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JORDAN STRUCTURES WITH APPLICATIONS. VI. JORDAN TRIPLE SYSTEMS AND JORDAN PAIRS IN ANALYSIS

by

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JORDAN STRUCTURES WITH APPLICATIONS. VI.

JORDAN TRIPLE SYSTEMS AND JORDAN PAIRS IN ANALYSIS.

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The first three Sections of this paper describe the equivalence of symmetric complex Banach manifolds with Hermitian Jordan triple systems and that of symmetric Hermitian manifolds with Hilbert triple systems. Properties of Siegel domains involving Jordan algebras (or pairs) are given in Section 4. Finally, this paper includes the construction of theta-functions for Jordan pairs, and gives results on differential equations in Jordan triple systems, and the Riccati equation in Jordan pairs.

§ 1. Jordan triple systems and Banach manifolds

Using the classification of simple complex Lie algebras, CARTAN [10] gave the classification of symmetric Hermitian complex manifolds of finite dimension. In 1969 KOECHER [29 b] and, more recently, LOOS [33 b] gave a Jordan theoretic approach, the main result being that bounded symmetric domains are in one-to-one correspondence with Hermitian Jordan triple systems (for definition, see below) for which a certain trace form is positive definite Hermitian.

In 1977 KAUP [27 a] studied symmetric complex Banach manifolds (possibly of infinite dimension) and proved an equivalence theorem with Hermitian Jordan triple systems (see Theorem 1.1. below). KAUP's result [27 c] on bounded symmetric domains in complex Banach spaces is also mentioned below (see Theorem 1.2).

Notation. Let E, F be two complex Banach spaces and denote by L(E,F) the Banach space of all bounded operators $E \longrightarrow F$ endowed with the norm $\| \ \|_\infty$ (for the definition of $\| \ \|_\infty$, see below). We shall write $L(E) := L(E,E)$, and $GL(E)$ will denote the group of all invertible elements of L(E).

Recall that if $\{U_i\}_{i \in I}$ is a family of real (or complex) Banach spaces, then $\bigoplus_{i \in I}^{p} U_i := \{u \in \prod_{i \in I} U_i \mid \|u\|_p < \infty\}$, where

$$\|u\|_p := \begin{cases} \left(\sum_{i \in I} \|u_i\|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{i \in I} \|u_i\|, & p = \infty, \end{cases}$$

is called the p-sum of U_i.

Definition. A complex Banach space U together with a continuous map $U \times U \longrightarrow L(U)$, $(\alpha, \beta) \longrightarrow \alpha \square \beta^*$, which is linear in the first argument and antilinear in the second, is called a Hermitian Jordan triple system (or J^*-triple) if, for all $\alpha, \beta, \gamma, x, y, z \in U$, the following conditions hold:

1. the triple product $\{\alpha\beta^*\gamma\} := \alpha \square \beta^*(\gamma)$ is symmetric in the variables α, γ;

2. $\{\alpha\beta^*\{xy^*z\}\} = \{\{\alpha\beta^*x\}y^*z\} - \{x\{\beta\alpha^*y\}^*z\} + \{xy^*\{\alpha\beta^*z\}\}$

3. $\alpha \square \alpha^* \in L(U)$ is a Hermitian operator on U.

Remark. An operator λ of L(U) is called Hermitian if $\exp(it\lambda) \in GL(U)$ is an isometry of U for all $t \in \mathbb{R}$.

Comments. KAUP [27 a] defined the notion of J^*-triple system in the following manner: let $L^2(E)$ be the Banach space of all continuous homogeneous polynomials $E \longrightarrow E$ of degree 2. For every

$q \in L^2(E)$, $\{wqz\} := \frac{1}{2}(q(w+z) - q(w) - q(z))$ is bilinear in $(w,z) \in E^2$, and for every $\alpha \in E$, $\alpha \square q := \{\alpha q z\}$ defines an operator in $L(E)$. Then a J^*-triple system $(U, *)$ is a complex Banach space U together with a conjugate linear (continuous) map $*$ (for every $\alpha \in U$, α^* will be written instead of $*(\alpha)$) such that, for all $\alpha, \beta, x, y, z \in U$, conditions (2) and (3) hold.

