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JORDAN STRUCTURES WITH APPLICATIONS. V. JORDAN ALGEBRAS IN ANALYSIS

by

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The principal results in Jordan-Banach agebras are discussed in this paper, Section 1 and 2. Section 3 deals with a class of partially ordered Jordan algebras which are non-associative generalizations of semifields, while complete normed Jordan algebras over the complex field are treated in Section 4. Section 5 is concerned with two kinds of structures: certain infinite-dimensional Jordan algebras admitting an inner product, and certain Banach spaces whose open unit balls are bounded symmetric homogeneous domains. Jordan algebras and positive projections on operator algebras are examined in Section 6, while theta-functions for Jordan algebras are constructed in Section 7. Section 8 gives several applications of Jordan algebras to the Riccati equation, soliton equations and Hua equations, as well as to Szegö kemel, and to the (reproducing) kernel functions. A large bibliography supplements the text.

§ 1. JB - and JB*-algebras

The relationship between formally real Jordan algebras, self-dual homogeneous cones and symmetric upper half-planes in finite dimensions due to KOECHER [87 a,b,d] is the background for the study of the infinite-dimensional case. The objects here are JB*-algebras and their real analogues (the so-called JB-algebras). Using the results reported by KAUP [84 a] together with more recent results, in the first two sections of this paper we shall discuss the principal developments of this topic.

A generalization of formally real Jordan algebras to the infinite-dimensional case was introduced and studied by ALFSEN, SHULTZ and STØRMER [6] as follows:

Definition. A (linear) real Jordan algebra $\mathcal{J}$ with unit element e which is also a Banach space and in which the product and the norm satisfy

(i) $\|xy\| \leq \|x\| \, \|y\|$,
(ii) $\|x^2\| = \|x\|^2$,
(iii) $\|x^2\| \leq \|x^2 + y^2\|$,

for all $x, y \in \mathcal{J}$ is called a Jordan Banach algebra (or, briefly, a JB-algebra).

Remark. In the finite-dimensional case, condition (iii) is equivalent to the fact that $\mathcal{J}$ is a formally real Jordan algebra.

Comments 1. The term JB-algebra arose as the Jordan analogue of B^*-algebra, much the same as JC-algebras and JW-algebras were termed after C^*- and W^*-algebras, respectively.

Comments 2. Nonunital JB-algebras have also been considered, but a nonunital JB-algebra can always be algebraically and isometrically embedded in a unital JB-algebra (see SMITH [125] and YOUNGSON [141 d]). Addition of unity was proved for JB-algebras as defined above by BEHNCKE [23 a].

Example. One of the most important examples of JB-algebras is the selfadjoint part of a real or complex C^*-algebra equipped with the Jordan product $xy := \frac{1}{2}(x \circ y + y \circ x)$.

Remark. One can prove that the norm on $\mathcal{J}$ is completely determined by the Jordan product in $\mathcal{J}$, which was, denoted in the example above by juxtaposition of elements.

Comments. HANCHE-OLSEN and STØRMER introduced [63] the concept of JB-algebra as follows: A Jordan Banach algebra is a real Jordan algebra A (not necessarily unital) equipped with a complete norm satisfying $\| a\, b\| \leq \|a\| \, \|b\|$, $a, b \in A$. A JB-algebra is a Jordan Banach algebra A in which the norm satisfies the following two additional conditions for $a, b \in A$,

$$1^{o}.\ \|a^2\| = \|a\|^2, \text{ and } \quad 2^{o}.\ \|a^2\| \leq \|a^2 + b^2\|.$$

Theorem 1.1. A real Jordan algebra of finite-dimension is a JB-albegra if and only if it is formally real.

The algebras $H_p(\mathbb{F})^{(+)}$ (see JSAI, § 1) can be extended to arbitrary cardinality p as follows. Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let H be a (right) $\mathbb{F}$-Hilbert space of dimension p over $\mathbb{F}$. Denote by $\mathcal{L}(H)$ the algebra of all bounded $\mathbb{F}$-linear operators on H. Then there exists a natural involution (the adjoint $*$) on $\mathcal{L}(H)$ and $\mathcal{H}_p(\mathbb{F}) := \mathcal{H}(H) := \{ \lambda \in \mathcal{L}(H) \mid \lambda^* = \lambda \}$ is a JB-algebra with respect to the operator norm.

Remark. For every compact topological space S and every JB-algebra $\mathcal{J}$, the algebra $\mathcal{C}(S, \mathcal{J})$ of all continuous functions $S \longrightarrow \mathcal{J}$ is also a JB-algebra. In particular, $\mathcal{C}(S, \mathbb{R})$ is an associative JB-algebra.

Proposition 1.2. Every associative JB-algebra is isometrically isomorphic to $\mathcal{C}(S, \mathbb{R})$ for some compact topological space S.

ALFSEN, SHULTZ and STØRMER [6] showed that $H_3(\mathbb{O})$ is essentially the only JB-algebra which cannot be realized as an algebra of selfadjoint operators on a complex Hilbert space, by proving the Gelfand-Neumark theorem for JB-algebras (see Theorem 1.3 below).

Theorem 1.3. Let $\mathcal{J}$ be a JB-algebra. There exists a complex space H and a compact topological space S such that $\mathcal{J}$ is

isometrically isomorphic to a closed subalgebra of $\mathcal{H}(H) \oplus$
$\oplus \mathcal{C}(S, H_3(\mathbb{O}))$.

Notation. Let $\mathcal{J}$ be a JB-algebra and denote by
$\mathcal{J}^2 := \{x^2 \mid x \in \mathcal{J}\}$ the positive cone in $\mathcal{J}$.

Remark. $\mathcal{J}^2$ is a closed convex cone and the interior Ω of $\mathcal{J}^2$ is an open convex cone which is not empty (one can show that Ω is the connected identity component of $\mathrm{Inv}(\mathcal{J})$).

Definition. The elements of Ω are called positive definite, and an ordering $<$ on $\mathcal{J}$ is defined by : $x<y$ for $x, y \in \mathcal{J}$ if and only if $y-x \in \Omega$.

Definition. A subcone $\mathcal{H}$ of $\mathcal{J}^2$ is said be a face of $\mathcal{J}^2$ if $\mathcal{H}$ contains all elements a of $\mathcal{J}^2$ such that $a \leq b$ for some $b \in \mathcal{H}$.

Definition. A JB-subalgebra B of $\mathcal{J}$ is said to be hereditary if its positive cone B^2 is a face of $\mathcal{J}^2$.

EDWARDS [45 a] proved the following results (see also [48a]):

Theorem 1.4. The norm-closed quadratic ideals of a JB-algebra $\mathcal{J}$ coincide with the hereditary JB-subalgebras B of $\mathcal{J}$, and the norm-closed faces of $\mathcal{J}^2$ are the positive cones B^2 of such subalgebras B.

Proposition 1.5. The norm closure of a face of $\mathcal{J}^2$ is a face of $\mathcal{J}^2$.

PUTTER and YOOD [106] generalized a number of well-known Banach algebra results to the Jordan algebra situation by appropriately modifying the proofs. They confined themselves to special JB-algebras.

Notation. Let $\mathcal{L}(\mathcal{J})$ be the algebra of all bounded operators in $\mathcal{J}$ and put, for every subset $A \subset \mathcal{J}$, $GL(A) := \{ g \in \mathcal{L}(\mathcal{J}) \mid g$ invertible and $g(A) = A \}$.

For every t of Ω the quadratic representation P(t) is in GL(Ω) and sends e to t^2, i.e. $P(\Omega) \subset GL(\Omega)$ generates a transitive linear transformation group on Ω; in particular Ω is a (linearly) homogeneous cone.

The isotropy subgroup of GL(Ω) at $e \in \Omega$, i.e. the subgroup of $g \in GL(\Omega)$ with g(e) = e, is the group Aut $\mathcal{J}$ of all algebra automorphisms of $\mathcal{J}$.

Remark. The uniqueness of the JB-norm on $\mathcal{J}$ implies that every algebra automorphism of $\mathcal{J}$ is an isometry and, in particular, a homeomorphism. Therefore $GL(\Omega) = P(\Omega)\,\mathrm{Aut}\,\mathcal{J}$.

Consider now the complexification $\mathcal{J}^{\mathbb{C}} := \mathcal{J} \oplus i\mathcal{J}$ of $\mathcal{J}$. $\mathcal{J}^{\mathbb{C}}$ is a complex Jordan algebra with involution $(x+iy)^* := x-iy$.

Definition. $D := D(\Omega) := \{ z \in \mathcal{J}^{\mathbb{C}} \mid \mathrm{Im}(z) \in \Omega \}$ is called the tube domain (generalized upper half-plane) associated with the cone Ω.

Let us mention here that TSAO [131] proved that under certain conditions the Fourier coefficients of the Eisenstein series for an arithmetic group acting on a tube domain are rational numbers. The proof involves a mixture of Lie groups, Jordan algebras, Fourier analysis, exponential sums, and L-functions.

From the results reported by ALFSEN, HANCHE-OLSEN, SHULTZ and STØRMER [6], [4], it follows that for each JB-algebra $\mathcal{J}$ there exist a canonical C^*-algebra $\mathcal{O}\!\mathcal{b}$ and a homomorphism $\psi : \mathcal{J} \longrightarrow \mathcal{O}\!\mathcal{b}$ such that $\psi(\mathcal{J})$ generates $\mathcal{O}\!\mathcal{b}$. The kernel of ψ is the exceptional ideal J in $\mathcal{O}\!\mathcal{b}$. Using Takesaki and Tomiyama's methods, BEHNCKE and BÖS showed [24] that J may be described as an $H_3(\mathbb{O})$-fibre bundle over its primitive ideal space.

Comments. As was observed by UPMEIER [133 b], a promising application of JB-algebras is to be found in complex analysis, based on the one-to-one correspondence between JB*-algebras and bounded symmetric domains in complex Banach spaces with tube realization (KOECHER [87 d], and BRAUN, KAUP and UPMEIER [38 b]).

If we identify $GL(\mathcal{J})$ in a natural way with a subgroup of $GL(\mathcal{J}^{\mathbb{C}})$, then the group of complex affine transformations

$$\mathrm{Aff}(D) := \left\{ z \longrightarrow \lambda z + t \;\middle|\; \lambda \in GL(\Omega),\ t \in \Omega \right\}$$

is transitive on D and, in particular, D is (holomorphically) homogeneous.

Remark. If $\mathcal{J} = \mathbb{R}$, then $\mathcal{J}^{\mathbb{C}} = \mathbb{C}$ and $D = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Im}(z) > 0 \right\}$ is the classical upper half-plane. The Cayley transformation $z \longrightarrow i(z-i)(z+i)^{-1}$ maps D biholomorphically onto the open unit disk $\Delta = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\}$.

Every $z \in D$ is invertible and $z \longrightarrow i\,(z-ie)\,(z+ie)^{-1}$ defines a biholomorphic map from D onto the bounded balanced and convex domain $\Delta \subset \mathcal{J}^{\mathbb{C}}$. Actually $\Delta := \left\{ z \in \mathcal{J}^{\mathbb{C}} \;\middle|\; |P(z)P(z^*)|_\sigma < 1 \right\}$, where $|\ |_\sigma$ denotes the spectral radius. Therefore,

$$(1.1) \qquad \| z \| := \| z \|_\infty := |P(z)P(z^*)|_\sigma^{1/4}$$

defines a complex norm on $\mathcal{J}^{\mathbb{C}}$ such that Δ is the open unit ball of $\mathcal{J}^{\mathbb{C}}$. This norm extends the JB-norm on $\mathcal{J}$.

Definition. The above Δ is called the generalized unit disk and $\sum := \left\{ z \in \mathcal{J}^{\mathbb{C}} \;\middle|\; z \text{ invertible},\ z^* = z^{-1} \right\}$ is called the generalized unit circle.

Proposition 1.6. An element z of $\mathcal{J}^{\mathbb{C}}$ is in $\sum$ if and only if $P(z)$ is a surjective isometry of $\mathcal{J}^{\mathbb{C}}$.

One can show that the extreme boundary $\check{S}$ of $\overline{\Delta}$ is a closed real analytic submanifold of $\mathcal{J}^{\mathbb{C}}$, and that $\sum$ is a union of certain connected components of $\check{S}$. Actually, $s \in \check{S}$ is in

Σ if and only if the tangent space to $\check{S}$ in s is a real form of $\mathcal{J}^{\mathbb{C}}$. The set $\exp(i\mathcal{J})$ is a connected subset of Σ, and for every connected component T of Σ (or, more generally, of $\check{S}$) $\overline{\Delta}$ is the closed convex hull $\overline{\mathrm{co}\, T}$ of T in $\mathcal{J}^{\mathbb{C}}$. In a norm on $\mathcal{J}^{\mathbb{C}}$ in which the unit ball $\overline{\mathrm{co}(\exp(i\mathcal{J}))}$, is closed, WRIGHT [139] showed that $\mathcal{J}^{\mathbb{C}}$ is a JB^*-algebra (= Jordan C^*-algebra). However, the norm (1.1) is equivalent to the JB^*-norm of Wright.

Definition[1]. A JB^*-algebra is a complex Jordan algebra $\mathcal{J}$ with unit element e, (conjugate linear) involution $*$, and complete norm such that

(i) $\| xy \| \leq \| x \| \; \| y \|$,

(ii) $\| P(z)z^* \| = \| z \|^3$,

for all $x,y,z \in \mathcal{J}$.

Comments. YOUNGSON [141 d] studied JB^*-algebras in the nonunital case. He stated, among other results, that nonunital JB^*-algebras are C^*-triple systems in the sense of KAUP [84 b] (see also, JSA VI, § 1).

Example. Every unital C^*-algebra endowed with a Jordan product is a JB^*-algebra.

Note. There exist JB^*-algebras which cannot be embedded in any C^*-algebra.

Remark. Every closed associative $*$ - subalgebra of a JB^*-algebra is a commutative C^*-algebra.

Proposition 1.7. The selfadjoint part of a JB^*-algebra is a JB-algebra.

1) The concept of JB^*-algebra was formulated by KAPLANSKY [82] and introduced as "Jordan C^*-algebra".

WRIGHT [139] proved the converse:

Theorem 1.8. For every JB-algebra $\mathcal{J}$ there exists a unique complex norm on $\mathcal{J}^{\mathbb{C}}$ such that $\mathcal{J}^{\mathbb{C}}$ is a JB*-algebra with selfadjoint part $\mathcal{J}$. The correspondence $\mathcal{J} \longleftrightarrow \mathcal{J}^{\mathbb{C}}$ defines an equivalance of the category of JB-algebras onto the category of JB*-algebras

RUSSO and DYE [112] proved that the closed unit ball of a C*-algebra with identity is the convex hull of its unitary elements. The same result was proved by WRIGHT and YOUNGSON [140 a] for JB*-algebras.

Using the fact that the extreme points of the positive unit ball in a JB-algebra are projections, WRIGHT and YOUNGSON first showed [140 b] that a surjective unital linear isometry between two JB-algebras is a Jordan isomorphism, and then used this to obtain the same result for JB*-algebras.

BONSALL [33] showed that if B is a real closed Jordan subalgebra of a complex unital Banach algebra A, containing the unit and such that $B \cap iB = \{0\}$ and $B \subset H(B) \oplus iH(A)$, where H(A) denotes the set of Hermitian elements of A, then $B \oplus iB$ is homeomorphically $*$-isomorphic to a JB*-algebra. Using WRIGHT's and YOUNGSON's results [139], [140 a], [141 a,b], MINGO [98] gave a JB*-analogue of a C*-algebra result STØRMER [127 d], as follows.

Proposition 1.9. Suppose A is a JB*-algebra and B is a real selfadjoint subalgebra with unit such that $B \cap iB = \{0\}$. Then $B \oplus iB$ is a JB*-algebra.

Mingo used Proposition 1.9 to prove the above-mentioned result of Bonsall, dispensing with the assumption $B \cap iB = \{0\}$, and also to prove that the isomorphism is an isometry.

Definition. A bounded domain B in a complex Banach space is called symmetric if for every a of B there exists a holomorphic map $s_a : B \longrightarrow B$ with $s_a^2 = Id_B$ and a an isolated fixed point.

(s_a is uniquely determined if it exists and is called the symmetry at a.)

Remark. The generalized unit disk Δ is (holomorphically) homogeneous and $s_0(z) = -z$ is the symmetry at 0, i.e. Δ is symmetric.

Definition. For every open cone C in a real Banach space X, the domain $T := \{ z \in X \oplus iX \mid \mathrm{Im}(z) \in C \}$ is called a symmetric tube domain if T is biholomorphically equivalent to a bounded symmetric domain.

Theorem 1.10. Let $\mathcal{J}$ be a JB-algebra and let $\mathcal{J}^{\mathbb{C}} = \mathcal{J} \oplus i\mathcal{J}$ be the corresponding JB*-algebra. Then $D := \{ z \in \mathcal{J}^{\mathbb{C}} \mid \mathrm{Im}(z) \in \Omega \}$ is a symmetric tube domain. The symmetry at the point $ie \in D$ is given by $s(z) = -z^{-1}$, and $z \longrightarrow i(z-ie)(z+ie)^{-1}$ maps D biholomorphically on the open unit ball Δ of $\mathcal{J}^{\mathbb{C}}$. In particular, Δ is a homogeneous domain.

BRAUN, KAUP and UPMEIER [38 a] proved

Theorem 1.11. If B is a real Banach space and D is the symmetric tube domain for $\mathcal{B}^{\mathbb{C}} := \mathcal{B} \oplus i\mathcal{B}$, then for every $e \in \Omega$ there exists a unique Jordan product on $\mathcal{B}$ such that $\mathcal{B}$ is a JB-algebra with unit e, and D is the upper half-plane.

