(\mathcal{A})$, $(X \circ Y)_q$ is the product of X_q by Y_q in the mutation $\mathcal{A}^q$ of $\mathcal{A}$ with respect to q^{-1}.

Remark 1. Since $(\mathcal{A}^p)^q$ for all $p, q \in \mathrm{Inv}_o(\mathcal{A})$, it follows that $V(\mathcal{A}^p) = V(\mathcal{A})$.

Remark 2. For the trace form λ_q on $\mathcal{A}^q$, $q \in \mathrm{Inv}_o(\mathcal{A})$, we have $\lambda_q(a,b) = \lambda(a, P^{-1}(q)\,b)$ for all $a, b \in \mathcal{A}$.

Definition. A bilinear form g on V $(\mathcal{A})$ is defined by $g(X,Y)_q := \lambda_q(X_q, Y_q)$.

Remark. It is obvious that g is an associative form on the Jordan algebra V $(\mathcal{A})$.

Definition. For every W of $\Gamma_o(\mathcal{A})$, $\Gamma_o(\mathcal{A})$ denoting the connected component of the identity in $\Gamma(\mathcal{A})$, and every X of V, one defines $X^W \in V$ by $(X^W)_q := W(X_{W_q^{-1}})$.

Remark. One can easily see that $X \longrightarrow X^W$ is an automorphism of the Jordan algebra V $(\mathcal{A})$.

Notations. Denote by D that affine connection of $Inv_o(\mathcal{A})$ for which $D_X Y = 0$ for all constant vector fields X and Y on $Inv_o(\mathcal{A})$. Another affine connection of $Inv_o(\mathcal{A})$ can be defined by $\nabla_X Y := D_X Y - X \circ Y$.

Remark. Since all elements $W \in Inv_o(\mathcal{A})$ are affine with respect to D and $X \longrightarrow X^W$ is an automorphism of V, it follows that W is also affine with respect to ∇.

Proposition 2.3. a) For every $X \in V$, ∇_X is an R-derivation of the Jordan algebra V $(\mathcal{A})$.

b) ∇ is torsion-free and g is parallel with respect to ∇;

c) for any $X, Y \in V$, we have

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = [L(Y), L(X)] ,$$

where L is the regular representation of V and [,] is defined as usual;

d) $Inv_o(\mathcal{A})$ is totally geodesic. The exponential map with respect to ∇ coincides, at every point q of $Inv_o(\mathcal{A})$, with the exponential map exp_q of the algebra $\mathcal{A}^q$;

e) for every $X \in V$ and $q \in Inv_o(\mathcal{A})$ let $\gamma(t) := exp_q(t X_q)$, $t \in \mathbb{R}$. Then, for every Y of V,

$$(\nabla_X Y)_q = \lim_{t \to 0} \frac{1}{t} (P_q(\gamma(-\tfrac{t}{2})) Y_{\gamma(t)} - Y_q).$$

The proof is a straightforward calculation (see [20 c, pp.322-323]).

1'. Suppose now that $\mathcal{A}$ is endowed with an involution $\mathcal{J}$ (i.e. $\mathcal{J} \in \mathrm{Aut}(\mathcal{A})$, $\mathcal{J}^2 = \mathrm{Id}$).

Notations. Write $\mathrm{Inv}(\mathcal{A}, \mathcal{J}) := \{a \mid a \in \mathrm{Inv}(\mathcal{A}),\ a^{-1} = \mathcal{J}a\}$ and denote by $\mathrm{Inv}_o(\mathcal{A}, \mathcal{J})$ the component of $\mathrm{Inv}(\mathcal{A}, \mathcal{J})$ containing e. For every q of $\mathrm{Inv}(\mathcal{A}, \mathcal{J})$ define $\mathcal{J}_q := P(q)\mathcal{J}$.

Remark 1. We have $\mathcal{J}_q = \mathcal{J}P(q^{-1}) = \mathcal{J}P^{-1}(q)$, and $\mathcal{J}_q$ is an involutive map. Since $\mathcal{J}_q(q) = q$, it follows that $\mathcal{J}_q$ is an automorphism of $\mathcal{A}^q$.

Remark 2. If we write $\Gamma(\mathcal{A}, \mathcal{J}) := \{W \mid W \in \Gamma(\mathcal{A}), W^* \mathcal{J} W = \mathcal{J}\}$, then $\Gamma(\mathcal{A}^q, \mathcal{J}_q) = \Gamma(\mathcal{A}, \mathcal{J})$.

Remark 3. For every q of $\mathrm{Inv}(\mathcal{A}, \mathcal{J})$, we have $\mathrm{Inv}(\mathcal{A}^q, \mathcal{J}_q) = \mathrm{Inv}(\mathcal{A}, \mathcal{J})$, and $\mathrm{Inv}_o(\mathcal{A}^q, \mathcal{J}_q)$ coincides with the component of $\mathrm{Inv}(\mathcal{A}, \mathcal{J})$ containing q.

Convention. The module of vector fields of class C^∞ on $\mathrm{Inv}_o(\mathcal{A}, \mathcal{J})$ will be denoted by $V^{\mathcal{J}}$ and will be identified with the module of all C^∞ - maps X from $\mathrm{Inv}_o(\mathcal{A}, \mathcal{J})$ to $\mathcal{A}$ with $X_q \in T_q$, $q \in \mathrm{Inv}_o(\mathcal{A}, \mathcal{J})$, where T_q is the (-1)-eigenspace of the involution $\mathcal{J}_q$ of $\mathcal{A}^q$. For $X, Y \in V^{\mathcal{J}}$, $X \circ y$ will be defined as above (see on page 15). $X \circ Y \notin V^{\mathcal{J}}$, since $(X \circ Y)_q$ is an element of the (+1) - eigenspace of $\mathcal{J}_q$.

Notation. For X,Y,Z of $V^{\mathcal{J}}$, write $L(X,Y)Z := X \circ (Y \circ Z) - Y \circ (X \circ Z)$, which is an element of $V^{\mathcal{J}}$.

For the connection ∇^{J} induced by ∇ on $Inv_o(\mathcal{A}, J)$ we have

Proposition 2.4. a) ∇^{J} is torsionfree and the bilinear form g^{J} induced by g on V^{J} is parallel with respect to ∇^{J};

b) for all $q \in Inv_o(\mathcal{A}, J)$, the restriction of exp_q to T_q coincides with the exponential map corresponding to ∇^{J} at q;

c) the curvature R^{J} corresponding to ∇^{J} is given by $(X,Y,Z) \longrightarrow L(X,Y)Z$.

The proof is immediat from Proposition 2.3.

1''. Consider now the case when $\mathcal{A}$ is central simple and $J \neq Id$.

Notation. Let $\mathcal{A}_+$, resp. $\mathcal{A}_-$, denote the eigenspace of J corresponding to the eigenvalue (+1), resp. (-1). Let the degree of $\mathcal{A}$ be denoted by s, let n denote the dimension of $\mathcal{A}$, and let d be given by the relation:

$$n = s + \frac{s}{2}(s-1)d. \qquad (2.1)$$

Theorem 2.5. $Inv_o(\mathcal{A}, J)$ is a pseudo-Einsteinian space (i.e. the Ricci tensor has the form νg^{J}, ν being a scalar). More precisely,

a) if $\mathcal{A}_+$ is also central simple, then

$$\nu = \frac{s}{n}\left(\frac{\alpha}{2} - \frac{m}{2}\right),$$

where $m := \dim \mathcal{A}_-$ and $\alpha := 1 + (s-2)\frac{d}{4}$;

b) if $\mathcal{A}_+$ is not central simple, then

$$\nu = \frac{s}{2n}\left(1 - \frac{d}{2} - \frac{sd}{4}\right).$$

2. Let $\mathbb{A}$ be a central simple Jordan algebra of degree $s \geqslant 3$ over a field of characteristic zero. Suppose that $\mathbb{A}$ is endowed with an involution J such that $J \neq \mathrm{Id}$.

Notation and definitions. Denote by n the dimension of $\mathbb{A}$, by r the degree of $\mathbb{A}_+$, and let d be given by (2.1). An idempotent of an algebra is called absolutely primitive if it is primitive in any ground field extension of that algebra. An R-module $\mathcal{M}$, R being a ring, is called faithful if $\mathrm{ann}\,\mathcal{M} := \{x \mid x \in R,\ xm=0 \text{ for all } m \in \mathcal{M}\}$ vanishes. Put $\mathrm{Der}\,(\mathbb{A},J) := \{D \mid D \in \mathrm{Der}(\mathbb{A}),\ DJ = JD\}$, where $\mathrm{Der}(\mathbb{A})$ denotes the Lie algebra of all derivations of $\mathbb{A}$.

Theorem 2.6. If $\mathbb{A}_+$ is reduced (i.e. contains a complete orthogonal system of absolutely primitive idempotents) and if $\mathbb{A}_-$ is anisotropic with respect to the trace form of $\mathbb{A}$, then $\mathbb{A}_-$ is a faithful and simple $\mathrm{Der}\,(\mathbb{A},J)$ - module except in the following cases:

(1) $\mathbb{A}_+$ is central simple, $d = 2$, and $r = s = 4$,

(2) $\mathbb{A}_+$ is not central simple, $d = 1$, and $s = 4$.

The proof of Theorem 2.6 given by HELWIG [80 c, pp.347-348] is based on results reported elsewhere [80 a] and [80 b].

3. Let $\mathcal{A}$ be a semisimple real Jordan algebra endowed with an involution J, and let $J \neq \mathrm{Id}$. Denote again by λ the trace form on $\mathcal{A}$.

Remark. For every W of $\Gamma(\mathcal{A})$, W^* coincides with the adjoint of W with respect to λ.

Notation. Write

$$\lambda_J(a,b) := \lambda(a, J(b)). \tag{2.2}$$

Then λ_J is a symmetric, nondegenerate bilinear form on $\mathcal{A}$.

Theorem 2.7. If $\mathcal{A}$ is central simple of degree $s \geq 3$, one (and only one) of the two forms λ and λ_J is positive definite, and neither condition (1) nor condition (2) of Theorem 2.6 holds, then $\mathrm{Inv}_o(\mathcal{A}, J)$ is an irreducible symmetric Riemannian space. The component $\Gamma_o(\mathcal{A}, J)$ of $\Gamma(\mathcal{A}, J)$ is a simple Lie group except when $\mathcal{A}_+$ is simple, $d = 2$, and $\lambda_J > 0$.

The proof follows from Theorem 2.6.

Remark. If $\mathcal{A}$ is simple, but $\mathcal{A}_+$ is not, then J is the Peirce reflection with respect to an idempotent $c \neq e$, whose length will be denoted by t.[1)]

Theorem 2.8. If $\mathcal{A}$ is formally real, then, under the hypothesis of Theorem 2.7, the symmetric Riemannian space $\mathrm{Inv}_o(\mathcal{A}, J)$ is a symmetric Hermitian space if and only if the following five conditions holds:

(I) $\mathcal{A}_+$ is not simple, $d = 2$;

(II) $\mathcal{A}_+$ is simple, $r = s$, $d = 4$;

(III) $\mathcal{A}_+$ is simple, $s = 2r$, $d = 1$;

(IV) $\mathcal{A}_+$ is not simple, $d = 1$, $t = 2$, $s \geq 5$;

(H) $\mathcal{A}_+$ is not simple, $d = t = 1$, $s = 3$.

Sketch of the proof. Theorem 2.7, together with Theorems 1.1 and 6.1 from [77] and Theorems 7,8 and 9 from [80 b], give the proof of the theorem.

Remark. In the case (H), $\mathrm{Inv}_o(\mathcal{A}, J)$ is a model of the upper half-plane. The other cases are denoted by (I) - (IV) so as to suggest Siegel's notation for the four main classes of bounded symmetric domains.

1) The primitive degree (the maximal number of idempotents in a complete orthogonal system which the algebra can possess) of the 1-Peirce-component of c is called the length of c.

4. Notation. In the real vector space $\mathcal{A}$ we define a new product of any two elements $a, b \in \mathcal{A}$, denote it by $a * b$, as follows:

$$2(a * b) := ab + a(J(b)) + b(J(a)) - J(ab),$$

and the real algebra defined by means of $*$ in the vector space $\mathcal{A}$ will be denoted by $\mathcal{A}_J$.

Remark. If $\mathcal{A}$ is a Jordan algebra, then the algebra $\mathcal{A}_J$ is also a Jordan algebra, $J \in \mathrm{Aut}\,(\mathcal{A}_J)$, $(\mathcal{A}_J)_J = \mathcal{A}$, and λ_J is the trace form of $\mathcal{A}_J$.

By Theorem 5.2 due to HELWIG [80 a], a real semi-simple Jordan algebra $\mathcal{A}$ has an involution J such that λ_J is positive definite (i.e. the Jordan algebra $\mathcal{A}_J$ is formally real). Suppose that $J \neq \mathrm{Id}$ and that $\mathcal{A}$ is not formally real. If $\mathcal{A}$ is simple, then it belongs to one of the following classes (see again [80 a]):

A. $\mathcal{A}$ is not reduced. Then $\mathcal{A}_+$ is simple and $2r = s$, where r and s are the degrees of $\mathcal{A}_+$ and $\mathcal{A}$, respectively.

B. $\mathcal{A}$ is reduced and $\mathcal{A}_+$ is simple. Then $r = s$ and every primitive idempotent of $\mathcal{A}_+$ is absolutely primitive in $\mathcal{A}$.

C. J is the Peirce reflection with respect to an idempotent $c \neq e$.

Theorem 2.9. The cut locus of a point $p \in \mathrm{Inv}_o(\mathcal{A}, J)$ consists of all points $q \in \mathrm{Inv}_o(\mathcal{A}, J)$, for which $p+q$ is not invertible.

Note. For the definitions of cut point and cut locus see, for instance, KOBAYASHI and NOMIZU [92, vol. II, pp. 96-100].

Remark 1. The (Riemannian) antipodal set A_p of a point $p \in \mathrm{Inv}_o(\mathcal{A}, J)$, i.e. the set A_p of all points of $\mathrm{Inv}_o(\mathcal{A}, J)$ at maximal Riemannian distance from p, is contained in the cut locus of p.

Remark 2. $\mathrm{Inv}_o(\mathcal{A}, J)$ has rank one only when A_p coincides with the cut locus of p.

Theorem 2.10 Every maximal torus of $Inv_o(\mathcal{A}, \mathcal{J})$ through p contains exactly one antipodal point of p.

Notation. Denote by K_o the biggest connected subgroup of K, as determined in Cartan's conjugacy theorem for maximal compact subgroups (see HELGASON [77, pp.214-218]).

Theorem 2.10 has two important corollaries:

Corollary 1. K_o is transitive on A_e.

Corollary 2. Distinct points of $Inv_o(\mathcal{A}, \mathcal{J})$ have distinct antipodal sets.

Notation. Denote by Fix $\mathcal{J}$ the set of all elements of $Inv_o(\mathcal{A}, \mathcal{J})$ which are invariant with respect to $\mathcal{J}$.

Convention. In the following theorem the length of an idempotent c is denoted by $\ell(c)$, c= 0 is regarded as an idempotent of length zero, and ϱ denotes the rank of $Inv_o(\mathcal{A}, \mathcal{J})$ (see HELWIG [80 c, p.329]).

Theorem 2.11. K_o is transitive on each component of Fix $\mathcal{J}$, and A_c is a component of Fix $\mathcal{J}$. More specifically,

(A) If $\mathcal{A}$ is not reduced, then Inv $(\mathcal{A}, \mathcal{J})$ is connected and $A_e = \{-e\}$;

(B_1) If $\mathcal{A}$ is reduced, $\mathcal{A}_+$ is simple and s even, then Fix $\mathcal{J}$ consists of the elements of the form 2 c-e, where c is an arbitrary idempotent of $\mathcal{A}_+$ with $\ell(c)$ even; $A_e = \{-e\}$, and Inv $(\mathcal{A}, \mathcal{J})$ has exactly two components. The distinct components of $Inv_o(\mathcal{A}, \mathcal{J})$ coincide with $Inv_o(\mathcal{A}^f, \mathcal{J}_f)$, f := 2c-e, and c is an arbitrary idempotent of $\mathcal{A}_+$ with $\ell(c)$ odd.

(B_2) If $\mathcal{A}$ is reduced, $\mathcal{A}_+$ is simple, and s odd, then Fix $\mathcal{J}$ consists of the elements of the form 2 c-e, where c is an arbitrary idempotent of $\mathcal{A}_+$ with $\ell(c)$ odd; Inv $(\mathcal{A}, \mathcal{J})$ has two distinct components, namely, $Inv_o(\mathcal{A}, \mathcal{J})$ and $-Inv_o(\mathcal{A}, \mathcal{J})$, and A_e consists of the elements of the form 2c-e, where c is an arbitrary primitive idempotent of $\mathcal{A}_+$.

(C) If J is the Peirce reflection with respect to an idempotent c of length $\rho \leq [s/2]$, then Inv $(\mathcal{A}, J)$ has exactly s+1 components; A_e consists of the elements of the form 2 d-e, ehere d is an arbitrary idempotent of $\mathcal{A}_0(c)$ with $\ell(d) = s - 2\rho$. The elements of Fix J are exactly those of the form $2(c' + c'') - e$ where c' and c'', are idempotents of $\mathcal{A}_1(c)$ and $\mathcal{A}_0(c)$, respectively, with $\ell(c'') = s - 2\rho + \ell(c')$.

<u>Definitions.</u> Let p be a point of $\mathrm{Inv}_0(\mathcal{A}, J)$, let $\gamma : t \longrightarrow \gamma(t)$, $t \in \mathbb{R}$, be a geodesic of $\mathrm{Inv}_0(\mathcal{A}, J)$ with $\gamma(0) = p$ and let $\dot\gamma(0)$ be of length 1. If there exists an $\ell > 0$ such that $\gamma(t+\ell) = \gamma(t)$ for all $t \in \mathbb{R}$, then γ is called <u>closed.</u> If, in addition, $\gamma(t) \neq \gamma(s)$ for all t,s with $0 < t < s \leq \ell$, then γ is called <u>simple closed,</u> ℓ is called the <u>length</u> of γ, and $\gamma(\ell/2)$ is called the <u>midpoint</u> of γ. The <u>midpoint locus</u> A'_p of p is the set of midpoints of all simple closed geodesics with minimal length.

<u>Theorem 2.12.</u> a) The midpoint locus A'_p coincides with the set of all cut points of p with minimal distance from p. For every maximal torus T of $\mathrm{Inv}_0(\mathcal{A}, J)$ with $p \in T$, the set $T \cap A'_p$ contains exactly ρ points.

b) $K \cup J(K_0)$ is transitive on the set of all simple closed geodesics of minimal length and starting from p; K_0 is transitive on A'_e.

c) Any two distinct points of $\mathrm{Inv}_0(\mathcal{A}, J)$ have distinct midpoint locus if and only if $\rho \neq 2$ or $s \neq 4$. In the case $\rho = 2$ and $s = 4$, the antipodal (-p) of p is the unique point $\neq$ p whose midpoint locus coincides with that of p.

<u>Proposition 2.13.</u> If $\mathcal{A}$ has degree 2 then $\mathrm{Inv}_0(\mathcal{A}, J)$ is a Euclidean sphere.

Proposition 2.14. If the antipodal set A_e of e is at least one dimensional, then $\mathrm{Inv}_o(\mathcal{A}, J)$ is simply connected if and only if A_e is.

Theorem 2.15. If $\mathcal{A}$ is central simple, $s \geq 3$, and neither condition in Theorem 2.6 holds, then $\mathrm{Inv}_o(\mathcal{A}, J)$ is symmetric Hermitian if and only if one of the following cases hold:

(I) $\mathcal{A}$ is not simple, $d = 2$;

(II) $\mathcal{A}$ is reduced, $\mathcal{A}_+$ is simple, $d = 4$;

(III) $\mathcal{A}$ is central, but not reduced, and $d = 1$.

Notation. Denote by n the dimension of $\mathcal{A}$ and let S^{n-1} be the set of all $q \in \mathcal{A}$ with $\mu(q,q) = n$, where $\mu := \lambda_J$.

Suppose that S^{n-1} is endowed with the Riemannian structure given by μ . Then $\mathrm{Inv}_o(\mathcal{A}, J)$ is a Riemannian submanifold of S^{n-1}. Identify the tangent space T'_q of S^{n-1} at point q with the space of all $a \in \mathcal{A}$ for which $\mu(q,a) = 0$. Then the normal space $T_q^{\perp}$ of $\mathrm{Inv}_o(\mathcal{A}, J)$ at $q \in \mathrm{Inv}_o(\mathcal{A}, J)$ consists of all elements of T'_q which are invariant with respect to J_q.

Notation. For all $q \in \mathrm{Inv}_o(\mathcal{A}, J)$ denote by η_q the normal of the mean curvature, and by η the mean curvature normal (for definitions, see KOBAYASHI and NOMIZU [92 , vol.II, pp.33-34]).

Theorem 2.16. a) If $\mathcal{A}_+$ is simple, then $\eta = 0$.

b) If J is the Peirce reflection with respect to an idempotent c of $\mathcal{A}$ with length t, then

$$\eta_q = k \left[\left(\frac{1}{2} - \frac{t}{s}\right)q + c - \frac{1}{2}\, e \right], \text{ where } m = \dim \mathcal{A}_- \text{ and } K := \frac{s^2 d}{nm}\left(\frac{t}{s} - \frac{1}{2}\right).$$

c) $\mathrm{Inv}_o(\mathcal{A}, J)$ is a totally geodesic subspace of S^{n-1} if and only if $\mathcal{A}$ has degree 2.

Remark. Theorem 2.16 gives a complete answer for the case when $\mathrm{Inv}_o(\mathcal{A}, J)$ is a minimal submanifold of S^{n-1}.

4'. Let $\mathcal{A}$ be simple, but not central, and let $\lambda_J > 0$. As a complex algebra, $\mathcal{A}$ is the complexification of the formally real Jordan algebra $\mathcal{A}_+$, J is the corresponding conjugation, and $\mathcal{A}_- = i\,\mathcal{A}_+$.

Notation. Let $\mathcal{A}_+'$ be the space of all $b \in \mathcal{A}_+$ with $\lambda(b) = 0$. In what follows, we will write $\mathcal{A}^*$ instead of $i\,\mathcal{A}_+'$.