<u>Definition</u>. A continuous linear map $\mu : U \longrightarrow V$ between two J^*-triples U and V such that $\mu\{z\alpha^* w\} = \{\mu z\,(\mu\alpha)^* \mu w\}$ for all α, z, $w \in U$, is called a J^*-<u>morphism</u>.

<u>Remark</u>. It is obvious that the class of all J^*-triples is a category.

<u>Definitions</u>. Let U be a J^*-triple. If, for every real number t, we define, a new triple product ${}^t\{xy^*z\} := t\{xy^*z\}$ on U for all $x,y,z \in U$, then U is again a J^*-triple with respect to this new triple product; it is denoted by ${}^tU$. For $t \neq 0$, U and ${}^tU$ are called <u>proportional</u>, and ${}^{-1}U$ is called the <u>dual</u> of U.

<u>Remark</u>. In general, a J^*-triple U and its dual ${}^{-1}U$ are not isomorphic.

We now consider a complex manifold[1] M, the tangent bundle functor T, the corresponding tangent bundle TM, and the canonical projection $\pi : TM \longrightarrow M$. If V is an open subset of a complex Banach space E, then the tangent bundle TV can be identified in a natural way with the direct product $V \times E$, and $\pi : TV \longrightarrow V$ is the canonical projection onto the first factor. Suppose that $\nu : TM \longrightarrow \mathbb{R}$ is a lower semicontinuous function. The function ν is called a <u>norm on</u> TM if the <u>restriction of</u> ν to every tangent space T_x, $x \in M$, is a norm on T_x with

1) By a <u>complex manifold</u> KAUP [27 a, p.43] means a Hausdorff manifold(possibly of infinite dimension) modelled locally over open subsets of complex Banach spaces <u>via</u> biholomorphic coordinate transformations.

the following property: there is a neighborhood V of $x \in M$ which can be realized (i.e. by a biholomorphic map) as a domain of a complex Banach space E such that $c\|a\| \leq \nu(u,a) \leq C\|a\|$ for all $(u,a) \in V \times E \cong$ $\cong$ T(V) and suitable constants $0 < c \leq C$.

Definitions. A complex manifold M together with a fixed norm ν on TM is called a complex Banach manifold. If every norm $\nu\big|_{T_x}$ is a Hilbert norm then M is called a complex Hilbert manifold (or, also, a Hermitian manifold).

Comments. KAUP [27 a] does not require that the restriction of ν to every tangent space T_x, $x \in M$, is a Hilbert norm (otherwise he would exclude many interesting examples such as the open unit ball in the Banach space L(H) of all bounded operators on a Hilbert space H with dim H = ∞). That is why KAUP's notion of Hermitian Jordan triple system cannot rely on trace forms.

Definition. Let M be a connected complex Banach manifold and L the Lie group of all biholomorphic isometries of M. Then M is called symmetric if there exists a point $a \in M$ such that:

(i) there is an involution $s \in L$ with a as isolated fixed point,

(ii) the map $L \longrightarrow M$ defined by $g \longrightarrow ga$ is a submersion.

Definition. Let M_1 and M_2 be two symmetric complex manifolds. A holomorphic map $h : M_1 \longrightarrow M_2$ is called a morphism of symmetric manifolds if $h \circ s_x = s_{hx} \circ h$ for all $x \in M_1$.

Remark. It is obvious that the class of all symmetric complex Banach manifolds is a category.

The main result established by KAUP [27 a] is

Theorem 1.1. The category of simply connected, symmetric, complex Banach manifolds with base point is equivalent to the category of Hermitian Jordan triple systems.

The notion of J^*-triple leads to a generalization of C^*-algebras as follows. A J^*-triple U is called a C^*-triple if: (i) it is positive (i.e. the spectrum $\sigma(\alpha \square \alpha^*) \geqslant 0$ for all $\alpha \in U$); and (ii) $\|\alpha \square \alpha^*\| = \|\alpha\|^2$ for all $\alpha \in$ U. It is obvious that every C^*-algebra is a C^*-triple. For details, see KAUP [27 a, pp.54-58].