Remark. It follows that JB-algebras, as well as JB*-algebras, are in one-to-one correspondence with symmetric tube domains.

In the theory of formally real Jordan algebras of finite dimension an important fact is the minimal decomposition of elements of such an algebra with respect to a complete orthogonal system of primitive idempotents $\{ e_1, \ldots, e_k \}$. The importance of the minimal decomposition follows from the fact that

$\{e_1, \ldots, e_k\}$ determine a Peirce decomposition of the algebra which, for instance, diagonalizes the operator $L(x)$, and hence also $P(x)$.

The analogue for an arbitrary JB-algebra $\mathcal{J}$ is the fact that for every $\alpha \in \mathcal{J}$ the unital closed subalgebra $C(\alpha)$ generated by α is isomorphic to some $\mathcal{C}(S, \mathbb{R})$, where S is a compact topological space. However, in case S is connected, e is the only nontrivial idempotent in $C(\alpha)$ and the Peirce decomposition cannot be applied.

SHULTZ [120] proved that the bidual of JB-algebra with the Arens product is also a JB-algebra. Hence, every JB-algebra is a norm-closed subalgebra of a JB-algebra which is a dual Banach space. Algebras of this type admit not only a continuous but also an L^{∞}-functional calculus.

<u>Remark</u> EDWARDS [45 c] showed how some of the results on multipliers and quasi-multipliers of C^*-algebras can be extended to JB-algebras.

§ 2. JBW-algebras

<u>Definition</u>. A JB-algebra $\mathcal{J}$ is called a JBW-<u>algebra</u> if $\mathcal{J}$ is a dual Banach space (i.e. there exists a Banach space $'\mathcal{J}$ with $\mathcal{J} = '\mathcal{J}'$ as dual Banach space; $'\mathcal{J}$ is uniquely determined by $\mathcal{J}$ (see SAKAI [113]) and is called the <u>predual</u> of $\mathcal{J}$).

<u>Example</u>. The selfadjoint part of a von Neumann algebra is a JBW-algebra.

<u>Remark</u>. For every α in the JBW-algebra $\mathcal{J}$ the w^*-closed unital subalgebra $W(\alpha)$ of $\mathcal{J}$ generated by α is a commutative von Neumann algebra, i.e. $W(\alpha) \approx \mathcal{C}(S, \mathbb{R})$ for S hyperstonian or, equivalently, $W(\alpha) \approx L^{\infty}(\mu)$, where μ is a localizable measure

(see SAKAI [113]).

Important remark. JBW-algebras (weakly closed analogues of JB-algebras) are the abstract analogues of von Neumann algebras in the Jordan case.

SHULTZ [123] proved the Gelfand-Neumark theorem for JBW-algebras:

Theorem 2.1. Every JBW-algebra $\mathcal{J}$ is a unique direct sum (as algebras) $\mathcal{J} = \mathcal{J}_{sp} \oplus \mathcal{J}_{ex}$, where $\mathcal{J}_{sp}$ is isomorphic to a weakly closed Jordan subalgebra of $\mathcal{H}(H)$ for some complex Hilbert space H, and $\mathcal{J}_{ex} \approx \mathcal{C}(S, H_3(\mathbb{O}))$, where S a is a compact hyperstonian space.

Comments. $\mathcal{J}_{sp}$ and $\mathcal{J}_{ex}$ are called the special part and the exceptional part of $\mathcal{J}$, respectively. Every JW-algebra (i. e. a weakly closed subalgebra of $\mathcal{H}(H)$, H a complex Hilbert space) is a JBW-algebra with $\mathcal{J} = \mathcal{J}_{sp}$. Let us mention in this respect AYUPOV's recent study [16 h], which starts from results given by EFFROS and STØRMER [48 a, b, 127 b,d]. The principal aim of [16 h] is to complete the research of connections between properties of JW-algebras and their enveloping von Neumann algebras in the general case. Ayupov obtained the types criteria for JW-algebras in terms of the existence of normal traces. These results are similar to those for von Neumann algebras. In particular, he considered the problem of the extension of traces from a JW-algebra to its enveloping von Neumann algebra. This problem is of interest in its own right and is also used in the above-mentioned results. Additionally, it gains in importance by applications in the theory of integration on Jordan algebras (see AYUPOV [16 c], the details being given by AYUPOV in [16 i].

Definition. A JBW-algebra $\mathcal{J}$ is called a factor if the centre of $\mathcal{J}$, defined by $Z(\mathcal{J}) := \{ x \in \mathcal{J} \mid [L(x), L(y)] = 0$ for all $y \in \mathcal{J} \}$, reduces to $\mathbb{R}e$ or, equivalently, if there does not exists a direct sum decomposition $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$ into proper subalgebras.

Remark. A good substitute for the minimal decomposition of the finite-dimensional case is the L^{∞}-functional calculus in JBW-algebras.

Proposition 2.2. If $\mathcal{J}$ is a JBW-algebra, then the unit circle $\Sigma = \{ z \in \mathcal{J}^{\mathbb{C}} \mid z^{*} = z^{-1} \}$ in $\mathcal{J}^{\mathbb{C}} = \mathcal{J} \oplus i\mathcal{J}$ is connected. Actually, $\Sigma = \exp(i\mathcal{J})$, and $GL(\Delta) = P(\Sigma)\,\mathrm{aut}\,\mathcal{J}$, Δ being the open unit ball in $\mathcal{J}^{\mathbb{C}}$.

Remark. The group Aut $\mathcal{J}$ is a real Banach Lie group. Its corresponding Banach Lie algebra will be denoted by aut $\mathcal{J}$. Note that aut $\mathcal{J}$ is the set of all derivation of the algebra $\mathcal{J}$.

For every $x, y \in \mathcal{J}$, the commutator $[L(x), L(y)]$ is contained in aut $\mathcal{J}$; every finite linear combination of such commutators is called an inner derivation of $\mathcal{J}$. The set int $\mathcal{J}$ of all inner derivations of $\mathcal{J}$ is an ideal of aut $\mathcal{J}$. In the finite-dimensional case we have aut $\mathcal{J}$ = int $\mathcal{J}$, while in general this is not true.

UPMEIER [133 b] proved

Theorem 2.3. If $\mathcal{J}$ is a JB-algebra, then aut $\mathcal{J}$ is the closure of int $\mathcal{J}$ in $\mathcal{L}(\mathcal{J})$ with respect to the strong operator topology (i.e. the topology of simple convergence on $\mathcal{J}$). If $\mathcal{J}$ is a JBW-factor that is not isomorphic to a factor of infinite dimension, then aut $\mathcal{J}$ is the norm closure in $\mathcal{L}(\mathcal{J})$ of int $\mathcal{J}$.

Remark. The methods and results given in [133 b] are applied by UPMEIER [133 a] to deduce fundamental algebraic properties of the Lie algebra aut $\mathcal{J}$ of a JB-algebra $\mathcal{J}$.

Recently, UPMEIER [133 c] showed that for a JBW-algebra $\mathcal{J}$ the connected identity component of Aut $\mathcal{J}$ is algebraically generated by involutions of the form P(s), where $s \in \mathcal{J}, s^2 = e$ and P denotes the quadratic representation (see JSA.I, § 1). The only exception to this result are spin factors of infinite dimensions; in this case, Aut $\mathcal{J}$ is only topologically generated by the transformations P(s). Applications to several algebraic and geometric objects associated with a JB-algebra $\mathcal{J}$ - like the positive cone of $\mathcal{J}$, the Jordan pair $(\mathcal{J}, \mathcal{J})$, etc. - as well as to dynamical systems are also given in [133 c].

<u>Definition</u>. A JBW-factor $\mathcal{J}$ is called of <u>type</u> I if $\mathcal{J}$ contains a primitive idempotent.

STØRMER [127 b] gave a complete classification of JBW-factors of type I :

<u>Theorem 2.4.</u> The JBW-factors of type I are precisely the following algebras, where p is an arbitrary cardinal number,

Type	algebra
I_1	$\mathbb{R}$
I_2	V_p , $p \geq 3$
I_p	$H_p(\mathbb{F})^{(+)}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and $p \geq 3$ ($p=3$ if $\mathbb{F} = \mathbb{O}$)[1].

<u>Definitions</u>. Let $\mathcal{J}'$ be the dual Banach space of a JB-algebra $\mathcal{J}$ and denote by $(\mathcal{J}^2)' := \{\lambda \in \mathcal{J}' \mid \lambda(\mathcal{J}^2) \geq 0\}$ the dual cone of $\mathcal{J}^2$. Then $K := \{\lambda \in (\mathcal{J}^2)' \mid \lambda(e) = 1\}$ is called the <u>state space of</u> $\mathcal{J}$, the elements of K being called <u>states on</u> $\mathcal{J}$.

<u>Remark</u>. K is a w^*-compact, convex subset and $\mathcal{J}$ can be identified (as a Banach space) with the space of all w^*-continuous

1) V_p, called the <u>spin factor</u> of dimension p, corresponds to $\mathcal{J}(Q)$ from JSA I, § 1.

affine functions on K. The bidual of $\mathcal{J}$ coincides with the set of all bounded affine functions on K.

In a comprehensive study of state spaces of a JB-algebra ALFSEN and SHULTZ [5] gave necessary and sufficient conditions for a compact convex set to be a state space of a JB-algebra. ARAKI [11] improved the characterization of state spaces of JB-algebras given in [5] to a form with more physical appeal in the simplified finite-dimensional case.

ALFSEN, HANCHE-OLSEN and SHULTZ [4] characterized the state spaces of C^*-algebras among the state spaces of all JB--algebras. Together, [5] and [4] give a complete characterization of the state spaces of C^*-algebras. As is shown in [4], a JB-algebra $\mathcal{J}$ is the selfadjoint part of a C^*-algebra if and only if $\mathcal{J}$ is of complex type and the state space of $\mathcal{J}$ is orientable. STACEY [126 c] showed that the state space of a JBW-algebra of complex type is orientable if and only if it is locally orientable. For local and global splittings in the state space of a JB-algebra see STACEY [126 b]. Recently, IOCHUM and SHULTZ [73] characterized normal state spaces of JBW-algebras among convex sets by proving that they alone are spectral and elliptic.

Every state $\lambda \in K$ defines by $(x \mid y)_\lambda := \lambda(xy)$ a positive inner product on $\mathcal{J}$ and, in particular, by $|x|_\lambda := \lambda(x^2)^{1/2}$ a seminorm on $\mathcal{J}$.

<u>Definition 1.</u> A state λ on a JB-algebra $\mathcal{J}$ is called <u>faithful</u> if $|\ |_\lambda$ actually is norm on $\mathcal{J}$, i.e. if $\lambda(x^2) = 0$ implies that $x = 0$.

<u>Definition 2.</u> A state λ on a JBW-algebra $\mathcal{J}$ is called <u>normal</u> if $\lim \lambda(x_\alpha) = \lambda(x)$ for every increasing net x_α in $\mathcal{J}$ with $x = \sup x_\alpha \in \mathcal{J}$.

Definition 3. A normal state λ on a JBW-algebra $\mathcal{J}$ is called a finite trace if it is associative in the sense that $\lambda((xy)z) = \lambda(x(yz))$ for all $x,y,z \in \mathcal{J}$.

Remark. The condition from Definition 3 above states that every $L(y)$, $y \in \mathcal{J}$, is selfadjoint with respect to the inner product $(\ |\ )_\lambda$.

A complete study of JBW-algebras with a faithful finite trace was undertaken by JANSSEN [78 b]. On the basis of this paper, JANSSEN [78 c, I] studied the properties of the lattice of idempotents in a finite weakly closed Jordan algebra. He proved that such an algebra admits a unique decomposition into a direct sum of a discrete Jordan algebra and a continuous Jordan algebra. JANSSEN [78 c, II] gave a completely description of the discrete finite weakly closed Jordan algebras by finite-dimensional simple formally real Jordan algebras and by simple formally real Jordan algebras of quadratic forms of real Hilbert spaces. A JBW-algebra of type I_2 is a direct sum of weakly closed finite Jordan algebras in the sense of JANSSEN [78 b, c].

STACEY [126 a] gave a structure theorem for JBW-algebras of type I_2. The case $n \geq 3$ was treated also by STACEY [126 d].

PEDERSEN and STØRMER [105] showed that the different definitions of trace on a Jordan algebra are all equivalent for JBW-algebras, and that conditions that do not involve projections are equivalent for JB-algebras. They have considered only finite traces. IOCHUM [69 a] extended the results to semifinite traces. By a suitable definition of semifiniteness, he showed that for any JBW-algebra we have a unique central decomposition in finite (semifinite) and proper-infinite (pure-infinite) parts exactly as in the case of von Neumann algebras (see Theorem V.1.6 [69 a]). Iochum proved also (for the semifinite case) the equivalence between the category of facially homogeneous self-dual cones and the category of JBW-algebras of selfadjoint derivations (see [69 a, Theorem V.5.1]), and([69 a, Chapter VII]) his main theorem, which establishes the equivalence between the category of facially homogeneous self-dual cones in Hilbert spaces and the category of JBW-algebras (see also [25 d]).

Assume now that $\mathcal{J}$ is a JBW-algebra with a faithful finite trace λ . Then λ is essentially uniquely determined (every other faithful finite trace is of the form $\lambda \circ P(h) = \lambda \circ L(h^2)$ for some $h > 0$ in the centre of $\mathcal{J}$), and $\mathcal{J}^{\mathbb{C}}$ is a complex pre-Hilbert space with respect to the inner product $(z \mid w) := (z \mid w)_\lambda := = \lambda(zw^*)$, where λ is extended $\mathbb{C}$-linearly to $\mathcal{J}^{\mathbb{C}}$.

<u>Notation</u>. Denote by $\hat{\mathcal{J}}^{\mathbb{C}}$ the completion of $\mathcal{J}^{\mathbb{C}}$ with respect to the norm $\| z \|_2 := \| z \|_\lambda := \lambda(zz^*)^{1/2}$, and consider the closures $\hat{\mathcal{J}}$ and $\hat{\mathcal{J}^2}$ of $\mathcal{J}$ and $\mathcal{J}^2$ in $\mathcal{J}^{\mathbb{C}}$.

The operators $L(z)$ and $P(z)$, $z \in \mathcal{J}^{\mathbb{C}}$, can be continuously extended to bounded operators on $\hat{\mathcal{J}}^{\mathbb{C}}$ satisfying $L(z)^* = L(z^*)$ and $P(z)^* = P(z^*)$. The cone $\hat{\mathcal{J}^2}$ is self-dual in $\hat{\mathcal{J}}$, satisfies a certain geometrical homogeneity condition, and has e as trace vector (i.e. as quasi-interior point of $\hat{\mathcal{J}^2}$ fixed by every connected set of isometries in $GL(\hat{\mathcal{J}})$). On the other hand, every cone of this type in a real Hilbert space is obtained in this way from a JB-algebra with faithful finite trace (see BELLISSARD and IOCHUM [25 a]). This result can be viewed as a generalization to the infinite-dimensional case of the following theorem of KOECHER: The self-dual cones with homogeneous interior in real Hilbert spaces of finite dimension are precisely (up to linear equivalence) the cones of squares in formally real Jordan algebras.

The connection between formally real Jordan algebras with a trace and cones in an infinite-dimensional Hilbert space was given in 1971 by JANSSEN [78 a]. He found a class of domains of positivity in pre-Hilbert spaces that are in one-to-one correspondence with the formally real Jordan algebras that have one-dimensional centres. Since Bellissard and Iochum held the opinion that facial homogeneity[1)] is a very crucial property in the category of cones, they have re-considered this problem in 1978, by giving a self-consistent exposition of the results [25 a]. Later on, they showed [25 b] that a JBW-algebra can be associated canonically with a facially homogeneous self-dual cone. This construction generalizes the case where there is a trace vector in the cone.

<u>Proposition 2.5.</u> The JB^*-norm $\| \ \|_\infty$ on $\mathcal{J}^{\mathbb{C}}$ satisfies

$$\| \ \|_2 \leq \| \ \|_\infty \text{ on } \mathcal{J}^{\mathbb{C}}, \text{ and } \Delta = \left\{ z \in \mathcal{J}^{\mathbb{C}} \;\middle|\; 1 - P(z)P(z)^* > 0 \right\},$$

$$\Sigma = \exp(i\mathcal{J}) = \left\{ z \in \mathcal{J}^{\mathbb{C}} \;\middle|\; P(z) \text{ unitary on } \mathcal{J}^{\mathbb{C}} \right\} = \left\{ z \in \bar{\Delta} \;\middle|\; \|z\|_2 = 1 \right\}.$$

<u>Proposition 2.6.</u> If $\mathcal{J}$ is a JB-algebra, then the following conditions are equivalent :

(i) there exists a maximal associative subalgebra of finite dimension in $\mathcal{J}$;

(ii) $\mathcal{J}$ is locally finite (i.e. every finitely generated subalgebra has finite dimension);

---

1) Recall that a cone in a real Hilbert space is <u>facially homogeneous</u> if for any face F, the operator $P_F - P_{F^\perp}$ is a derivation Here P_F (resp., $P_{F^\perp}$) is the projector onto the closed linear subspace generated by F (resp., $F^\perp$, the orthogonal face of F).