Every element a of $\mathcal{A}_-$ has the form $a = i(\alpha_1 c_1 + \dots + \alpha_r c_r)$, $\alpha_j \in \mathbb{R}$, where $c_1, \dots, c_r$ is a complete orthogonal system of primitive idempotents of $\mathcal{A}_+$, and $a \in \mathcal{A}^*$ if and only if $\sum_{j=1}^{r} \alpha_j = 0$

Notation. Let M be given by $M := \exp \mathcal{A}^*$.

Remark. The totally geodesic subspace M of $\mathrm{Inv}_o(\mathcal{A}, J)$ is compact and of rank r-1. As a totally geodesic subspace, M is invariant with respect to all J_q, $q \in M$. This implies that M depends only on the set of mutations $(\mathcal{A}^q, J_q)$, $q \in M$.

Theorem 2.17. M is simply connected.

5. Now we give HELWIG's classification (see HELWIG [80 c, pp.343-349]).

Convention. $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ will be denoted by $\mathbb{F}$.

For $r \geq 2$, let $M_r(\mathbb{F})^+$ be the real Jordan algebra of all $(r \times r)$-matrices with elements from $\mathbb{F}$. The Jordan product xy of $x, y \in M_r(\mathbb{F})^+$ is then given by $xy = \frac{1}{2}(x \circ y + y \circ x)$, where $x \circ y$ is the matrix product.

Remark. The elements of $M_r(\mathbb{F})^+$ act on the left on $\mathbb{F}^r$ (considered as right vector space over $\mathbb{F}$).

Notation. Let σ be a nondegenerate (and a non negative definite) Hermitian form on $\mathbb{F}^r$, and let $H(\mathbb{F}^r, \sigma)$ be the Jordan algebra of all Hermitian matrices of $M_r(\mathbb{F})^+$.

Remark. The set I_t of all idempotents of $H(\mathbb{F}^r, \sigma)$ with length t coincides with the set $\{ c \mid c \in H\ (\mathbb{F}^r, \sigma),\ \mathrm{Tr}\ c = t \}$.

Notation. For fixed $t \leq \left[\frac{r}{2}\right]$, denote by $G_t(\mathbb{F}^r, \sigma)$ the set of all t-dimensional subspaces of $\mathbb{F}^r$ on which the restriction of σ is nondegenerate. For all $c \in I_t$, let V_c be the t-dimensional subspace of $\mathbb{F}^r$ of all $v \in \mathbb{F}^r$ with $cv = v$.

Remark. The subspace V_c belongs to $G_t(\mathbb{F}^r, \sigma)$ and c is completely determined by V_c. Conversely, if $V \in G_t(\mathbb{F}^r, \sigma)$, then the element c of $M_r(\mathbb{F})^+$ which is identity on V and zero on $V^\perp$ (the orthogonal complement of V with respect to σ), is an element of I_t (for the definition of I_t, see the above remark). The identification of I_t and $G_t(\mathbb{F}^r, \sigma)$ which follows from the above considerations is compatible with the symmetric structures given on I_t and $G_t(\mathbb{F}^r, \sigma)$.

Two special cases are considered :

a) σ is the usual form $(v,w) \longrightarrow \bar{v}'w$. In this case $G_t(\mathbb{F}^r) := G_t(\mathbb{F}^r, \sigma)$ is the well-known Grassmann manifold of all t-dimensional subspaces of $\mathbb{F}^r$ and $H(\mathbb{F}^r) := H(\mathbb{F}^r, \sigma)$ is formally real;

b) let e_t be the unit element of $M_r(\mathbb{F}^r)$,

$$f := \begin{pmatrix} e_t & 0 \\ 0 & -e_{r-t} \end{pmatrix},$$

and let $\sigma' : (v,w) \longrightarrow \sigma(v, fw)$.

Remark. The space $\mathbb{M}_t(\mathbb{F}^r)$ of all elements of $G_t(\mathbb{F}^r, \sigma')$ on which the restriction of σ' is positive definite is the non-compact space associated with $G_t(\mathbb{F}^r)$.

Note. Throughout the remainder of this section, we shall use HELGASON's notation [7, Chapter IX, § 4].

Proposition 2.18. All noncompact spaces of type $\mathbf{BDI}$ are contained in the form $\mathbb{H}_t(\mathbb{R}^r)$, all of type $\mathbf{AIII}$ are contained in the form $\mathbb{H}_t(\mathbb{C}^r)$, and all of type $\mathbf{CII}$ are contained in the form $\mathbb{H}_t(\mathbb{H}^r)$. In particular, $\mathbb{H}_t(\mathbb{C}^r)$ gives the bounded symmetric domains of Siegel type $\mathbf{I}$ of all complex $t \times (r-t)$-matrices z with $e_{r-t} - \bar{z}'z > 0$, and $\mathbb{H}_2(\mathbb{R}^r)$, $r \geqslant 5$, gives the bounded symmetric domain of Siegel type $\mathbf{IV}$ of complex dimension $r-2$.

Let $\mathcal{A}$ be a simple formally real Jordan algebra of degree 2, endowed with an involution J. Then, for $J \neq \mathrm{Id}$, $\mathrm{Inv}_o(\mathcal{A}_J, J)$ is a sphere and any sphere can be obtained in this manner, by means of a Peirce reflection.

Proposition 2.19. All Grassmann manifolds, as well as all compact symmetric spaces of rank one, are contained in the form $\mathrm{Inv}_o(\mathcal{A}_J, J)$, where $\mathcal{A}$ is a simple formally real Jordan algebra and J is a Peirce reflection of $\mathcal{A}$. The noncompact spaces associated with the above-mentioned spaces have the form $\mathrm{Inv}_o(\mathcal{A}, J)$.

Comments. In [11 a], BARBILIAN is mainly concerned with the real Grassmann manifold $G_n(\mathbb{R}^{2n+1})$ of all n-dimensional vector subspaces of $\mathbb{R}^{2n+1}$. So, after establishing a certain group isomorphism, Barbilian uses this isomorphism to give geometrical criteria for the direct identification on $G_n(\mathbb{R}^{2n+1})$ of the subsets corresponding to the algebras with minimal representation in a given complete matrix ring, as well as criteria for finding the order of the characteristic equation, and the ranks of the centre and the radical.

Open problem. To reconsider in the Jordan algebra setting BARBILIAN's results [11 a] concerning $G_n(\mathbb{R}^{2n+1})$.

Comments. As is well known, the simplices form a natural class of polyhedra in the (real) projective spaces. GEL'FAND and MacPHERSON [61] generalized them to polyhedra in real Grassmann manifolds. These new objects, which are called Grassmannian simplices, are then studied in terms of their combinatorial structure. ←—— Next, they establish a relation between these objects and harmonic differential forms on real Grassmann manifolds. This relation is subsequently used to obtain results about some new differential forms (one of which is the classical dilogarithm). However, as was mentioned in [61, p.50], it is still an open question to relate the forms $S^{n,m}$ to the Grassmannian geometry developed in [61]. As the author suggested to MacPherson in 1980, it would be fruitful to make use of the Jordan algebra description for Grassmann manifolds in the study undertaker in [61].

Open problem. (Suggested to Gel'fand on the occasion of OAGR Conference, Neptun - Romania, 1980.) To study the so called "double fibrations" of Gel'fand et al. (see, for instance, [59]) using the above-mentioned Jordan algebraic description of projective spaces and Grassmann manifolds.

Comments 1. Solutions of Yang-Mills equations were interpreted by MANIN [108] in terms of so-called super-Grassmannians and flag superspaces. The ground structures of these new geometrical objects are so-called "superspaces". In the section dealing with flag superspaces of classical type and the exotic Minkowski superspace, Manin defines Π - symmetric Grassmannians and isotropic Grassmannians (see [108 , § 10]). For details, the reader is referred to MANIN's paper [108], in which a supergeometry is developed. It would be interesting to reconsider Manin's concepts in a Jordan algebra setting in order to give a further development of Yang-Mills theory in quantum mechanics as well as in mathematics.

Comments 2. It would also be worth while to use the Jordan algebra setting for the results of BEREZIN [18b] on instantons and Grassmann manifolds.

Remark. Another open problem of related interest is that pointed out to Sato in 1983 and reported in § 8 of JSA IV.

Question. (CRAIOVEANU [36]). It is possible to simplify the difficult calculation of the cohomology of Grassmann manifolds by using the Jordan algebra description given by Proposition 2.19 ?

The Grassmann manifolds appeared in physics in field theoretical models (see, for instance, BEREZIN [18a]), as well as in instanton theory (see, for instance, PERELOMOV [135a], and recently, they are mainly involved in KP hierarchy and string theories (see Section 8 of JSA.V and Section 7 of JSA.VIII). In other physical situations, more general objects, which generalize the Grassmann manifolds, are considered. For example, BERCEANU and GHEORGHE, using group-theoretical methods, constructed in [16a] perfect Morse functions (see MORSE [121], BERCEANU [15]) on compact manifold of coherent states (see PERELOMOV [135b] admiting a Kählerian C-space structure (see WANG [200]), considering linear Hamiltonians in the generators of the group of symmetry.

In 1989, BERCEANU and GHEORGHE [16b] restricted themselves to the complex Grassmann manifolds and showed that linear Hamiltonians in bifermion operators lead to energy functions which satisfy the Morse-Bott inequalities as equalities.

In ref. [16c] BERCEANU and GHEORGHE studied the motion problem on the complex Grassmann manifold. The equations of motion are a first order system of differential equations, the right hand side being a second degree polynomial. These equations can be put in the form of matrix Riccati equations. The solution, in a given chart, is written explicitely for the case of energy function associated to the linear Hamiltonian in bifermion operators. The globalization problem of the solution (see SCHNEIDER [168a,b] is discussed, noting that the matrix Riccati equation is a flow on the Grassmann manifold (see HERMANN & MARTIN [81], SHAYMAN [170]).

Open problem. Taking into account, on one hand, of the relationship between the matrix Riccati equation and the Grassmann manifolds (see HERMANN and MARTIN [81], SHAYMAN [170], BERCEANU & GHEORGHE [16c]), and, on the other hand, of the Jordan description of Grassmann manifolds (see HELWIG [80c]) and the relationship between the matrix Riccati equation and Jordan pairs (see BRAUN [23c], WALCHER [199a,b]), give a unified treatment in Jordan structure terms.

WET [203] used the products of particlelike representations of the homogeneous Lorentz group in order to construct the degrees of spin angular momentum of a composite system of protons and nentrons. Ground-state energy levels are calculated for all the even-even nuclei by using a differentiable manifold that is spin-graded and gauge-invariant by construction. It is shown that this manifold is a Grassmann manifold. Hence, it would be useful to reconsider WET's results in a Jordan algebra setting.

In order to get quantitative results in the study of the controllability of the linear system $\dot{x} = Ax + Bu$, DRAGER, FOOTE and MARTIN [47] defined the geometric object $\sum_k (\mathbb{R}^n)$, which is a generalization of the Grassmannian, called the splitting space, by

$$\sum_k(\mathbb{R}^n) := \left\{(S,W) \in G_k(\mathbb{R}^n) \times G_{n-k}(\mathbb{R}^n) \;\middle|\; S \oplus W = \mathbb{R}^n\right\}.$$

It is easy to see that $\sum_k(\mathbb{R}^n)$ is a homogeneous space of $GL(\mathbb{R}^n)$. This space derives its geometry from $GL(\mathbb{R}^n)$ in the same way that $G_k(\mathbb{R}^n)$ derives its geometry from $O(\mathbb{R}^n)$ - technically, $\sum_k(\mathbb{R}^n)$ is a reductive homogeneous space of $GL(\mathbb{R}^n)$. The special group $SL(\mathbb{R}^n)$ induces a pseudo-Riemannian structure on $\sum_k(\mathbb{R}^n)$ in the same way that $O(\mathbb{R}^n)$ induces a Riemannian structure on $G_k(\mathbb{R}^n)$.

Open problem. To describe in the Jordan algebra terms the above-mentioned splitting spaces and to rewrite algebraically the qualitative geometric conditions for the controllability of the system $\dot{x} = Ax + Bu$ given by Drager - Foote - Martin in [47].

It is well-known that the controllability of the system $\dot{x} = Ax + Bu$ is related to the one — parameter family of operators $e^{At}B$. Drager-Foot-Martin used this to give a proof of the classical controllability conditions in terms of the differential geometry of certain curves in $\mathbb{R}^n$. Then they considered $\gamma(t) := \mathrm{Im}\,(e^{At}B)$ as a curve in an appropriate Grassmannian and proved that, in local coordinates, γ is an integral curve of the flow induced by a matrix Riccati equation.

Comments. Solving the above-mentioned open problem, use the connections between Jordan pairs and the matrix Riccati equation (see BRAUN [23c] and WALCHER [199a,b]) to the matrix Riccati equation appearing in the Drager-Foote-Martin approach [47].

Remark. Let us mention that Grassmannians are mainly involved in the theory of circuits. As HELTON [79, p.30] emphasized, "philosophically from an engineering point of view, Grassmannians actually are more natural than input-output operators. Indeed, there is a senior level circuits book by KUH and ROHER [99] which takes this point of view. Practical engineers don't use the Grassmannian at all and pay the price of having to change coordinates very often. Many senior circuit books have a full chapter on these changes of coordinates".

DABROWSKI and TRAUTMAN [38] studied spinor structures on projective spaces. The natural spinor connections on these spaces may be interpreted as simple gauge configurations, but they postponed their description to another work. In [38] they restrict themselves to the construction of the spinor structures. Their approach is differential geometric and Lie-group theoretic. It yields an explicit construction of all spaces and maps occuring in the description of spinor structures on projective spaces. It can be extended to other homogeneous spaces, such as the Grassmannians, as well as to pseudo-Riemannian manifolds.

Open problem. To reconsider in the Jordan algebra setting the above-mentioned study of D browski and Trautman (including its possible extension).

TALLINI [182] characterized the Grassmann space related to a projective space as a partial linear space which fulfills some suitable axioms. MISFELD, TALLINI and ZANELLA [117] defined topological Grassmann spaces and showed that every space of this kind is isomorphic to the Grassmann space related to a topological projective space.

NICOLESCU and PRIPOAE [126], [148] studied some algebraic structures defined by certain geometrical properties, Jordan algebras being involved.

Comments. It would also be interesting to reconsider in terms of Jordan algebras, the geometrical constructions corresponding to operations of the algebra of m-planes in projective (2 m + 1)-space, given by ROZENFEL'D, KUZNETSOVA, HANTURINA [155], as well as the results on line manifolds in Grassmann manifolds obtained by VASHKAS and NAVITSKIS [195]. Worthy of re-assessing in terms of Jordan algebras are also the results due to TELEMAN [184a], HSIANG and SZCZARBA [83], HANGAN [74 a,b], and ISHIHARA [85], as well as the results on Grassmannians or rank one symmetric Riemannian spaces contained in the paper collected in Bibliography.

Remark. Other open problems have been formulated on pages 7-8.

Now, let $\mathcal{A}$ be a simple formally real Jordan algebra endowed with an involution J, $J \neq \mathrm{Id}$, for which $\mathcal{A}_+$ is simple.

Remark. $\mathrm{Inv}_0(\mathcal{A}_J, J) \approx G_1(\mathbb{H}^4)$, and, therefore, $\mathrm{Inv}_0(\mathcal{A}, J) = H_1(\mathbb{H}^4)$.

Notation. For $r \geqslant 2$, let j_r be given by

$$j_r := \begin{pmatrix} 0 & e_r \\ -e_r & 0 \end{pmatrix} .$$

Denote by s the degree of $\mathcal{A}$ and assume that $s \geqslant 3$.

C I (Siegel type III). If $\mathcal{A} = H(\mathbb{R}^s)$, $s \equiv 0(2)$, $r := \frac{s}{2}$, and $J : x \longrightarrow j_r x j_r^{-1}$, then $\mathrm{Inv}_0(\mathcal{A}, J)$ is the space of symmetric elements of $\mathbf{Sp}(r, \mathbb{R})$, and $\mathrm{Inv}_0(\mathcal{A}_J, J) = \mathbf{Sp}(r)/\mathbf{U}(r)$.

$\mathcal{C}_r (r \geqslant 3)$. If $\mathcal{A} = H(\mathbb{C}^s)$, $s \equiv 0(2)$, $r := \frac{s}{2}$, and $J : x \longrightarrow j_r x' j_r^{-1}$, then $\mathrm{Inv}_0(\mathcal{A}, J)$ is the space of symmetric elements of $\mathbf{Sp}(r, \mathbb{C})$. Therefore, $\mathrm{Inv}_0(\mathcal{A}, J) = \mathbf{Sp}(r, \mathbb{C})/\mathbf{Sp}(r)$ and $\mathrm{Inv}_0(\mathcal{A}_J, J) = \mathbf{Sp}(r)$.

$\mathcal{b}_r$, $\mathcal{d}_r$($r = [s/2]$). If $\mathcal{A} = H(\mathbb{C}^s)$ and $J : x \longrightarrow x'$, then $\mathrm{Inv}_o(\mathcal{A}, J)$ is the space of symmetric elements of $\mathbf{SO}(s,\mathbb{C})$, and $\mathrm{Inv}_o(\mathcal{A}_J, J) = \mathbf{SO}(s)$.

D III (Siegel type **II**). Identify $\mathcal{A} := H(\mathbb{H}^s)$ with the algebra of all $x \in H(\mathbb{C}^{2s})$ with $x\, j_s = j_s\, x'$. Let J be given by $x \longrightarrow x'$. Then $\mathrm{Inv}_o(\mathcal{A}, J)$ coincides with the space of symmetric elements of $\mathbf{SO}^*(2s)$, and hence with $\mathbf{SO}^*(2s)/\mathbf{U}(s)$. We have $\mathrm{Inv}_o(\mathcal{A}_J, J) = \mathbf{SO}(2s)/\mathbf{U}(s)$.

Finally, let $\mathcal{A}$ be of the form $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}$, where $\mathcal{B}$ is a <u>simple formally real</u> Jordan algebra. Let J be the principal involution of $\mathcal{A}$. Then $\mathrm{Inv}_o(\mathcal{A}, J)$ can be identified with $\mathrm{Inv}_o(\mathcal{B})$. We have $\mathrm{Inv}_o(\mathcal{B}) = \mathbb{R}^+ \times Y$, where $Y := \exp \mathcal{B}'$, and $\mathcal{B}'$ consists of all elements of $\mathcal{B}$ on which the trace form of $\mathcal{B}$ vanishes.

A I. If $\mathcal{B} = H(\mathbb{R}^s)$, then Y is the space of symmetric elements of $\mathbf{SL}(2,\mathbb{R})$.

a_{s-1}. If $\mathcal{B} = H(\mathbb{C}^s)$, then Y is the space of symmetric elements of $\mathbf{SL}(2,\mathbb{C})$.

A II. If $\mathcal{B} = H(\mathbb{H}^s)$, then Y is the space of symmetric elements of $\mathbf{SU}^*(2s)$.

If $\mathcal{B} = H(\mathbb{C}^3)$, then Y is noncompact of type **E IV**.

6. Let us present succintly some of the results obtained by MARTINELLI [112], MARCHIAFAVA [109 ab], MARCHIAFAVA and ROMANI [110 ab], OPROIU [131 a-f] and IORDANESCU [84 e,g,h] concerning quaternionic structures in order to show their connection with the above-mentioned Jordan algebra results.

Let $L_n^{\mathbb{H}}$ (resp., $U_n^{\mathbb{H}}$) be the homogeneous linear (resp., unitary) quaternionic group acting on the left in the right quaternionic vector space $\mathbb{H}^n$ of dimension n. Denote by $\Xi :=$ $:= (\xi^\alpha)$ ⟵—— — — the matrix of a vector ξ of $\mathbb{H}^n$, and by $A := (a^\alpha_\beta)$ an $(n \times n)$-matrix over $\mathbb{H}$ (i.e. $a^\alpha_\beta \in \mathbb{H}$). Then a transformation T of $L_n^{\mathbb{H}}$ (resp., $U_n^{\mathbb{H}}$) is given by

$$(2.3) \qquad T : \Xi \longrightarrow A\,\Xi \,,$$

where A is an invertible (resp., unitary) matrix.

Definition. A real differentiable manifold V_{4n} is endowed with a (right) almost (resp., almost Hermitian) quaternionic structure if its structure group is $L_n^{\mathbb{H}}$ (resp., $U_n^{\mathbb{H}}$).

Note. Recall that a G-structure on a real differentiable m-dimensional manifold V_m is defined by a subbundle with structure group G of the tangent bundle $T(V_m, \mathbb{R}^m, L_m)$, where G is a certain subgroup of the homogeneous linear group L_m. The spaces $\mathbb{R}^{4n}$ and $\mathbb{H}^n$ are assumed tobe canonically identified as real vector spaces.

MARTINELLI defined [112] a generalized (right) almost (resp., almost Hermitian) quaternionic structure on a real differentiable manifold V_{4n} as a structure with structure group $\tilde{L}_n^{\mathbb{H}}$ (resp., $\tilde{U}_n^{\mathbb{H}}$). The latter consists of all transformations T given by

$$(2.4) \qquad T : \Xi \longrightarrow A\,\Xi\, b \,,$$

where A is an invertible matrix and $b \in \mathbb{H} - \{0\}$ (resp., A is a unitary matrix and $b\bar{b} = 1$).

Remark. An important example of manifolds endowed with an integrable generalized almost-quaternionic structure is given by quaternionic projective space. This example suggested the above-mentioned generalization to Martinelli.

MARCHIAFAVA proved, [109 a], that in case the structure (2.4) is integrable the coordinate transformations are of quaternionic projective type (i.e. linear fractional maps). OPROIU has studied, [131 b,c], the same problem in terms of the general theory of G-structures (see also GHEORGHIEV and OPROIU [62]). He proved that the integrability is equivalent to quaternionic projective (Q-projective) flatness.

Definitions. Two vectors from $\mathbb{H}^n$ are called equivalent if they differ by a factor λ, $\lambda \in \mathbb{H} - \{0\}$. An equivalence class of vectors from $\mathbb{H}^n$ is called a direction in $\mathbb{H}^n$. The image in $\mathbb{R}^{4n}$ of a direction from $\mathbb{H}^n$ by canonical identification is called a quaternionic characteristic facet (q.c.f.).

Proposition 2.20. A generalized almost quaternionic structure determines, for every point of the manifold V_{4n} on which it is defined, a system of q.c.f. in the tangent space of V_{4n} at that point.

Proposition 2.21. A generalized almost-Hermitian quaternionic structure determines on the manifold V_{4n} on which it is defined a Hermitian quaternionic metric.