The following result has been recently obtained by KAUP [27 c]:

Theorem 1.2. Every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a C^*-triple. Thus, in this way, the category of all bounded symmetric domains with base point is equivalent to the category of C^*-triples.

Comments. BRAUN, KAUP and UPMEIER [9] proved that the open unit balls of (unital) JB^*-algebras are precisely (i.e. up to a biholomorphic map) those bounded symmetric domains which admit a realisation as the upper half-plane. Consequently, the C^*-triples from [27 a] are called JB^*-triples in [27 c]. The main result in [27 c] is as follows: Every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB^*-triple; hence, in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB^*-triples.

A problem emerging from [27 a] is to find under what conditions a Jordan triple system is isomorphic to a bounded symmetric domain. In the finite-dimensional case LOOS [33 b] gave a necessary and sufficient condition (a certain Hermitian form has to be positive definite). For the infinite-dimensional case, VIGUÉ [50] gave a condition in terms of a certain spectral property.

§ 2. Classification of Hermitian Jordan triple systems

As is well known, the notion of idempotent is of much importance in the structure theory of algebras. The analogue notion for triple system is that of tripotent:

Definition. An element $c \neq 0$ of a Hermitian Jordan triple system U is called tripotent if $\{cc^*c\} = \sigma c$ with $\sigma = \pm 1$. The coefficient $\sigma = \sigma(c)$ is called the sign of c; c is called positive (negative) if its sign is +1 (-1).

Remark. The element $d := \sigma(c)c$ is also a tripotent. It is called the tripotent associated with c.

Let c be a tripotent of U and let d be its associated tripotent. One can define on U a Jordan algebra structure by means of the product $xy := \{xd^*y\}$, and c is an idempotent with respect to this product. The corresponding Peirce decomposition is $U = U_1 \oplus U_{1/2} \oplus U_0$, where for every $\nu \in \mathbb{R}$ the ν-space of c and d^* is denoted by $U_\nu := U_\nu(c) = \{x \mid x \in U,\ cx = \nu x\}$. The 1-component U_1 is a complex Jordan algebra with respect to the above-defined product on U.

Definitions. Let c be a tripotent of U. If the Jordan algebra $U_1(c)$ is associative, then c is called abelian. If $U_1(c) = \mathbb{C}c$, then c is called minimal, while in the case when $U_0(c) = 0$, the tripotent c is called maximal. A tripotent c is called primitive if there does not exists a decomposition $c = \mu + \nu$ with $\mu \in U$, $\nu \in U_0(\mu)$.

Definition. A set $\mathcal{E}$ of tripotents in U for which $\{\alpha\alpha^*\beta\} = 0$ for all $\alpha \neq \beta$ from $\mathcal{E}$ is called an orthogonal system of tripotents. Such a system is called complete if $U_{00} = 0$ (For U_{00}, see below.)

For all α, β from an orthogonal system $\mathcal{E}$, the following $U_{\alpha\beta}$ are defined

$$U_{\alpha\alpha} := U_1(\alpha)$$

$$U_{\alpha\beta} := U_{1/2}(\alpha) \cap U_{1/2}(\beta) \quad \text{when} \quad \alpha \neq \beta$$

$$U_{\alpha 0} := U_{0\alpha} := U_{1/2}(\alpha) \cap \Big(\bigcap_{\gamma \in \mathcal{E} - \{\alpha\}} U_0(\gamma) \Big)$$

$$U_{00} := \bigcap_{\gamma \in \mathcal{E}} U_0(\gamma).$$

If $\mathcal{E} = \emptyset$, then $U_{00} := U$. Let us denote by $\hat{\mathcal{E}}$ the set of all unordered pairs (α, β) with $\alpha, \beta \in \mathcal{E} \cup \{0\}$ such that $U_{\alpha\beta} = U_{\alpha\beta}(\mathcal{E})$ is well defined for every $(\alpha, \beta) \in \hat{\mathcal{E}}$ and

$$P(\mathcal{E}) := \bigoplus_{(\alpha,\beta)\in\hat{\mathcal{E}}} U_{\alpha\beta}$$

is an algebraic direct sum in U (called the Peirce sum with respect to $\mathcal{E}$).