(iii) for every $a \in \mathcal{J}$ the operator $L(a) \in \mathcal{L}(\mathcal{J})$ satisfies a polynomial equation over $\mathbb{R}$;

(iv) there exists a natural number r such that every $a \in \mathcal{J}$ admits a representation $a = \alpha_1 e_1 + \ldots + \alpha_r e_r$, where $\{e_1, \ldots, e_r\}$ is a set of orthogonal idempotents and $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$;

(v) there exists a faithful finite trace λ on $\mathcal{J}$ such that the corresponding Hilbert norm $\| x \|_2 = \lambda(x^2)^{1/2}$ on $\mathcal{J}$ is equivalent to the JB-norm $\| x \|_\infty$;

(vi) $\mathcal{J}$ is reflexive.

Definition. A JB-algebra $\mathcal{J}$ is called of finite rank if one of the conditions from Proposition 1.16 is satisfied. The number $r =: r(\mathcal{J})$ from condition (iv) is uniquely determined, and is called the rank of $\mathcal{J}$.

The classification of JBW-factors of type I (see Theorem 2.4) implies

Theorem 2.7. Every JB-algebra $\mathcal{J}$ of finite rank is a unique direct sum $\mathcal{J} = \mathcal{J}_1 \oplus \ldots \oplus \mathcal{J}_k$ of JBW-factors $\mathcal{J}_j$ with $r(\mathcal{J}) = r(\mathcal{J}_1) + \ldots + r(\mathcal{J}_k)$. The JBW-factors of finite rank are precisely the following algebras:

$\mathcal{J}$	$r(\mathcal{J})$	
$\mathbb{R}$	1	
V_p	2 ,	p an arbitrary cardinality $\geqslant 3$
$H_3(\mathbb{O})$	3	
$H_p(\mathbb{F})$	p ,	$3 \leq p < \infty$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Remark. JB-algebras of finite rank, except for spin factors, are formally real Jordan algebras.

KAUP [84 a] gives

Theorem 2.8. Every symmetric tube domain D in a reflexive Banach space is linearly equivalent to a direct product

(2.1) $$D(\mathcal{J}_1) \times D(\mathcal{J}_2) \times \dots \times D(\mathcal{J}_k),$$

where $\mathcal{J}_1, \dots, \mathcal{J}_k$ are uniquely determined (up to order) algebras from the list in Theorem 1.17 and $D(\mathcal{J}_j)$ is the upper half-plane associated with $\mathcal{J}_j$. On the other hand, every domain of the form (2.1) is linearly equivalent to a symmetric tube domain in a complex Hilbert space.

CHU [41 a] studied the Radon-Nikodým property (for definition see below) in the context of JBW-algebras.

Definition. A (real or complex) Banach space X is said to possess the Radon-Nikodým property if for any finite measure space (Ω, Σ, μ) and μ-continuous vector measure $L: \Sigma \longrightarrow X$ of bounded total variation, there exists a Bochner integrable function $g: \Omega \longrightarrow X$ such that $L(E) = \int_E g\,d\mu$ for all E in Σ.

Using Theorem 2.1, due to SHULTZ [123], CHU [41 a] established the following result.

Theorem 2.9. Let $\mathcal{J}$ be a JBW-algebra. Then its dual $\mathcal{J}'$ has the Radon-Nikodým property if and only if $\mathcal{J}$ is a finite direct sum of Jordan algebras, each of which is one of the following algebras:

(i) Jordan $(n \times n)$ - matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$;

(ii) spin factors;

(iii) the exceptional Jordan algebra of Hermitian (3×3) matrices over $\mathbb{O}$.

Recently, CHU [41 b] proved that the dual of a JB-algebra $\mathcal{J}$ possesses the Radon-Nikodým property if and only if the state space of $\mathcal{J}$ is the σ- convex hull of its pure states. Namely, he proved

Theorem 2.10. If $\mathcal{J}$ is a JB-algebra with state space K, then the following conditions are equivalent:

(i) K is σ-convex hull of the pure states (i.e.

$$K = \left\{ \sum_{n=1}^{\infty} \lambda_n k_n \;\middle|\; \sum_{n=1}^{\infty} \lambda_n = 1, \ \lambda_n \geqslant 0, \ k_n - \text{pure states} \right\});$$

(ii) $\mathcal{J}'$ has the Radon-Nikodým property;

(iii) $\mathcal{J}''$ is a direct sum of type I JBW-algebras (i.e. JBW--algebras which contains a (non-zero) minimal idempotent).

A generalization of the commutant of a von Neumann algebra in the more general setting of JBW-algebras was investigated by KING [86 a]. King's formulation seems to be of advantage in the study of the foundations of the quantum theory, as it only involves objects and operations that have a physical interpretation.

In general, a JBW-algebra does not have a concrete realization as an operator algebra acting on some Hilbert space; hence the notion of commutant should be revised from the start.

King did this as follows. Let $\mathcal{J}$ be a JBW-algebra and λ a normal state; he studied the order ideal generated by λ in the dual of $\mathcal{J}$. This order ideal was denoted by V_λ. When $\mathcal{J}$ is the selfadjoint part of a von Neumann algebra $\mathcal{N}$, then V_λ can be identified in a natural manner with the selfadjoint part of the commutant of the GNS representation of $\mathcal{N}$ with respect to λ.

Note. KING's [86 a] conjecture is that when λ is a faithful normal state, there exists an order isomorphism of $\mathcal{J}$ onto V_λ.

In [86 a, Chapter II] it was shown that this conjecture is true for atomic JBW-algebras (see Theorem 2.11). King proved the conjecture to be true for a faithful normal trace, and, as an application, proved a Radon-Nikodým theorem for normal traces (see Theorems 2.12 and 2.13). For an interpretation of these results in terms of physical concepts, see EMCH and KING [50].

Definitions. An idempotent p in a JBW-algebra $\mathcal{J}$ is called an atom if it is minimal (i.e. if $0 \leq q \leq p$ with $q^2=q$ implies that $q = 0$ or $q = p$). $\mathcal{J}$ is said to be atomic if every idempotent is the least upper bounded of orthogonal atoms.

Notation. If λ is a state on the JBW-algebra $\mathcal{J}$, then $V_\lambda := \{ f \mid f \in \mathcal{J}', \exists\, a \in \mathbb{R}_+ \text{ with } -a\lambda \leq f \leq a\lambda \}$.

Theorem 2.11. Let $\mathcal{J}$ be an atomic JBW-algebra and let λ be a faithful normal state on $\mathcal{J}$. Then there exists an order isomorphism $\varphi : V_\lambda \longrightarrow \mathcal{J}$ with $\varphi(\lambda) = e$.

Theorem 2.12. Suppose that a JBW-algebra $\mathcal{J}$ admits a faithful normal trace λ (i.e. for all idempotents p,q we have $\lambda(U_p q - U_q p) = 0$, where $U_p q := (pq)q - (qp)q + q^2 p$). Then there exists an order isomorphism $\varphi : V_\lambda \longrightarrow \mathcal{J}$ with $\varphi(\lambda) = e$. Moreover for every positive μ (i.e. $\mu \in (\mathcal{J}^2)'$) from V_λ, there exists a positive element y in $\mathcal{J}$ such that $\mu(x) = \lambda(U_y x)$.

Theorem 2.13. Let $\mathcal{J}$ be a JBW-algebra satisfying the quadratic Radon-Nikodým property (i.e. for any $f,g \in \mathcal{J}'$ with $0 \leq f(x^2) \leq g(x^2)$ for every $x \in \mathcal{J}$, there exists a positive y in $\mathcal{J}$ such that $f(x) = g(U_y x)$ for every $x \in \mathcal{J}$) and let μ and ν be faithful normal states on $\mathcal{J}$. Then V_μ and V_ν are order isomorphic.

Corollary. Let $\mathcal{J}$ be as in Theorem 2.13 and suppose that $\mathcal{J}$ admits a faithful normal trace. Let λ be a faithful normal state on $\mathcal{J}$. Then V_λ is order isomorphic to $\mathcal{J}$.

Recently, HAAGERUP and HANCHE-OLSEN [61] attempted to generalize the Tomita-Takesaki theory to JBW-algebras. As was noticed by HANCHE-OLSEN [62 a]: "When trying to extend the notions of Tomita-Takesaki's theory to Jordan algebras, one immediately runs into a major obstacle: One cannot, in general, associate a "GNS-representation" with a state. So, there is no Hilbert space on which to define the operators of Tomita-Takesaki theory. Moreover, if the product in a von Neumann algebra is reversed, then the modular automorphism group is also reversed; hence it cannot be determined in terms of Jordan structure alone. The 'symmetrization' $\theta_t = (\sigma_t + \sigma_{-t})/2$, however, is left unharmed by this reversal of product". An analogue of θ_t can be defined on any JBW-algebra with a faithful normal state. The definition of θ_t, given by Haagerup and Hanche-Olsen, uses the structure theory of JBW-algebras; however, the characterization of θ_t does not depend on this structure. In particular, this yields a new characterization of $\sigma_t + \sigma_{-t}$ in the von Neumann case.

In order to recall below the main result of Haagerup and Hanche-Olsen, let us first give the following two definitions.

Definition. A family $(v_t)_{t\in\mathbb{R}}$ of linear operators on a linear space M satisfying $v_o = I$ and the cosine identity

$$2\, v_s v_t = v_{s+t} + v_{s-t} ,$$

is called a (one-parameter) cosine family on M.

Remark. If (u_t) is a one-parameter group, then $((u_t + u_{-t})/2)$ is a cosine family.

Definition. Let $\mathcal{J}$ be a JBW-algebra and λ a normal state on $\mathcal{J}$. A bilinear, symmetric, positive semidefinite form s on $\mathcal{J}$ satisfying

(i) $s(a,b) \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$,

(ii) $s(1,a) = \lambda(a)$, $a \in \mathcal{J}$,

(iii) if $0 \leq \mu \leq \lambda$, there is $0 \leqslant b \leqslant 1$ so that

$$\mu(a) = s(a,b), \quad a \in \mathcal{J},$$

is called a <u>self-polar form associated with</u> λ.

<u>Remark</u>. There exists at most one self-polar form associated with λ.

<u>Theorem 2.14.</u> Let $\mathcal{J}$ be a JBW-algebra and let λ be a faithful normal state on $\mathcal{J}$. Then there exists a unique cosine family (θ_t) of positive, unital linear mappings of $\mathcal{J}$ into itself, having the following properties:

(i) for each $a \in \mathcal{J}$, $t \longrightarrow \theta_t(a)$ is weakly continuous;

(ii) $\lambda(\theta_t(a) \circ b) = \lambda(a \circ \theta_t(b))$;

(iii) $s(a,b) := \int \lambda(a \circ \theta_t(b)) \cosh(\pi t)^{-t} dt$ defines a self-polar form associated with λ.

Finally, let us mention the following result of STØRMER [127 g], related to JW-algebras.

Let M be a von Neumann algebra and let α be a central involution of M, i.e. α is a $*$-antiautomorphism of order 2 leaving the centre of M elementwise fixed. Then the set $M^{\alpha} := \{x \in M \mid x = x^* = \alpha(x)\}$ is a JW-algebra with Jordan product $xy := 1/2\,(x.y + y.x)$. STØRMER studied the relationship between M^{α} and M^{β} for two central involutions α and β. The main result states that α and β are (centrally) conjugate, i.e. there exists a $*$-automorphism ϕ of M leaving the centre elementwise fixed, such that $\beta = \phi \alpha \phi^{-1}$ if and only if M^{α} and M^{β} are isomorphic as Jordan algebras <u>via</u> an isomorphism which leaves the centre elementwise fixed. Now M^{α} generates M as a von Neumann algebra (except in a few simple cases) and there are von Neumann algebras with many conjugate classes of central involutions.

Thus there may be many, even an uncountable number, of non-isomorphic JW-algebras which generate the same von Neumann algebra. Such examples may be found in [127 g, Section 5].

§ 3. OJ-algebras

A class of partially ordered Jordan algebras, called OJ-algebras (see below for definition), was introduced in 1979 (see AYUPOV [16 a], SARYMSAKOV and AYUPOV [116]) and studied by AYUPOV [16 b-j]. These algebras are nonassociative generalizations of semifields and can be used for an algebraic approach to quantum probability theory (for the notion of semifield, see SARYMSAKOV [115]). This section deals with the results given by AYUPOV in [16 a-j].

Notation. Let A be a Jordan algebra over $\mathbb{R}$.

Definition. Two elements $a,b \in A$ are called compatible (denoted by $a \longleftrightarrow b$), if the Jordan subalgebra $A(a,b)$ generated by these elements is strongly associative (i.e. $(xy)z = x(yz)$ for all $x,z \in A(a,b)$ and $y \in A$).

Definition. A partial order $\geqslant$ on A is said to be consistent with the algebraic operations if

(i) $a \geqslant b$ implies that $a + c \geqslant b + c$ for all $c \in A$;

(ii) $a \geqslant b$ implies that $\lambda a \geqslant \lambda b$ for all $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$;

(iii) $a \geqslant \theta$, $b \geqslant \theta$, $a \longleftrightarrow b$ implies that $ab \geqslant \theta$;

(iv) $a^2 \geqslant \theta$ for all $a \in A$,

where θ is the null element of A.

Definition. A real Jordan algebra A with unit element e is called an OJ-algebra if it is equipped with a partial order consistent with the algebraic operations and such that: 1) if (x_α) is an arbitrary monotone increasing net in A bounded from above, then $x = \sup x_\alpha$ exists and $x \longleftrightarrow y$ for all $y \in A$ compatible with every x_α; 2) every maximal strongly associative subalgebra of A is a lattice ordered with respect to the induced order.

Note. OJ-algebras were introduced for the sake of an axiomatic approach to quantum probability theory. The elements of an OJ-algebra can be interpreted as observables, its idempotents form a complete orthomodular lattice and can be interpreted as events. This OJ-algebra approach is a synthesis of SARYMSAKOV's approach [115] to classical probability theory and the algebraic approach to quantum mechanics suggested by JORDAN [80 a,b,c] and JORDAN, von NEUMANN and WIGNER [81]. The class of OJ-algebras is sufficiently large and contains as particular cases semifields (the associative case), selfadjoint parts of von Neumann algebras, and Jordan algebras; these appear in the formalism of the quantum mechanics. Additionally, OJ-algebras are endowed with a structure rich enough to obtain analogues of theorems of classical probability theory in OJ-algebras.

Examples of OJ-algebras. 1. Every algebra of measurable functions on a measure space is an associative OJ-algebra.

2. Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, let $M(\mathcal{A})$ be the $*$-algebra of measurable operators affiliated with $\mathcal{A}$. Then the family of all selfadjoint operators from $M(\mathcal{A})$ forms an OJ-algebra with respect to the natural order and the symmetrized product $a{\cdot}b := \frac{1}{2}(a \circ b + b \circ a)$, where $a \circ b$ is the ordinary associative product of a and b.

3. Every JW-algebra is an OJ-algebra. In particular, the selfadjoint part of a von Neumann algebra is an OJ-algebra.

4. An arbitrary JBW-algebra B is an OJ-algebra if the order defined by the cone $B^2 := \{ b^2 \mid b \in B \}$.

<u>Theorem 3.1</u>. The set of all idempotents (i.e. $c^2 = c$) of an OJ-algebra, endowed with the induced order and with orthogonal complement defined as $c^{\perp} := e-c$, is a complete orthomodular lattice.

<u>Definition</u>. An element $a \in A$ is said to be (order) <u>bounded</u> if $-\lambda e \leq a \leq \lambda e$ for some positive $\lambda \in \mathbb{R}$.

<u>Definition</u>. A family $\{ e_\lambda \}_{\lambda \in \mathbb{R}}$ of idempotents of A is called a <u>spectral family</u> if

(i) $e_\lambda \leq e_\mu$ for $\lambda \leq \mu$;

(ii) $\inf \{ e_\lambda \} = \theta$, $\sup \{ e_\lambda \} = e$;

(iii) $e_\mu = \sup\limits_{\lambda < \mu} \{ e_\lambda \}$ for every $\mu \in \mathbb{R}$.

<u>Theorem 3.2</u>. Given an element a in an OJ-algebra A, there exists a unique spectral family $\{ e_\lambda \}$ such that $a = \int_{-\infty}^{\infty} \lambda \, de_\lambda$, and for $b \in A$ we have $b \longleftrightarrow a$ if and only if $b \longleftrightarrow e_\lambda$ for all $\lambda \in \mathbb{R}$.

<u>Definition</u>. An OJ-algebra containing only bounded elements is called an OJB-<u>algebra</u>.

<u>Theorem 3.3</u>. Every OJB-algebra is a JB-algebra with respect to the norm $\| x \| := \inf \{ \lambda > 0 \mid -\lambda e \leq x \leq \lambda e \}$.

<u>Theorem 3.4</u>. Every monotone complete JB-algebra is an OJB-algebra with respect to the order defined by the cone of all squares.

Remark. From Theorem 3.4., it follows that the class of OJB-algebras coincides with the class of all monotone complete JB-algebras. In particular, it contains the class of all JBW-algebras.

Theorem 3.5. For every OJB-algebra A there exists a central idempotent c ∈ A such that the OJB-algebra cA admits a separating set of normal states and for the OJB-algebra (e-c)A there exists no normal state on it.

Remark 1. An OJB-algebra is a JBW-algebra if and only if it admits a separating set of normal states.

Remark 2. Every OJB-algebra is the direct sum of two ideals, one of which is a JBW-algebra, while the other has no normal states.

The following nonassociative analogue of the Vitali-Hahn-Saks theorem holds:

Theorem 3.6. Let A be an OJB-algebra, let $\{\varphi_n\}$ be a sequence of normal states on A pointwise converging to φ (i.e. $\lim \varphi_n(a) = \varphi(a)$ for all $a \in A$). Then φ is also a normal state on A.