IORDANESCU [84 e] considered the quaternionic Grassmann manifold $G_p(\mathbb{H}^{p+q})$ of all p-dimensional right vector subspaces of $\mathbb{H}^{p+q}$, endowed with Pontryagin coordinates (see PONTRYAGIN [142]), and proved that its structure group $\mathcal{G}$ consists of all transformation T given by

$$(2.5) \qquad T : \Xi \longrightarrow (A \otimes Id_q)\, \Xi\, (B \otimes Id_p),$$

where A and B are invertible quaternionic matrices of order p, resp. q, Id_q and Id_p are the unit matrices and Ξ is a coloum vector.

Definition. Let V_{4pq} be a C^{∞} paracompact real manifold of dimension 4pq ($p,q \geqslant 1$, integers). If, with respect to a suitable open covering, there exist systems of admissible quaternionic coordinates transforming each into another, in common domains, by transformations of the group $\mathcal{G}$, then V_{4pq} is called a locally Grassmann quaternionic manifold.

MARCHIAFAVA [109 b] proved that all locally Grassmann quaternionic manifolds and only these are quaternionic manifolds admiting an integrable tensor product structure (see [109b, p.84]).

Open problem. To find the kind of differentiable manifolds that are endowed with structures whose structure groups are similar to , but in which the factors $(A \otimes Id_q)$, $(B \otimes Id_p)$ are replaced by other (more general, e.g, unitary, invertible) matrices. Study their (eventual) description in terms of Jordan algebras.

Comments. Before IORDANESCU [84 e], the group $\tilde{L}_n^{\mathbb{H}}$ (resp., $\tilde{U}_n^{\mathbb{H}}$) was considered only as a generalization of the group $L_n^{\mathbb{H}}$ (resp., $U_n^{\mathbb{H}}$).

By considering the group $\tilde{L}_n^{\mathbb{H}}$ (resp., $\tilde{U}_n^{\mathbb{H}}$) as a subgroup of , MARCHIAFAVA and ROMANI [110 b, pp.131-132] obtained topological theorems related to "generalized quaternionic fibre bundles" and their Stiefel-Whitney classes.

Open problem . To undertake a topological research like that given in [110 b] by considering, instead of the group $\tilde{L}_n^{\mathbb{H}}$ (resp., $\tilde{U}_n^{\mathbb{H}}$), the group $\mathcal{G}$ or groups from the above-mentioned open problem.

OPROIU [131 d,f] has proved that Pontryagin classes of quaternionic manifolds are powers of the Pontryagin class of the fundamental vector bundle.

Open problem. To obtain results similar to those of OPROIU [131 d,f] for the generalized quaternionic vector bundles considered by MARCHIAFAVA and ROMANI [110 a,b].

By using the tensor product structure of the tangent bundle of a real Grassmann manifold, OPROIU [131 a,c] has obtained nonembedding theorems in Euclidean spaces similar to those for projective spaces.

Open problem . To obtain similar results for complex and quaternionic Grassmann manifolds (by eventually using a Jordan algebra setting).

Definitions. Two vectors from $\mathbb{H}^{pq}$, written as $(q \times p)$--matrices, are called equivalent if they differ by a "factor" $\Lambda = (\lambda_i^j)$, Λ being an invertible quaternionic matrix of order p. An equivalence class of vectors from $\mathbb{H}^{pq}$ is called a generalized direction in $\mathbb{H}^{pq}$. If $\Theta \neq 0$ is a vector representing a generalized direction, then the subspace

$$D_{\Theta} := \left\{ \Theta\Lambda \;\middle|\; \Lambda \text{ an arbitrary quaternionic } (p \times p)\text{-matrix} \right\}$$

of $\mathbb{H}^{pq}$ is called the associated subspace of the generalized direction under consideration. The image of D_{Θ} in $\mathbb{R}^{4pq}$ under the canonical identification $\mathbb{H}^{pq} \equiv \mathbb{R}^{4pq}$ is called a generalized quaternionic characteristic facet (g.q.c.f.).

Proposition 2.22. The subspaces D_{Θ} are invariant under the group $\mathcal{G}$, and

$$\dim_{\mathbb{H}} D_{\Theta} = rp ,$$

where r, $1 \leq r \leq \min(p,q)$, is the rank of the $(q \times p)$-matrix Θ (i.e., the maximum number of linear independent rows on the right).

Remark. If $p \geqslant q$ then, in general, Θ has the rank q, and $\dim_{\mathbb{H}} D_{\Theta} = pq$; therefore in that case the subspace D_{Θ} coincides with the whole tangent space.

Comments. Taking into account the subspaces D_{Θ}, MARCHIAFAVA and ROMANI defined in a natural manner the "characteristic subspaces" of a quaternionic structure [110 b, p.134]. These subspaces were useful in later topological constructions [110 b, pp.140-142].

The group $\mathcal{G}$ with invertible (resp.,unitary) matrices A, B has led to the following generalization of MARTINELLI's results [112 , pp.356-357] (herein recalled as Propositions 2.20 and 2.21) as follows (see IORDANESCU [84 g]):

Proposition 2.23. Let V_{4pq} be a manifold endowed with a tensor product quaternionic structure. Then, at every point of V_{4pq} this structure determines a system of g.q.c.f. in the tangent space of V_{4pq} at that point.

Proposition 2.24. A tensor product quaternionic structure with unitary matrices A,B determines on the manifold V_{4pq} on which it is defined a Hermitian quaternionic metric.

Notation. Denote by $\mathcal{S}(\mathbb{H}^{pq})$ the set of all subspaces of $\mathbb{H}^{pq}$.

Proposition 2.25. Consider $\mathbb{H}^{p+q}$ with $p < q$. By means of the map $\Phi : \mathbb{H}^{pq} \longrightarrow \mathcal{S}(\mathbb{H}^{pq})$ defined by $\Phi(\Theta) := D_{\Theta}$, to every Grassmann manifold $G_p(\mathbb{H}^{p+q})$ from $\mathbb{H}^{p+q}$ corresponds the set of Grassmann manifolds $\left\{ G_{rp}(\mathbb{H}^{pq}) \right\}_{r=1,2,\ldots,p}$ from $\mathbb{H}^{pq}$.

Comments. It would be interesting to find relations between the map Φ and various correspondences between Grassmann manifolds studied by GEL'FAND et al. (see [59]).

From the considerations given in subsection 5 above and Proposition 2.25, we obtain.

Proposition 2.26. Let $H_p(\mathbb{H})^+$, $p \geqslant 2$, be the Jordan algebra of all Hermitian $(p \times p)$-matrices over $\mathbb{H}$, (see Theorem 1.8 from JSA.I), and let q be a natural number grater than p. Then, to every set I_p of all idempotents of $H_{p+q}(\mathbb{H})$ with length p corresponds, by Φ (see Proposition 2.25), the set $\left\{ I_{rp} \right\}_{r=1,2,\ldots,p}$, where I_{rp} is the set of all idempotents of $H_{pq}(\mathbb{H})$ with length rp.

Comments. Taking into account that I_p (resp., I_{rp}) in Proposition 2.26 coincides with the set of all Hermitian $(p+q) \times$ $\times$ $(p+q)$-matrices of trace p (resp., $(pq \times pq)$-matrices of trace rp), it would be interesting to find algebraic properties of Φ.

7. Finally, let us mention the use of Jordan algebra structures in the explicit description of the orbit structure of all irreducible compact Hermitian symmetric spaces $X_c = G_c/K$ under the action of the corresponding noncompact form G_o of the compact Lie group G_c. DRUCKER's monograph [49 b] contains a unified detailed exposition of the results of WOLF [205] for the classical spaces X_c and DRUCKER [49 a] for the two exceptional spaces.

§ 3. Symmetric space of idempotents in a real Jordan algebra

In this section we shall deal with some of the results established by NEHER [125].

Notation. Let $\mathcal{A}$ be a real Jordan algebra with unit element e. Consider the Peirce reflection acting on Idemp ($\mathcal{A}$) $\times$ $\times$ Idemp ($\mathcal{A}$), with values in Idemp ($\mathcal{A}$), and denote it by σ, i.e. $\sigma(c,d) = P(2c-e)\, d =: c.d$, where $c,d \in$ Idemp ($\mathcal{A}$). That σ is well defined can be easily proved by making use of Proposition 1.2, JSA.I.

Theorem 3.1. a) Let d be an idempotent of $\mathcal{A}$ and let

$$d = d_0(c) + d_{1/2}(c) + d_1(c)$$

be the decomposition of d given by the Peirce decomposition of $\mathcal{A}$ with respect to c. Then

$$c.d = d_0(c) - d_{1/2}(c) + d_1(c).$$

b) Idemp ($\mathcal{A}$) is a reflection space (in the sense of LOOS [106 b]) with respect to the multiplication given by σ, i.e. for c, $c_i \in$ Idemp ($\mathcal{A}$), the following identities hold:

$$c.c = c$$
$$c_1.(c_1.c_2) = c_2,$$
$$c_1.(c_2.c_3) = (c_1.c_2).(c_1.c_3).$$

Remark. Theorem 3.1. holds in a more general setting, namely for Jordan algebras with unit element over a field of characteristic different from two (see NEHER [125, Chapter I, Theorem 2.1]).

As a real vector space, the Jordan algebra $\mathcal{A}$ is a differentiable manifold.

Theorem 3.2. Suppose that Idemp ($\mathcal{A}$) is endowed with the induced topology from $\mathcal{A}$. Then Idemp ($\mathcal{A}$) is a symmetric space (in the sense of the definition from § 2) with respect to the multiplication given by σ. As a manifold, Idemp ($\mathcal{A}$) is a regular submanifold of $\mathcal{A}$.

Remark. As was proved by NEHER [125, Chapter I, Theorem 3.5], the set of all idempotents of $\mathcal{A}$ with given reduced trace t ($0 \leq t \leq$ degree of $\mathcal{A}$), as well as, the set of all idempotents of given length s ($0 \leq s \leq$ primitive degree of $\mathcal{A}$), are symmetric subspaces of Idemp ($\mathcal{A}$).

Notations. For every c of Idemp ($\mathcal{A}$) we denote by Y_c the connected component of c in Inv ($\mathcal{A}_1(c)$) with respect to the induced topology from $\mathcal{A}$. For a regular submanifold I of Idemp ($\mathcal{A}$) we put $X(I) := \bigcup_{c \in I} Y_c$ and

$$E_\nu(I) := \{(c,x) \mid (c,x) \in I \times \mathcal{A},\ x \in \mathcal{A}_\nu(c)\},\ \nu = 0,\ 1/2, 1.$$

Theorem 3.3. For $\nu = 0,\ 1/2,\ 1$, the map $pr_1 : E_\nu(I) \longrightarrow I$, given by $pr_1(c,x) := c$, is a regular subbundle of the trivial vector bundle $pr_1 : I \times \mathcal{A} \longrightarrow I$, and $I \times \mathcal{A} = E_0(I) \oplus E_{1/2}(I) \oplus E_1(I)$.

Theorem 3.4. Let I be a symmetric subspace of Idemp ($\mathcal{A}$) such that $\sigma(I,I) \subset I$. Put $F(I) := \{(c,x) \mid (c,x) \in E_1(I),\ x \in Y_c\}$. Then $pr_1 : F(I) \longrightarrow I$ is a fibre bundle and F(I) is an open submanifold of $E_1(I)$.

Notation. Denote by $\pi(I)$ the map $\pi(I) : X(I) \longrightarrow I$ defined by $\pi(I)(x) := c$, where c is the index of the component Y_c containing x.

Theorem 3.5. Let I be a symmetric subspace of Idemp ($\mathcal{A}$) such that $\sigma(I,I) \subset I$. Then $\pi(I) : X(I) \longrightarrow I$ is a fibre bundle and X(I) is a submanifold of $\mathcal{A}$. The map $pr_2 : F(I) \longrightarrow X(I)$, given by $pr_2(c,x) := x$, is a fibre bundle isomorphism (for the definition of F(I) see Theorem 3.4).

Remark. For the case when $\mathcal{A}$ is formally real, NEHER has given [125, Chapter II, § 2] a detailed descriptions of the fibres from X (Idemp ($\mathcal{A}$)). Several results obtained by HIRZEBRUCH [82 b,IV,4] for simple formally real Jordan algebras were rediscovered, by Neher but by using different methods.

Note. From now on we will assume that I is an open submanifold of Idemp ($\mathcal{A}$), so that $\sigma(I,I) \subset I$ and c an element of I for which $\mathcal{A}_{1/2}(c) \neq \{0\}$.

Theorem 3.6. If we denote by Exp the exponential map of the symmetric space I at the point c, then, for every v of $\mathcal{A}_{1/2}(c)$ (canonically identified with the tangent bundle T_cI) we have

$$Exp(v) = \frac{e}{2} - \frac{1}{2}(e-2c)\cos 2v + \frac{1}{2}\sin 2v,$$

where $\frac{e}{2} - \frac{1}{2}(e-2c)\cos 2v \in \mathcal{A}_0(c) \oplus \mathcal{A}_1(c)$ and $\frac{1}{2}\sin 2v \in \mathcal{A}_{1/2}(c)$.

Remark. If $\mathcal{A}$ is a simple formally real Jordan algebra and $I = Idemp_1(\mathcal{A})$, the results given by HIRZEBRUCH in [82 b] (see also § 1) are obtained.

Theorem 3.7. Let u,v,w be three nonzero elements of $\mathcal{A}_{1/2}(c) (=T_cI)$. Then

a) $Exp(tv) = (c - cv^2 + \frac{v^2}{2}) + (\cos 2t)(cv^2 - \frac{v^2}{2}) + \frac{1}{2}(\sin 2t)v$ is equivalent to $v^3 = v$. ~~In this case the geodesic through c in~~ the direction of v has the shape of an ellipse.

b) $Exp(tu) = (c - cu^2 - \frac{u^2}{2}) + (ch\ 2t)(cu^2 - \frac{u^2}{2}) + \frac{1}{2}(sh\ 2t)u$ is equivalent to $u^3 = -u$. In this case the geodesic through c in the direction of u has the shape of a hyperbola.

c) $Exp(tw) = c + t^2(w^2 - 2cw^2) + tw$ is equivalent to $w^3 = 0$.

In this case the geodesic through c in the direction of w has the shape of a parabola (in the case $w^2 = 0$ it is a line).

Definitions. A geodesic of I is called elliptic, hyperbolic, or parabolic if, by a linear transformation of the parameter, its shape becomes as in a), b), or c), Theorem 3.7, respectively. A geodesic is called elementary if it is elliptic, hyperbolic, or parabolic.

Theorem 3.8. Let γ be a geodesic of I. Then γ is elementary if and only if the dimension of the algebra $\mathcal{A}(\gamma(0), \dot{\gamma}(0))$ generated by $\gamma(0)$ and $\dot{\gamma}(0)$ is at most four.

Definitions. Let α be an involutive automorphism of a Jordan algebra $\mathcal{a}$. If the bilinear form $(u,v) := RS^{\mathcal{a}}(u, \alpha(v))$, $u,v \in \mathcal{a}$, is positive definite, then α is called a Cartan involution of $\mathcal{a}$ 1). If α is a Cartan involution of a Jordan algebra $\mathcal{a}$, then we write $\mathcal{a}_{\pm} := \{ x \mid x \in \mathcal{a},\ \alpha(x) = \pm x \}$. It follows that $\mathcal{a} = \mathcal{a}_+ \oplus \mathcal{a}_-$. This is called a Cartan decomposition of $\mathcal{a}$.

Theorem 3.9. Let γ be an elementary geodesic of I. If we write $\gamma(0) =: c$, $\dot{\gamma}(0) =: v$, and denote by $\mathcal{A}(c,v)$ the algebra generated by c and v, then:

a) γ is elliptic if and only if the reduced trace $RS^{\mathcal{A}(c,v)}$ of $\mathcal{A}(c,v)$ is positive definite (i.e. $\mathcal{A}(c,v)$ is formally real);

b) γ is hyperbolic if and only if $P(2c-e)\big|_{\mathcal{A}(c,v)}$ is a Cartan involution of $\mathcal{A}(c,v)$;

c) γ is parabolic if and only if $\mathcal{A}(c,v)$ has nonzero radical.

Remark. In case c) of Theorem 3.9, unlike the cases a) and b), the algebra $\mathcal{A}(c,v)$ is not semisimple.

1) $RS^{\mathcal{a}}$ denotes the reduced trace ("reduzierte Spur") of $\mathcal{a}$, a generalization of the linear trace form of $\mathcal{a}$ (see BRAUN and KOECHER [24, p.827).

Theorem 3.10. Let α be a Cartan involution of $\mathcal{A}$ which fixes a point c of I. Put

$$\mathcal{B}_- := (\mathcal{A}_0(c) \cap \mathcal{A}_+) \oplus (\mathcal{A}_{1/2}(c) \cap \mathcal{A}_-) \oplus (\mathcal{A}_1(c) \cap \mathcal{A}_+),$$

where $\mathcal{A}_\pm$ are as in the definitions above. Then the geodesics of I are elliptic or hyperbolic, namely: elliptic (resp.,hyperbolic) when the dimension of the symmetric space $I \cap \mathcal{A}_+$(resp.,$I \cap \mathcal{B}_-$) at the point c is not zero.

Corollary. If $\mathcal{A}$ is semisimple, then the geodesics of I are elliptic or hyperbolic.

Proposition 3.11. If $\mathcal{A}$ is formally real, then every geodesic of $\mathrm{Idemp}_1(\mathcal{A})$ is elliptic.

Note. From now on we assume that $\mathcal{A}$ is semisimple and thus $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_k$, where $\mathcal{A}_i$ $(1 \leq i \leq k)$ are simple ideals of $\mathcal{A}$. Hence, for every x of $\mathcal{A}$, we have $x = x_1 \oplus \ldots \oplus x_k$, $x_i \in \mathcal{A}_i$. In addition we will assume that I is connected.

Theorem 3.12. The symmetric space I is the direct product of its subspaces $I \cap \mathcal{A}_i$, $i=1,2,\ldots,k$, i.e. $I = (I \cap \mathcal{A}_1) \times \ldots \times (I \cap \mathcal{A}_k)$ as manifolds, and $c.d = (c_1.d_1) \ldots (c_k.d_k)$.

Theorem 3.13. If $\mathcal{A}$ is simple, then

a) I is a symmetric Riemannian space only if at least one of the following conditions is fulfilled:

(i) $\mathcal{A}$ is formally real;

(ii) the mutation $\mathcal{A}_{2c-e}$ is formally real;

(iii) $\mathcal{A}$ is not central simple, $\dim_{\mathbb{R}} \mathcal{A} = 6$, and the primitive degree of $\mathcal{A}$ is two;

(iv) $\mathcal{A}$ is central simple, but not reduced, $\dim_{\mathbb{R}} \mathcal{A} = 10$, and the primitive degree of $\mathcal{A}$ is two;

b) I is compact if and only if $\mathcal{A}$ is formally real.

Corollary. Suppose that $\mathcal{A}$ is semisimple and that I is a symmetric Riemannian space. Then I is of rank one if and only if all geodesics of I are elementary.

Comments. Consider on the Grassmann manifold $G_p(H^{p+q})$ a point given by its local Pontryagin coordinates. Suppose that this point describes a geodesic passing through it. An interesting problem is to characterize the image of a current $\Theta := (d\xi^k_i)$, i=1,2, ...,p, k=1,2, ...,q under the map Φ (defined in Proposition 2.25) in terms of the rank of Θ.

In the opinion of GÜNAYDIN [72], the construction of symmetric Riemannian spaces over Jordan algebras can be extended so as to define and realize supersymmetric spaces over Jordan superalgebras. As he mentioned in the Introduction to [72], "we are presently working on the realization of this program and their physical applications".

§ 4. Central simple real Jordan algebras and spaces with constant affine connection.

A simple method for associating a space with constant affine connection to every real finite-dimensional algebra by taking the structure constants as coefficients of the connection was given by MOISIL and VRANCEANU in 1958 (for details, see [198 d] and was subsequently developed by VRANCEANU [198 e]. This method has been extensively used by Vranceanu and his followers. In 1966, Vranceanu suggested, independently of KOECHER [93 e] (see also § 2, Theorem 2.2), to some of his co-workers the study of spaces with constant affine connection associated to finite-dimensional real Jordan algebras. Hence, spaces with constant affine connection (for definition and details, see VRANCEANU [198 b, Chapter VI, § 4, and also 198 d]) associated to real Jordan algebras of types A–D (see JSA.I, § 1) have been studied by TURTOI [189 a], POPOVICI and TURTOI [145] and IORDANESCU [84 a–d]. (All these results have been presented more systematically in [144]).

The first objective of this study was to investigate reducibility of the system of infinitesimal operators

$X_{k\ell} := \Gamma^{i}_{jk\ell}\, x^{j} \dfrac{\partial}{\partial x^{i}}$ corresponding to the spaces with constant affine connection Γ^{i}_{jk} associated with Jordan algebras of type A–D[1].

<u>Definition.</u> A space A_n with affine connection is called <u>irreducible</u> (<u>reducible</u>) at a point P if the system of operators $X_{k\ell}$ acting on the tangent space of A_n at P is irreducible (reducible). If A_n is irreducible at all its points, it is called irreducible.

<u>Theorem 4.1.</u> The spaces A_n with constant affine connection associated to real Jordan algebras of types A, B, C or D are reducible, but admit distributions of rank n-1 which are invariant and irreducible with respect to $X_{k\ell}$.

<u>Remark.</u> The distributions from Theorem 4.1 have a purely algebraic construction.

<u>Comments.</u> All considerations made in [144] are of <u>local</u> type, and so a global treatment need to be performed.

The second objective of this study was to find which spaces with constant affine connection associated to real Jordan algebras of types A–D admit a (pseudo-) Riemannian metric.

All considerations made were of a <u>local</u> nature and the algebraic tools employed (for types A–C) were maximal gradings

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1) As usual, $\Gamma^{i}_{jk\ell}$ denote the components of the [illegible]

(i.e. with one-dimensional sumands). POPOVICI has reconsidered the notion of space with constant affine connection in a global setting and has generalized [143 a] the results from [144] concerning the existence of (pseudo-) Riemannian metrics. In particular, he proved

Theorem 4.2. Let g be a nonnull symmetric tensor of type (0,2) which is parallel with respect to the constant affine connection associated to a real Jordan algebra $\mathcal{J}$ of type A-E. Then $\mathcal{J}$ is of type D and g is a (pseudo-)Riemannian metric.