Definitions and notation. If a complete orthogonal system of minimal (resp.,abelian) tripotents exists in the J^*-triple, then U is called atomic (resp.discrete). The minimal cardinality of a complete orthogonal system $\mathcal{E}$ of minimal (resp.,abelian) tripotents in U is called the rank (resp.,degree) of U; it it is denoted by r(U) (resp., d(U)).For a complete orthogonal system of tripotents $\mathcal{E}$ in U one defines

$$a(\mathcal{E}) := \inf \left\{ \dim U_{\alpha\beta} \;\middle|\; \alpha,\beta \in \mathcal{E}, \alpha \neq \beta \right\},$$

$$b(\mathcal{E}) := \sup \left\{ \dim U_{\alpha 0} \;\middle|\; \alpha \in \mathcal{E} \right\}$$

(if no $\alpha, \beta \in \mathcal{E}$ with $\alpha \neq \beta$ exist,then one takes $a(\mathcal{E}) = 2$).

For atomic U we also defines

$$a(U) := \sup a(\mathcal{E}),$$

$$b(U) := \inf b(\mathcal{E}),$$

where $\mathcal{E}$ runs over all complete orthogonal systems of minimal tripotents in U. If U is indecomposable, then, by means of r, a, and b, we can define another invariant, namely

$$g := 2 + (r-1)a + b.$$

The construction of J^*-triples of types I - VI. Let H, K be complex Hilbert spaces of arbitrary dimensions n,m, respectively.

Notation. The adjoint of an operator $\lambda \in L(H,K)$ will be denoted by λ^*.

For $x \in K$, $y \in H$, by $z \longrightarrow (z \mid y)x$ an operator $x \otimes y^* \in L(H,K)$ is defined. We have $\| x \otimes y^* \|_\infty = \| x \|_\infty \| y \|_\infty$ and $(x \otimes y^*)^* = y \otimes x^*$.

Convention. The Hilbert spaces H and K are, in a natural way, identified with $L(\mathbb{C}, H)$ and $L(\mathbb{C}, K)$, respectively; the operator $x \otimes y^*$ is written as xy^*.

Type I . Let $U := L(H,K)$. Define

$$\{xy^*z\} := \frac{1}{2}\{xy^*z + zy^*x\} \tag{2.1}$$

for all $x,y,z \in U$. Then U is a J^*-triple with respect to (2.1). Following HARRIS [22 b] (see also KAUP [27 b,I]), it is called a Cartan factor of type I, and it is denoted by $I_{n,m}$.

Remark. As $I_{n,m}$ and $I_{m,n}$ are isometrically isomorphic, it is sufficient to consider the case $n \leq m$ only.

Type II. Let $x \longrightarrow \bar{x}$ be a conjugation (i.e. an isometric, antilinear, involutive endomorphism) of the complex Hilbert space H. Define a $\mathbb{C}$-linear transposition $z \longrightarrow z'$ in $L(H)$ by $z'(x) := \overline{z^*(\bar{x})}$. Consider $U := \{ z \mid z \in L(H),\ z' = -z \}$. Then U is a J^*-triple with respect to the norm of $L(H)$ and the triple product (2.1). The J^*-triple U is called a Cartan factor of type II, and it is denoted by II_n.

Type III. Using the notation as in the case of type II above, we define $U := \{ z \mid z \in L(H),\ z' = z \}$. The U is called a Cartan factor of type III, and it is denoted by III_n.