As was mentioned in the Note on page 25, the elements of a given OJ-algebra are interpreted as observables, and its idempotents, which form a logic, correspond to the quantum mechanical events. Furthermore, the order bounded elements in the given OJ-algebra correspond to the physical states. Therefore, Theorem 2.1 can be considered as a representation theorem for the algebra of bounded observables. AYUPOV's [16] main result is a representation theorem for arbitrary OJ-algebras of observables, containing unbounded elements as well.

In order to formulate the above-mentioned representation theorem of Ayupov, we give some definitions.

Definition. Let A be a JW-algebra in a Hilbert space $\mathcal{H}$. A projection p in A is said to be modular if the projection lattice $[0,e] := \{ f \in A \mid f \leq p \}$ is modular (i.e. $(f \vee g) \wedge h = f \vee (g \wedge h)$ for all $f,g,h \in [0,p]$, $f \leq h$; see [130]).

Definition. A selfadjoint operator T(not bounded, in general) in the Hilbert space $\mathcal{H}$ is said to be affiliated with the JW-algebra A if all its spectral projections P_λ (in the spectral resolution $T = \int_{-\infty}^{\infty} \lambda \, d P_\lambda$) lie in A.

Definition. A self-adjoint operator T affiliated with a JW-algebra A is called:

a) measurable if the projections $P_{\lambda_0}^{\perp}$ and P_{λ_0} are modular for some positive $\lambda_0 \in \mathbb{R}$;

b) locally measurable if there exists a sequence (q_n) of central projections in A monotonically increasing to e and such that $q_n T$ is a measurable operator for all $n = 1,2, \ldots$.

Definition. A Jordan subalgebra A_1 in an OJ-algebra A is called a solid-OJ-subalgebra if it is an OJ-algebra with respect to the induced partial order and if for $x,y \in A$, $0 \leq x \leq y \in A_1$ implies that $x \in A_1$.

Definition. An OJ-algebra A is said to be universal if, given any spectral family $\{ e_\lambda \}$ in A, the integral $\int_{-\infty}^{\infty} \lambda \, de_\lambda$ exists .

Definition of the exceptional OJ-algebra $\mathcal{S}(S,H_3(\mathbb{O}))$.

Let S be a hyperstonean space, and let $\tilde{H} := H_3(\mathbb{O}) \cup \{\infty\}$ be the one-point compactification of the finite-dimensional JBW-algebra $H_3(\mathbb{O})$. Consider the set $\mathcal{S}(S, H_3(\mathbb{O}))$ of all continuous maps $f : S \longrightarrow \tilde{H}$ such that $f^{-1}(\infty)$ is nowhere dense in S. The algebraic operations and a partial order on this set are defined as follows. Let $f, g \in \mathcal{S}(S, H_3(\mathbb{O}))$; then $Y := S - \{ f^{-1}(\infty) \cup g^{-1}(\infty) \}$ is an open dense subset in S, and $f(x), g(x) \in H_3(\mathbb{O})$ for $x \in Y$. Define the map f+g from Y to $\tilde{H}$ by $(f+g)(x) := f(x) + g(x)$, $x \in Y$. Since Y is a hyperstonean space, it coincides with the Stone-Chech compactification of every dense subset $Y \subset S$. Therefore f+g can be uniquely extended to a continuous map $f+g : S \longrightarrow \tilde{H}$ and, obviously, $f+g \in \mathcal{S}(S, H_3(\mathbb{O}))$. Similarly we can define other algebraic operations on $\mathcal{S}(S, H_3(\mathbb{O}))$. Since $H_3(\mathbb{O})$ is a Jordan algebra, $\mathcal{S}(S, H_3(\mathbb{O}))$ also becomes a Jordan algebra.

Remark. If one considers on $\mathcal{S}(S, H_3(\mathbb{O}))$ the pointwise partial order $\leq$ (i.e. $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in S - \{ f^{-1}(\infty) \cup g^{-1}(\infty) \}$), then the Jordan algebra $\mathcal{S}(S, H_3(\mathbb{O}))$ becomes a universal OJ-algebra, and its subalgebra of bounded elements is isomorphic to the JBW-algebra $\mathcal{C}(S, H_3(\mathbb{O}))$ (see [16 j, Theorem 5].

Theorem 3.7. Let A be an arbitrary OJ-algebra, let B be the JBW-algebra of bounded elements in A. Then, there exists a unique central idempotent $c \in A$ such that

(i) the algebra cB of bounded elements in the OJ-algebra c A is isomorphic to a JW-algebra $\mathcal{J}$, and the OJ-algebra c A is isomorphic to a solid OJ-subalgebra of the OJ-algebra $S(\mathcal{J})$ of all locally measurable selfadjoint operators affiliated with $\mathcal{J}$;

(ii) the algebra (e-c) B of bounded elements in the OJ-algebra (e-c) A is isomorpic to the JBW-algebra $\mathcal{C}(S, H_3(\mathbb{O}))$, and the OJ-algebra (e-c) A is isomorphic to a solid OJ-subalgebra of the universal OJ-algebra $\mathcal{S}(S, H_3(\mathbb{O}))$.

Note. As was noticed by AYUPOV [16 k], Theorem 3.9 was given in [117] only for a particular case, because it was proved in the general case only after the manuscript had been sent to the publishers.

Definition. Let A be an OJ-algebra and let τ be a Hausdorff topology on A such that (A, τ) is a topological vector space. Then (A, τ) is called a topological OJ-algebra, and τ is said to be an $\mathbb{R}$-topology, if the following conditions are satisfied:

1. For every neighbourhood U of zero in A there exists a "normal" (or solid) neighbourhood $V \subset U$ i.e. such that

1_a. $\theta \leq x \leq y \in V$ implies that $x \in V$;

2_a. $y \in V$ implies that $P_{(p)} y \in V$ for every idempotent or symmetry p in A ;

2. if $\{c_\alpha\}$ is an arbitrary net of idempotents monotonically decreasing to θ, then $c_\alpha \xrightarrow{\tau} \theta$;

3. if $\{c_\alpha\}$ is an arbitrary net of idempotents τ-converging to θ, then for every net $\{x_\alpha\}$ from A net $\{x_\alpha c_\alpha\}$ also τ-converges to θ.

Definition. Let A be an OJ-algebra, let ∇ be the logic of idempotents of A, and let E be a topological semifield. A function $d : \nabla \longrightarrow E$ is called a measure if

1) $d(c) \geqslant 0_E$ for all $c \in \nabla$, and $d(c) = 0_E$ if and only if $c = \theta$;

2) $d(c_1) = d(c_2)$ if c_1 and c_2 are equivalent [1];

---

1) Two idempotents c_1 and c_2 of A are said to be equivalent if there exists a finite family of symmetries $s_1, \ldots, s_n$ (i.e. $s_i^2 = e$, $i = 1, \ldots, n$) such that $P(s_1) \ldots P(s_n) c_1 = c_2$, where P is the quadratic representation of A (see JSA. I, § 1

3) d is completely additive.

Here 0_E is the null element of E.

Construction of the topology τ_d. Let A be an OJ-algebra, let B be the JB-algebra of bounded elements of A, let ∇ be the logic of idempotents in A, and let d be a measure on ∇ with values in a topological semifield E. Consider the topology τ_d of convergence in the measure d in which the base neighborhoods of zero consist of sets of the form

$$U(\varepsilon, V) := \left\{ x \in A \;\middle|\; \exists p \in \nabla, d(p^{\perp}) \in V, P_{(p)}\, x \in B, \| P_{(p)}\, x \| \leq \varepsilon \right\},$$

where ε is a positive number and V is a neighborhood of zero in E.

Theorem 3.8. Let A be an OJ-algebra and let τ_d be defined as above. Then (A, τ_d) is a topological Jordan algebra (i.e. the linear operations and the product are continuous in both variables simultaneously in the topology τ_d).

Theorem 3.9. The above-defined topology τ_d is an $\mathbb{R}$-topology.

Theorem 3.10. Let A be an OJ-algebra. The following conditions are equivalent:

1) A admits an $\mathbb{R}$-topology;

2) A is a finite OJ-algebra (i.e. every orthogonal family of mutually equivalent idempotents is finite), and its centre is a topological semifield;

3) the lattice of idempotents of A is modular, and the Boolean algebra of its central idempotents is a topological Boolean algebra.

Theorem 3.11. If A is an OJ-algebra, and τ_1 and τ_2 are two $\mathbb{R}$-topologies on A, then $\tau_1 = \tau_2$.

Theorem 3.12. Every $\mathbb{R}$-topology on an OJ-algebra A is the topology of convergence in measure with values in the centre of A.

Theorem 3.13. For every topological OJ-algebra (A, τ) there exists a unique (up to isomorphism) universal topological OJ-algebra $(\hat{A}, \hat{\tau})$ such that A is isomorphic to a subalgebra of $\hat{A}$, and such that the JB-algebras of bounded elements in A and $\hat{A}$ are isomorphic.

Remark. Further results due to AYUPOV on topological OJ-algebras may be found in [16 d].

We now recall some results concerning several aspects of probability theory on JBW-algebras.

Given a JBW-algebra A with a finite faithful normal trace, AYUPOV [16 g] constructed the OJ-algebra $\hat{A}$ of measurable elements for A and defined the space of integrable and square-integrable elements from $\hat{A}$. Then analogues of convergence in measure and almost everywhere convergence (i.e. convergences in probability and almost sure convergence) are introduced. Mean and pointwise ergodic theorems for Markov operators on Jordan probability space are also given.

In [16 f] Ayupov gave some results on martingale convergence and strong laws of large numbers in a JBW-algebra A. Let us mention that some of these results are nonassociative generalizations of almost sure convergence theorems for martingales and, in the particular case when the JBW -algebra A is isomorphic to the selfadjoint part of a von Neumann algebra, they coincide with the results obtained by CUCULESCU [43].

Let us finally mention that the recent AYUPOV's book [16 r] may be considered as a good supplement to the well-known monograph of HANCHE-OLSEN and STØRMER [63].

§ 4. JV-algebras

We first recall some

Definitions. A nonassociative real or complex algebra $\mathcal{A}$ with unit element e in which a norm $\| \ \|$ is defined and satisfies $\|ab\| \leq \|a\| \|b\|$ and $\|e\| = 1$, for all $a, b \in \mathcal{A}$, is called a unital normed algebra. If a is an element of such an algebra $\mathcal{A}$, then the numerical range of a, denoted by $V(\mathcal{A}, a)$, is defined by

$$V(\mathcal{A}, a) := \left\{ f(a) \;\middle|\; f \text{ from the dual of } \mathcal{A},\ \|f\| = f(e) = 1 \right\};$$

the numerical radius of a, denoted by v(a), is defined by

$$v(a) := \sup \left\{ |c| \;\middle|\; c \in V(\mathcal{A}, a) \right\};$$

and the numerical index of $\mathcal{A}$, denoted $n(\mathcal{A})$, is defined by

$$n(\mathcal{A}) := \inf \left\{ v(a) \;\middle|\; a \in \mathcal{A},\ \|a\| = 1 \right\}.$$

Definition. A complex unital complete normed algebra $\mathcal{A}$ satisfying $\mathcal{A} = H(\mathcal{A}) \oplus i\ H(\mathcal{A})$, where $H(\mathcal{A})$ denotes the set of its Hermitian elements (i.e. elements whose numerical ranges are subsets of $\mathbb{R}$) is called a (Vidav) V-algebra.

Definition. A V-algebra which is also a Jordan algebra is called a JV-algebra.

MARTINEZ [93 a] founded a general theory of complete normed Jordan algebras similar to that of Banach algebras. The complex case was considered in detail. It has been proved that the set of invertible elements of a complete normed Jordan algebra $\mathcal{A}$ over $\mathbb{C}$ is open, that the map $a \longrightarrow a^{-1}$ is differentiable. MARTINEZ [93 a] showed that the resolvent function

$z \longrightarrow (a-ze)^{-1}$ is analytic, that the spectrum of an element $a \in \mathcal{A}$ (i.e. $\{z \mid z \in \mathbb{C},\ ze - a \notin \mathrm{Inv}(\mathcal{A})\}$) is a (nonempty) compact subset of $\mathbb{C}$ and established a "spectral mapping theorem" [93 a, Chapter II, § 6]. A theorem for the uniqueness of the norm on semi-simple Jordan algebras of type $\mathcal{A}^{(+)}$, $\mathcal{A}$ being a real or complex associative algebra, has also been given (compare with BALACHANDRAN and REMA's results [22].) Other results from [93 a] are the following two theorems.

Theorem 4.1. Let $\mathcal{A}$ be a JV-algebra and let a be an element of $H(\mathcal{A})$ such that $a^2 = b + ic$, where $b, c \in H(\mathcal{A})$. Then $V(\mathcal{A}, a^2) \subseteq \mathbb{R}_+$, and so $a^2 \in H(\mathcal{A})$.

Theorem 4.2. Let $\mathcal{A}$ be a JV-algebra. Then the mapping $a + ib \longrightarrow a - ib$, $a, b \in H(\mathcal{A})$, is an algebra involution on $\mathcal{A}$ which converts it into a JB^*-algebra.

Remark. For Theorems 4.1 and 4.2, see also MARTINEZ [93 c].

By analogy with the associative case, MARTINEZ [93 b] defined the notion of compact elements in a complete normed Jordan algebra. He proved that the set of compact elements of such an algebra is quadratic ideal. For the principal results of Riesz-Schauder theory (concerning compact operators) for this case, see also [93 b].

RODRIGUEZ [111 a] continued the study of noncommutative JV-algebras first undertaken by MOJTAR KAIDI [101]. The basic tool is a theorem of Vidav-Palmer type due to MARTINEZ [93 a] and YOUNGSON [141 a, b]. It shows that the class of (commutative) JV-algebras coincides with that of the JB^*-algebras. In particular, the following theorems were established (see Theorems 8 and 10 in [111 a]):

Theorem 4.3. If $\mathcal{A}$ is a V-algebra, the following statements are equivalent:

a) $\mathcal{A}$ is a noncommutative Jordan algebra;

b) the natural involution $*$ of $\mathcal{A}$ is multiplicative (i.e. $(ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$);

c) the natural involution $*$ of $\mathcal{A}$ is isometric.

Remark. For a complete and simplified proof of Theorem 4.3. see [95].

Recently, RODRIGUEZ [111 b] proved

Theorem 4.4. The natural involution of a V-algebra is always multiplicative.

Comments. It follows that the class of general nonassociative V-algebras coincides with the class of unital noncommutative JB^*-algebras. In particular, every V-algebra is a noncommutative Jordan algebra.

Theorem 4.5. Let $\mathcal{A}_i$, i=1,2, be two noncommutative JV-algebras with unit elements e_i, and let F be a linear bijection from $\mathcal{A}_1$ onto $\mathcal{A}_2$. The following statements are equivalent:

a) F is a symmetric (i.e. $F(a^*) = (F(a))^*$) homomorphism from $\mathcal{A}_1^{(+)}$ onto $\mathcal{A}_2^{(+)}$;

b) F is isometric and $F(e_1) = e_2$;

c) the numerical range of any a_1 in $\mathcal{A}_1$ coincides with the numerical range of $F(a_1)$ in $\mathcal{A}_2$.

Remark. The equivalence of a) and b) has been proved by WRIGHT and YOUNGSON [141 b] in the commutative case.

Subsequently RODRIGUEZ [111 a] showed that the bidual of a given noncommutative JV-algebra is again a noncommutative JV-algebra with the Arens product in which any multilinear identity from the given algebra holds. Consequently, the bidual of a JB^*-algebra is also a JB^*-algebra. The numerical index of a noncommutative JV-algebra is shown to be 1 when this algebra is associative and commutative and 1/2 when it is not so.

As a generalization of C^*-algebras (including JB^*-algebras), PAYÁ, PÉREZ and RODRIGUEZ [104 a] defined the concept of noncommutative JB^*-algebra :

Definition. A complete normed complex noncommutative Jordan algebra $\mathcal{A}$ with a multiplicative involution $*$ satisfying

$$\| U(a)a^* \| = \| a \|^3 , \qquad a \in \mathcal{A},$$

where $U(a)b := a(ab + ba) - a^2 b$, $a,b \in \mathcal{A}$, is called a noncommutative JB^*-algebra (or, briefly, an n.c. JB^*-algebra).

Comments. The class of n.c. JB^*-algebras contains the class of JB^*-algebras introduced by KAPLANSKY [82] in the unital case (see JSA.V , § 1), and studied by YOUNGSON [141 d] in the non-unital case. In [104 a] a normed algebra $\mathcal{A}$ is understood to mean, as in the definition at the beginning of this section, an algebra $\mathcal{A}$ with compatible norm $\| \ \|$ such that $\| ab \| \leq \| a \| \ \| b \|$ for all a,b in $\mathcal{A}$, so that this axiom is no longer required by the above definition. A characterization (in terms of the numerical range) of unital n.c. JB^*-algebras was recently given by MARTINEZ, MENA, PAYÁ and RODRIGUEZ [94].

Remark. If $\mathcal{A}$ is a n.c. JB^*-algebra, then $\mathcal{A}^{(+)}$ is a commutative JB^*-algebra. Hence many results for the new concept can be reduced to the well-known commutative case.

In [104 a] PAYÁ , PEREZ and RODRIGUEZ deal with those results for n.c. JB^*-algebras which cannot be easily reduced to the commutative case. Let us mention some of these:

Theorem 4.6. The bidual of an n.c. JB^*-algebra $\mathcal{A}$ is a unital n.c. JB^*-algebra satisfying all multilinear identities satisfied by $\mathcal{A}$.