Definition. A (pseudo-)Riemannian space whose metric has the form

$$ds^2 = e^{a_k x^k} c_{ij}\, dx^i\, dx^j, \quad (i,j,k = 1,2, \ldots, n),$$

a_k, c_{ij} being constants, is called a Vagner space.

Recall the following result (see VRANCEANU [198 e]): Let α be a nonzero real number and let V_n be a Vagner space whose vector $(a_1, \ldots, a_n)$ is $\neq 0$. Then there exist a real number β and a linear transformation $y^i = a^i_j x^j$ such that the metric of V_n is given by

$$ds^2 = \beta e^{\alpha y^1} \left[(dy^1)^2 + \varepsilon_2 (dy^2)^2 + \ldots + \varepsilon_n (dy^n)^2 \right],$$

$$\varepsilon_i = \pm 1, \quad i = 2,3, \ldots, n,$$

or by

$$ds^2 = \beta e^{\alpha y^1} \left[2dy^1 dy^2 + \varepsilon_3 (dy^3)^2 + \ldots + \varepsilon_n (dy^n)^2 \right],$$

$$\varepsilon_i = \pm 1, \quad i = 3,4, \ldots, n,$$

according to whether the vector $(a_1, \ldots, a_n)$ is nonisotropic or isotropic with respect to the metric $d\sigma^2 = c_{ij} dx^i dx^j$.

Theorem 4.3. The class of all tensors g from Theorem 4.2 is in one-to-one correspondence with the class of nonisotropic

Vagner spaces. Moreover, g is determined, up to a constant factor, by the corresponding constant connection Γ^i_{jk}.

Remark. For a local description (motion groups, geodesics, etc.) of nonisotropic Vagner spaces see [144], and for a global description see POPOVICI [143 b].

Comments. VRANCEANU [198 a] proved the following theorem: If a Riemannian space is irreducible then its metric is determined, up to a constant factor, by the connection Γ^i_{jk}. TELEMAN [184 b] proved that, under the same hypothesis, the metric is determined, up to a constant factor, by the curvature tensor Γ^i_{jkl}. The nonisotropic Vagner spaces are reducible and with metrics (which may or may not be positive definite) determined, up to a constant factor, by the connection Γ^i_{jk}. Thus they are examples of (pseudo-)Riemannian spaces satisfying the above-mentioned theorems of VRANCEANU and TELEMAN, but under weaker hypotheses.

Open problem. To reconsider the above-mentioned results in view of Theorem 2.2, and, to subsequently generalize Vranceanu and Teleman's theorems on the uniqueness of the metric.

Remark. As ANASTASIEI [4] has righteously noted, KULKARNI's results [100] could be useful in solving the second part of the open problem above.

Comments. Some authors have determined the automorphism groups of simple and quasi-simple Jordan algebras of types A-E over particular fields (see, FREUDENTHAL [56], SPRINGER and VELDKAMP [174], ROZENFEL'D and KARPOVA [154], PERSITS [136 a,b]; more general results for simple Jordan algebras over fields of zero characteristic follow from BRAUN and KOECHER's book [24 , pp.282-286]). As has been proved by

ROZENFEL'D and ZAMANOVSKI [156], the orbits of the automorphism groups of simple and quasi-simple Jordan algebras of types A,B,C, or E give models of non-Euclidean, quasi-non-Euclidean, projective, symplectic, and quasi-symplectic spaces. ZAMANOVSKI has considered [209] Jordan algebras of the remainder type D, has proved that the orbits of the automorphism groups give models of non-Euclidean spaces, and that the spinor representation of the group of motions of the non-Euclidean spaces obtained in this way corresponds to the representation of a Jordan algebra of type D in a Clifford algebra[1].

Open problem. Connect the above-mentioned results of Zamanovski to those of Iordanescu-Popovici-Turtoi in this section.

Comments. TURTOI [189 b]has recently considered algebraic bundles having a central simple Jordan algebra of type D as standard fibre. Then she studied idempotent sections of this kind of bundles and, making use of a decomposition theorem of Peirce type , identified a differential distribution on the ground manifolds.

PREMA and KIRANAGI [147] constructed a Lie algebra bundle for a given locally trivial Jordan bundle in which each fibre has a unit element. Then, they proved the rigidity theorem for Jordan algebras, which is of independent interest, to establish the local triviality of a semisimple Jordan bundle. They also proved the unique decomposition theorem for a semisimple Jordan bundle using an ideal bundle of the Lie bundle constructed and this generalizes the Albert's theorem on semisimple Jordan algebras.

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1 A more general case of modelling symmetric spaces as orbits of arbitrary groups of transformations of Jordan algebras was considered by SEMYANISTII [169].

Finally, let us mention that PICCINNI [138 a] extended the Chern-Weil theorem on the representation of characteristic classes to $\mathbb{H}$ by means of curvature forms. This extension is based on algebraic results previously established [138 b], where a kind of Diendonné determinant is defined. Using some of the information furnisched by IORDANESCU [84 f], PICCINNI [138 c] illuminated an interesting connection between his determinant and the central simple (finite-dimensional) Jordan algebras.

§ 5. Domains of positivity and applications to relativity

Now we survey the principal facts about domains of positivity and their relations to Jordan algebras (see BRAUN [23 b] and KOECHER [93 b,c,e]) and we consider some applications to Minkowski space (see TILGNER [185]). Some of CHEREMISIN's results on causal symmetric spaces (see [32 a,b]) are also presented.

Recall that domains of positivity were introduced by KOECHER [93 a] in a generalization of the cone of positive definite matrices studied by SIEGEL [171] and others. Another important class of such domains is given by the cone of positive definite Hermitian matrices (see BRAUN [23 a]).

We begin with a brief survey of the principal facts about domains of positivity and their relations to Jordan algebras, resorting to some of TILGNER's formulations [185].

Notation. Denote by X a real vector space of finite dimension n and by ν a nondegenerate symmetric bilinear form on it.

Definition. A subset Y of X is said to be convex if $x, y \in Y$ and $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, imply that $\alpha x + (1-\alpha)y \in Y$.

Definition. A subset Y of X is called a cone if $x \in Y$ and $\alpha \in \mathbb{R}$, $\alpha > 0$, imply that $\alpha x \in Y$.

Notation. For $a \in X$ consider the map $a \longrightarrow a^{\nu}$ given by $a^{\nu}(x) := \nu(a,x)$.

Remark. The map $a \longrightarrow a^{\nu}$ is an isomorphism of X onto its dual space X^{*}.

Definition. Let Y be an open convex cone in X. Then Y^{*} defined by

$$Y^{*} := \left\{ f \;\middle|\; f \in X^{*},\ f(x) > 0 \text{ for all } x \in \bar{Y},\ x \neq 0 \right\},$$

is an open convex cone in X^{*}, called the dual cone of Y. (By $\bar{Y}$ we denote the closure of Y in the given topology on X.)

Definition. The image of Y^{*} in X by the inverse isomorphism $a^{\nu} \longrightarrow a$ is $Y^{\nu} = \left\{ x \;\middle|\; x \in X,\ \nu(x,y) > 0 \text{ for all } y \in \bar{Y},\ y \neq 0 \right\}$. It is called the ν-dual cone of Y.

Remark. Obviously, Y is called self-dual if $Y = Y^{\nu}$. However, not for every open convex cone there exists a bilinear form with respect to which it is self-dual.

Definition. An open convex cone Y with $Y^{*} \neq \emptyset$ is called a domain of positivity in X with respect to ν if it is self-dual.

Notation. A domain of positivity in X with respect to ν will be denoted by $Pos(X,\nu)$.

Theorem 5.1. An open nonempty subset Y of X is a domain of positivity with respect to ν if and only if

(a) $x,y \in Y$ implies that $\nu(x,y) > 0$;

and

(b) $\nu(x,y) > 0$ for all $y \in \bar{Y}$, $y \neq 0$, implies that $x \in Y$.

Theorem 5.2. If Y is a domain of positivity then

(a) $x \in Y$ if and only if $\nu(z,x) > 0$ for all $z \in \bar{Y}$, $z \neq 0$;

and

(b) $x \in \bar{Y}$ if and only if $\nu(z,x) \geq 0$ for all $z \in Y$.

Notation. Let Y be an open convex cone in X with $Y^* \neq \emptyset$. A partial order relation in X is introduced as follows: $x \leq y$ if and only if $y-x \in \bar{Y}$.

Remark. The relation $\leq$ is Archimedean, and is compatible with the vector space structure of X; any ϑ on X is monotone.

Definition. Two points $x, y \in \bar{Y}$ are called equivalent if there exist $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, with $x \leq \beta y$ and $y \leq \alpha x$.

Remark. The Archimedean property of the relation $\leq$ implies that Y itself is a full equivalence class. In general, its boundary decomposes into several classes.

Definition. For an open convex cone Y in X, a transformation $A \in GL(X, \mathbb{R})$ is called an automorphism of Y if $A(Y)=Y$. The set of all automorphisms of Y is denoted by Aut Y.

Remark. Aut Y is a closed subgroup of $GL(X, \mathbb{R})$ and Aut $\bar{Y}$ = Aut Y.

Definition. For $A \in GL(X, \mathbb{R})$ the adjoint transformation A^{ϑ} of A with respect to ϑ is defined by $\vartheta(A^{\vartheta}(x), y) = \vartheta(x, A(y))$.

Remark. $(Y^{\vartheta})^{\vartheta} = Y$ implies that Aut $(Y^{\vartheta}) = (\text{Aut } Y)^{\vartheta}$. Hence, in a domain of positivity $A \in \text{Aut Pos}(X, \vartheta)$ implies that $A^{\vartheta} \in \text{Aut Pos}(X, \vartheta)$.

Theorem 5.3. $\text{Pos}(X, \vartheta)$ is a domain of positivity with respect to the nondegenerate bilinear form τ if and only if there exists an $A \in \text{Aut Pos}(X, \vartheta)$ with $A^{\vartheta} = A$ and $\tau(x,y) := \vartheta(A(x), y)$.

Remark. There may be several bilinear forms for which Y is a domain of positivity.

Given $t \in X$ with $\vartheta(t,t) = 1$, the bilinear form $\tau(x,y) := 2\vartheta(t,x)\vartheta(t,y) - \vartheta(x,y)$ is symmetric and nondegenerate. The subset $Y := \{ y \mid y \in X,\ \vartheta(t,y) > 0,\ \vartheta(y,y) > 0 \}$ of X is open and not empty since it contains t. Choosing a basis in X in which the matrix of ν is diagonal, it is easy to see that sign $\nu=(n_1,n_2)$ if and only if sign $\tau=(n_2+1,\ n_1-1)$. Thus τ is positive definite if and only if ν has Lorentz signature; in this case Y is the interior of the forward light cone and Y is a domain of positivity for ν and τ. (The boundary of Y is the forward light cone.)

Remark. Aut Y is exactly the group which preserves the order relation $\leq$. This is Zeeman's result [115, p.101], [172 , p.2], which expresses the idea of causality in Minkowski space.

Let $\mathcal{A}$ be a formally real Jordan algebra of finite dimension. Denote by e its unit element and by λ its trace form (see § 1, JSA.I.)

Theorem 5.4. $Inv_0(\mathcal{A})$ is a homogeneous open convex cone that is self-dual with respect to λ.[1]

Theorem 5.5. $Pos(\mathcal{A}, \lambda) = \{\exp x \mid x \in \mathcal{A}\} = \{x^2 \mid x \in Inv\ (\mathcal{A})\} =$

$= \{y \mid y \in \mathcal{A},\ L(y)$ positive definite with respect to $\lambda\} =$

$=$ component of e of $\{y \mid y \in \mathcal{A}, P(y)$ positive definite with respect to $\lambda\}$,

and $\overline{Pos}\ (\mathcal{A}, \lambda) = \{x^2 \mid x \in \mathcal{A}\} =$

$= \{y \mid y \in \mathcal{A}\ ,\ L(y)$ positive semidefinite with respect to $\lambda\}$.

Theorem 5.6. If Y is a homogeneous domain of positivity in a real vector space X, then there exists a formally real Jordan algebra $\mathcal{O}$ in X such that $Y = Inv_0(\mathcal{O})$.

Consider again a real vector space X of finite dimension endowed with a nondegenerate symmetric bilinear form ν.

Definition. The vector space X endowed with the product $xy := \nu(x,t)y + \nu(y,t)x - \nu(x,y)t$ becomes a Jordan algebra for every $t \in X$. Assuming that $\nu(t,t) = 1$, t is the unit element of this Jordan algebra which is denote by $\mathcal{A}(X, \nu, t)$.

1) If $Y \neq \emptyset$ is an open convex cone, then Y is called homogeneous if Aut Y acts transitively on Y.

Remark 1. $\mathcal{A}(X, \nu, t)$ is semi-simple and also formally real if and only if ν has Lorentz signature. In fact, $\mathcal{A}(X, \nu, t)$, which appears in BRAUN & KOECHER [24, Chapter VI, § 5, p.193], is central simple.

Remark 2. If ν has Lorentz signature, then (and only then) the interior of the forward light cone is the component of t in this space and hence the (homogeneous) domain of positivity of the Jordan algebra $\mathcal{A}(X, \nu, t)$.

Remark 3. KOECHER's construction of a formally real Jordan algebra for every domain of positivity can be applied to ω-domains (a generalization of domains of positivity[1]). The resulting Jordan algebra is semi-simple. Conversely, $Inv_o(\mathfrak{a})$, $\mathfrak{a}$ a semi-simple Jordan algebra with unit element e, is such a ω-domain. An ω-domain is a domain of positivity if and only if it is convex. Applying this construction to the component of some t in X (endowed with ν) with $\nu(t,t)=1$ in the interior of the null cone $\{y \mid y \in X, \nu(y,y) = 0\}$, we obtain the Jordan algebra $\mathcal{A}(X, \nu, t)$. Conversely, $Inv_o(\mathcal{A}(X, \nu, t))$ is an example of an ω-domain, which is convex if and only if ν has the Lorentz signature.

In 1966, ROTHAUS [153 a] used VINBERG's results [194] on left-symmetric algebras to show that every homogeneous convex cone can be constructed from such cones of lower dimension. However, no description is given for the infinitesimal automorphism of the cone with respect to its low dimensional constituents [153 a] and, on the other hand, this construction is not unique. In 1979, DORFMEISTER [44 b] used ROTHAUS' [153 b] and KOECHER's results [93 f], ← those obtained by himself and KOECHER [45 a, b], and his own results [44 a] to show how to build up a homogeneous convex cone from lower-dimensional ones in a unique way[2].

1) See BRAUN and KOECHER [24, Chapter VI, § 8].

2) He showed how to construct each homogeneous cone inductively, using domains of positivity as building blocks. This construction turned out to be unique.

The infinitesimal automorphisms of such a cone and its associated left-symmetric algebra can be described in the lower-dimensional constituents of the cone. Several equivalent conditions for a cone to be self-dual have been given. DORFMEISTER's investigations on homogeneous convex cones are summed up in [44 c], in which also a classification of homogeneous convex cones is given.

Notation. Let V be a finite-dimensional real vector space and let A be a Jordan algebra on it.

Definition. The Jordan algebra A defines a Jordan structure on a regular cone (i.e. a nonempty open convex cone not containing an entire straight line) C in V if all left multiplications of A are infinitesimal automorphisms of C.

In [153 c], ROTHAUS showed that A defines a Jordan structure on C if and only if the quadratic vector field given by A is an infinitesimal automorphism of the tube domain associated with C.

Definition. A Jordan algebra is called ordered if the interior C of the cone generated by squares is a regular cone.

ROTHAUS [153 c] established a close connection between regular cones with a Jordan structure and ordered Jordan algebras namely.

Theorem 5.7. The cone C of an ordered Jordan algebra A has the Jordan structure given by A, and, conversely, in essence all regular cones with a Jordan structure arise from ordered Jordan algebras.

Another result [153 c] is that each homogeneous regular cone is the cone of an ordered Jordan algebra.

Remark. As was observed by Rothaus, it must be possible to generalize his results to Siegel domains of the second kind by using some results of KAUP [87].

In a study of the relations between multilattice theory and space-time geometry, CHEREMISIN [32 a] related partially ordered Jordan algebras (for a description, see, CHEREMISIN [32 c]) to space-time geometry.[1] Let us mention that BENADO proved [14] that the Minkowski world is a multilattice.

Definition. Let X be a topological space in which no one element subset is open, and suppose that the set X is endowed with a partial order relation $\leq$. If, for every x of X, the sets $X_x^+ := \{ y \mid y \in X, x \leq y \}$ and $X_x^- := \{ y \mid y \in X, y \leq x \}$ are closed domains, then X is called a causal space.

Definition. A causal space which is a partially ordered real vector space of finite dimension and with the simplicial topology as its topology is called a causal vector space.

Definition. A causal space X endowed with a multiplication satisfying the conditions

(i) $x \circ x = x$,

(ii) $x \circ (x \circ y) = y$,

(iii) $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$

(iv) $x \circ X_y = X_{x \circ y}$, $x \circ X_y^o = X_{x \circ y}^o$,

for all x,y,z from X, where $X_y := X_y^+ \cup X_y^-$ and $X_y^o := X - X_y$, is called a reflection causal space.

Notation. Consider the Jordan algebra $\mathcal{A}(V, \nu, t)$, where V is a causal vector space of dimension $n \geq 3$ and ν has Lorentz signature, and denote it by $\mathcal{A}$. Define in V a (nonreflexive) order relation $\prec$ by: $x \prec y$ if and only if $y \in$ Int (V_x^+), where Int (V_x^+)

1) By space-time geometry, Cheremisin understands the geometry of the ground space of general relativity and its generalizations to arbitrary dimensions.

denotes the interior of V_x^+. As a topological space, Inv($\mathcal{A}$) has three connected components, which will be denoted as follows:

$$I^+(\mathcal{A}) := \{x \mid x \in \mathcal{A},\ 0 < x\},\quad I^-(\mathcal{A}) := \{x \mid x \in \mathcal{A},\ x < 0\},\ \text{and}$$
$$I^0(\mathcal{A}) := \mathcal{A} - (\overline{I^+(\mathcal{A})} \cup \overline{I^-(\mathcal{A})}).$$

Theorem 5.8. $I^+(\mathcal{A})$, $I^-(\mathcal{A})$, and $I^0(\mathcal{A})$ are reflection causal spaces. If $\mathcal{A}$ is of rank 4, then $I^+(\mathcal{A})$ and $I^-(\mathcal{A})$ are symmetric Riemannian spaces with respect to KOECHER's metric ([93 e] and, also, § 2 above, while $I^0(\mathcal{A})$ is a symmetric pseudo-Riemannian space.

Remark. Inv ($\mathcal{A}$) is not a reflection causal space.

Recently, CHERMISIN [32 b] has given a description of the partially ordered (with respect to maximal orders) formally real Jordan algebras of finite dimension and the corresponding causal symmetric spaces (for a definition, see below). Such algebras are finite direct sums of formally real algebras $\mathcal{A}$ and Jordan algebras of J - symmetric matrices, J being the standard involution, $H_n(\mathbb{R})^+$, $H_n(\mathbb{C})^+$, $H_n(\mathbb{H})^+$, $H_3(\mathbb{O})^+$.

Definition. A causal space X which is also a symmetric space (the multiplication of which is denoted by $\circ$) so that $x \circ X_y = X_{x \circ y}$ and $x \circ X_y^0 = X_{x \circ y}^0$ for all $x, y \in X$, where $X_y := X_y^+ \cup X_y^-$, $X_y^0 := X - X_y$, is called a causal symmetric space.

Theorem 5.9. The connected causal symmetric spaces given by the partially ordered formally real Jordan algebras of finite dimension are finite cartesian products of connected components of the sets of invertible elements of algebras $\mathcal{A}$, $H_n(\mathbb{R})^+$, $H_n(\mathbb{C})^+$, $H_n(\mathbb{H})^+$, $H_3(\mathbb{O})^+$.

Comments. It would be interesting to use results of HELWIG and NEHER on geodesics (see [80 c] and [125], and §§ 2,3 above) in order to obtain new geometrical properties of causal symmetric spaces from Cheremisin's studies [32 ab].

Remark. In [32 d], CHEREMISIN has given a complete description of "Jordan kinematics".

§ 6. A characterization of quasi-symmetric domains in terms of curvature

ZELOW [211 b] has characterized the quasi-symmetric domains (i.e, domains biholomorphic to quasi-symmetric Siegel domains in the sense of SATAKE [167]) among the bounded homogeneous domains. For this he used PYATETSKII-SHAPIRO's [149] description of bounded homogeneous domains in terms of so-called "j-algebras". In [211 a] the j-algebra conditions are translated into more geometric conditions (involving the curvature of the Bergman metric).

We first recall [211 a] (see also [149] and [211e]) some notation and results connected with j-algebras, and the associated description of bounded homogeneous domains.

Let D be a bounded homogeneous domain, which is assummed to be indecomposable (i.e. not to be a product of other bounded homogeneous domains). There exists a simply connected subgroup $\mathcal{G}$ of Aut D, with Lie algebra $\mathfrak{g}$, acting simply transitively on D by holomorphic automorphisms. Choose a base point in D and denote it by o; the Bergman metric on D gives an invariant metric on $\mathcal{G}$, and, in particular, we have a metric on $\mathfrak{g}$ that can be written in the form

$$\langle X, Y\rangle = \mathrm{Re}\ h(X, Y) = \omega\left[jX, Y\right] ,$$

where h is the Bergman metric at σ, j is the complex structure on the tangent space $T_\sigma D$ (identified with $\mathfrak{g}$), and ω is a linear form on $\mathfrak{g}$ (see [149]).

The invariant metric on $\mathcal{G}$ (translation of $\langle , \rangle$ by left actions) defines a Riemannian connection ∇ on $\mathcal{G}$ (identified with $\mathfrak{g}$), and by considering elements of $\mathfrak{g}$ as left invariant vector fields on $\mathcal{G}$, at the point σ we have

$$2 \langle \nabla_X Y, Z \rangle = - \langle X, [Y,Z] \rangle - \langle Y, [X,Z] \rangle + \langle Z, [X,Y] \rangle.$$

Then $\nabla_X Y = \frac{1}{2}\{ [X,Y] - (\mathrm{ad}\, X)' Y - (\mathrm{ad}\, Y)' X \}$ is an element of $\mathfrak{g}$ (identified with $T_\sigma D$) for every $X, Y \in \mathfrak{g}$, where $'$ denotes the transpose with respect to $\langle , \rangle$.