Type IV. Consider again a conjugation $x \longrightarrow \bar{x}$ on H and suppose $n \geqslant 3$. Then $U := H$ is a J^*-triple with respect to the triple product

$$\{xy^*z\} := \frac{1}{2}((x \mid y)z + (z \mid y)x - (x \mid \bar{z})\bar{y}).$$

A subset $\mathcal{E}$ of U is a complete orthogonal system of minimal tripotents if $\mathcal{E} = \{\alpha, \beta\}$, where α, β are unitary vectors with $(\alpha \mid \beta) = 0$

and $\beta = \bar{\alpha}$. For all $z \in U$ there exists a decomposition $z = s\alpha + t\beta$ with $s, t \geq 0$, and α, β orthogonal tripotents. By taking $\| z \|_{\infty} :=$ $:= \max (s, t)$, we define on U another (equivalent) norm and U is again a J^*-triple. This new norm has the following explicit form:

$$\| z \|^2_{\infty} = \frac{1}{2} ((z \mid z) + \sqrt{(z \mid z)^2 - |(z \mid \bar{z})|^2}).$$

U endowed with this norm is called a <u>Cartan factor of type IV</u> (or <u>complex spin factor</u>), and it is denoted by IV_n.

<u>Type V (resp.,VI).</u> Let U be the triple system which corresponds to the exceptional bounded symmetric domain in $\mathbb{C}^{16}$ (resp., $\mathbb{C}^{27}$) - see [11 b]. Then U is a J^*-triple with respect to the spectral norm $\| \ \ \|_{\infty}$ (see [11 b]) and it is denoted by V (resp.,VI).

The types I-VI mentioned above occur only once in the following list:

	r(U)	a(U)	g(U)	b(U)	
$I_{n,m}$	n	2	n+m	m-n[1)]	$m \geq n \geq 1$
II_n	$\left[\frac{n}{2}\right]$	4	2n-2	$\begin{cases} 2 & n \in 2N+1 \\ 0 & \text{other ways} \end{cases}$	$n \geq 4$
III_n	n	1	n+1	0	$n \geq 2$
IV_n	2	n-2	n	0	
V	2	6	12	4	
VI	3	8	18	0	

The following classification theorem holds (see CARTAN [10], KOECHER [29 b], LOOS [33 a,b]).

<u>Theorem 2.1.</u> Every J^*-triple U of finite dimension admits a unique (up to the order) decomposition as direct sum $U = U_0 \oplus U_1 \oplus \ldots$ $\ldots \oplus U_s$, where U_0 is a trivial ideal in U and $U_1, \ldots, U_s$ run over the simple ideals of U.The simple J^*-triples of finite dimension are (without repetition and up to a J^*-isomorphism) exactly the systems

1) For $n \leq m$, m-n is the least cardinal number q with n+q=m.

$I_{n,m}$ with $1 \leq n \leq m < \infty$; II_n with $4 < n < \infty$;

III_n with $1 < n < \infty$; IV_n with $4 < n < \infty$;

V and VI,

and their dual systems.

Let us recall two results on J^*-triples of finite rank (see KAUP [27 b, I]).

<u>Theorem 2.2.</u> Every atomic J^*-triple U of finite rank admits a unique (up to the order) decomposition $U = U_1 \oplus \ldots \oplus U_s$ where $U_1, \ldots, U_s$ are simple ideals of U. We have $r(U)=r(U_1)+\ldots+r(U_s)$. The simple atomic J^*-triples of finite rank are (up to an isomorphism) exactly the triples of types I-VI of finite rank and their dual triples.

<u>Remark.</u> Every J^*-algebra of finite rank in the sense of HARRIS (see [22 a] and, also, JSA.V, § 5) is isomorphic to a finite direct sum of Cartan factors of types I-IV.

To formulate the second result on J^*-triples of finite rank we need the following preliminaries:

Let U be a (not necessarily simple) atomic J^*-triple of finite rank r. Then there exists a complete orthogonal system $\mathcal{E} = \{ e_i \mid i=1,2, \ldots, r \}$ of minimal tripotents in U. All tripotents e_i are positive and have norm 1. The corresponding Peirce spaces $U_{e_i e_j}$ will be denoted U_{ij}, where $e_o := 0$. Then every $x \in U$ has a unique decomposition $x = \sum_{o \leq i \leq j} x_{ij}$ with $x_{ij} \in U_{ij}$. $F := \mathbb{R} e_1 \oplus \ldots \oplus \mathbb{R} e_r$ is a maximal $\mathbb{R}$-linear subspace of U with the property that $\{\alpha \beta^* \gamma\} = \{\beta \alpha^* \gamma\}$ for all $\alpha, \beta, \gamma \in F$. We define

$$\Sigma := \{ \delta \in \mathrm{Inn}\,(U) \mid \delta(F) = F \},$$

where Inn(U) is the group of all inner automorphisms of U. The following result holds: There exists a one-to-one correspondence between the set of all norms Φ of U with respect to which U is a J^*-triple and the set of all $\sum$ - invariant norms on F.