Theorem 4.7. The product of an n.c. JW^*-algebra (an n.c. JB^*-algebra which is a dual Banach space) is w^*-continuous in each variable.

Proposition 4.8. Every derivation on an n.c. JW^*-algebra is w^*-continuous.

Theorem 4.9. Let $\mathcal{A}_i$, $i = 1,2$, be two n.c. JW^*-algebras. Then every isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$ is w^*-continuous.

Theorem 4.10. The closed two-sided ideals of an n.c. JB^*-algebra $\mathcal{A}$ are exactly the M-ideals[1)] in the underlying Banach space of $\mathcal{A}$.

Theorem 4.11. A primitive M-ideal in an n.c. JB^*-algebra $\mathcal{A}$ is the kernel of a finite representation of $\mathcal{A}$. Hence, $\mathcal{A}$ has a faithful family of factor representations (a factor is an n.c. JW^*-algebra without central projections).

Recently, BRAUN [37 a] proved

Theorem 4.12. If $\mathcal{A}$ is a unital n.c. JB^*-algebra, then $\mathcal{A}$ has a faithful family of type I factor representations, namely that of $\mathcal{A}^{(+)}$ (a type I factor is a factor with minimal projections).

Remark. Theorem 4.12 reduces the study of n.c. JB^*-algebras to that of factors.

In order to formulate (see Theorem 4.13) the main result established by BRAUN [37 a], we first give some

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1) The concept of M-ideal was introduced by ALFSEN and EFFROS [3] as follows:
A closed subspace Y of a real Banach space X is said to be an L-ideal if there exists a closed subspace Y' such that $X = Y \oplus Y'$, and for respective elements y and y' of Y and Y' we have $\|y+y'\| = \|y\| + \|y'\|$. A closed subspace Z of a real Banach space X is said to be an M-ideal if $Z^{\perp}$ is an L-ideal in the dual of X. These concepts were extended in a natural way to complex Banach spaces by taking the real restriction.

Definitions. Let $\mathcal{A}$ be an arbitrary nonassociative $\mathbb{K}$-algebra and let $\lambda \in \mathbb{K}$. Then the λ-mutation $\mathcal{A}^{(\lambda)}$ of $\mathcal{A}$ is the vector space $\mathcal{A}$ endowed with the product $x \circ_\lambda y := \lambda xy + (1-\lambda)yx$, $x,y \in \mathcal{A}$.

A complex complete normed nonassociative $*$-algebra $\mathcal{A}$ is called a quasi C^*-factor if there exists a real number λ with $0 \leq \lambda \leq 1$ and a C^*-algebra $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}^{(\lambda)}$ are isometrically $*$-isomorphic and $\mathcal{B}$ is a factor. An n.c. JB^*-factor is called quadratic if there exist a complex linear form $t : \mathcal{A} \longrightarrow \mathbb{C}$ and a symmetric $\mathbb{C}$-bilinear form $n : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}$ such that for all elements z in $\mathcal{A}$ we have $z^2 - 2t(z)z + n(z,z)1 = 0$.

Theorem 4.13. If $\mathcal{A}$ is an n.c. JB^*-factor then either: 1) $\mathcal{A} = \mathbb{C}1$; 2) $\mathcal{A}$ is quadratic; 3) $\mathcal{A}$ is a quasi C^*-factor; or 4) $\mathcal{A}$ is a commutative JB^*-factor.

Comments. More on the same topic can be found in PAYÁ, PEREZ and RODRIGUEZ [104 b], in which also a concrete construction of n.c. JB^*-algebras which are quadratic algebras is given.

In an n.c. JB^*-algebra B is in fact alternative, then the axiom $\| U(a)a^* \| = \| a \|^3$ is equivalent to $\| a^*a \| = \| a^2 \|$, and PAYÁ, PEREZ and RODRIGUEZ [104 a] call B and alternative B^*-algebra. For alternative B^*-algebras they proved

Theorem 4.14. An alternative factor is either associative or the B^*-algebra of complex octonions.

Theorem 4.15. The primitive M-ideals of an alternative B^*-algebra $\mathcal{A}$ are exactly the primitive ideals (in the algebraic sens) of $\mathcal{A}$.

Comments. RODRIGUEZ's study [111 a] arose from the interest in C^*-alternative algebras. By definition, a C^*-alternative algebra A is a unital nonassociative complex algebra A with conjugate linear involution $*$: A $\longrightarrow$ A such that the alternative laws $a^2b = a(ab)$, $ab^2 = (ab)b$, $a,b \in A$ are satisfied, and such that A is a complex Banach space with respect to a norm satisfying the C^* - norm condition $\| a^*a \| = \| a \|^2$ for all a in A. Recently, BRAUN [37 b] proved a Gelfand-Neumark theorem for such algebras. Let us mention in this respect that C^*-alternative algebras appear also in complex analysis as induced substructures associated with symmetric Siegel domains of the second kind in Banach spaces (see BRAUN, KAUP and UPMEIER [38 b], and KAUP and UPMEIER [85]).

OCAÑA [103] focussed principally on two problems for a JW^*-algebra endowed with the weak topology, namely: the continuity of the involution (in Chapter II), and the separated continuity of the product (in Chapter III). He gave an affirmative answer to both problems (see [103, Theorem 5.4., Chapter II and Theorem 3.7, Chapter III]). Also, OCAÑA [103] established theorems for the stability of the structure and a characterization of weakly closed ideals, the set of projections of a JW^*-algebra being previously studied. The establishment of the abovementioned results, led OCAÑA develop some aspects of the theory of JV-algebras. In particular, he obtained a characterization of the positive linear functionals, the existence and unicity of the positive roots of order n for positive elements, and, hence, the orthogonal decomposition of Hermitian elements (see [103, Chapter II, § 3]).

Finally we have AUPETIT's [13] recent result concerning the equivalence of complete norms in semi-simple JB-algebras, as well as the characterizations of associative JB-algebras and of $\mathbb{C}$ given by AUPETIT and ZRAÏBI [15].

§ 5. J^*-algebras

Note. There are two different notions related to the Jordan structures referred to as J^*-algebras. In the first part of this section we deal with J^*-algebras in the sense of VIOLA DEVAPAKKIAM, and denote them by $\mathfrak{J}^*$-algebras, while in the second part we shall treat J^*-algebras in the sense of HARRIS, and denote them by $\mathcal{J}^*$-algebras.

1. VIOLA DEVAPAKKIAM and REMA [134], [135] studied the structure of certain infinite-dimensional Jordan algebras admitting an inner product. These algebras, called $\mathfrak{J}^*$-algebras, had already been considered by BALACHANDRAN and REMA [22] in connection with the norm uniqueness problem for nonassociative algebras.

Definition. A linear Jordan algebra $\mathfrak{J}$ over the complex field $\mathbb{C}$ is called a $\mathfrak{J}^*$-algebra if

(i) $\mathfrak{J}$ is equipped with an inner product (,) under which it is a Hilbert space,

(ii) for each element $x \in \mathfrak{J}$ there exists an associated element x^*, called the adjoint of x, such that $(xy,z) = (y,x^*z)$ for all y,z in $\mathfrak{J}$.

Remark. $\mathfrak{J}^*$-algebras are the Jordan analogues of SCHUE's L^*-algebras [122] (or AMBROSE's H^*-algebras [9]).

Definition. A $\mathfrak{J}^*$-algebra $\mathfrak{J}$ is called semisimple if $\{a \in \mathfrak{J} \mid ax = 0 \text{ for all } x \in \mathfrak{J}\} = 0$ and simple if it is semisimple and has no closed ideals other than $\{0\}$.

Definition. A closed Jordan subalgebra of a $\mathfrak{J}^*$-algebra $\mathfrak{J}$ which is also closed for the adjoint, is called a $\mathfrak{J}^*$-subalgebra of $\mathfrak{J}$.

Remark. Any H^*-algebra gives rise to a $\mathcal{J}^*$-algebra H^+, the multiplication in H^+ being the usual Jordan multiplication $xy := \frac{1}{2}(x \circ y + y \circ x)$, the $*$-operation and inner product being the same as in H. $\mathcal{J}^*$-subalgebras of H^+ arising from a semisimple H^*-algebra H are called special. Every special $\mathcal{J}^*$-algebra is semisimple.

Proposition 5.1. Every finite-dimensional semisimple Jordan algebra is a $\mathcal{J}^*$-algebra.

Theorem 5.2. (Wedderburn Structure Theorem.) Every semisimple $\mathcal{J}^*$-algebra $\mathcal{J}$ is the orthogonal sum (=closure of the algebraic direct sum with pairwise orthogonal components) of its simple ideals, and, further, any ideal of $\mathcal{J}$ is the orthogonal sum of a subcollection of these simple ideals.

Notation. Let $\mathcal{J}$ be a semisimple $\mathcal{J}^*$-algebra. For an idempotent e_i of $\mathcal{J}$ put $\mathcal{J}_{ii} := \{x \in \mathcal{J} \mid x\, e_i = x\}$, and for a pair of orthogonal idempotents e_i, e_j, put $\mathcal{J}_{ij} := \{x \in \mathcal{J} \mid x\, e_i = \frac{1}{2} x = x\, e_j\}$.

Theorem 5.3. (Peirce decomposition.) Let $\mathcal{J}$ be a semisimple $\mathcal{J}^*$-algebra. Then for a maximal family $\{e_i\}$, $i \in I$, of orthogonal selfadjoint idempotents, $\mathcal{J}$ is the orthogonal sum $\mathcal{J} = \bigoplus \mathcal{J}_{ij}$, where summation is over all distinct subspaces $\mathcal{J}_{ij}$, $i, j \in I$.

Theorem 5.4. Any semisimple $\mathcal{J}^*$-algebra $\mathcal{J}$ is the closure of the union of a net of $\mathcal{J}^*$-subalgebras with an identity.

Proposition 5.5. A special $\mathcal{J}^*$-algebra $\mathcal{J}$ is finite dimensional if and only if it has an identity element.

Using ANCOCHEA's result [10, Theorem 12.5], VIOLA-DEVAPAKKIAM and REMA [135] proved that if $M_n(\mathbb{C})^{(+)}$ denotes the Jordan algebra of all complex $(n \times n)$ - matrices and φ is a

Jordan isomorphism of $M_n(\mathbb{C})^{(+)}$ onto a special $\mathcal{J}^*$-algebra $\mathcal{J}$, then $M_n(\mathbb{C})$ is an H^*-algebra such that φ is a $*$-preserving isomorphism of $M_n(\mathbb{C})^{(+)}$ onto $\mathcal{J}$. This result permits the explicit construction of canonical bases for special simple finite-dimensional $\mathcal{J}^*$-algebras isomorphic to Jordan algebras of types A, B, and C from Albert's classification (see JSAI, § 1), considered over $\mathbb{C}$.

Notation. Let H be a complex Hilbert space with (,) as inner product. Denote by $\mathcal{H}$ the (simple) H^*-algebra of all Hilbert-Schmidt operators on H. A conjugate linear map K on H is called a conjugation or anti-conjugation according to whether $K^2 = Id$ or $K^2 = -Id$, Id denoting the identity operator on H.

The following three types of $\mathcal{J}^*$-algebras have been shown to be simple (see [135, pp.313-314]):

I. The Jordan algebra $\mathcal{H}^+$ of all Hilbert-Schmidt operators on H.

II. The Jordan algebra of all Hilbert-Schmidt operators T on H such that $T^*K = KT$ for a fixed conjugation K on H.

III. The Jordan algebra of all Hilbert-Schmidt operators T on H such that $T^*K = KT$ for a fixed anticonjugation K on H.

VIOLA DEVAPAKKIAM and REMA [135] considered infinite-dimensional separable (i.e., the underlying Hilbert space is separable in the topological sense) special $\mathcal{J}^*$-algebras and showed that any (infinite-dimensional) simple separable special $\mathcal{J}^*$-algebra is of type, I, II, or III.

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2. HARRIS [64 a] considered a large class of Banach spaces whose open unit balls are bounded symmetric homogeneous domains. These Banach spaces, to which Harris referred as $\mathcal{J}^*$-algebras, are linear spaces of operators mapping one Hilbert space into another and a kind of Jordan triple product structure (for definition see

below). In particular, all Hilbert spaces and all B^*-algebras are $\mathcal{J}^*$-algebras. Moreover, all four types of classical Cartan domains and their infinite-dimensional analogues are the open unit balls of $\mathcal{J}^*$-algebras, and the same holds for any finite or infinite product of these domains. Thus, this is a setting in which a large number of bounded symmetric homogeneous domains may be studied simultaneously. A particular advantage of this setting is the interconnection existing between function-theoretic problems and problems of functional analysis.

Notation. Let H and K be complex Hilbert spaces and let $\mathcal{L}(H,K)$ denote the space of all bounded linear operators from H to K with the operator norm. For each $A \in \mathcal{L}(H,K)$ there exists a uniquely determined operator $A^* \in \mathcal{L}(K,H)$ such that $(Ax,y)=(x,A^*y)$ for all $x \in H$ and $y \in K$.

Definition. A closed complex subspace $\mathcal{A}$ of $\mathcal{L}(H,K)$ such that $AA^*A \in \mathcal{A}$ whenever $A \in \mathcal{A}$ is called a $\mathcal{J}^*$-algebra.

~~Definition. A closed complex subspace of (H,K) such that AA^*A whenever A is called a * algebra.~~

Comments. KAUP [84 b] gave an abstract version of $\mathcal{J}^*$-algebras, called C^*-triple systems by him (see also JSA.VI, § 1).

Examples. The sets $\mathcal{L}(H,K)$, $\{A \mid A \in \mathcal{L}(H),\ A^t = A\}$, and $\{A \mid A \in \mathcal{L}(H,K),\ A^t = -A\}$, where $x \longrightarrow \overline{x}$ is a given conjugation on H and $A^t x := \overline{A\overline{x}}$ for all $x \in H$, are $\mathcal{J}^*$-algebras. They are called Cartan factors of type I, II, and III, respectively. Also, any closed complex subspace $\mathcal{A}$ of $\mathcal{L}(H)$, such that $A^* \in \mathcal{A}$ and $A^2 \in \mathbb{C}\, I$ whenever $A \in \mathcal{A}$, is called a Cartan factor of type IV, unless $\dim \mathcal{A} = 2$. Recall that a C^*-algebra (resp. JC^*-algebra) is a closed complex subspace $\mathcal{A}$ of $\mathcal{L}(H)$ such that $\mathcal{A}$ contains products (resp. squares) and adjoints of each of its elements. <u>A ternary algebra</u> is a closed complex subspace of $\mathcal{A}$ of $\mathcal{L}(H,K)$ such that $AB^*C \in \mathcal{A}$ whenever $A,B,C \in \mathcal{A}$. Every C^*-algebra is a JC^*-algebra and a ternary algebra, and from [64 b] it follows that JC^*-algebras and ternary algebras are $\mathcal{J}^*$-algebras.

<u>Comments.</u> $\mathcal{J}^*$-algebras are not algebras in the ordinary sense. However, they contain certain symmetrically formed products of their elements. E.g., if A, B, and C are elements of a $\mathcal{J}^*$-algebra $\mathcal{A}$ and p is an arbitrary polynomial, then $AB^*C + CB^*A$, $A(B^*A)^n = (AB^*)^n A$, $p(AB^*)C + Cp(B^*A)$, and $p(AB^*)Cp(B^*A)$ belong to $\mathcal{A}$.

<u>Proposition 5.6.</u> The open unit ball of any $\mathcal{J}^*$-algebra is a bounded symmetric homogeneous domain.

HARRIS [64 a] gave an explicit algebraic formula for the Möbius transformations of these balls and showed that the origin can be mapped to any desired operator in the ball by Möbius transformation. He has also proved that the open unit balls of two $\mathcal{J}^*$-algebras are holomorphically equivalent if and only if the $\mathcal{J}^*$-algebras are isometrically isomorphic under a mapping preserving the $\mathcal{J}^*$-structure. Another result is that the open unit ball of a $\mathcal{J}^*$-algebra is holomorphically equivalent to a product of balls if and only if the $\mathcal{J}^*$-algebra is isometrically isomorphic to a product of $\mathcal{J}^*$-algebras (see [64, a § 3]).

<u>Convention.</u> If the open unit ball of a $\mathcal{J}^*$-algebra $\mathcal{A}$ is holomorphically equivalent to a Cartan domain of type I-IV, then $\mathcal{A}$ is called a <u>finite-dimensional Cartan factor.</u> We denote by C(S) the space of all continuous complex-valued functions on a locally compact Hausdorff space S vanishing at infinity. A map L between two $\mathcal{J}^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be a $\mathcal{J}^*$-isomorphism if it is a bounded linear bijection of $\mathcal{A}$ onto $\mathcal{B}$ such that $L(AA^*A) = L(A)L(A)^*L(A)$ for all $A \in \mathcal{A}$

Theorem 5.7. Every finite-dimensional J^*-algebra is isometrically J^*-isomorphic to a product of finite-dimensional Cartan factors. In particular, every J^*-algebra of dimension less than 16 is isometrically J^*-isomorphic to a product of finite-dimensional Cartan factors of types I-IV.

Theorem 5.8. The open unit balls of the following J^*-algebras are not holomorphically equivalent to a product of balls: all Cartan factors of types I-IV, except the 2-dimensional one of type IV; all W^*-algebras which are factors in the ordinary sense ; all spaces C(S), where S is compact and connected.

HARRIS [64 c] showed that the infinite-dimensional analogues of the classical Cartan domains of different types are not holomorphically equivalent and introduced an invariant Hermitian metric on a class of bounded symmetric domains in Banach spaces (including all classical bounded symmetric domains) which yields the best constant in the Schwarz-Pick inequality. The domains considered were just the open unit balls of J^*-algebras.