The following decomposition holds (see [149]):

$$(6.1) \qquad \mathfrak{g} = \mathfrak{h} \oplus \sum_\alpha k_\alpha = \mathfrak{l} \oplus j\mathfrak{l} \oplus \mathcal{U}.$$

Here $\mathfrak{h} := [\mathfrak{g}, \mathfrak{g}]^\perp$ (orthogonal complement with respect to $\langle , \rangle$), and $[\mathfrak{g}, \mathfrak{g}] = \sum_\alpha k_\alpha$ with

$$k_\alpha = \{ X \mid X \in [\mathfrak{g}, \mathfrak{g}], \ [H,X] = \alpha(H) X, \text{ for any } H \text{ of } \mathfrak{h} \},$$

where the root α is a linear form on $\mathfrak{h}$. Further, if $\alpha_1, \dots, \alpha_p$ are all roots α with $jk_\alpha \subset \mathfrak{h}$, then $\mathfrak{h} = jk_{\alpha_1} \oplus \dots \oplus jk_{\alpha_p}$, $\dim \mathfrak{h} = p$; using a proper enumeration, all roots are of the form α_k, $\frac{1}{2}\alpha_k$ with $1 \leq k \leq p$ and $\frac{1}{2}(\alpha_k \pm \alpha_m)$ with $1 \leq k < m \leq p$. We have

$$jk_{\frac{1}{2}(\alpha_k + \alpha_m)} = k_{\frac{1}{2}(\alpha_k - \alpha_m)} \text{ and } jk_{\frac{1}{2}\alpha_k} = k_{\frac{1}{2}\alpha_k}.$$

If we put

$$k_k := k_{\alpha_k}, \quad k_{(k,\pm m)} := k_{\frac{1}{2}(\alpha_k \pm \alpha_m)} \text{ and } \mathcal{U}_k := k_{\frac{1}{2}\alpha_k},$$

then the second decomposition in (6.1) is given by

$$\ell := \sum_{k=1}^{p} k_k \oplus \sum_{1 \leq k < m \leq p} k_{(k,m)} \text{ and } \mathcal{U} := \sum_{k=1}^{p} \mathcal{U}_k .$$

We have $[k_\alpha, k_\beta] \subset k_{\alpha+\beta}$ ($= \{0\}$ if $\alpha+\beta$ is no root), $k_\alpha \perp k_\beta$ for $\alpha \neq \beta$, $h \perp k_\alpha$, dim $k_k = 1$, and there exists a unique non-zero element $E_k \in k_k$ with $[jE_k, E_k] = E_k$. Also, h is an abelian subalgebra, ℓ an abelian ideal of $\mathfrak{g}$, $j\ell$ a subalgebra, $[\mathcal{U}, \mathcal{U}] \subset \ell$, $[j\ell, \mathcal{U}] \subset \mathcal{U}$, and $[\ell, \mathcal{U}] = \{0\}$.

Making use of the properties of the j-algebra $\mathfrak{g}$, one can prove the following propositions (see ZELOW [211 d], D'ATRI [39 a,c]).

<u>Proposition 6.1.</u> The action of ∇ on $\mathfrak{g}$ satisfies

1) $\nabla_H = 0$ for $H \in h$;

2) $\nabla_Y Y = |Y|^2 \, H_\alpha \in h$ for $Y \in k_\alpha$, where H_α is defined by $\alpha(H) = \langle H, H_\alpha \rangle$ for any H of h ;

3) $\nabla_Y H = -\alpha(H) Y \in k_\alpha$ for $Y \in k_\alpha$, $H \in h$;

4) $\nabla_{E_k} E_\ell = \begin{cases} 0 & \text{if } k \neq \ell , \\ jE_k & \text{if } k = \ell . \end{cases}$

The curvature is then given by the usual formula $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

<u>Remark.</u> If $X, Y \in \mathfrak{g}$, then, by the invariance of ∇, $\nabla_Y Z$, etc. are also elements of $\mathfrak{g}$. Hence one can use Proposition 6.1 repeatedly when calculating the curvature.

<u>Proposition 6.2.</u> $R(X,H) = \alpha(H) \nabla_H$ as operators on $\mathfrak{g}$, where $X \in k_\alpha$, $H \in h$.

Let $Y = \sum_{k=1}^{p} a_k jE_k$ be an element of h, and write

$\varkappa_k := |jE_k|^2 = |E_k|^2$. Then Proposition 6.2 implies that

$$R(Y, jY) = \sum a_k R(E_k, Y) = \sum a_k \alpha_k(Y) \nabla_{E_k},$$

where $\alpha_k(Y) = a_k$ and, by Proposition 6.1, $\nabla_{E_k} E_\ell = \delta_{k\ell}\, jE_k$, where $\delta_{k\ell}$ is the Kronecker symbol. It follows that

$$R(Y, jY)jY = -\sum a_k^3 \; jE_k ,$$

and hence the holomorphic sectional curvature given by Y is

$$(6.2) \qquad K(Y) := \langle R(Y, jY)jY \, , \, Y \rangle = -\sum_{1}^{p} \varkappa_k \, a_k^4 \; ,$$

where $1 = |Y|^2 = \sum_{1}^{p} \varkappa_k \, a_k^2$.

The stationary points of K(Y) on h under the constraint $|Y| = 1$ are obtained from $\nabla(\sum \varkappa_k \, a_k^4) = \lambda \nabla(\sum \varkappa_k \, a_k^3)$, where ∇ denotes the gradient with respect to $(a_1, \ldots, a_p)$, and λ is a Lagrange multiplier. One gets

$$2\varkappa_k \, a_k^3 = \lambda \varkappa_k \, a_k \; , \qquad k = 1, \ldots, p.$$

Then $\lambda = 2\, a_k^2$ for all $a_k \neq 0$, and the corresponding $|a_k|$'s are therefore equal at a stationary point. The converse is also true, hence $(a_1, \ldots, a_p)$ is a stationary point (under the same constraint) if and only if all non-zero $|a_k|$'s are equal.

Consider a stationary point $(a_1, \ldots, a_p)$ with

$|a_{i_1}| = \ldots = |a_{i_t}| =: a \neq 0$ for some sequence $i_1 < \ldots < i_t$ of indices, while the other a_k's are equal to zero, and let

$\varkappa := \min\{\varkappa_1, \ldots, \varkappa_p\}$, then

$$\max_{Y\in h, |Y|=1} |K(Y)| = \varkappa^{-1} \text{ and } \min_{Y\in h, |Y|=1} |K(Y)| = (\varkappa_1 + \ldots + \varkappa_p)^{-1}.$$

In [211e] a condition for quasi-symmetry is

$$\text{(C)} \qquad \varkappa_1 = \ldots = \varkappa_p \quad (= \varkappa).$$

Remark. Condition (C) is satisfied if ond only if $p\varkappa = \varkappa_1 + \ldots + \varkappa_p$. Therefore condition (C) is equivalent to the following condition

$$\text{(C')} \qquad \max_{Y\in h, |Y|=1} |K(Y)| = \dim h \min_{Y\in h, |Y|=1} |K(Y)|.$$

One can see that

$$k_{(a,b)} = \left\{ X \,\middle|\, X\in \mathfrak{g},\ R(X, jE_k)jE_\ell = \begin{cases} -X/4 & \text{if } k,\ell = a,b, \\ 0 & \text{otherwise} \end{cases} \right\},$$

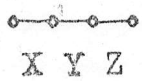
$$\text{(6.3)} \qquad \mathcal{U}_a = \left\{ X \,\middle|\, X\in \mathfrak{g},\ R(X, JE_k)jE = \begin{cases} -X/4 & \text{if } k = \ell = a, \\ 0 & \text{otherwise} \end{cases} \right\},$$

and so if (C') is satisfied a method can be found to find the decomposition $[\mathfrak{g}, \mathfrak{g}] = \sum_\alpha k_\alpha$ in $\mathfrak{g} = h \oplus [\mathfrak{g}, \mathfrak{g}]$ (see [211e , p.5]).

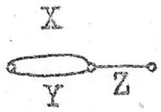
In [211e] ZELOW proved that a necessary condition for quasi-symmetry is that ℓ is a formally real Jordan algebra with the product

$$XY := T_X Y = T_Y X, \text{ where } T_Y := \frac{1}{2} \{ \operatorname{ad} jY + (\operatorname{ad} jY)' \},$$

the transpose ' being with respect to $\langle , \rangle$. It has also been established that this is the case if and only if (under condition (C)) the following conditions hold:

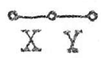
(A) For elements connected as 

we have (XY)Z=X(YZ) ,

(B) For elements connected as 

we have

$$X(YZ) + Y(XZ) = (XY)Z, \text{ i.e. } X(YZ) + Y(XZ) = \frac{1}{4\varkappa}\langle X,Y\rangle Z.$$

(The diagrams show that $X,Y,Z \in \sum_{1\leq k<m\leq p} \mathfrak{k}_{(k,m)}$ and that, for instance, X and Y are connected as 

if $X \in \mathfrak{k}_{(k,m)}$ and $Y \in \mathfrak{k}_{(\ell,n)}$, $\{k,m,\ell,n\}$ being a set of three different letters.)

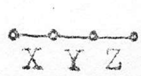
<u>Definition.</u> A subgroup $\mathcal{G}$ of Aut D (i.e. the group of biholomorphic automorphisms of D) which is a simply connected solvable group acting simply transitively on D, and such that ad X has only real characteristic roots for X in the Lie algebra of $\mathcal{G}$, is called a <u>triangular</u> subgroup of Aut D.

By translating his results [211e] into curvature conditions, ZELOW obtains (see [211d , pp.14-15])

<u>Theorem 6.3.</u> Let D be an indecomposable bounded homogeneous domain, and let $\mathcal{G}$ be a triangular subgroup of Aut D with Lie algebra $\mathfrak{g}$. Choose a base point o of D and specify on $\mathfrak{g}$ a j-algebra structure (by the identification $\mathfrak{g} \equiv T_o D$), a complex structure on $T_o D$, and a Bergman metric on $T_o D$. Then D is quasi-symmetric if and only if the conditions (C'), (A'), (B'), ($\tilde{A}'$), (D), ($\tilde{D}$) below hold.

(C') $\max_{Y\in\mathfrak{h},\,|Y|=1} |K(Y)| = \dim \mathfrak{h} \min_{Y\in\mathfrak{h},\,|Y|=1} |K(Y)|$, where

K(Y) is the holomorphic sectional curvature defined by (6.2)

(A') $R(X,Z)Y = 0$ for $X,Y,Z \in \sum \mathfrak{k}_{(a,b)}$ connected as 

,

R being the curvature tensor.

(B') $K(X,Z) = -\frac{1}{8}|X|^2|Z|^2 \max_{Y\in\mathfrak{k},\, |Y|=1} |K(Y)|$ for $X, Z \in \mathfrak{k}_{(a,b)}$

connected as $\underset{X\ \ Z}{\circ\!-\!\circ\!-\!\circ}$, where $K(X,Z) := \langle R(X,Z)Z, X\rangle$.

($\tilde{A}'$) $R(X,u)v = -R(X,ju)jv$ for $u\in\mathfrak{U}_m, v\in\mathfrak{U}_a$, $X\in\mathfrak{k}_{(a,b)}$, with

$m \neq a$. (It suffices to restrict to $a \neq m < b$).

(D) $\sum_{\ell=1}^{\dim\mathfrak{h}} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_\ell+\alpha_k)}$ is independent of k.

($\tilde{D}$) $\dim \mathfrak{U}_k$ is independent of k.

In [211 f], which is a sequel of [211 d,e], ZELOW translates (some of) the conditions set in [211 e] into curvature conditions: symmetric domains are in particular quasi-symmetric, and they are characterized among quasi-symmetric domains by the vanishing of ∇R, the covariant derivative of curvature. Now quasi-symmetric domains are "almost" symmetric, and so ZELOW focusses on ∇R. It turns out [211e] that some of the quasi-symmetry conditions are equivalent to the vanishing of ∇R on certain subspaces (see [211 f ,pp.10-11]).

Comments. Let us recall in this respect that SAGLE and SCHUMI showed [158]how real nonassociative algebras arise from multiplications on certain homogeneous spaces. Subsequently, these algebras are used to obtain an invariant connection on the homogeneous space and some applications of nonassociative algebras to these topics are given. Conversely, every real finite-dimensional nonassociative algebra arises from an invariant connection and a local multiplication on a homogeneous space. Hence, much of the basic theory of nonassociative algebras can be formulated in terms of multiplications and connections, and conversely. Let H be a closed (Lie)subgroup of a connected Lie group G.

Making use of the correspondence between G-invariant connections on the reductive homogeneous space G/H and certain nonassociative algebras, SAGLE computed [157a] pseudo-Riemannian connections in terms of a Jordan algebra J of endomorphism. If G and H are semisimple Lie groups, then, as was proved by SAGLE, J is a semisimple Jordan algebra.

SAGLE [157c] indicated how a G-invariant mechanical system involving the equations of motion in covariant form may be expressed in terms of Jordan algebras and reductive algebras which generalize Lie-admissible algebras. The connections on the homogeneous configuration space are given by the reductive algebras. Consequently, the mechanical system, geodesics, conserved quantities, etc., may be analyzed in terms of differential equations written in these algebras. Thus, the free n-dimensional rigid body motion is extended to free G-rigid motion by using the Nambu mechanics expressed in terms of the reductive algebras. In ref. [157d], SAGLE used the nonassociative algebras to describe the interrelated nature of an invariant Lagrangian mechanical system and its differential geometry. He also discussed pseudo-metric connection algebras in terms of Jordan algebras, and noted that the pseudo-Riemannian connection algebra satisfies a particular identity.

Open problem. As SAGLE suggested [157d , pp.485-486], a geometrical classification of algebras using geodesics would be interesting to be given.

Bibliography

1. ABEASIS, S., On the Plücker relations for the Grassmann varieties, Adv. in Math.36 (1980), No.3, 277-282.
2. AKBAR-ZADEH,H., On the holomorphically projective transformations of the Hermitian and Kählerian manifolds, C.R. Acad.Sci.Paris 304 (1987), No.12, 335-338 (see J.Math.Pures Appl.67 (1988), No.3, 237-262).
3. AKIVIS,M.A., Webs and almost Grassmannian structures (in Russian), Sibirsk.Mat.Zh. 23 (1982), No.6, 6-15.
4. ANASTASIEI,M., private communication in December 1983.
5. ATHORNE,C., FORDY,A., Generalized KdV and MKdV equations associated with symmetric spaces, J.Phys.A 20 (1987), No.6, 1377-1386.
6. ATIYAH,M.F., WARD,R.S., Instantons and algebraic geometry,Comm. Math.Phys. 55 (1977), 117-124.
7. ATSUYAMA,K., Symmetric spaces derived from algebras,Kōdai Math. J. 3 (1980), 3, 358-363.
8. AZUKAWA,K.a)Curvature operator of the Bergmann metric on a homogeneous bounded domain, Tôhoku Math.J. 37 (1985), 2, 197-223.

   b) Negativity of the curvature operator of a bounded domain, Tôhoku Math.J. 39 (1987), 281-285.

   c) Criteria for quasi-symmetricity and the holomorphic sectional curvature of a homogeneous bounded domain, Tôhoku Math.J. 41 (1989), No.3, 489.
9. BALAS,A., Compact Hermitian manifolds of constant holomorphic sectional curvature, Math.Z. 189 (1985), 2, 193-210.

10. BANNAI,E., HOGGAR,S,G.,On tight t-designs in compact symmetric spaces of rank one, Proc.Japan Acad. 61(1985), No.1, 78-82.

11. BARBILIAN,D., a) Grundiss einer geometrischen Algebrenlehre, Bull.Math.Soc.Roumaine Sci. 40 (1939), 2.
b) Finslerian natural J-metrics (in Romanian), Stud. Cerc.Mat.11 (1960), 1, 7-44.

12. BARBILIAN,D., RADU,N., Finslerian natural J-metrics and Riemann's representation function (in Romanian), Stud. Cerc.Mat. 12 (1962), 1, 21-36.

13. BENABDALLAH, A.S., Quelques propriétés des opérateurs invariants sur les espaces symmetriques, C.R.Acad.Sci. Paris 275 (1975), 49-51.

14. BENADO,M., On a problem of Garrett Birkhoff (in Romanian), Bull.Sci.6 (1954), 4, 703-735.

15. BERCEANU,S., Lectures on Morse theory, Institute of Physic and Nuclear Engineering preprint FT-23, Bucharest, 1984.

16. BERCEANU,S., GHEORGHE,A., a) On the construction of perfect Morse functions on compact manifolds of coherent states, J. Math.Phys. 28 (1987), No.12, 2889-2907;
b) Perfect Morse functions on the manifold of Slater determinants, Revue Roumaine Phys. 34 (1989), No.2, 125-146.
c) On equations of motion on complex Grassmann manifold, Institute of Physics and Nuclear Engineering preprint FT - 347, Bucharest, 1989.

17. BERESTOVSKII, V.N., Generalized symmetric spaces (in Russian), Sibirsk. Mat.Zh. 26 (1985), No.2, 3-17 (For his previous papers, see Sibirsk.Mat.Zh.23 (1982), No.2, 3-15 and Dokl.Akad.Nauk SSSR 247 (1979), No.3, 525-528).

18. BEREZIN, F.A., a) Models of Gross-Neveu type are quatization of a classical mechanics with nonlinear phase space, Comm.Math.Phys.63 (1978), 131-153.

b) Instantons and Grassmann manifolds (Russian), Funkcional.Anal i Prilozen. 13 (1979), 2, 75-76.

19. BERZI, V., RENI, M., Weak Riemannian structures on gauge-group orbits, Internat.J.Theor.Phys. 26 (1987), No.2, 151-174.

20. BLAIR, D.E., LEDGER, A.J., A characterization of oriented Grassmann manifolds, Rocky Mountain J.Math. 14 (1984), 3, 573-584.

21. BLOCH, A.M., Lax-type flows on Grassmann manifolds, in "Differential Geometry: The interface between pure and applied mathematics," M.Luksic, C.Martin, W.Shadwick (eds.), Contemporary Math.,vol.68, Amer.Math.Soc., 1988.

22. BRADA,C., PÉCAUT-TISON,F., Explicit calculation of the curvature and canonic 8-form of the projective plane of Cayley octaves, C.R.Acad.Sci.Paris, 301 (1985), 2, 41-44.

23. BRAUN,H., a) Hermitian modular functions III, Ann.of Math. (2) 53 (1951), 1, 143-160.

b) Analytische Aspekte der Jordan-Algebren,Lecture Notes Univ.Hamburg, 1969.

c) The Riccati differential equation in Jordan pairs, Tagungsbericht 35/1979, Jordan-Algebren,Oberwolfach, 1979.

24. BRAUN,H., and KOECHER,M., Jordan-Algebren, Springer Verlag, Berlin-Heidelberg-New-York, 1966.

25. Le BRUN,C., Quaternionic-Kähler manifolds and conformal geometry, Math.Ann.284 (1989), No.3, 353-376.

26. BURAU, W., Le varietà Grassmanniane $G_{n,k}$ definite in modo sintetico come proiezioni di speciali varietà di Segre, Rend.Sem.Mat.Brescia 7 (1984), 169-180.

27. BURSTALL, F.E., A twistor description of harmonic maps of a 2-sphere into a Grassmannian, Math.Ann. 274 (1986), 1, 61-74.

28. CAHEN, M., GUTT, S., Star-product for a symmetric Riemannian space, C.R.Acad.Sci.Paris 291 (1980), No.9, 543-546.

29. CAMPANA,F., Geometric algebraicity of moduli spaces of compact Kähler symplectic manifolds, J.reine angew.math. 397(1989), 202 - 213.

30. CANUTO,G., Associated curves and Plücker formulas in Grassmannians, Invent. Math. 53 (1979), No.1, 77.

31. CHEN, B.Y., DEPREZ,J., VERHEYEN,P., Immersions in a Euclidean space of a compact symmetric space of rank one with simple geodesics, C.R.Acad.Sci.Paris 304 (1987), No.18, 567-570.

32. CHEREMISIN, A.I., a) Vector multilattices and space-time geometry(in Russian), Novosibirsk, 1976 (VINITI, No.1419-77).

b) Formally real Jordan algebras and causal symmetric spaces (Russian), Novosibirsk, 1980 (VINITI, No-4932-80).

c) Partially ordered Jordan algebras of symmetric bilinear forms (in Russian), Siberian Math.J. 22 (1981), No.1, 205-209.

d) Jordan kinematics (in Russian), Sibirsk.Mat.Zh. 24 (1983), No.4, 189-198.

33. CHI,Q.-S., a) Curvature properties of rank one symmetric spaces, Ph.D.Thesis, Stanford Univ.,1986.
b) Quaternionic Kähler manifolds, talk at the 837[th] Meeting of Amer.Math.Soc.,Univ.of Nebraska, 1987.

34. CIFUENTES,P., H^p-classes on rank one symmetric spaces of non-compact type 2. Nontangential maximal function and area integral, Bull.Sci.Math.108 (1984), 4, 355-372.

35. COWEN,M.J., Automorphisms of Grassmannians, Proc.Amer.Math. Soc.106 (1989), No.1, 99-106.

36. CRAIOVEANU ,M., private communication in December 1983.

37. CRAIOVEANU,M., PUTA,M., On a class of operators on compact symmetric spaces (to appear).

38. DABROWSKI, L., TRAUTMAN,A., Spinor structures on spheres and projective spaces, J.Math.Phys. 27 (1986), No.8, 2022-2028.

39. D'ATRI,J.E., a) Holomorphic sectional curvatures of bounded homogeneous domains and related questions, Trans. Amer. Math.Soc.256 (1979), 405-413 (Erratum in Trans.Amer.Math.Soc.271 (1982), No.1, 349).
b) The curvature of homogeneous Siegel domains, J.Differential Geom.15 (1980), No.1, 61-70.
c) Sectional curvatures and quasi-symmetric domains J.Differential Geom. 16 (1981), No.1, 11-18.

40. D'ATRI, J.E., MIATELLO, I.D.A., A characterization of bounded symmetric domains by curvature, Trans.Amer.Math.Soc. 276 (1983), 531-540.

41. DICK,R., Locally symmetric Riemannian spaces and solutions of Yang-Mills equations, Lett.Math.Phys.14 (1987), No.1, 63-68.