We now consider the case when U is simple. The $\sum$ - invariant norms Φ on F with $\Phi(e_i) = 1$ correspond to the norms Φ on $\mathbb{R}^r$ with the following properties

(i) $\| \ \|_\infty \leq \Phi \leq \| \ \|_1$;

(ii) $\Phi(|t|)=\Phi(t)$ for all t, where $|t| :=(|t_1|, \ldots, |t_r|)$;

(iii) $\Phi(s) = \Phi(t)$ if s and t differ only by a permutation of coordinates.

A norm Φ on $\mathbb{R}^r$ with the properties (i)-(iii) is called a <u>symmetric gauge function</u> (see SCHATTEN [43, p.61]). By the above-mentioned correspondence such a function Φ on $\mathbb{R}^r$ yields an Inn(U)-invariant complex norm Φ on U. If we consider U endowed with the norm Φ, then it is again a J^*-triple; it will be denoted by U^Φ. For $t \in \mathbb{R} - \{0\}$, ${}^tU$ denotes a J^*-triple proportional to U^Φ.

<u>Theorem 2.3</u>. Every simple J^*-triple of finite rank r is isometrically isomorphic to a J^*-triple ${}^tU^\Phi$, where $t \in \mathbb{R} - \{0\}$, Φ is a symmetric gauge function on $\mathbb{R}^r$ and U is a J^*-triple of type I-VI of rank r. The set (r,a,g,t, Φ) is a complete system of invariants for the set of all isometric isomorphism classes of simple J^*-triples of finite rank.

§ 3. Hilbert triple systems and Hermitian manifolds

We first recall a result [27 b, II] (see Theorem 3.1 below) on atomic, triple systems of infinite rank. In order to formulate this result, some preliminary considerations and notations, given by KAUP [27 b, II, pp.62-63], are necessary.

The space of all real sequences $t = (t_1, t_2, \dots)$ with $t_n = 0$ for almost all n is a real algebra with respect to componentwise multiplication. For every $t \in \ell$, put $|t| := (|t_1|, |t_2|, \dots)$ and $\tau(t) := \sum_{n=1}^{\infty} t_n$. A norm ϕ on ℓ is called a <u>symmetric gauge function</u> if

$$\| \ \|_\infty \leq \phi \leq \| \ \|_1 ;$$

$$\phi(t) = \phi(|t|), \text{ for all } t;$$

and $\phi(s) = \phi(t)$, for all $s, t \in \ell$ which differ only by a permutation of coordinates.

Using ϕ we define another symmetric gauge function ϕ' by

$$\phi'(t) := \sup \left\{ \tau(st) \ \middle| \ s \in \ell, \ \phi(s) = 1 \right\}.$$

Let E be a complex Hilbert space. Then L(E) is an associative $*$-algebra. We denote by $\mathcal{R}$ the ideal of all operators of finite rank and by $L_1(E)$ the ideal of all trace operators. Let $\tau : L_1(E) \longrightarrow \mathbb{C}$ be the usual trace (i.e. $\tau(x) := \sum (x e_i \mid e_i)$ for every orthonormal basis $\{e_i\} \subset E$). For every symmetric gauge function ϕ on ℓ and every $x \in \mathcal{R}$ put $\phi(x) := \phi(t_1, t_2, \dots, t_r, 0, 0, \dots)$, where $t_1, t_2, \dots, t_r$ are nonnull eigen values of $|x| = \sqrt{x^* x}$. Then ϕ can be extended to a function $\phi : L(E) \longrightarrow \mathbb{R} \cup \{\infty\}$, as follows:

$$\phi(x) := \sup \left\{ \tau(xy) \ \middle| \ y \in \mathcal{R} \text{ and } \phi'(y) = 1 \right\}.$$

ϕ is a complete norm on $L_\phi(E) := \{ x \mid x \in L(E), \ \varphi(x) < \infty \}$. For $\| \ \|_p$ we shall also write $L_p(E)$; in particular, $L_2(E)$ is the ideal of Hilbert-Schmidt operators.