Definitions. An operator B of a J^*-algebra $\mathcal{A}$ is called a minimal element of $\mathcal{A}$ if for each $A \in \mathcal{A}$ there exists a $\lambda \in \mathbb{C}$ with $BA^*B = \lambda B$. Two operators $A, B \in \mathcal{A}$ are called orthogonal if $AB^* = B^*A = 0$. The rank of a J^*-algebra is the maximum number of mutually orthogonal non-zero minimal elements which can be found in the J^*-algebra.

The Schwarz-Pick inequalities are proved by HARRIS [64 c] for the open unit balls of J^*-algebras having finite rank. The rank of a finite-dimensional J^*-algebra coincides with the rank of its open unit ball as a Hermitian symmetric space. On each J^*-algebra of finite rank r, Harris defined an inner product, in terms of the minimal elements, and showed that it induces an invariant infinitesimal Hermitian metric on the open unit ball of the $\mathcal{J}^*$-algebra with Schwarz constant $\sqrt{r}$. He also showed that any other infinitesimal metric having these properties must be one of its scalar multiples. This contradicts the results obtained by Look and Korányi for the Bergman metric. Finally, Harris gave necessary and sufficient conditions for each of Korányi's inequalities to hold for a classical bounded symmetric domain and obtained an expression for the integrated form of an arbitrary infinitesimal Hermitian metric on such a domain.

Recently, HARRIS [64 d] explored an algebraic theory parallel to that of C^*-algebras for $\mathcal{J}^*$-algebras. In particular he established the following spectral theorems.

Theorem 5.9. Let $\mathcal{A}$ be a $\mathcal{J}^*$-algebra and let $A \in \mathcal{A}$, $A \neq 0$. Then the spectrum of A^*A has no nonzero limit point if and only if there exists a set $\{V_n\}$ of mutually orthogonal nonzero partial isometries in $\mathcal{A}$, and a set $\{\alpha_n\}$ of distinct positive numbers, such that $A = \sum_n \alpha_n V_n$. Moreover, this representation of A is unique (up to the order of terms) when it exists, and $\{\alpha_n\}$ is the set of square roots of the nonzero eigenvalues of A^*A.

Theorem 5.10. Let $\mathcal{A}$ be a $\mathcal{J}^*$-algebra which is supposed to be W^*-closed Let $A \in \mathcal{A}$ and let Σ be the set of square roots of the nonzero elements of the spectrum of A^*A. Then there exists a unique $\mathcal{A}$-valued spectral measure $\sigma \longrightarrow V(\sigma)$ on Σ such that if $\sigma_1, \dots, \sigma_n$ is a partition of Σ consisting of Borel sets and satisfying diam $\sigma_k < \varepsilon$, and $\alpha_k \in \sigma_k$ for $k=1, \dots, n$, then

$$\left\| A - \sum_{1}^{n} \alpha_k V(\sigma_k) \right\| < \varepsilon$$

. Moreover, $V(\sigma)$ commutes with A

for each Borel subset σ of $\sum$ and $V(\sum)$ convers A.

Theorem 5.11. Let $\mathcal{A}$ be a $\mathcal{J}^*$-algebra, suppose that $\mathcal{A}$ has semifinite rank (i.e. the spectrum of A^*A has no nonzero limit point for each $A \in \mathcal{A}$), and let $A \in \mathcal{A}$ with $A \neq 0$. Then there exists a set $\{V_n\}$ of mutually orthogonal nonzero minimal partial isometries in $\mathcal{A}$, and a, possibly finite, sequence $\{a_n\}$ of positive numbers, such that $A = \sum_n a_n V_n$. In fact, the terms of the sequence $\{a_n\}$ are precisely the square roots of all nonzero eigenvalues of A^*A with at most finitely many repetitions.

HARRIS [64 d] also discussed indecomposability, transitivity and irreducibility (see [64 d, §§ 4 and 5]). An extended notion of irreducibility is used, since there are many $\mathcal{J}^*$-algebras and even JC^*-algebras which cannot be faithfully represented by any transitive $\mathcal{J}^*$-algebra. A theory of ideals for $\mathcal{J}^*$-algebras is then presented. A number of equivalent conditions for a $\mathcal{J}^*$-algebra to have finite rank are also given [64 d, § 6]. These include reflexivity, a variant of von Neumann regularity and the existence of finitely many mutually orthogonal minimal elements whose sum is an extremal element . Necessary and sufficient conditions for a $\mathcal{J}^*$-algebra to be $\mathcal{J}^*$-isomorphic to a JC^*-algebra, to an adjoint closed $\mathcal{J}^*$-algebra, or to a complexified spin factor are also given. Finally, Harris listed eight open problems which suggest some directions for further development of the structure theory of $\mathcal{J}^*$-algebras.

§ 6. Jordan algebras and positive projections on operator algebras

Let B be a unital C^*-algebra and let $P : B \longrightarrow B$ be a unital positive projection, i.e. $P \geq 0$, $P(1) = 1$, and $P^2 = P$. As is known (CHOI and EFFROS [40] and TOMIYAMA [129]) the image of B is a C^*-algebra under the product $a.b := P(ab)$ if and only if P is completely positive. Thus, when B acts on a Hilbert space H, P(B) is a C^*-algebra if and only if there exists a linear isometry V of H on a Hilbert space K and a $*$-representation π of B on K such that

(6.1) $$P(x)=V^* \pi (x) V,$$

for all $x \in B$.

EFFROS and STØRMER [48 b] showed that with the product $a * b := P(a \circ b)$, where $a \circ b := \frac{1}{2}(ab+ba)$, in the image A of the self-adjoint part B_h of B under P, A has a faithful representation as a JC-algebra (i.e. a norm closed Jordan algebra of selfadjoint operators on a Hilbert space). [48 b] may be regarded as an attempt to place ARAZY and FRIEDMAN's monograph [12] in a general setting. The latter authors have characterized the ranges of contractive projections in the algebra of compact operators on a separable Hilbert space. A closer inspection of their results seems to indicate that what they are doing is to classify certain Jordan and Lie algebras of operators. EFFROS and STØRMER's approach [48 b] might explain the unexpected occurrence of Jordan algebras.

If $N := \{ a \in B_h \mid P(a^2) = 0 \}$, then A+N is a JC-subalgebra of B_h, and P restricted to A+N is a Jordan homomorphism of A+N onto A with kernel N. As there exist hosts of examples in which A is not the selfadjoint part of a C^*-algebra (see [48 b]), in general we cannot expect that P is completely positive. However, we might expect that if one symmetrizes the definition of π in (6.1) to be a Jordan homomorphism, then a decomposition like (6.1) might hold. In this case, P is decomposable in the sense of STØRMER [127 e], namely:

<u>Definition</u>. Let B be a C^*-algebra and let H be a (complex) Hilbert space. A positive linear map φ of B into the bounded operators B(H) on H is called <u>decomposable</u> if there exist a Hilbert space K, a bounded linear operator V of H into K, and a Jordan $*$-homomorphism π of B into B(K) such that $\varphi(x) = V^*\pi(x)V$, for all $x \in B$.

<u>Remark</u>. For such maps the term "Jordanian type map" is also used (see WORONOWICZ [138]).

STØRMER characterizes [127 e] those projections P which are decomposable, the characterization being in terms of the JC-algebra A+N.

<u>Definition</u>. A JC-algebra is called <u>reversible</u> if it is closed under symmetric products $a_1a_2\ldots a_n + a_n\ldots a_2a_1$, when the a_i's lie in the algebra.

The main result given by STØRMER [127 c] is the following theorem (Størmer's extra asumptions on A or P lead to the "less technical" form given below).

<u>Theorem 6.1.</u> Let B be a unital C^*-algebra and let P be a unital positive projection of B into itself. Let $A = P(B_h)$ and let $N := \{a \in B_h \mid P(a^2) = 0\}$. If A is a JC-subalgebra of B_h then P is decomposable if and only if A is reversible. If the restriction of P to the C^*-algebra generated by A is faithful, then P is decomposable if and only if A+N is a reversible JC-subalgebra of B_h.

Remark 1. A unital positive projection onto a spin factor whose real dimension exceeds six is never decomposable (for examples, see [48 b]).

Remark 2. If A is the set of fixed points in B_h under a family of Jordan automorphisms of B, then a positive projection onto A is automatically decomposable.

Remark 3. Let φ be a normal unital positive linear map of a von Neumann algebra M onto itself and let $A := \{a \in M_h \mid \varphi(a)=a\}$. As was noted by Connes (see [48 b, Corollary 1.6]) there exists a positive projection of M into itself with $P(M_h) = A$, hence A has a faithful representation as a weakly closed JC-algebra. It can be easily seen that if φ is decomposable then so is P, hence the representation is onto a reversible JC-algebra. Since A is the eigenspace in M_h for the eigenvalue 1, an algebraic condition on one of the eigenspaces, necessary in order that φ be decomposable was thus obtained. This result, which was for a long time an open question, shows that a future theory of spectral subspaces of positive maps might be extremely fruitful.

RODRIGUEZ [111 b] proved that every unital n.c. JB^*-algebra can be linearly and isometrically embeded into the algebra of bounded linear operators on a suitable complex Hilbert space (moreover, with preservation of the unit). From this it is deduced that unit-preserving positive linear maps between n.c. JB^*-algebras have norm one. This, together with Theorem 4.4., is used to generalize the results of CHOI and EFFROS [40], and EFFROS and STØRMER [48 b] on ranges of unit preserving positive projections in associative B^*-algebras and JB-algebras, respectively.

Starting from the remark that if α is an anti-automorphism then the elements of B satisfying $\alpha(x) = -x$ form a Lie algebra with respect to the natural Lie product $[x,y] := xy - yx$, ROBINSON and STØRMER [110] explore relations between this Lie structure, the Jordan structure of the fixed points, and positive projections associated with anti-automorphisms. They show that there exists a pairing between the classical Lie algebras and their infinite-dimensional analogues, and the irreducible reversible Jordan algebras of selfadjoint operators or the scalars.

As has already been noted on page 48, ARAZY and FRIEDMAN characterized the contractive projections on the C^*-algebra of compact operators on a separable Hilbert space with contractive complements (see [12, Proposition 7.7 and Corollary 7.8]). Recently, ROBERTSON and YOUNGSON [109] studied positive contractive projections with contractive complements on unital JB-algebras.

Remark. A unital projection P (i.e. $P(1) = 1$, where 1 denotes the unit) on a JB-algebra is positive if and only if it is contractive.

Let us recall the following two theorems of ROBERTSON and YOUNGSON [109]:

Theorem 6.2. Let P be a unital positive projection on a JB-algebra $\mathcal{J}$. Then Id - P is contractive if and only if $\| P(x^2) \| = \| x^2 \|$, whenever $P(x) = 0$, $x \in \mathcal{J}$.

Remark. This problem has been studied extensively, and fairly general results have been proved. In the C^*-case, STØRMER [127 f] showed that, for a C^*-algebra A, a unital positive projection $P : A \longrightarrow A$ is of the form $P = \frac{1}{2}(Id + \theta)$, θ a Jordan automorphism of order 2, if and only if Id - P is contractive. This result was extended by Friedman and Russo to Jordan triple systems, their proof being rather involved. A very short proof was given by KAUP [84 c].

Theorem 6.3. Let $\mathcal{J}$ be a nontrivial JBW-factor and let $P : \mathcal{J} \longrightarrow \mathcal{J}$ be a unital positive projection with contractive complement. Suppose that $P(\mathcal{J})$ is commutative and atomic (that is, the identity is the sum of minimal idempotents in $P(\mathcal{J})$). Then $\mathcal{J}$ is a spin factor and $P(\mathcal{J})$ is at most two-dimensional.

Comments. FRIEDMAN and RUSSO (see [57]) showed that Jordan triple systems appear naturally as the fixed point set of a contraction on an operator algebra. Namely, they proved that if P is a contractive projection on a C^*-algebra, then the range of P is a Jordan triple system. The fact that the range of a unital positive projection on a C^*-algebra is a Jordan algebra follows from this result.

As we have seen (Theorem 6.1), STØRMER [127 e] linked the decomposability of positive projections to the theory of JC-algebras. ROBERTSON [108 a] further elaborated this connection. E.g., he considered automorphisms of a 6-dimensional spin factor embedded in the C^*-algebra of complex (4×4)-matrices. In fact he proved that automorphisms not lying in the connected component of the identity do not even extend to decomposable positive linear maps (in the sense of [127 a]). The aim of another recent paper by ROBERTSON [108 b] is to show that this result can be generalized to spin factors of dimension $4k + 2$, $k=1,2,\ldots$, but not to those of any other dimension.

§ 7. Theta functions for Jordan algebras

In 1975 RESNIKOFF [107 b] introduced theta functions intrinsically associated with formally real finite-dimensional

Jordan algebras. If the Jordan algebra consists of the real symmetric matrices of a certain order, then the theta functions introduced by SIEGEL [124] are the "nullwerte" of the theta functions associated with that algebra, whereas if the algebra consists of the Hermitian complex matrices of a certain order, then the "nullwerte" of the associated theta functions are the theta functions introduced by BRAUN [35] and used by FREITAG [56].

As was observed by RESNIKOFF [107 b], theta functions can be considered as functions of variables of two types: the first, called "toroidal" variable by Resnikoff, is an element of $\mathbb{C}^n$ for some n; the second, called "modulus", characterizes a given lattice in $\mathbb{C}^n$. The above-mentioned "nullwerte" are the functions of the modulus which result when the toroidal variable is 0. For theta functions associated with Jordan algebras as defined below, the toroidal variable and the modulus are elements of the algebra, with the consequence that the modulus can be made to act on the toroidal variable. For the Jordan algebra of real symmetric matrices of a certain order, this action is the classically known action of the modulus on the toroidal variable for "abelian theta functions" (i.e. the theta functions studied by Riemann. These suffice to represent arbitrary abelian functions).

In 1978 DORFMEISTER [44] gave another approach, which is more general since algebras of degree 2 are included. He showed how the main properties of theta functions, proved by RESNIKOFF [107 b], carry over to the more general situation.

In order to recall the main results established by RESNIKOFF [107 b], we first give some

<u>Notation</u>. Let $\mathcal{A}$ be a (finite-dimensional) formally real Jordan algebra with unit element e. We denote by σ the redu-

ced trace form associated with $\mathcal{A}$. If we put $Z(\mathcal{A}) := \mathcal{A} + i \exp \mathcal{A} \subset \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$, then $Z(\mathcal{A})$ is biholomorphically equivalent to a bounded symmetric domain of tube type (see KOECHER [87 b]). Set $Z(\mathcal{A}_{1/2}) := \mathcal{A}_{1/2} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathcal{A}_{1/2}$ is the 1/2-Peirce component of $\mathcal{A}$ with respect to an idempotent c. Denote by $\mathcal{L}_{1/2}$ a lattice in $\mathcal{A}_{1/2}$ (i.e. a free $\mathbb{Z}$-module of maximal rank in the $\mathbb{R}$-linear structure of $\mathcal{A}_{1/2}$). Define the dual of $\mathcal{L}_{1/2}$ in $\mathcal{A}_{1/2}$ by

$$\hat{\mathcal{L}}_{1/2} := \left\{ \hat{n} \in \mathcal{A}_{1/2} \,\middle|\, \sigma(\hat{n}, n) \in 2\mathbb{Z} \text{ for all } n \in \mathcal{L}_{1/2} \right\}.$$

<u>Definition</u>. Let c be an idempotent of $\mathcal{A}$, $c \neq o, e$. Let $\mathcal{L}_{1/2} \subset \mathcal{A}_{1/2}$ be a lattice, and let $v \in \exp \mathcal{A}_o$. The <u>theta function</u> of order v associated with c and $\mathcal{L}_{1/2}$ is

$$\theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2}; v) := (\mathrm{vol}\, \mathcal{L}_{1/2})^{\frac{1}{2}} \sum_{n \in \mathcal{L}_{1/2}} \exp i\pi\sigma(P(n)z_1 + 2nz_{1/2}, v)$$

where $(z_1, z_{1/2}) \in Z(\mathcal{A}_1) \times Z(\mathcal{A}_{1/2})$ and vol $\mathcal{L}_{1/2}$ denotes the volume of a fundamental domain for $\mathcal{L}_{1/2}$ relative to a Haar measure on the $\mathbb{R}$-linear space $\mathcal{A}_{1/2}$.

Let $\mathcal{A} = \bigoplus_{i=1}^{N} \mathcal{A}^{(i)}$ be the decomposition of $\mathcal{A}$ into simple summands. If $x \in \mathcal{A}$, then $x = \bigoplus x^{(i)}$ with $x^{(i)} \in \mathcal{A}^{(i)}$. If $s \in \mathbb{C}^N$, write $|x|^s = \prod_{i=1}^{N} |x^{(i)}|^{s_i}$, where $|x^{(i)}|$ denotes the reduces norm of $x^{(i)}$ in the simple algebra $\mathcal{A}^{(i)}$.

<u>Notation</u>. For a semisimple algebra $\mathcal{A}$, with the decomposition into simple summands as denoted above, we introduce a vector $q(\mathcal{A}) \in \frac{1}{2}\mathbb{Z}^N$ by $q_i(\mathcal{A}) := \dim \mathcal{A}^{(i)} / \mathrm{rank}\, \mathcal{A}^{(i)}$.