42. van DIJK, G., Invariant eigendistributions on the tangent space of a rank one semisimple symmetric space, Math. Ann. 268 (1984), 405-416.

43. DOMIC, A., TOLEDO, D., The Gromov norm of the Kaehler class of symmetric domains, Math. Ann. 276 (1987), No. 3, 425-432.

44. DORFMEISTER, J., a) Peirce-Zerlegung und Jordan-Strukturen zu homogenen Kegeln, Math. Z., 169 (1979), No. 2, 179-194.

b) Inductive construction of homogeneous cones, Trans. Amer. Math. Soc. 252 (1979), 321-349.

c) Algebraic description of homogeneous cones, Trans. Amer. Math. Soc. 255 (1979), 61-89.

d) Simply transitive groups and Kähler structures on homogeneous Siegel domains, Trans. Amer. Math. Soc. 288 (1985), 1, 293-306.

e) Homogeneous Kähler manifolds, C. R. Acad. Sci. Paris, Serie I, 301 (1985), 4, 97-98.

f) Homogeneous Kähler manifolds admitting a transitive solvable group of automorphisms, Ann. Sci. Ècole Norm. Sup. 18 (1985), 143-180.

g) The radical conjecture for homogeneous Kähler manifolds, in "Lie algebras and related topics", Conference Proceedings, vol. 5, Canadian Math. Soc., Amer. Math. Soc., Providence, 1986, Reprinted 1987, 189-208.

h) Automorphisms of homogeneous Kähler manifolds, Preliminary report, 837th Meeting of Amer. Math. Soc. Univ. of Nebraska, 1987.

45. DORFMEISTER,J., KOECHER,M., a) Relative Invarianten und nicht-assoziative Algebren, Math.Ann. 228 (1977), 147-186.

b) Reguläre Kegel, Jahresber.Deutsch.Math.-Verein, 81 (1978-79), 3, 109-151.

46. DORFMEISTER,J., NAKAJIMA,K., The fundamental conjecture for homogeneous Kähler manifolds, C.R. Acad.Sci.Paris 303 (1986), No.8, 335-338.

47. DRAGER, L.D., FOOTE, R.L., MARTIN, C.F., Controllability of linear system, differential geometry of curves in Grassmannians and generalized Grassmannians, and Riccati equations, in " Differential Geometry: The interface between pure and applied mathematics", M.Luksic, C.Martin, W.Shadwick (eds.), Contemporary Math.,vol.68, Amer.Math.Soc., 1988.

48. DRINFELD, V.G., MANIN, Ju.I., Instantons and sheaves on CP^3 (in Russian), Funkcional Anal.i Prilozen.13 (1979), 2, 59-74.

49. DRUCKER,D., a) Orbit structure of the exceptional Hermitian symmetric spaces I, II, Bull.Amer.Math.Soc.80 (1974), 285-289, 1225-1229.

b) Exceptional Lie algebras and the structure of Hermitian symmetric spaces, Mem.Amer.Math.Soc.16 (1978), No.208.

50. ELMENDORF,A.D., The Grassmannian geometry of spectra, J.Pure Appl.Algebra 54 (1988), No.1, 37-94.

51. ESCOBALES Jr., R.H., Riemannian foliations of the rank one symmetric spaces, Proc.Amer.Math.Soc. 95 (1985),3, 495-498.

52. FAIBUSOVICH, L.E., Generalized Toda flows, Riccati equation on Grassmannian, and QR-algorithm (in Russian). Funktsional. Anal.i Prilozhen.21 (1987), No.2, 88-89.

53. FAVA,F., Sulle varietà pseudo-riemanniane a connessione costante di genere uno e due, "G.Ţiţeica and D.Pompeiu", Symposium of Geometry and Global Analysis, Bucharest, 1973.

54. Di FIORE,L., FRENI, S., Sulle Grassmanniane contenute în una Grassmanniana assegnata, Ricerche Mat. 30 (1981), No.1, 145-152.

55. FREED, D.S., Flag manifolds and Kähler geometry, in Proc. Infinite-dimensional Lie groups (Berkeley, 1989) - to appear.

56. FREUDENTHAL, H., Oktaven, Ausnahmengruppen und Oktavengeometrie, Math.Inst.Rijksuniversiteit Utrecht,Utrecht, 1951 (New revised edition, 1960).

57. FULTON,W., KLEIMAN,S., PIENE,R., TAI,H., Some intrinsic and extrinsic characterisations of the projective space, Bull.Soc.Math.France, 113 (1985), 2, 205-210.

58. GALBURA,G., Formes différentielles extérieures sur la variété de Grassmann quaternionique, Travaux du IV<u>ème</u> Congrés des mathematiciens roumains, Bucarest, 1956.

59. GEL'FAND,I.M., GINDIKIN, S.G., GRAEV,M.I.,Integral geometry in affine and projective spaces (in Russian), Itogi Nauki i Tehniki, VINITI, Sovrem. Prob.Mat. 16, (1980), 53-226.

60. GEL'FAND, I.M.,GONCHAROV, A.B.,On a characterization of Grassmann manifolds (in Russian), Dokl. Akad. Nauk SSSR 289 (1986),No.5, 1047-1052.

61. GEL'FAND, I.M., MacPHERSON, R.D., Geometry in Grassmannians and a generalization of the dilogarithm, Preprint IHES, June 1981.

62. GHEORGHIEV, G., OPROIU, V., Remarkable G-structures and structures of higher order (in Romanian), Memoirs of the Scientific Sections, Romanian Academy (IV), 3, (1980), 1, 51-121.

63. GHERARDELLI, F., Varietà complesse a struttura grassmanniana, Atti Acc. Lincei, Rend. fis. mat. natur. 76 (1984), 3, 182-188.

64. GLAZENBROOK, J.F., a) On isotropic harmonic maps to real and quaternionic Grassmannians, in "Complex Differential Geometry and Nonlinear Differential Equations" Ed. Y.T.Sin, Contemporary Math. vol 49, Amer. Math. Soc., Providence, 1986, pp. 51-61.

b) Harmonic maps of Riemann surfaces to quaternionic projective spaces and Grassmannians, in "The Lefschetz Centennial Conference", Part III, Ed. A.Verjovski, Contemporary Math. vol. 58, 3, Amer. Math. Soc., Providence, 1987.

65. GLOVER, H.H., HOMER, W.D., STONG, R.E., Splitting the tangent bundle of projective space, Indiana Math. J. 31 (1982), 161-166.

66. GLUCK, H., Calibrated geometries and minimal surfaces in Grassmann manifolds, ABSTRACTS Amer. Math. Soc. 6 (1985), 6, 433.

67. GRAHAM, P.J., LEDGER, A.J., s-Regular manifolds, in "Differential Geometry", in honour of Kentaro Yano, Tokyo, 1972, pp. 133-144.

68. GRECU, E., Sur les espaces de Grassmann réels, complexes et quaternioniques, Publ.Inst.Math. (Beograd)(N.S.) 33 (47) (1983), 73-82.

69. GOLDBERG,V.,V., Almost Grassmannizable and almost algebraizable webs and webs of maximum 2 rank, Albany Meetings, August 8-11, 1983 (NOTICES Amer.Math. Soc.Aug.1983, p.518).

70. GONZALES, F.B., Radon transforms on Grassmann manifolds, J. Funct.Anal.71 (1987), No.2, 339-362.

71. GUEST,M.A., Geometry of maps between generalized flag manifolds, J.Differential Geom.25 (1987), No.2, 223-247.

72. GÜNAYDIN,M., Quadratic Jordan formulation of quantum mechanics and construction of Lie (super) algebras from Jordan (super) algebras, Ann.Israel Phys.Soc. 3 (1980), 279-296.

73. GUYOT,M., Caractérisation par l'uniformité des fibrés universels sur la Grassmannienne, Math.Ann.270 (1985),1, 47-62.

74. HANGAN ,T., a) Géométrie différentielle grassmannienne, Rev. Roumaine Math.Pure Appl. 11 (1966), 5, 519-531.
b) Tensor product tangent bundles, Arch.Math.(Basel) 19 (1968), 4, 436-440.

75. HARVEY,R., MORGAN,F., The faces of the Grassmannian of the three-planes in $\mathbb{R}^7$ (Calibrated geometries on $\mathbb{R}^7$), Invent.Math.83 (1986), 2, 191-228.

76. HEFEZ,A., THORUP,A., Reflexivity of Grassmannians and Segre varieties, Comm. Algebra 15 (1987), No.6,1095-1108.

77. HELGASON,S., Differential geometry and symmetric spaces, Academic Press, New York, 1962.

78. HELMKE, U., SHAYMAN, M.A., The biflag manifold and the fixed points of a unipotent transformation on the flag manifold, Linear Alg.Appl. 92 (1987), 125-160.

79. HELTON, J.W., Operator theory, analytic functions, matrices, and electrical engineering, Regional conference series in mathematics, No.68, Amer.Math.Soc., Providence, 1987.

80. HELWIG, K.-H., a) Halbeinfache reelle Jordan-Algebren, Math.Z. 109 (1969), 1-28.

b) Involutionen von Jordan-Algebren, Manuscripta Math. 1 (1969), 211-229.

c) Jordan-Algebren und symmetrische Räume, I. Math.Z. 115 (1970), 315-349.

81. HERMANN, R., MARTIN, C., Lie and Morse theory for periodic orbits of vector fields and matrix Riccati equations, I General Lie-theoretic methods, Math. Systems Theory 15 (1982), No.3, 277-284.

82. HIRZEBRUCH, U., a) Halbräume und ihre holomorphen Automorphismen, Math.Ann. 153 (1964), 395-417.

b) Über Jordan-Algebren und kompakte Riemannsche symmetrischen Räume von Rang 1, Math.Z. 90 (1965), 339-354.

c) Über Jordan-Algebren und beschränkte symmetrische Gebiete, Math. Z. 94 (1966), 387-390.

83. HSIANG, W.C., SZCZARBA, R.H., On the tangent bundle of a Grassmann manifold, Amer.J.Math. 86 (1964), 698-704.

84. IORDANESCU, R., a) L'étude des opérateurs infinitésimaux associés aux algèbres de Jordan simples, Ann.Univ. Bucureşti Mat. 19 (1970), 83-90.

b) Géometrie différentielle sur les formes réelles de Jordan de type A_I, Bull.Un.Mat. Ital., 4 (1970)

585-594.

c) Metriques sur les formes réelles de Jordan, Rev.Roumaine Math.Pures Appl. 15 (1970), 1437-1444.

d) Sur les graduations spéciales réellés simples, Rev.Roumaine Math.Pures Appl. 16 (1971), 691-700.

e) Sulle strutture quaternionali grassmanniane, Boll.Un.Mat.Ital. 10 (1974), 406-411.

f) Jordan algebras with applications, Preprint INCREST, Bucharest, No.79/1979, ISSN 0250-3638.

g) Grassmann manifolds and Jordan algebras (in Romanian) INCREST, Bucharest, 1980.

h) Symmetric spaces investigated by means of Jordan algebras, INCREST, Bucharest, 1981.

i) On two recent geometrical approaches from quantum mechanics, IFTAR, Bucharest, 1983.

85. ISHIMARA,T., On tensor-product structures and Grassmann structures, J.Math.Tokushima Univ.4 (1970), 1-17.

86. KAPRANOV,M.M., On the derived category of coherent sheaves on Grassmann manifolds, Math.URSS - Izv. 48 (1984), 1, 183-192.

87. KAUP,W., Einige bemerkungen über polynomiale Vectorfelder, Jordan Algebren und die Automorphismen von Siegelschen Gebieten, Math.Ann. 204 (1973), 131-144.

88. KIMELFELD,B., Homogeneous domains on flag manifolds, J.Math. Anal.Appl. 121 (1987), No.2, 506-588.

89. KIRWAN,F., On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles, Ark.Mat.24 (1986), No.2, 221-275.

90. KLEIMAN,S., Geometry of Grassmannians and applications to splitting bundles and smoothing cycles, Publ. IHES 36 (1969), 281-297.

91. KNOP,F., MENZEL,G., Dual varieties of flag manifolds, Comment. Math. Helv. 62 (1987), No.1, 38-61.

92. KOBAYASHI,S., NOMIZU,K., Foundations of differential geometry I,II, Interscience Publishers, New York and London, 1963, 1969.

93. KOECHER,M., a) Positivitätsbereiche im $\mathbb{R}^n$, Amer.J.Math. 79 (1957), 3, 575-596.

b) Jordan algebras and their applications, Lecture Notes Univ. Minneapolis, 1962.

c) Konvexe Kegel, Lecture Notes Univ. München, 1968.

d) An elementary approach to bounded symmetric domains, Lecture notes of Rice Univ., Houston, Texas, 1969.

e) Jordan algebras and differential geometry, Actes Congrès Internat. Math. I, (Nice, 1970), 279-283.

f) Eine Konstruktion von Jordan-Algebren, Manuscripta Math. 23 (1978), 387-425.

94. KOSTADINOV,G.D., Almost holomorphic functions on the generalized symmetric spaces, Dokl. Akad. Nauk BSSR 32 (1988), No.8, 677-680.

95. KOSTERS,M.T., Spherical distributions on rank one symmetric spaces, Ph.D. Thesis, Univ. of Leiden, 1984.

96. KOSTERS, W.A., Harmonic analysis on symmetric spaces, Ph.D. Thesis, Univ. of Leiden, 1984.

97. KOWALSKI, O., a) Smooth and a ffine s-manifolds, Period.Math. Hungar. 8 (1977), No.3-4, 299-311.
b) Generalized affine symmetric spaces, Math.Nachr 80 (1977), 205-208.
c) Generalized symmetric spaces, Lecture Notes in Math.805 Springer, Berlin-Heidelberg-New York, 1980.

98. KOWALSKI, O., KULICH,I., Generalized symmetric submanifolds of Euclidean spaces, Math.Ann. 277 (1987), No.1, 67-79.

99. KUH,E., ROHER, Theory of linear active networks, Holden-Day, San Francisco, 1967.

100. KULKARNI,R.S., Curvature structures and conformal transformations, J.Differential Geom. 4 (1970), 425-451.

101. LASSALLE,M., On the orbital structure of a compact Hermitian symmetric space, C.R. Acad.Sci.Paris 302 (1986), 1,1-4.

102. LEDGER,A.J., OBATA, M., Affine and Riemannian s-manifolds, J.Differential Geom. 2 (1968), 451-459.

103. LEDGER,A.J., VANHECKE,L., Symmetries on Riemann manifolds, Math.Nachr. 136 (1988), 81-90.

104. LEMOINE,C., Fourier transforms of homogeneous distributions, Ann.Sc. Norm. Sup. Pisa, Ser.III, 26 (1972), 117-149.

105. LIAO, R.J., a) Scalar curvature of Kähler submanifolds of a complex projective space, Kexue Tongbao 32 (1987), No.13, 865-869.
b) Sectional curvature of Kähler submanifolds of a complex projective space, Chinese Ann.Math. Series B 9 (1988), No.3, 292-296.

106. LOOS,O., a) Spiegelungsräume und homogene symmetrische Mannigfaltigkeiten, Dissertation, München, 1966.

b) Spiegelungsräume und homogene symmetrische Räume. Math.Z. 99 (1967), 141-170.

c) Symmetric spaces I : General theory, New York and Amsterdam, Benjamin, 1969.

d) Kantor triple systems and symmetric quotients of contact manifolds, Tagnungsbericht Jordan-Algebren (4.8.-10.8.85) Oberwolfach.

107. MACPHERSON,R., Grassmannian cohomology, IRMA Journées de Géométrie en l'honneur de M.H.Schwartz, Arril 1986, pp.75-96.

108. MANIN,Yu.I., a) Holomorphic supergeometry and Yang-Mills superfields (in Russian), Preprint, 1983.

b) Grassmannians and flag manifolds in supergeometry, in "Fourteen Papers Translated from the Russian", Ed.A.A.Panchishkin et al, Amer.Math.Soc. Translations, Series 2, vol.137, 1987.

109. MARCHIAFAVA,S., a) Sulle varietà a struttura quaternionale generalizzata, Rend.Mat.(VI) 3 (1970), 1, 529-545.

b) Varietà localmente grassmanniane quaternionali, Atti Accad.Naz.Lincei Rend.Cl.Sci.Fis.Mat.Natur., 57 (1974), 80-89.

110. MARCHIAFAVA,S., ROMANI,G., a) Classi caratteristiche dei fibrati quaternionali generalizzati, Atti Accad. Naz.Lincei Rend.Cl.Sci.Fis.Mat.Natur., 56 (1974), 899-906.

b) Sui fibrati con struttura quaternionale generalizzata, Ann.Mat.Pura Appl., 107 (1975), 131-157.

111. MARINOSCI, R.A., Generalized pointwise symmetric Riemannian spaces - A classification, Math.Proc.Cambridge Philos. Soc. 104 (1988), No.3, 505-520.

112. MARTINELLI, E., Varietà a struttura quaternionale generalizzata, Atti Accad.Naz.Lincei Rend.Cl.Sci.Fis.Mat. Natur. 26 (1959), 353-362.

113. MASHIMO, K., Degree of the standard isometric minimal immersions of the symmetric spaces of rank one into spheres, Tsukuba J. Math. 5 (1981), No.2, 291-297.

114. MATSUMOTO,M., Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Otsu-Shi, Tokyo, 1986.

115. MICHEL, L., Applications of group theory to quantum physics. Algebraic aspects, in "Group representations in mathematics and physics" , Lecture Notes in Physics, 6, Springer, Berlin-Heidelberg-New York, 1970, pp.36-149.

116. MIRON,R., private communication by letter in January 1984.

117. MISFELD, J., TALLINI,G., ZANELLA, C., Topological Grassmann spaces, Rendiconti di Mat. (VII) 8 (1988), No.2, 223-240.

118. MISLIN, G., The homotopy classification of self-maps on infinite quaternionic projective space, Quart. J.Math. 38 (1987), No.150, 245-

119. MOK, N., a) Uniqueness theorems of Hermitian metrics of seminegativ curvature on quotients of bounded symmetric domains, Ann.Math. 125 (1987), No.11, 105-152.

b) Uniqueness theorems of Kähler metrics of semi-positive bisectional curvature on compact Hermitian symmetric spaces, Math.Ann.276 (1987),No.2, 177-204.

120. MORENO, C., Geodesic symmetries and invariant star products on Kähler symmetric spaces, Lett.Math.Phys. 13 (1987), No.3, 245-258.

121. MORSE,M., The calculus of variations in the large, Amer.Math. Soc.18 , Princeton Univ.Press., Providence, 1934.

122. NAITOH,H., Symmetric submanifolds of compact symmetric spaces, Tsukuba J.Math. 10 (1986), No.2, 215-242.

123. NAKAJIMA, K., a) On j-algebras and homogeneous Kähler manifolds, Hokkaido Math.J. 15 (1986), No.1, 1-20.

b) Homogeneous Kähler manifolds of nonpositive Ricci curvature, J.Math.Kyoto Univ. 26 (1986), No.4, 847-858.

124. NAKAJIMA,K., SE-ASHI,Y., On totally geodesic Hermitian symmetric submanifolds of Kähler manifolds, Proc. Japan Acad. 58A (1982), 163-166.

125. NEHER, E., Differentialgeometrische Aspekte von Idempotenten in reellen Jordan-Algebren, Diplomarbeit, Münster, 1975.

126. NICOLESCU, L., PRIPOAE, G., Structures algebriques générées par certaines propriétés géométriques, Ann.Univ. Bucureşti, Mat.35 (1986), 57-64.

127. NITTA,T., a) Connections for vector bundles over quaternionic Kähler manifolds, Proc.Japan Acad.A 63 (1987), No.1, 23-26.

b) Moduli spaces of B_2 - connections over quaternionic Kähler manifolds, Proc.Japan Acad. A 63 (1987), No.8, 327-329.

c) Vector bundles over quaternionic Kähler manifolds, Tôhoku Math.J. 40 (1988), No.3, 425-440.

128. OGIUE, K.,TAKAGI,R., A geometric meaning of the rank of Hermitian symmetric spaces, Tsukuba J.Math. 5 (1981) No.1, 33-37.

129. OHNITA,Y, a) Stable minimal submanifolds in compact rank one symmetric spaces, Tôhoku Math.J. 38 (1986), No.2, 199-218.

b) On stability of minimal submanifolds in compact symmetric spaces, Compositio Math. 64 (1987), No.2, 157-189.

130. OKUBO,S., Nonassociative differential geometry, Algebras Groups Geom. 4 (1987), No.2, 215-263.

131. OPROIU, V., a) Some non-embedding theorems for the Grassmann manifolds $G_{2,n}$ and $G_{3,n}$, Proc.Edinburgh Math.Soc. 20 (1976-77), 177-185.

b) Almost quaternal structures, Ann.Ştiinţ.Univ. "Al.I.Cuza" Iaşi,Sec.I Mat.(N.S.), 23 (1977), 2, 287-298.

c) Some results concerning the non-embedding codimension of Grassmann manifolds in Euclidean spaces, Rev.Roumaine Math.Pures Appl. 26 (1981), 275-286.

d) Pontryagin forms of quaternion Kaehler manifolds of constant Q-sectional curvature, Tensor, N.S. 38 (1982), 298-302.

e) Integrability of almost quaternal structures, An.Ştiinţ.Univ."Al.I.Cuza" Iaşi, Sec. I.Mat (N.S.) (to appear).

f) Pontryagin formes of quaternion manifolds, Atti Accad.Naz.Lincei,Cl.Sci.Fis.Mat.Natur.76 (1984), No.1, 11-27.

132. OTTAVIANI, G., Splitting criteria for vector bundles on Grassmannians and quadrics, C.R. Acad.Sci.Paris 305 (1987), No.6, 257-260.

133. PAK, J.I., KANG, T.H., Planar geodesic submanifolds in a quaternionic projective space, Geom.Dedicata 26 (1988), No.2, 139-155.

134. PARK, K., Transnormal systems on projective spaces, Ph.D.Thesis, Univ.of Maryland, College Park, 1988.

135. PERELOMOV, A.M., a) Instanton-like solutions in chiral models (Review paper), Physica D 4 (1981), No.1, 1-25.

b) Coherent states for arbitrary Lie group, Comm. Math.Phys. 26 (1972), 222-236.

136. PERSITS, D.B., a) Geometries over degenerate octaves (in Russian), Dokl.Akad.Nauk SSSR 173 (1967), 1010-1013;

b) Geometries over degenerate octaves and degenerate antioctaves (in Russian), Trudy Sem. Vektor.Tenzor.Anal.15 (1970), 165-187.