Let U be a Cartan factor of type I, II or III (see KAUP [27 b, I]). We embed U in an L(E), E being a complex Hilbert space, as follows:

Type I. Choose H,K such that $E = H \oplus K$. Then $U:=L(H,K)$ can be naturally identified with the space of all $x \in L(E)$ such that $x(K)=0$ and $x(H) \subset K$. Suppose we have a conjugation preserving H and K defined on E. The corresponding transposition $z \longrightarrow z'$ of L(E) generates then U.

Type II (resp., III). In this case we have (see [27 b, I, Section 3]) $U:=\{z \mid z \in L(E),\ z'=-z\}$ (resp. $U:=\{z \mid z \in L(E), z'=z\}$).

In all cases, $U \subset L(E)$ is w^*-closed and $\mathcal{F} := U \cap \mathcal{R}$ is a J^*-ideal in U. For every symmetric gauge function ϕ on ℓ, let $U^{\phi} := U \cap L_{\phi}(E)$ be endowed with the norm $c\,\phi \big|_{U^{\phi}}$, where the factor $c > 0$ is defined such that the norm of every minimal tripotent in U is 1. The norm on U^{ϕ} thus obtained will be also denoted by ϕ. Let U_{ϕ} be the convex hull of $\mathcal{F}$ in U^{ϕ}.

Theorem 3.1. Let W be a simple atomic J^*-triple of infinite rank. Then W is proportional to a U_{ϕ}, where U is of type I, II, or III, and ϕ is a symmetric gauge function on ℓ, if and only if there exists a complete orthogonal system $\mathcal{E} \subset W$ of minimal tripotents in W for which the Peirce sum $P(\mathcal{E})$ is dense in W.

Comments. In the case $\phi = \| \ \|_{\infty}$ we have $U=U^{\phi}$ for every Cartan factor of type I, II, III, and U_{ϕ} is the subtriple of all compact operators in U. In the case $\phi = \| \ \|_2$, for every $t \neq 0$ and every J^*-triple U of types I-VI, the triple ${}^tU^{\phi}$ is well defined.

Convention . In what follows we shall write ${}^t\tilde{U}$ instead of ${}^tU^{\| \ \|_2}$, and $\tilde{U}$ instead of $U^{\| \ \|_2}$.

Definition. A J^*-triple U is called a Hilbert triple system (or JH^*-triple) if the complex Banach space U is a complex Hilbert space.

Remark. JH^*-triples are generalizations of JH^*-algebras : A JH^*-algebra is a complex Hilbert space H together with a complex Jordan algebra structure and a continuous involution $x \longrightarrow x^*$ such that the Jordan product is continuous and $L(a)^* = L(a^*)$ for every $a \in H$, L being left multiplication. By defining $\alpha \square \beta^* := L(\alpha\beta^*) + [L(\alpha), L(\beta^*)]$, H becomes a JH^*-triple.

KAUP [27 b, II] proved the following classification theorem for Hilbert triple systems.

Theorem 3.2. Every Hilbert triple system U is isometric to an orthogonal sum $\bigoplus^2_{i \in I} U_i$, with $0 \in I$, U_o is a trivial ideal in U, and U_i, $i \neq 0$, run over the simple closed ideals of U. The simple Hilbert triple systems are (up to an isometric isomorphism) the triples ${}^t\widetilde{W}$, where $t \neq 0$ are real numbers, and W runs over the triple systems of type I-VI. The set (r,a,g,t) is a complete system of invariants for the classes of isometrically isomorphic simple Hilbert triple systems.