Theorem 7.1. Put $c^{\perp} := e - c$. Then

$$\theta_{\hat{\mathcal{L}}_{1/2}}(-z_1^{-1}, 4z_1^{-1}(z_{1/2}v) : v^{-1}) =$$

$$= \left|-iz_1 + c^{\perp}\right|^{q(\mathcal{A})-q(\mathcal{A}_1)} \left|v+c\right|^{q(\mathcal{A})-q(\mathcal{A}_0)} \exp i\pi\sigma(P(z_{1/2})z_1^{-1}, v) \times$$

$$\times \theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} : v).$$

Theorem 7.2. Let $W \in \operatorname{End}\mathcal{A}$ satisfy $P(Wa) = W\,P(a)W^{\sigma}$ for $a \in \mathcal{A}$, where W^{σ} denotes the σ-adjoint of W. Further assume that $W|_{\mathcal{A}_i} \in \operatorname{End}\mathcal{A}_i$, that $W|_{\mathcal{A}_0}$ is an automorphism of $\mathcal{A}_0$, and finally that W^{σ} preserves $\mathcal{L}_{1/2}$. Then

$$\theta_{\mathcal{L}_{1/2}}(Wz_1, Wz_{1/2} : Wv) = \theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} : v).$$

Definitions. A lattice $\mathcal{L} \subset \mathcal{A}$ is called a Jordan lattice if there exists a $\varrho \in \mathbb{Z}^+$ such that $n \in \mathcal{L}$ implies that $\varrho n^2 \in \mathcal{L}$. (see HELWIG [65 a, II, p.172]). If $\mathcal{L}$ is a Jordan lattice, then $\varrho\mathcal{L}$ is a Jordan lattice such that $n \in \varrho\mathcal{L}$ implies that $n^2 \in \varrho\mathcal{L}$. A lattice having the latter property is called a standard Jordan lattice (see RESNIKOFF [107 b, p.90]).

Remark. HELWIG (see [65 a, II, § 3]) showed the existence of Jordan lattices for formally real Jordan algebras.

Theorem 7.3. Let $\mathcal{L} \subset \mathcal{A}$ be a standard Jordan lattice. Put $\mathcal{L}_i := \mathcal{L} \cap \mathcal{A}_i$. Set $\hat{\mathcal{L}} = \left\{ \hat{n} \in \mathcal{A} \,\middle|\, n \in \mathcal{L} \text{ implies } \sigma(\hat{n}, n) \in 2\mathbb{Z} \right\}$. If $v \in \mathcal{L}_0$, then $\hat{n}_1 \in \hat{\mathcal{L}}_1$ implies that $\theta_{\mathcal{L}_{1/2}}(z_1 + \hat{n}_1, z_{1/2} : v) =$

$$= \theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} : v).$$

The following theorem gives the functional equations, describing the periodicity of the toroidal variable.

Theorem 7.4. If $\mathcal{L} \subset \mathcal{A}$ is a standard Jordan lattice, $\mathcal{L}_i = \mathcal{L} \cap \mathcal{A}_i$, and $v \in \mathcal{L}_o$, then

a) $\theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} + \hat{n}_{1/2} : v) = \theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} : v)$ if $\hat{n}_{1/2} \in \hat{\mathcal{L}}_{1/2}$;

b) $\theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} + 2\, z_1\, n_{1/2} : v) =$

$= \exp i\pi\sigma(P(n_{1/2})z_1 + 2\, n_{1/2} z_{1/2}, v)\theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} : v)$

if $n_{1/2} \in \mathcal{L}_{1/2}$.

Remark. If $\mathcal{O} := \mathcal{O}(Z(\mathcal{A}_1))$ denotes the ring of holomorphic functions on $Z(\mathcal{A}_1)$ and $\mathcal{M}$ denotes the set of holomorphic functions on $Z(\mathcal{A}_1) \times Z(\mathcal{A}_{1/2})$ which satisfy the functional equations a) and b) in Theorem 7.4, then $\mathcal{M}$ is an $\mathcal{O}$-module. Moreover, the following theorem holds.

Theorem 7.5. For $\mathcal{L} \subset \mathcal{A}$ a standard Jordan lattice, put $\mathcal{L}_i = \mathcal{L} \cap \mathcal{A}_i$, and suppose $v \in \mathcal{L}_o$. Then

$$\dim_{\mathcal{O}} \mathcal{M} = (\mathrm{vol}\ 2v\mathcal{L}_{1/2})/(\mathrm{vol}\ \mathcal{L}_{1/2}),$$

where $2v\mathcal{L}_{1/2} = \{2vn \mid n \in \mathcal{L}_{1/2}\}$. A basis for $\mathcal{M}$ over $\mathcal{O}$ is given by the functions

$$\exp i\pi\sigma(n_{1/2})\theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2} + 4\, v^{-1}(z_1 n) : v),$$

where n runs through the $\dim_{\mathcal{O}} \mathcal{M}$ elements of a fundamental domain for $2\, v\mathcal{L}_{1/2}$ in $\mathcal{L}_{1/2}$.

Corollary. If $v = c^{\perp}$ and $\varphi \in \mathcal{M}$, then there exists a function $h \in \mathcal{O}(Z(\mathcal{A}_1))$ such that $\varphi(z_1, z_{1/2}) = h(z_1)\theta_{\mathcal{L}_{1/2}}(z_1, z_{1/2}: c^{\perp})$.

Definition. Denote by $\varphi : Z(\mathcal{A}) \longrightarrow \mathbb{C}$ a holomorphic function possessing a Fourier expansion of the form

$$\varphi(z) = \sum a(n) \exp i\pi\sigma(n) ,$$

where the argument n takes nonnegative values and belongs to a lattice. Suppose further that φ satisfies the equation

$$\varphi(-z^{-1}) = k\,|z|^{w}\varphi(z) ,$$

where k is a constant and $w \in \mathbb{R}^{\operatorname{rank}\mathcal{A}}$. If $a(n) \neq 0$ implies that $|n| = 0$, then φ is said to be a singular form of weight w.

Theorem 7.6. Suppose that $\mathcal{A}$ is a simple Jordan algebra. Let $\mathcal{L}$ be a standard Jordan lattice in $\mathcal{A}$ and put $\mathcal{L}_i = \mathcal{L} \cap \mathcal{A}_i$. Further assume that $\widehat{\mathcal{L}}_{1/2} = \mathcal{L}_{1/2}$. Then $Z(\mathcal{A}_1) \ni z_1 \longrightarrow \theta_{\mathcal{L}_{1/2}}(z_1, 0 : c^{\perp})$ is a singular form of weight $q(\mathcal{A}) - q(\mathcal{A}_1)$ if and only if $2 \operatorname{rank}\mathcal{A}_1 > \operatorname{rank}\mathcal{A}$. For every possible singular weight (i.e. weight of a singular form) associated with $\mathcal{A}_1$, there exists a singular theta function of that weight.

RESNIKOFF [107 b] also proved that, apart from theta functions in one toroidal variable, two theta functions coincide after a linear transformation of their arguments if and only if the corresponding Jordan algebras are isomorphic. Finally, he showed how the theta functions associated with (formally real) Jordan algebras can be realized as restrictions of abelian theta functions to submanifolds of, in general, large codimension in the space of the modulus variable.

In 1973, RESNIKOFF [107 a] showed that singular forms have singular weights. Two years later, he proved [107 c] the converse, which is substantially more difficult and requires the introduction of the entire apparatus of the theory of theta functions associated with formally real Jordan algebras.

Remark. Resnikoff defined a theta function for each formally real Jordan algebra which is a Peirce-1-space of another formally real Jordan algebra. This condition is not satisfied - apart from some exceptions - for algebras of degree two.

In 1978, DORFMEISTER [44] showed how the above-mentioned restriction can be removed. He changed the setting, and, instead of dealing with a Peirce-1-space of a formally real Jordan algebra, he considered a formally real Jordan algebra together with a representation of that algebra. So, Dorfmeister's setting is very close to that of the classical theory of theta functions.

We shall recall now the results given by DORFMEISTER [44].

Notation. If U is a real finite-dimensional vector space and τ is a positive definite symmetric bilinear form on U, then $\mathrm{Sym}(U, \tau) := \{ A \mid A \in \mathrm{End}_{\mathbb{R}} U,\ A^{\tau} = A \}$, where A^{τ} denotes the adjoint endomorphism to A with respect to τ.

Remark. As is well known (see BRAUN and KOECHER [36, Chapter XI]), $\mathrm{Sym}(U, \tau)$ is a formally real Jordan algebra with respect to the Jordan product $AB := \frac{1}{2}(A.B + B.A)$, $A, B \in \mathrm{Sym}(U, \tau)$, where the dot denotes the ordinary product in $\mathrm{End}_{\mathbb{R}} U$.

Definition. Let $\mathcal{A}$, τ, and φ be as follows: $\mathcal{A}$ is a (finite-dimensional) formally real Jordan algebra, τ is a positive definite symmetric bilinear form on a finite-dimensional $\mathbb{R}$-vector space U, φ is an injective homomorphism of Jordan algebras,

$\varphi : \mathcal{A} \longrightarrow \mathrm{Sym}(U, \tau)$, which satisfies $\varphi(e) = \mathrm{Id}$ (e is the unit of $\mathcal{A}$, which exists by [36 Theorem 3.4, Chapter XI]). Then $(\mathcal{A}, \varphi, \tau)$ is called a Θ - triple.

Proposition 7.7. If $\mathcal{A}$ is a special formally real Jordan algebra, then there exist φ and τ such that $(\mathcal{A}, \varphi, \tau)$ is a Θ - triple.

Notation. The set of all invertible elements of $\mathcal{A}$ will be denoted by $\mathrm{Inv}(\mathcal{A})$, while $\mathrm{Inv}_o(\mathcal{A})$ will denote the connected component of $\mathrm{Inv}(\mathcal{A})$ containing e.

Proposition 7.8. Let $(\mathcal{A}, \varphi, \tau)$ be a Θ - triple and let $(W, \hat{W}) \in GL_{\mathbb{R}}\mathcal{A} \times GL_{\mathbb{R}}U$ such that $\varphi(Wx) = \hat{W}\varphi(x)\hat{W}^{\tau}$ for all $x \in \mathcal{A}$. Then $W\,\mathrm{Inv}_o(\mathcal{A}) = \mathrm{Inv}_o(\mathcal{A})$.

Definition. Let $(\mathcal{A}, \varphi, \tau)$ be a Θ - triple and let $A \in \mathrm{End}_{\mathbb{R}}U$ be a positive definite self-adjoint endomorphism with respect to τ such that $A\varphi(x) = \varphi(x)A$ for all $x \in \mathcal{A}$. Then A is called a weight for $(\mathcal{A}, \varphi, \tau)$.

Conventions. If $(\mathcal{A}, \varphi, \tau)$ is a Θ - triple, then τ is extended to a $\mathbb{C}$-bilinear form on the complexification $U^{\mathbb{C}}$ of U. This extension is again called τ. Further, φ is extended to a map of $\mathcal{A}^{\mathbb{C}}$ to $\mathrm{End}_{\mathbb{C}}\,U^{\mathbb{C}}$ by linearity. This map is again written as φ. We put $Z(\mathcal{A}) := \mathcal{A} \oplus i\,\mathrm{Inv}_o(\mathcal{A})$.

Definition. Let $(\mathcal{A}, \varphi, \tau)$ be a Θ -triple and let A be a weight for $(\mathcal{A}, \varphi, \tau)$. If $\mathcal{L}$ is a lattice in U, then, for all $z \in Z(\mathcal{A})$ and all $u \in U^{\mathbb{C}}$, we define

$$\Theta_{(\mathcal{A}, \varphi, \tau, \mathcal{L})}(z, u, A) := (\mathrm{vol}\,\mathcal{L})^{\frac{1}{2}} \sum_{l \in \mathcal{L}} \exp i\pi\sigma(\varphi(z)l + 2u, Al),$$

where $\mathrm{vol}\,\mathcal{L}$ denotes the volume of a fundamental domain for $\mathcal{L}$ relative to a Haar measure on U. The function $(z,u) \longrightarrow$.

$\rightarrow \Theta_{(\mathcal{A},\varphi,\tau,\mathcal{L})}(z,u,A)$ is called a Θ -<u>function for</u> $\mathcal{A}$ (with weight A).

Proposition 7.7 implies

<u>Theorem 7.9</u>. For each special formally real Jordan algebra there exists a Θ - function.

<u>Remark</u>. Let $\mathcal{B}$ be a formally real Jordan algebra with unit b, left multiplications L(x), and reduced trace form β. Let $c \neq o$, be an idempotent in $\mathcal{B}$. Consider the Peirce decomposition $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_{1/2} \oplus \mathcal{B}_o$ of $\mathcal{B}$ relative to c, and define

$$\mathcal{A} := \mathcal{B}_1, \quad \varphi(x) := 2\,L(x)\big|_{\mathcal{B}_{1/2}},\ x \in \mathcal{B}_1,\ U := \mathcal{B}_{1/2},\ \tau := \beta\big|_{\mathcal{B}_{1/2}}.$$

From the properties of a Peirce decomposition it follows that $(\mathcal{A},\varphi,\tau)$ is a Θ - triple, that $A := L(y)\big|_{\mathcal{B}_{1/2}}$ is a weight for $(\mathcal{A},\varphi,\tau)$ for all $y \in \mathrm{Inv}_o(\mathcal{B}_o)$, and that $\tau(P(n)z_1 + 2\,nz_{1/2}, v) = \tau(\varphi(z_1)n + 2\,z_{1/2}, An)$ for all $z_1 \in Z(\mathcal{A})$, $z_{1/2} \in U^{\mathbb{C}}$, $n \in U$, $v \in \mathrm{Inv}_o(\mathcal{B}_o)$ and $A = L(v)\big|_{\mathcal{B}_{1/2}}$. Now for $\mathcal{L} = \mathcal{L}_{1/2}$, Resnikoff's definition of theta functions coincides with Dorfmeister's definition of Θ -functions.

<u>Notation</u>. Consider a fixed Θ -triple $(\mathcal{A},\varphi,\tau)$, a weight A for $(\mathcal{A},\varphi,\tau)$, and a lattice $\mathcal{L}$ in U. Put

$$\mathcal{L}^{\tau} := \{x \mid x \in U, \quad \tau(x,l) \in 2\,\mathbb{Z} \text{ for all } l \in \mathcal{L}\}.$$

<u>Proposition 7.10</u>. Let $(W,\hat{W}) \in GL_{\mathbb{R}}\mathcal{A} \times GL_{\mathbb{R}}U$ satisfy $\varphi(Wx) = \hat{W}\varphi(x)\hat{W}^{\tau}$ for all $x \in \mathcal{A}$. Then for all $z \in Z(\mathcal{A})$, $u \in U^{\mathbb{C}}$, we have

$$\Theta_{\mathcal{L}}(Wz,\ \hat{W}u,\ A) = (\det \hat{W})^{-1/2}\ \Theta_{\hat{W}^{\tau}\mathcal{L}}(z,u,\hat{W}^{\tau}A(\hat{W}^{\tau})^{-1}).$$

Theorem 7.11. For all $z \in Z(\mathcal{A})$ and all $u \in U^{\mathbb{C}}$ we have

$$\Theta_{\mathcal{L}}(-z^{-1}, 2\varphi(z^{-1})Au, \tfrac{1}{4}A^{-1}) =$$

$$=(\det 2A)^{1/2}(\det \varphi(-iz))^{1/2} \exp i\pi\sigma(\varphi(z^{-1})u, Au). \Theta_{\mathcal{L}}(z,u,A).$$

Proposition 7.12. Let $z \in Z(\mathcal{A})$, $u \in U^{\mathbb{C}}$ and $n \in \mathcal{A}^{\mathbb{C}}$ such that $\tau(\varphi(n)\ell, A\ell) \in 2\mathbb{Z}$ for all $\ell \in \mathcal{L}$. Then $\Theta_{\mathcal{L}}(z+n,u,A) = \Theta_{\mathcal{L}}(z,u,A)$.

Proposition 7.13. Let $z \in Z(\mathcal{A})$ and $n \in U^{\mathbb{C}}$. Then

a) $\Theta_{\mathcal{L}}(z,u+n,A) = \Theta_{\mathcal{L}}(z,u,A)$ for all $n \in \frac{1}{2} A^{-1}\mathcal{L}^{\tau}$;

b) $\Theta_{\mathcal{L}}(z,u+\varphi(z)m,A) = \exp - i\pi\tau(\varphi(z)m+2u, Am)\, \Theta_{\mathcal{L}}(z,u,A)$ for all $m \in \mathcal{L}$.

Note. Assume now that $2A\mathcal{L} \subset \mathcal{L}$.

Notation. Denote by $\mathcal{O} := \mathcal{O}(Z(\mathcal{A}))$ the ring of holomorphic functions on $Z(\mathcal{A})$ and by $\mathcal{M}$ the $\mathbb{C}$-vector space of holomorphic functions on $Z(\mathcal{A}) \times U^{\mathbb{C}}$ which satisfy the functional equations in Proposition 7.13 for all $z \in Z(\mathcal{A})$, $u \in U^{\mathbb{C}}$, $n \in \mathcal{L}^{\tau}$, and all $m \in \mathcal{L}$.

Remark. From $2A\mathcal{L} \subset \mathcal{L}$ it follows that $\frac{1}{2}A^{-1}\mathcal{L}^{\tau} \supset \mathcal{L}^{\tau}$. Hence, by Proposition 7.13, the function $\Theta_{\mathcal{L}}$ is an element of $\mathcal{M}$.