137. PICASSO, E., Connessioni e varietà in ambiente proiecttivo, Rend.Mat. (6) 3 (1970), 97-132.

138. PICCINNI, P., a) Quaternionic differential forms and symplectic Pontryagin classes, Ann.Mat.Pura Appl.(4), 129 (1981), 57-68.

b) Dieudonné determinant and invariant real polynomials on gl (n,$\mathbb{H}$), Rend.Mat.(VII), 2 (1982), 1, 31-45.

c) private communication in October 1981.

139. PICKRELL, D., a) Measures on infinite-dimensional Grassmann manifolds, J.Funct.Anal.70 (1987), No.2, 323-356.

b) On the support of quasi-invariant measures on infinite-dimensional Grassmann manifolds, Proc. Amer.Math.Soc.100 (1987), No.1, 111-116.

140. PISO,M., SARU,D., Jordan sheaves of symmetric bilinear forms (in Romanian), Progresses in Physics, 10th Session, Constanţa, 1988, pp.228-229.

141. PONTIER,M., SZPIRGLAS,J., Filtering with observations on a Riemannian symmetric space, SIAM J.Control Optimization 26 (1988), No.3, 609-627.

142. PONTRYAGIN,L.S., Characteristic cycles on differentiable manifolds (in Russian), Mat.Sb.(N.S.) 21 (1950), 233-284.

143. POPOVICI,I., a) Spaces with constant connection (in Romanian), in "Connections on differentiable manifolds", Univ. Bucureşti, 1979, pp.153-204.

b) Sur les espaces à connexion constante attachés aux algèbres de Jordan simple et centrales, An. Ştiinţ. Univ."Al.I.Cuza" Iaşi, Sec.I.Mat.(N.S.) 25 (1979), 365-380.

144. POPOVICI,I., IORDANESCU,R., TURTOI,A., Simple Jordan and Lie gradings considered in differential geometry (in Romanian), Edit.Acad.RSR, Bucureşti, 1971.

145. POPOVICI,I., TURTOI,A., Propriétés métriques des formes réelles de Jordan de type A_{II}, Rev.Roumaine Math. Pures Appl. 15 (1970), 303-314.

146. PORTA,H., RECHT,L., Minimality of geodesics in Grassmann manifolds, Proc.Amer.Math.Soc. 100 (1987), No.3, 464-466.

147. PREMA, G., KIRANAGI, B.S., Lie algebra bundles defined by Jordan algebra bundles, Bull.Math.Soc.Sci.Math. Roumanie 31 (1987), No.3, 255-264.

148. PRIPOAE, G., Algèbras de Tzitzeica, Bull.Math.Soc.Sci.Math. Roumanie 29 (1985), No.4, 365-370.

149. PYATETSKII-SHAPIRO, I.I., Geometry of classical domains and theory of automorphic functions, Gordon & Breach, New York, 1967.

150. RAN, Z., Surfaces of order 1 in Grassmannians, J.Reine Augew. Math. 368 (1986), 119.

151. RAPINCHUK, A.S., The Hasse principle for symmetric spaces (in Russian), Dokl.Akad.Nauk.Belorus.SSR 31 (1987), No.9, 773-776.

152. ROGOV, V.B.K., Beltrami-Laplace operator on homogeneous symmetric spaces of rank one, Math.Notes 28 (1980), No.1-2, 493-496.

153. ROTHAUS, O.S., a) The construction of homogeneous convex cones, Ann. Math. (2) 83 (1966), 358-376.
b) Automorphisms of Siegel domains, Trans.Amer. Math.Soc. 162 (1971), 351-382.
c) Ordered Jordan algebras, Amer.J.Math. 100 (1978), 5, 925-939.

154. ROZENFEL'D, B.A., KARPOVA, L.M., Flag groups and reductive Lie groups (in Russian), Trudy Sem.Vekt.Tenzor.Anal. 13 (1966), 168-202.

155. ROZENFEL'D, B.A., KUZNETSOVA, V.A., HANTURINA, S.A., Geometric construction of operations of the algebra of m-planes in projective (2m+1)-space (in Russian). Problems of group theory and homological algebra, No.1 (in Russian), pp.211-219, Jaroslav.Gos.Univ., Yaroslavl, 1977.

156. ROZENFEL'D, B.A., ZAMAHOVSKI, M.P., Simple and quasisimple Jordan algebras (in Russian), Ivz.Vyss. Ucebn. Zaved. Matematika, 1971, No.8 (111), 111-121; Letter to the editors; Izv.Vyss.Ucebn.Zaved. Matematika 1976, No.2 (165), 123.

157. SAGLE, A.A., a) Jordan algebras and connections on homogeneous spaces, Trans.Amer.Math.Soc. 187 (1974), 1, 405-427.

b) Mutation algebras related to connections on homogeneous spaces, Algebras Groups Geom. 1 (1984), 164-175.

c) Nonassociative algebras and invariant mechanical syptems I,II, Hadronic J. 7 (1984), 1259-1302; 8 (1985).

d) Nonassociative algebras and Lagrangian mechanics on homogeneous spaces, Algebras Groups Geom. 2 (1985), 478-494;

e) Invariant Lagrangian mechanics, connections, and nonassociative algebras, Algebras Groups Geom. 3 (1986), No.2, 199-263.

158. SAGLE, A.A., SCHUMI, J.R., Multiplications on homogeneous spaces, nonassociative algebras and connections, Pacific J. Math. 48 (1973), 1, 247-266.

159. SAKAI, T., a) On the structure of cut loci in compact Riemannian symmetric spaces, Math.Ann. 235 (1978), 129-148.

b) Cut loci of compact symmetric spaces, in "Minimal submanifolds and geodesics", North-Holland Amsterdam-New York, 1979, pp.193-207.

160. SAKAMOTO, K., Helical immersions into a unit sphere, Math. Ann. 261 (1982), 63-80.

161. SAKANE, Y., On non-singular hyperbolic sections of some Hermitian symmetric spaces, Osaka J. Math. 22 (1985), 1, 107-122.

162. SAKANE, Y, TAKEUCHI, M., On defining equations of symmetric submanifolds in complex projective spaces, J. Math. Soc. Japan 33 (1981), No.2, 267-280.

163. SALAMON, S.M., a) Quaternionic Kähler manifolds, Invent. Math. 67 (1982), 143-171.

b) Quaternionic manifolds, Symposia Math. 26 (1982), 139-151.

164. SÁNCHEZ, C.V., a) k-symmetric submanifolds of $\mathbb{R}^N$, Math. Ann. 270 (1985), 297-316.

b) Extrinsic 2r-symmetric submanifolds of $\mathbb{R}^N$; general structure, Geom. Dedicata 22 (1987), No.1, 39-47.

165. SANCHEZ, O.V., a) On supergeometric structures, Ph.D. Thesis, Harrard Univ., 1986.

b) Remarks on Grassmannian supermanifolds, Trans. Amer. Math. Soc. 307 (1988), No.2, 597-614.

166. SANKARAN, P., Which Grassmannians bound? Arch. Math. (Basel) 50 (1988), No.5, 474-476.

167. SATAKE, I., On classification of quasi-symmetric domains, Nagoya Math. J. 62 (1976), 1-12.

168. SCHNEIDER, C.R., a) Global aspects of the matrix Riccati equation, Math. Systems Theory 7 (1973), No.3, 281-286.

b) On the periodic solutions of the matrix Riccati equation, Math. Systems Theory 16 (1983), No.3, 297-306.

169. SEMYANISTII, V.I., Symmetric domains and Jordan algebras (in Russian), Dokl.Akad.Nauk SSSR, 190 (1970), 4, 788-791; Soviet.Math.Dokl. 11 (1970), 1, 215-218.

170. SHAYMAN, M.A., Phase portrait of the matrix Riccati equation, SIAM J.Control Optim. 24 (1986), 1-65.

171. SIEGEL, C.L., Symplectic geometry, Amer.J.Math. 65 (1943),1, 1-86.

172. SIMMS,D., Lie groups in quantum mechanics, Springer Lecture Notes in Math. 52 , Berlin-Heidelberg-New York, 1968.

173. SMITH, A.J., Symplectic Kähler manifolds, Ph.D.Thesis, Univ. of California, Berkeley, 1986.

174. SPRINGER, T.A., VELDKAMP, F.D., Elliptic and hyperbolic octave planes, Nederl.Akad.Wetensch.Proc., A 66 = Indag. Math. 25 (1963), 413-451.

175. SRIVASTAVA, J.P., Almost quaternion structure on cross-section in the cotangent bundle, J.Univ.Kuwait-Sci. 16 (1969), No.1, 13-20.

176. STETKAER, H., Scalar irreducibility of eigenspace representations associated to a symmetric space, Math. Scand. 57 (1985), 2, 289-292.

177. STONG, R.E., Splitting the universal bundles over Grassmannians, in "Algebraic and differential topology--Global differential geometry", Tenbner-Texte zur Mathematik, vol.70, pp.275-287.

178. STRICHARTZ, R.S., Harmonic analysis on Grassmannian bundles, Trans. Amer.Math.Soc. 269 (1986), 1, 387-410.

179. STURMFELS, B., On the matroid stratification of Grassmann varieties, specialization of coordinates, and a problem of N.White, Advances Math. 75 (1989), No.2, 202-211.

180. SUMITOMO, T., TANDAI, K., On the centralizer of the Laplacian of $P_n(\mathbb{C})$ and the spectrum of complex Grassmann manifold $G_{2,n-1}(\mathbb{C})$, Osaka J. Math. 22 (1985), 1, 123-156.

181. TAKEUCHI, M., a) On conjugate and cut loci of compact symmetric spaces, I, II, Tsukuba Math. J. 2 (1978), 35-68; 3 (1979), 1-29.

b) Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, Tôhoku Math. J. 36 (1984), 293-314.

182. TALLINI, G., On a characterization of the Grassmann manifold representing the lines in a projective space, in Finite geometries and designes (Proc. Conf. Chelwood Gate 1980) pp. 354-358, London Math. Soc., Lecture Notes Ser. 49, Cambridge Univ. Press, Cambridge, 1981.

183. TASAKI, H., a) The cut locus and the diastasis of a Hermitian symmetric space of compact type, Osaka J. Math. 22 (1985), No.4, 863-870.

b) Quaternionic submanifolds in quaternionic symmetric spaces, Tôhoku Math. J. 38 (1986), No.4, 513-538.

184. TELEMAN, C., a) Sur les variétés de Grassmann, Bull. Math. Soc. Sci. Math. Phys. RPR 2 (1958), 205-224.

b) On a theorem of Borel-Lichnerowicz (in Russian) Rev. Roumaine Math. Pures Appl. 3 (1958), 107-115.

185. TILGNER, H., Symmetric spaces in relativity and quantum mechanics, in "Group theory in non-linear problems", Reidel Publishing Company, Dordrecht, 1974.

186. TILLIER, A., a) Sur les idempotents primitifs d'une algebre de Jordan formelle réelle, C.R.Acad.Sci.Paris 280 (1975), 767-769.

b) Quelques applications géometriques des algèbres de Jordan, Publ.Dép.Math.(Lyon) 14 (1977), 3, 59-128.

c) Les groupes de transvections des espaces symmetriques associés à une algèbre de Jordan de forme réelle, Publ.Dép.Math. (Lyon) 14 (1977), 4, 27-56.

187. TSUJI, T., On the curvature of affine homogeneous convex domains, ABSTRACT Amer.Math.Soc. 6 (1985), 6, p.435.

188. TSUKADA, K., a) Parallel Käehler submanifolds of Hermitian symmetric spaces, Math.Z. 190 (1985), 1, 129-

b) Parallel submanifolds in a quaternion projective spaces, Osaka J.Math. 22 (1985), 2, 187-242.

c) Helical geodesic immersions of compact rank one symmetric spaces into spheres, Tokyo J.Math. 6 (1983), 2, 267-285.

189. TURTOI, A., a) Espaces de Vagner et formes réelles de Jordan de type D, Rev.Roumaine Math.Pures Appl. 16 (1971), 111-119.

b) On Jordan algebraic fibrations (in Romanian), Stud.Cerc.Mat. 37 (1985), No.5, 484-492.

190. UDAGAWA, S., a) Spectral geometry of Kähler submanifolds of a complex projective space, J.Math.Soc.Japan 38 (1986), No.3, 453-472.

b) Einstein parallel Kähler submanifolds in a complex projective space, Tokyo J.Math. 9 (1986), No.2, 335-340.

191. UHLMANN, A., Die Cartan-Algebra der äusseren Differentialformen als Supermanigfaltigkeit: Automorphismen und ihre Realisierung als Diffeomorphismen, Z. Analysis und ihre Anwendungen 4 (1985), 6, 481-492.

192. UPMEIER, H., Symmetric Banach manifolds and Jordan C^*-algebras, North-Holland Publ. Co., Amsterdam-New York, 1985 (for symmetric analitic manifolds of compact type, i.e., infinite-dimensional Grassmann manifolds and generalizations).

193. VANNINI, W., RUBINSTEIN, J.H., Symmetric cut loci in Riemannian manifolds, Proc. Amer. Math. Soc. 94 (1985), 2, 317-320.

194. VINBERG, E.B., The theory of homogeneous conex cones, Trudy Moskov. Mat. Obsc. 15 (1963), 303-358; Trans. Moscow Math. Soc. 12 (1963), 340-403.

195. VASHKAS, P., NAVITSKIS, K., Geometry of line manifolds and nets theory, Liet. matem. rink, 20 (1980), 3, 77-81.

196. VASIL'EV, V.A., GEL'FAND, I.M., ZELEVINSKII, A.V., Generalized hypergeometric functions on complex Grassmannians (în Russian), Funct. Anal. i prilozh. 21 (1987) No.1, 23-38.

197. VDOVIN, V.V., a) Simple projective planes, Arch. Math. (Basel) 47 (1986), No.5, 469.
b) Homomorphisms of freely generated projective planes, Comm. Algebra 16 (1988), No.11, 2209-2230.

198. VRANCEANU, G., a) Sur la représentation géodésique des espaces de Riemann, Rev. Roumaine Math. Pures Appl. 1 (1956), 147-165.
b) Leçons de géometrie différentielle, II, Edit. Acad. RPR, Bucureşti, 1957.

c) Spazi a connessione affine e le algebre di numeri ipercomplessi, Ann.Sc.Norm.Sup.Pisa 12 (1958), 5-20.

d) Propriété globales de espaces à connexion affine, Bull.Math.Soc.Sci.Math.Phys.RPR 2 (1958), 475-478.

e) Sopra gli spazi di Riemann a connessione constante, Ann.Mat.Pura Appl.(4) 58 (1962).

199. WALCHER, S., a) A characterization of regular Jordan pairs and its application to Riccati differential equations, Comm.Algebra 14 (1986), No.10, 1967-1978.

b) Über polynomiale, insbesondere Riccatische, Differentialgleichungen mit Fundamentallösungen, Math.Ann.275 (1986), No.2, 269-280.

200. WANG,H.C., Amer.J.Math.76 (1954), 1-31.

201. WAWRZYNEZYK, A., Spectral analysis and synthesis on symmetric spaces, J.Math.Anal. Appl. 127 (1987), No.1, 1-17.

202. WEGRZYNOWSKI,S., Representation of generalized affine symmetric spaces by S-structures, Demonstratio Math. 9 (1976), No. 4, 707-727.

203. De WET, J.A., Nuclear structure on a Grassmann manifold, Found. Phys.17 (1987), No.10, 993-1018.

204. WILCZYNSKI, D.M.,Group actions on the complex projective plane, Trans.Amer.Math.Soc.303 (1987),No.2, 707-732.

205. WOLF, J.A., Symmetric spaces (Short Courses, Washington Univ., St.Louis, 1969-1970), pp.271-357, Pure and Appl. Math. 8 , Dekker, 1972.

206. WOLFSON, J.G., Minimal two-spheres in Grassmannians, ABSTRACTS Amer.Math.Soc. 6 (1985), 6, 434.

207. WOLTER, F.-E., Distance function and cut loci on a complete Riemannian manifold, Arch.Math.(Basel) 32 (1979), 92-96.

208. WRIGHT, P.E., Darboux transformations, algebraic subvarieties of Grassmann manifolds, commuting flows and bispectrality Ph.D.Thesis, Univ. of California, Berkeley, 1988.

209. ZAMAHOVSKI, M.P., Models of non-Euclidean and quasi-non-Euclidean spaces in the form of orbits and of quasi-simple Jordan algebras of class D (in Russian), Geometry and topology, No.2 (in Russian), pp.94-98, Leningrad, Gos.Ped.Inst., Leningrad, 1974.

210. ZAMAHOVSKII, M.P., ORLOVSKAYA, T.G., Bireductive spaces, Jordan algebras and spinor representations of non-Euclidean and quasi-non-Euclidean motions (in Russian), Gravitatsiya i Teor. Otnositel'nosti No.8 (1971), 100-108.

211. ZELOW, R., a) On the geometry of some Siegel domains, Nagoya Math.J. 73 (1979), 175-195.

b) Curvature of quasi-symmetric Siegel domains, J.Differential Geom. 14 (1979), 629-655.

c) Holomorphic sectional curvature of quasi-symmetric domains, Proc.Amer.Math.Soc. 76 (1979), No.2, 299-301.

d) Quasi-symmetric domains and curvature, Inst.of Math., Univ.Oslo, Preprint No.12, 1980.

e) Quasi-symmetric domains and j-algebras, Inst. of Math.Univ.Oslo, Preprint No.14, 1980.

f) Quasi-symmetric domains and derivative of curvature, Inst.of Math.,Univ.Oslo, Preprint No.15, 1980.

212. ZHENG,Y., Harmonic maps into Grassmann manifolds, Ph.D.Thesis Washington Univ.,1988 .

213 ZILLER,W., Homogeneous Einstein metrics on spheres and projective spaces, Math.Ann.259 (1982), 3, 351-358.

214 ZOU,Y.M., Minimal immersions of rank 1 compact symmetric spaces into spheres, Scientia Sinica A 28 (1985), 3, 263-272.

JORDAN STRUCTURES WITH APPLICATIONS IV. JORDAN TRIPLE SYSTEMS IN DIFFERENTIAL GEOMETRY.

Radu IORDANESCU

This paper deals with the connection between Jordan triple systems and symmetric R-spaces, and gives applications of Jordan triple systems to hypersurfaces in spheres. Let us mention in this respect that some of Cartan's results are now being successfully re-examined against the background of the theory of Jordan algebras.

§ 1. Jordan triple systems and symmetric R-spaces.

R-spaces constitue an important class of homogeneous submanifolds in the Euclidean spheres. This class includes many examples appearing in differential geometry of submanifolds. For example, all homogeneous hypersurfaces and all parallel submanifolds in spheres are realized as R-spaces.

FERUS [7] has characterized the R-spaces as compact symmetric submanifolds of Euclidean spaces.

The connection between Jordan algebras and symmetric R-spaces was first illuminated by KANTOR, SIROTA and SOLODOVNIKOV [14], KOECHER [16] and LOOS [17 a].

Definition (see TAKEUCHI [30 a]). A symmetric R-space is a compact symmetric space on which there exists a group of transformations containing the group of motions as a proper subgroup.

There exists a one-to-one correspondence between compact Jordan triple systems and symmetric R-spaces, as was established by LOOS [17 a] - see Theorem 1.1. below. A simple geometric characterisation of the symmetric R-spaces among compact symmetric spaces is also given in [17 a] (see Theorem 3 in [17 a, p.559]). The noncompact dual of a symmetric R-space can be realized as a bounded domain D in a real vector space. LOOS [17 a] proved that there is a one-to-one correspondence between boundary component of D and idempotents of the corresponding Jordan triple system (see Theorem 1.3 below).

We first recall some of the results given by LOOS [17 a], and we then deal with results of KANTOR [13], MAKAREVICH [18 a,c], and RIVILIS [26].

A compact Jordan triple system T becomes a Euclidean vector space with the scalar product $(x,y) := \mathrm{Tr}L(x,y)$. By the second equality in the definition of a Jordan triple system (see § 1 of JSA.I) the vector space $\mathcal{H}$ spanned by $\{L(x,y) \mid x,y \in T\}$ is a Lie algebra of linear transformations of T, which is closed under taking transposes with respect to (,). The contragradient $\mathcal{H}$-module T' of T can thus be identified with T as a vector space, and $X.v' = -{}^tX(v')$ for $X \in \mathcal{H}$ and $v' \in T'$. The map $\tau : X \longrightarrow {}^tX$, $x \in \mathcal{H}$, $v \longrightarrow v'$, $v' \longrightarrow v$, is a Cartan involution of the Lie algebra $\mathcal{L} := T \oplus \mathcal{H} \oplus T'$, and $\sigma\big|_{\mathcal{H}} = \pm 1$, $\sigma\big|_{T\oplus T'} = -1$ defines an involutive automorphism σ of $\mathcal{L}$ commuting with τ. Recall that by a result of KOECHER, $\mathcal{L} = T \oplus \mathcal{H} \oplus T'$ becomes a semisimple Lie algebra, adopting

$$[X,Y] := XY - YX, \quad [X,v] := -[v,X] := X.v$$

for $X,Y \in \mathcal{H}$ and $v \in T \cup T'$

$$[T,T] := [T',T'] := 0 \quad [u,v'] := -2L(u,v)$$

for $u \in T$ and $v' \in T'$. Also Koecher's result states that $Z = -\mathrm{Id}_T$ is an element of $\mathcal{H}$, $(\mathrm{ad}\, Z)^3 = \mathrm{ad}\, Z$, and the -1-, 0-, +1 -

-eigenspaces of adZ are T, $\mathcal{H}$, T'. Let L be the centre free connected Lie group with Lie algebra $\mathcal{L}$, let H be the centralizer of Z in L, let U be a maximal compact subgroup of L determined by τ, and let $K := U \cap H$. Then K lies between the full set of fixed points of σ in U and the identity component of U. If we denote by P the normalizer of T in L, then P is parabolic and $U/_K \cong L/_P$. It follows that $M := U/_K$ is a symmetric R-space.

<u>Theorem 1.1.</u> The map $T \longrightarrow M$ establishes a one-to-one correspondence between isomorphism classes of compact Jordan triple systems and symmetric R-spaces.

If G denotes the connected component of the set of fixed points of $\sigma\tau$ in L, then $M^* := G/_{K_0}$ is the non-compact dual of M. There exists an imbedding $\zeta : M^* \longrightarrow T$ such that

$g \equiv \exp(\zeta(gK_0)) \pmod P$, for $g \in G$, and the image $D := \zeta(M^*)$ is a bounded domain in T (see NAGANO [20], TAKEUCHI [30 a]) which inherits a Riemannian metric from M^*. Following PYATETSKII-SHAPIRO [25], two points $x, y \in \overline{D}$ (= the closure of D in T) are called <u>equivalent</u> if there exist sequences (x_n), (y_n) in D converging to x, y, respectively, such that the Riemannian distance of x_n and y_n remains bounded. The equivalence classes are the <u>metric boundary components</u> of D. A subset F of $\overline{D}$ is called an <u>affine boundary component</u> if: 1) every segment (i.e. a set of the form $\{a + tb \mid a, b \in T,\ 0 < t < 1\}$) contained in $\overline{D}$ and meeting F is contained in F; and 2) no proper nonempty subset of F satisfies 1). Recall that an element c of T is called <u>idempotent</u> if $\{ccc\} = c$.

<u>Theorem 1.2.</u> a) There exists a Peirce decomposition $T = T_1(c) \oplus T_{1/2}(c) \oplus T_0(c)$, where $T_i(c)$ is the eigenspace of $L(c,c)$ corresponding to the eigenvalue i ;

b) putting $x \circ y := \{x\ c\ y\}$, $T_1(c)$ is a real semisimple Jordan algebra with unit element c. The map $x \longrightarrow \bar{x} := \{cxc\}$ is a Cartan involution of $T_1(c)$; in particular, $T_1^+(c) := \{x \in T_1(c) \mid \bar{x} = x\}$ is a formally real Jordan algebra;

c) with the induced multiplication, $T_o(c)$ is a compact Jordan triple system.

Remark. Part of this theorem is due to MEYBERG [unpublished] - see LOOS [17 a, p.560].

We can now describe the relation between boundary components and idempotents.

Theorem 1.3. Metric and affine boundary components coincide; they are precisely the sets $F_c := c \oplus (D \cap T_o(c))$ where c is an idempotent. $D_c := D \cap T_o(c)$ is the bounded symmetric domain belonging to the compact Jordan triple system $T_o(c)$.

In 1968 KANTOR [13] related Jordan algebras and symmetric R-spaces as follows: let $\mathcal{A}$ be either a semisimple Jordan algebra or a subalgebra $\mathcal{A}' \subset \mathcal{A}$ possessing regular idempotents, and let L be the group of (local) transformations of $\mathcal{A}$ generated by the structure group, the translations and the inversion J. The neighborhood of zero in $\mathcal{A}$ is the local homogeneous space of L. The group L is a semisimple Lie group of noncompact type possessing a parabolic stationary subgroup P, and thus $M = L/_P$ is a symmetric R-space. The space M can be locally realized as a neighborhood of zero in $\mathcal{A}$, and for its global construction we must complete $\mathcal{A}$ with ideal points; in the last case $\mathcal{A}$ is open and everywhere dense in M (see MAKAREVICH [18 b].

All reductive groups having open symmetric orbits in symmetric R-spaces were described by MAKAREVICH [18 a] and

RIVILIS [26] as follows: let $\mathcal{A}$ be a semisimple Jordan algebra and let B be a self-adjoint transformation from the structure group with respect to the trace form of $\mathcal{A}$. Then the Lie algebra of all reductive Lie groups has the form $G(\mathcal{A}, B) :=$ $=\{k + a + K(Ba)\}$, where k is an element of the structure group commuting with B, and $a \in \mathcal{A}$. All groups $G(\mathcal{A}, B)$, up to conjugation in symmetric R-spaces, were described by MAKAREVICH in [18 a]. From the description of the groups having open symmetric orbits in the spaces related to nonsemisimple Jordan algebras, it follows that if the space is different from complex or real projective space, then all reductive Lie groups also have the form $G(\mathcal{A}, B)$. In this case k and B are elements of the structure groups of the semisimple algebras $\tilde{\mathcal{A}} \supset \mathcal{A}$, $kB = Bk$, k preserves $\mathcal{A}$, and B transforming $\mathcal{A}$ into $R\mathcal{A}$, where R is the involution which isolates a maximal compact subalgebra of $\tilde{\mathcal{A}}$ and such that $K(Ba)b \in \mathcal{A}$ for $a, b \in \mathcal{A}$.

MAKAREVICH [18 c] introduced an aritmetic invariant, called Takeuchi indice , and, in terms of the Jordan algebra $\mathcal{A}$ and the transformation B, proved that two points of a symmetric R-space belong to the same orbit of the group $G(\mathcal{A}, B)$ if and only if their Takeuchi indices coincide (see Theorem 1.4 below).

<u>Theorem 1.4.</u> Two points in a symmetric R-space M belong to an orbit of an irreducible group $G(\mathcal{A}, B)$ if and only if their Takeuchi indices p, q, s coincide in the case of a Weil group of type C and the Takeuchi indices p, q, s_1, s_2 coincide in the case of a Weil group of type A. In the case of a Weil group of type D all orbits for which $s < r$ (and in the case C - also the set of points with Takeuchi indices $s = r$) consists of two connected components, each of which is an orbit of $G(\mathcal{A}, B)$.

Comments. A theorem concerning the orbits of the group of reduced type is also given by MAKAREVICH [18 c] (see Theorem 2 in [18 c, p.174]).

Remark. In the special case when the dual space is one of the orbits, this problem was solved by TAKEUCHI [30 b]. Another special case is the one discovered by WOLF [31].

Comments 1. MAKAREVICH's [18 a] results are exhaustive with one exception: he does not considered homogeneous spaces associated with special Jordan algebras.

Comments 2. MAKAREVICH remarked at the end of [18 c] that some formulas established in it are also valid for Grassmann manifolds. It would be interesting to relate these results to results on Grassmann manifolds given by GEL'FAND and Mac PHERSON [9], and by IORDANESCU [12 a,b].

NAITOH [21 a] has defined orthogonal Jordan triple systems and has constructed pseudo-Riemannian symmetric R-spaces from orthogonal Jordan triple systems with a certain condition. They are extensions of compact Jordan triple systems and Riemannian symmetric R-spaces.

Recall now the results given by NAITOH in [21 b]. Let M be a complete connected parallel submanifold of a pseudo-Euclidean space E satisfying that, for any point $p \in M$, (1) the normal space at p is linearly spanned by the image of the second fundamental form and (2) there exists a normal vector at p such that its shape operator is the identity map. Then, (A) all such spaces M are exhausted by pseudo-Riemannian symmetric R-spaces.

Moreover assume that E is a pseudo-Hermitian space and M is a totally real submanifold of E such that 2 dim M = dim E. Then, (B) all such spaces M are exhausted by pseudo-Riemannian symmetric

R-spaces constructed from orthogonal Jordan triple systems such that the underlying Jordan triple systems are associated with Jordan algebras with unity.

Next, (C) pseudo-Riemannian symmetric R-spaces are imbedded as minimal submanifolds of some pseudo-Riemannian hyperspheres in E if and only if the associated orthogonal Jordan triple systems are non-degenerate Jordan triple systems with the trace forms. Lastly, NAITOH's paper [21 b] lists up pseudo-Riemannian symmetric R-spaces with simple Jordan triple systems.

Comments 1. As KANTOR remarked [Math.Rev. # 81 e :17008], the HIRZEBRUCH's construction [11], "aside from algebraic interest, is of interest for the study of symmetric R-spaces".

Comments 2. SATAKE's method [27] can be applied to the determination of Q - forms of real semisimple Lie algebras corresponding to symmetric R-spaces.

Recently, for a compact (connected) symmetric space M, CHEN and NAGANO [5] introduced a Riemannian geometric invariant, called the two-number ϑ (M), as follows: points $p, q \in M$ are said to be "antipodal" to each other, if p=q or there exists a closed geodesic of M on which p and q are antipodal to each other; a subset A of M is called an "antipodal subset" if every pair of points of A are antipodal to each other; now the two-number ϑ (M) is defined as the maximum possible cardinality of an antipodal subset A of M.

Remark . The two-number is finite.

TAKEUCHI [30 d] proved the following

Theorem 1.5. If M is a symmetric R-space, then

$$\vartheta(M) = \dim H(M, \mathbb{Z}_2),$$

where $H(M, \mathbb{Z}_2)$ denotes the holonomy group of M with coefficients $\mathbb{Z}_2$.

Finally, let us mention that BACKES [1] and BACKES and RECKZIEGEL [2] applied Jordan triple systems to the classification of pseudoumbilical symmetric submanifolds of hyperbolic space, and to the classification of all isometric immersions with parallel second fundamental form into standard spaces, respectively.

§ 2. Isoparametric triple systems

Several recent papers of DORFMEISTER and NEHER [6 a,b,c] deal principally with isoparametric hypersurfaces in spheres and show that homogeneous examples with four distinct principal curvatures are closely related to certain Jordan triple systems (see Theorem 2.1 below). Then isoparametric triple systems (see below for definition) of certain types are studied.

We first recall some facts about isoparametric hypersurfaces [6 d], which are one of the three important classes of oriented submanifolds of codimension 1 of a sphere.

Let V be a Euclidean vector space and let S be its unit sphere. Then the inner product $\langle , \rangle$ of V induces on S, as well as on every hypersurface M of S, a Riemannian metric $\langle , \rangle^S$, resp. $\langle , \rangle^M$, which determines uniquely the covariant derivative ∇^S, resp. ∇^M. If ζ is a normal vector field for M in S, then

$$\nabla^M_X Y = \nabla^S_X Y - (X,Y)\zeta$$

for all vector fields X and Y of M, where (X,Y) is a symmetric bilinear form. Thus, for each point p of M we get a selfadjoint endomorphism S(p) of the tangent space to M at the point $p \in M$, such that

$$(X,Y)_p = \langle S(p)X_p, Y_p\rangle^M$$

holds. The endomorphism S(p) is called the <u>second fundamental form</u> of M in S. A hypersurface of S is uniquely determined (up to a rigid motion) by the Riemannian metric and the second fundamental form (see [1]). It follows that various classes of hypersurfaces of S can be determined by special properties of S(p), $p \in M$. Thus, the <u>totally geodesic</u> hypersurfaces are characterized by the condition $S(p) = 0$, $p \in M$, the <u>umbilical</u> hypersurfaces by the condition $S(p) = \lambda(p)\, Id$, $p \in M$, and, finally, the <u>isoparametric</u> hypersurfaces by the condition that for all points p of M the endomorphism S(p) has the same eigenvalues with the same multiplicities. Hence, isoparametric hypersurface are hypersurfaces with constant principal curvatures.

The earliest research into isoparametric hypersurfaces is due to CARTAN [4 a,b]. Apparently they have been forgotten down to the papers [22 a] and [22 b] of NOMIZU. In [29] TAKAGI and TAKAHASHI gave the classification of all homogeneous hypersurfaces in spheres, which includes the description of all isoparametric hypersurfaces with at most 3 distinct principal curvatures, since CARTAN had shown that all such hypersurfaces are homogeneous. MÜNZNER proved [19] that the number g of distinct principal curvatures of an isoparametric hypersurface in a sphere is 1, 2, 3, 4 or 6. Moreover, Münzner showed that each such hypersurface is an open submanifold of a level surface of a homogeneous polynomial of degree g, and characterized these polynomials by two differential equations. It remains to consider the cases $g = 4$ and $g = 6$ and to classify the corresponding polynomials. The first examples of nonhomogeneous isoparametric hypersurfaces with $g = 4$ were given by OZEKI and TAKEUCHI [24] and have recently been generalized by FERUS, KRACHER, and MÜNZNER [8].

Adopting the description of isoparametric hypersurfaces in terms of homogeneous polynomials given by MÜNZNER [19], DORFMEISTER and NEHER [6 b] showed that the theory of the geometric object "isoparametric hypersurface in a sphere with $g = 4$" is equivalent to the theory of the algebraic object "isoparametric triple system".

To understand the meaning of Theorem 2.1 (see below), recall the following.

Definitions. A compact Jordan triple system of real rank 2 is a real finite-dimensional Jordan triple system for which the reduced trace σ is positive definite and if every system of orthogonal tripotents contains at most two elements. Recall [19] that an isoparametric hypersurface M of the sphere in a Euclidean vector space (V $\langle\, ,\, \rangle$) is called homogeneous if there exists a subgroup of $O(\langle\, ,\, \rangle)$ which leaves M invariant and acts transitively on M.

Theorem 2.1. There exists a one-to-one correspondence between the congruence classes of homogeneous isoparametric hypersurfaces with $g = 4$ and the isomorphism classes of compact Jordan triple systems $\mathcal{T}$ of real rank 2 for which $\dim \mathcal{T}(c) \geq 2$ and $P(c)\big|_{\mathcal{T}_2(c)} \neq \mathrm{Id}$ for every minimal tripotent c of $\mathcal{T}$. (As always, $P(x)$ denotes the quadratic representation of the Jordan triple system).

Definition. Let V be a finite-dimensional Euclidean vector space with inner product denoted by $\langle\, ,\, \rangle$, and suppose that V is endowed with a trilinear map $V \times V \times V \longrightarrow V$, denoted by

$$(x,y,z) \longrightarrow \{xyz\} =: T(x,y)z,$$

such that for all x_i, x, $y \in V$ $(i=1,2,3)$ the following conditions hold:

(1) $\{x_1 \, x_2 \, x_3\} = \{x_{\sigma(1)} \, x_{\sigma(2)} \, x_{\sigma(3)}\}$ for any permutation σ ;

(2) $\langle \{x_1 x_2 x\}, y \rangle = \langle x, \{x_1 \, x_2 \, y\} \rangle$;

(3) $$\{xx \, \{xx\}\} - 6\langle x,x\rangle \{xxx\} - 3\langle \{xxx\}, x\rangle x + 18\langle x,x\rangle^2 x = 0 ;$$

(4) there are positive integers m_1, m_2 such that $\mathrm{Tr}\, T(x,y) = 2(3+2m_1+m_2)\langle x,x\rangle$ and $\dim V = 2(1+m_1+m_2)$.

Then V is called an <u>isoparametric triple system</u>.

<u>Remark</u>. Condition (1) means that the trilinear map is totally symmetric, while condition (2) means that the endomorphism $T(x,y)$ is selfadjoint with respect to $\langle \, , \rangle$.

As there is no classification of all isoparametric hypersurfaces in spheres available yet, special types of hypersurfaces, i.e. special types of isoparametric triple system, were investigated. Thus, DORFMEISTER and NEHER classified [6 a] isoparametric triple systems of <u>algebra type</u>. Such triples correspond uniquely to the isoparametric hypersurfaces satisfying condition (A) of [24, I] but not necessarily the additional condition (B) of [24, I]. Another subclass of isoparametric triple systems, closely related to Jordan triple systems and therefore called the class of triples with <u>Jordan composition</u> (or, briefly, the class of <u>JC-type</u>) is considered in [6 b]. Isoparametric triple systems which correspond to the example constructed by FERUS, KARCHER and MÜNZNER [8], and hence called isoparametric triple systems of <u>FKM-type</u> by Dorfmeister and Neher, are investigated in [6 c] and relations with triple systems of algebra type are given.

<u>Open problem</u>. To prove DORFMEISTER and NEHER's conjecture (see [6 d, p.18]): "Every isoparametric triple system is either homogeneous (i.e. the corresponding isoparametric hypersurface is homogeneous) or equivalent to a triple of FKM-type".

References

1. BACKES, E., Geometric applications of Euclidean Jordan triple systems, Manuscripta Math. 42 (1983), No. 2-3, 265-272.

2. BACKES, E., RECKZIEGEL, H., On symmetric submanifolds of spaces of constant curvature, Math. Ann. 263 (1983), No. 4, 419-433.

3. BISHOP, R., CRITTENDEN, R., Geometry of manifolds, Academic Press, New York and London, 1964.

4. CARTAN, É., a) Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, Math. Z. 45 (1939), 335-367.

b) Sur quelques familles remarquables d'hypersurfaces C.R. Congrès Math. Liège, 30-41, 1939.

5. CHEN, B.-Y., NAGANO, T., A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc. 308 (1988), 273-297.

6. DORFMEISTER, J., NEHER, E., a) Isoparametric triple systems of algebra type, Osaka J. Math. 20 (1983), No. 1, 145-175.

b) An algebraic approach to isoparametric hypersurfaces in spheres I, II, Tôhoku Math. J. 35 (1983), 187-224; 225-247.

c) Isoparametric triple systems of FKM-type I, II, III, Abh. Math. Sem. Univ. Hamburg 53 (1983), 191-216; Manuscripta math. 43 (1983), No. 1, 13-44; Algebra Groups Geom. 1 (1984), 305-343.

d) Isoparametric hypersurfaces in spheres and Peirce decomposition (a survey) (to appear).

e) Isoparametric hypersurfaces, case g=6, m = 1, Comm. Algebra 13 (1985), 11, 2299-2368.

7. FERUS, D., Symmetric submanifolds of Euclidean space, Math. Ann. 247 (1980), 81-93.

8. FERUS, D., KARCHER, H., MÜNZNER, H.F., Cliffordalgebren und neue isoparametrische hyperflächen, Math. Z. 177(1981), No.4, 479-562.

9. GEL'FAND, I.M. MacPHERSON, R.D., Geometry in Grassmannians and a generalization of the dilogarithm, Preprint IHES, Paris, June 1981.

10. HAHN, J.M., Homogene Hyperflächen in der pseudoriemannchen Geometrie, Dissertation, Bonn, 1986.

11. HIRZEBRUCH, U., A generalization of Tits'construction of Lie algebras by Jordan algebras to Jordan triple systems, Nederl. Akad. Wetensch. Indag. Math. 40 (1978), No.4, 456-459.

12. IORDANESCU, R., a) Sulle strutture quaternionali grassmanniane, Boll. Un. Mat. Ital. 10 (1974), 406-411.

b) Symmetric spaces investigated by means of Jordan algebras, INCREST, Bucharest, 1981.

13. KANTOR, I.L., Generalized norm of Jordan algebras and the geometry of certain transformations groups (in Russian), Trudy Sem. Vektor. Tenzor. Anal. 14 (1968), 114-143.

14. KANTOR, I.L., SIROTA, A.I., SOLODOVNIKOV, A.S., A class of symmetric spaces with extended motion group and a generalization of Poincare's model (in Russian), Dokl. Akad. Nauk SSSR 173 (1967), 511-514.

15 KOBAYASHI, S., TAKEUCHI, M., Minimal imbeddings of R-spaces, J. Differential Geom. 2 (1968), 203-215.

16. KOECHER, M., Imbedding of Jordan algebras into Lie algebras, II, Amer. J. Math. 90 (1968), No. 2, 476-510.

17. LOOS, O., a) Jordan triple systems, R-spaces and bounded symmetric domains, Bull. Amer. Math. Soc. 77 (1971), 558-561.

b) Charakterisierung symmetrischer R-Räume durch ihre Einheitsgitter, Math. Z. 189 (1985), No. 2, 211-226.

18. MAKAREVICH, B. O., a) Open symmetric orbits of the reductive groups in symmetric R-spaces (in Russian), Mat. Sb. (N.S.) 91 (1973), 3, 389-402.

b) Ideal points of semisimple Jordan algebras (in Russian) Mat. Zametki 15 (1974), No. 2, 295-305.

c) Jordan algebras and orbits in symmetric R-spaces (in Russian), Trudy Moscov. Obshch. 39 (1979), 157-179.

19. MÜNZNER, H. F., Isoparametrische Hyperflächen in Sphären, I, II, Math. Ann. 251 (1980), 1, 57-71, 256 (1981), 3, 215-232.

20. NAGATO, T., Transformation groups on compact symmetric spaces, Trans. Amer. Math. Soc. 118 (1965), 428-453.

21. NAITOH, H., a) Parallel submanifolds of complex space forms I, II, Nagoya Math. J. 90 (1983), 85-117; 91 (1983), 119-149.

b) Pseudo-Riemannian symmetric R-spaces, Osaka J. Math. 21 (1984), 733-764.

22. NOMIZU, K., a) Élie Cartan's work on isoparametric hypersurfaces, Symp. Pure Math. Diff. Geom. 27, II, Stanford, 1973.

b) Some results in É. Cartan's theory of isoparametric famillies of hypersurfaces, Bull. Amer. Math. Soc. 79 (1973), No. 6, 1184-1188.

23. OHNITA,Y., The degrees of the standard imbeddings of R-spaces, Tôhoku Math.J. 35 (1983), No.4,499-502.

24. OZEKI,H., TAKEUCHI,M.,On some types of isoparametric hypersurfaces in spheres, I,II, Tôhoku Math.J. 27 (1975), 515-559, 28 (1976), 7-55.

25. PYATETSKII-SHAPIRO,I.I.,Geometry of classical domains and theory of automorphic functions, Fizmatghiz., Moscow, 1961;French edition, Dunod,Paris, 1966; English edition Gordon and Breach, New York,1967.

26. RIVILIS,A.A., Homogeneous locally symmetric domains in homogeneous spaces associated with semisimple Jordan algebras (in Russian), Math.Sb.(N.S.) 82 (1970),No.2, 262-290.

27. SATAKE,I., Q-forms of symmetric domains and Jordan triple systems, in "Algebraic number theory" (Kyoto Internat.Sympos., Res.Inst.Math.Sci.,Univ.Kyoto, Kyoto, 1976), pp.163-176, Japan Soc.Promotion Sci., Tokyo, 1977.

28. TAKAGI,H., A condition for isoparametric hypersurfaces of S^n to be homogeneous, Tôhoku Math.J. 37 (1985),No.2, 241-250.

29. TAKAGI,H., TAKAHASHI,T., On the principal curvatures of homogeneous hypersurfaces in a sphere, Diff.Geom.in honor of K.Yano, Kinokunija, Tokyo, 1972,469-481.

30. TAKEUCHI,M., a) Cell-decompositions and Morse equalities on certain symmetric spaces, J.Fac.Sci.Univ.Tokyo 12 (1965), 81-192.

b) On orbits in a compact Hermitian space,Amer.J. Math.90 (1968),No.3, 657-680.

c) Basic transformation of symmetric R-spaces, Osaka J.Math.25 (1988), No.2, 259-298.

d) Two-number of symmetric R-spaces, Nagoya Math. J.115 (1989), 43-46.

31. WOLF,J.A., The action of a real semisimple group on a complex flag manifold I: Orbit structure and holomorphic arc components, Bull.Amer.Math.Soc.75 (1969),No.6, 1121-1237.