Theorem 7.14. a) $\mathcal{M}$ is a free $\mathcal{O}$-module;

b) $\dim_{\mathcal{O}} \mathcal{M} = (\mathrm{vol}\, 2A\mathcal{L})(\mathrm{vol}\,\mathcal{L})^{-1}$;

c) a basis for $\mathcal{M}$ over $\mathcal{O}$ is given by the functions

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$\exp i\pi\tau(q,u)\, \Theta_{\mathcal{L}}(z, u + \frac{1}{2}A^{-1}\varphi(z)q, A)$, where q runs through the $\dim_{\mathcal{O}}\mathcal{M}$ elements of a fundamental domain for $2A\mathcal{L}$ in $\mathcal{L}$.

Remark. FARAUT and TRAVAGLINI [53] studied Bessel functions on the symmetric cone associated with a special formally real Jordan algebra. They extended the classical results of radial Fourier analysis and proved an asymptotic formula.

Finally, let us mention a classical result of NEUMANN [102]: Let P_k be the Legendre polynomial of degree k; the domain of convergence of the Legendre series $f(z) := \sum_{k=0}^{\infty} a_k P_k(z)$ is bounded by an ellipse. LASSALLE [90 a] has given a group-theoretic interpretation of this result. FARAUT [51] presented the results obtained in collaboration with KORÁNYI (see [52 a]) concerning expansions in series that generalize Legendre series.

§ 8. Differential equations, kernel functions, and Jordan algebras

This section is concerned with applications of Jordan algebras to the Riccati differential equation (see 1), to soliton equations (see 2), to Hua equations and Szegö kernel (see 3), and to the (reproducing) kernel functions (see 4).

1. The Riccati differential equation

$$\dot{x} = p(x), \tag{8.1}$$

$x \in \mathbb{R}^n$ and $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ homogeneous and quadratic, plays an important role in biology, genetics, ecology, and chemistry. KOECHER [87 c,d,e] and MEYBERG [97] studied the relations of this equation with nonassociative algebras, in particular with Jordan algebras.

We consider a commutative algebra $\mathcal{A}$ over $\mathbb{R}^n$ with product $xy := \frac{1}{2}(p(x+y) - p(x) - p(y))$, and let $\mathcal{A}_a$ be the mutation of $\mathcal{A}$ with respect to a (see JSA. I, § 1).

Notation. Denote by $\mathcal{R}_n$ the vector space of power series in $\mathbb{R}^n$ converging in a neighborhood of zero.

For $p, q \in \mathcal{R}_n$ we define $p \cdot q \in \mathcal{R}_n$ by

$$\left[(p \cdot q)(u)\right]_i := \sum_{j=1}^{n} \frac{\partial p_i(u)}{\partial u_j} q_j(u).$$

Remark. The vector space $\mathcal{R}_n$ with the product $(p,q) \longrightarrow p \cdot q$ becomes a nonassociative algebra over $\mathbb{R}$.

Now define $g_{\mathcal{A}}(u) \in \mathcal{R}_n$ by $g_{\mathcal{A}}(u) := \sum_{m=0}^{\infty} \frac{1}{m!} g_m(u)$, where

$g_0(u) := u$, $g_{m+1} := g_m \cdot p$, and $p(u) := u^2$. (Powers in $\mathcal{A}$ are defined as follows: $u^1 := u$, $u^{m+1} := uu^m$.)

The elements $f \in \mathcal{R}_n$ such that $f(x(\xi))$ is a solution of the Riccati equation (8.1) whenever $x(\xi)$ is a solution, form a group $S(\mathcal{A})$ under composition, the solution-preserving group of (8.1).

Notation. Let $\mathcal{J}(\mathcal{A})$ denote the subspace of all $a \in \mathbb{R}^n$ satisfying $2u(u(ua)) + u^3 a = 2u(u^2 a) + u^2(ua)$ for all $u \in \mathbb{R}^n$.

Theorem 8.1. If $\mathcal{A}$ has a unit element, then $a \longrightarrow g_{\mathcal{A}_a}$ is an isomorphism of the additive group $\mathcal{J}(\mathcal{A})$ to $S(\mathcal{A})$.

Theorem 8.2. If $\mathcal{A}$ is a commutative algebra over $\mathbb{R}^n$, then $\mathcal{J}(\mathcal{A})$ is a Jordan subalgebra of $\mathcal{A}$.

Moreover, the following theorem holds.

Theorem 8.3. If $\mathcal{A}$ is a finite-dimensional commutative algebra over a field of characteristic different from two or three, then $\mathcal{J}(\mathcal{A})$ is a Jordan subalgebra of $\mathcal{A}$.

2. In the winter of 1980-1981, SATO proved that the totality of solutions of the Kadomtsev-Petviashvili (KP) equation,

$$3u_{yy} + (-4\, u_t + u_{xxx} + 12\, uu_x)_x = 0,$$

forms an infinite-dimensional Grassmann manifold GM (see [119 a]).

Note. Let us recall that the KP equation was discovered 1970 in an effort to understand the propagation of long, shallow waves in plasma (see B.KADOMTSEV and V.PETVIASHVILI, Dodl.Akad. Nauk SSSR 192 (1970), No.4, 753).

The evolution of u in the variables x,y,t is interpreted as a dynamical motion of a point on GM by the action of a three- (or more) parameter subgroup of the group of automorphisms of GM. Generic points of GM give generic solutions to the KP equation, whereas points on particular submanifolds of GM give solutions of particular type. E.g. , rational solutions correspond to points on finite-dimensional Grassmann submanifolds of GM. Also, different kinds of submanifolds of GM give rise to generic solutions of other soliton equations, such as the Korteweg - de Vries (KdV) equation, the modified KdV equation, the Boussinesq equation, the Sawada-Kotera equation, the non-linear Schrödinger equation, the Toda lattice, the equation of self-induced transparency, the Benjamin-Ono equation, as well as to solutions of particular type of these soliton equations.

Moreover, a multicomponent generalization of the theory shows that solutions of other soliton equations (such as the equation for three-wave interaction , the multi-component non-linear Schrödinger equation, the sine-Gordon equation, the Lund-Regge equation, and the equation for intermediate long wave) also constitute submanifolds of GM.

SATO [119 a] conjectured that any soliton equation, or completely integrable system, is obtained in this way. It follows that the classification of solition equations would be reduced to the classification of submanifolds of GM which are stable under the subgroup of automorphisms of GM describing space-time evolution.

Let us recall the construction of GM given by SATO [119 a]. (A treatment using a Young diagram can be found in [120].)

Consider the Grassmann manifold $G^{\mathbb{C}}_{m,n}$ of m-dimensional subspaces of $\mathbb{C}^{m+n}$. Since such a subspace is spanned by an m-frame $\xi := (\xi^{(1)}, \ldots, \xi^{(m)})$, consisting of m linearly independent vectors $\xi^{(1)}, \ldots, \xi^{(m)} \in \mathbb{C}^{m+n}$, we have

$$G^{\mathbb{C}}_{m,n} := \{\text{m-frames in } \mathbb{C}^{m+n}\}/GL(m) = GL(m+n)/_{GL(m,n)} ,$$

where GL(m,n) denotes the subgroup of GL(m+n) consisting of elements of the form $g = \begin{pmatrix} g_1 & g_2 \\ o & g_4 \end{pmatrix}$ with $g_1 \in GL(m)$, $g_4 \in GL(n)$.

We also set

$$\tilde{G}^{\mathbb{C}}_{m,n} := (\{\text{m-frames}\} \times GL(1))/GL(m) \quad (= \{\text{m-frames}\}/_{SL(m)} ,$$

if $m > 0$), $\Lambda^m(\mathbb{C}^{m+n})$:= m-th exterior product space of $\mathbb{C}^{m+n}$, and hence we have the following situation:

$$(8.2) \qquad \begin{array}{ccc} \tilde{G}^{\mathbb{C}}_{m,n} & \hookrightarrow & \Lambda^m(\mathbb{C}^{m+n}) - \{0\} \\ \downarrow GL(1) & & \downarrow GL(1) \\ G^{\mathbb{C}}_{m,n} & \hookrightarrow & \text{projective space of dimension } \binom{m+n}{m} - 1 \end{array}$$

where the embedding of the upper line is defined by letting $\overline{\xi} \in G^{\mathbb{C}}_{m,n}$, represented by an m-frame $\xi = (\xi^{(1)}, \ldots, \xi^{(m)})$, correspond to $\xi^{(1)} \wedge \ldots \wedge \xi^{(m)} \in \Lambda^m(\mathbb{C}^{m+n}) - \{0\}$.

Remark. The diagram (8.2) gives the standard way of embedding $G^{\mathbb{C}}_{m,n}$ into projective space.

Definition. $\tilde{G}^{\mathbb{C}}_{m,n}$ is called the standard line bundle over $G^{\mathbb{C}}_{m,n}$.

Denoting by $\xi_{\ell_1,\ldots,\ell_m}$ the minors of ξ (i.e. the determinants of $(m \times m)$-matrices consisting of ℓ_1-th, ..., ℓ_m-th rows of the $(m+n) \times m$-matrix ξ), we have

$$\xi^{(1)} \wedge \ldots \wedge \xi^{(m)} = \sum_{1 \leq \ell_1 < \ldots < \ell_m \leq m+n} \xi_{\ell_1 \ldots \ell_m} e_{\ell_1} \ldots e_{\ell_m},$$

where $e_1, \ldots, e_{m+n}$ denote the unit column vectors in $\mathbb{C}^{m+n}$.

Definition. $\xi_{\ell_1,\ldots,\ell_m}$ are called the Plücker coordinates of $\bar{\xi}$.

They satisfy the Plücker identities

$$(8.3) \qquad \sum_{i=1}^{n+1} (-1)^i \, \xi_{\ell_1 \ldots \ell_{m-1} k_i} \, \xi_{k_1 \ldots \hat{k}_i \ldots k_{m+1}} = 0.$$

Remark. In (8.2), $G^{\mathbb{C}}_{m,n}$ coincides with the intersection of the quadrics defined by (8.3) in projective space.

The infinite-dimensional Grassmann manifold GM and its standard line bundle $\widetilde{GM}$ (needed to parametrize the solutions of Kadomtsev-Petviashvili hierarcy) are obtained as the topological closure of the inductive limit of $G^{\mathbb{C}}_{m,n}$ and $\tilde{G}^{\mathbb{C}}_{m,n}$ as m and n tend to ∞.

Explicitly, GM is defined by

$$GM := \left\{ \mathbb{N}^c\text{-frames} \right\} / _{GL(\mathbb{N}^c)} ,$$

where by an $\mathbb{N}^c$-frame we mean an infinite-sized matrix

$$\xi := (\xi_{\mu\nu})_{\mu \in \mathbb{Z}, \nu \in \mathbb{N}^c},$$

whose rows and columns are labeled by integers $\mathbb{Z}$ and strictly negative integers $\mathbb{N}^c := \mathbb{Z} - \mathbb{N}$, respectively, satisfying the condition that there exists an $m \in \mathbb{N}$ such that: (i) $\xi_{\mu\nu} = \delta_{\mu\nu}$ for $\mu < -m$; and (ii) m column vectors for $\nu = -m, -m+1, \ldots, -1$ are linearly independent. $GL(\mathbb{N}^c)$ consists of the matrices

$$h := (h_{\mu\nu})_{\mu,\nu \in \mathbb{N}^c}$$

satisfying a condition similar to the above.

Remark. SATO [119 b] defined the notion of universal Grassmann manifold (UGM), which is a canonical form of GM.

Open problem (suggested to Sato on the occasion of the OATE Conference, Buşteni - Romania, 1983). To find an algebraic description of GM (as well as of $\widetilde{GM}$ and UGM) analogous to the Jordan algebra description of finite-dimensional Grassmann manifolds given by HELWIG [65 b] and recalled in JSA.III,§ 2 . This is an open problem of mathematical interest, but by solving it one obtains an algebraic description for the solutions of soliton equations which could be useful in quantum mechanics.

Comments. Rational solutions of the KP equation can be described in terms of Jordan algebras. It would be inte esting to make use of this description in soliton theory.

Note. As we have predicted in 1983 (talks given at the Universities of Timişoara and Iaşi), SATO's result [119 a] is an outstanding contribution with a deep impact in physics (For details, see JSA.VIII, § 7).

Let us recall that GERBER [59 a] clarified a Jordan algebra structure in the study of solitary wa es. As is known, a large number of physically relevant differential equations exhibiting solitary wave behaviour have been successfully analysed by the inverse scattering method. A key step in the development of this method was the finding that the transformation $v=u_x+u^2$ maps the modified KdVequation

$$u_t - 6u^2u_x - u_{xxx}=0$$

into the KdV equation

$$v_t - 6vv_{xx}=v_{xxx} .$$

In order to extend this method to systems (equations in $\mathbb{R}^n$), Gerber set conditions of f, H, and $\hat{H}$ such that

$$(8.4) \qquad v=f(u,u_x)$$

maps

$$(8.5) \qquad u_t=u_{xxx} + H(u)\, u_x$$

into

$$(8.6) \qquad v_t=v_{xxx} + \hat{H}(v)v_x .$$

After giving precise assumptions on (8.4), (8.5), (8.6), he proved a theorem in which a Jordan algebra structure basically participates.

3. If $M = G/K$ is a bounded symmetric domain and $S = {}^K/L$ is its Shilov boundary, then one can define a Poisson kernel on $M \times S$ and the Poisson integral for any hyperfunction on S. An open problem, formulated ten years ago by Stein, is to characterize these integrals as solutions of a system of differential equations, established for certain cases by HUA (see [68]).

In [90 a], LASSALLE dealt with bounded symmetric domains of tube type. Poisson integrals oven the Shilov boundary are then characterized by the system of differential equations given by

JOHNSON and KORÁNYI [79]. Results of BERLINE and VERGNE [32] for the domain $(I)_{n,n}$ and those of KORÁNYI and MALLIAVIN [89] for the Siegel disc of dimension two prove that the JOHNSON-KORÁNYI system [79] has, for these particular cases, too many equations. Lassalle proved that this is a general property. In particular, he established that among the dim K differential equations of the Johnson-Korányi system, a subsystem of dim S equations is sufficient to characterize the bounded functions on M which are Poisson integrals of a function on S. This new characterization has a very natural interpretation in terms of Jordan algebras (see LASSALLE [90 a, pp.326-327]).

As LASSALLE [90 b] proved, such an interpretation is also possible if M is a symmetric Hermitian space of tube type with Shilov boundary S and can be realized as a bounded symmetric domain. The main idea is to formulate the Hua differential equations [90 c] in terms of "polar coordinates" with respect to S.

Consider, again, a bounded symmetric domain D, S its Shilov boundary and let S(z,u) be the Szegö kernel on D × S . HUA [68] was the first to calculate explicitly the expression of S(z,u) for each of the four series of irreducible domains. Later Korányi gave a general proof that made Hua's case-by-case calculations unnecessary. However, Korányi's proof is not direct; it uses the unbounded realization of D as a "generalized half-plane"; in unbounded realization the Szegö kernel is given by an integral, which could be calculated by methods of Bochner and Gindikin. In bounded realization, on the other hand, the Szegö kernel is given by a Fourier series, and for this no methods of calculation had been devised.

LASSALLE's result [90 h] offers a solution to this problem: he manages to calculate the Fourier series defining S(z,u) directly, without going through the unbounded realization of D. In fact Lassalle is in position to solve the following much more difficult problem: For every positive real number λ, what is the Fourier series expansion of $(S(z,u))^{\lambda}$? This difficult problem had been open for nearly thirty years. The only known solution had been given implicitly by HUA [68, p.25] in the particular case of an irreducible domain of type $I_{n,m}$. But the solution was unknown in all the other cases, including each of the three other series of irreducible domains.

LASSALLE's goal in [90 h] is to present a general answer to this problem, one independent of any classification argument. What is noteworthy in [90 h] is that the framework and tools of Lassalle's proof are provided by Jordan algebra theory. In particular, his central result is a "binomial formula" in the complexification of a formally real Jordan algebra. The solution thus obtained is particularly simple and natural.

4. Whereas the study of Toeplitz operators for the strongly pseudoconvex domains uses methods of partial differential equations, their structure and Toeplitz C^*-algèbres over symmetric domains is closely related to the Jordan algebraic structure underlying these domains (see UPMEIER [133 f, Section 2]). Relation between bounded Toeplitz operators and Weyl operators of boson quantum mechanics was examined by BERGER and COBURN in [31].

As UPMEIER pointed out in [133 f, p.42] even though finite-dimensional bounded symmetric domains have been classified and their geometry is totally understood, there are still many open problems concerning their analysis (i.e., the structure of function spaces defined over these domains). Since symmetric domains are homogeneous under a (semisimple) Lie group, these problems are related to harmonic analysis and the theory of group representations. On the other hand, the occurring function spaces are often Hilbert spaces of holomorphic functions which give rise to (reproducing) kernel functions. These kernel functions can be defined in terms of certain basic "norm functions" derived from the Jordan algebraic structure.

ION and SCUTARU [75], and ION [74 a,b] introduced new scattering theories via optimal states. These states are reproducing kernels in the Hilbert spaces of scattering matrices, just as the coherent states are reproducing kernels in the Hilbert spaces of wave functions.

Using reproducing kernels, UPMEIER [133 f, Section 6] outlined a quatization procedure for certain curved phase spaces of possibly infinite-dimension, namely the "symmetric Hilbert domains". In the finite-dimensional setting, BEREZIN [30 a,b] has considered quatizations for more general complex (Kähler) manifolds.

The general formalism for quantum fields on any reproducing kernel Hilbert space is presented by SCHROECK [121], along with a discussion of the operator and distribution properties of those fields. Galilean and Poincaré examples are given along with considerations of the general relativistic cases.

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