The main theorem proved by KAUP [27 b, II] is

Theorem 3.3. Every symmetric Hermitian manifold M admits a unique decomposition as orthogonal product $M = M_o \times M'$ of symmetric Hermitian manifolds, where M_o is flat and M' is nondegenerate and simply connected. The category of nondegenerate symmetric Hermitian manifolds with base point is equivalent to the category of non-degenerate JH^*-triples.

Recently, FRIEDMAN and RUSSO [21 e] generalized the well-known Gel'fand-Neumark embedding theorem for JB-algebras due to ALFSEN, SHULTZ, and STØRMER [1]. The main result shows that every JB^*-triple can be isometrically embedded into the direct sum of a JC^*-triple (of Hilbert space operators, with triple product $\{ab^*c\} = (ab^*c + cb^*a)/2$) and two exceptional JB^*-triples related to the exceptional Jordan triple systems of 16 and 27 dimensions. In particular, there are only two exceptional JB^*-triple factors.

Remark. The above-mentioned result of Friedman and Russo is of fundamental importance to the structure theory of JB^*-triples and is also of interest to infinite-dimensional holomorphy, via the relationship between JB^*-triples and bounded symmetric domains (cf. paper of KAUP [27 c]).

Last but not least, let us mention the recent research monograph [35] by NEHER, where a theory of grids (i.e. special families of tripotents in Jordan triple systems) is presented. Among the applications given are structure theories for Hilbert triples and JBW^*-triples.

§ 4. Jordan algebras (or pairs) and Siegel domains

This section deals mainly with the results of DORFMEISTER [18 a,d], LOOS [33 b], and SATAKE [41] on Siegel domains connected with certain Jordan structures (algebras or pairs).

We shall firstly recall the principal results on homogeneous Siegel domains, as given in [18 c].

Notation. Let X be a finite-dimensional real vector space and let Y be a regular cone in X. As usually, $X^{\mathbb{C}} := X \oplus iX$ denotes the complexification of X. Re is the real part, Im represents the imaginary part, and $z \longrightarrow \bar{z}$ is the conjugation. Let U be a finite-dimensional complex vector space and let $S : U \times U \longrightarrow X^{\mathbb{C}}$ be a Y-Hermitian form.

Definition. $D(Y,S) := \left\{ (z,u) \in X^{\mathbb{C}} \times U \;\middle|\; \operatorname{Im} z - S(u,v) \in Y \right\}$ is called a Siegel domain.

Definition. Let D(Y,S) be a Siegel domain and let $\operatorname{Aut} D(Y,S) := \left\{ f \mid f : D(Y,S) \longrightarrow D(Y,S) \text{ biholomorphic} \right\}$. If Aut D(Y,S) acts transitively on D(Y,S), the Siegel domain D(Y,S) is called homogeneous.

<u>Note</u>. All Siegel domains that shall be considered in this section are assumed to be homogeneous.

We denote by $\mathcal{G} = \mathcal{G}(Y,S)$ the Lie algebra of the Lie group Aut D(Y,S). If $\mathcal{G}$ is assumed to be identified, as usually, with the Lie algebra of complete holomorphic vector fields on D(Y,S), then its elements are holomorphic maps of D(Y,S) into $X^{\mathbb{C}} \times U$, and are therefore represented by a pair of maps, $\mathcal{X}(z,u) = (\mathcal{X}_1(z,u), \mathcal{X}_2(z,u))$ where $\mathcal{X}_1(z,u) \in X^{\mathbb{C}}$ and $\mathcal{X}_2(z,u) \in U$.

As was shown by KAUP, MATSUSHIMA and OCHIAI [28], the elements of $\mathcal{G}$ are polynomials and there exists a canonical grading

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_{-1/2} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{1/2} \oplus \mathcal{G}_1 ,$$

where $[\mathcal{G}_\lambda, \mathcal{G}_\mu] \subset \mathcal{G}_{\lambda+\mu}$,

and $\mathcal{G}_{-1} = \{(a,o) \mid a \in X\}$,

$$\mathcal{G}_{-1/2} = \{(2i\ S(u,d),d) \mid d \in U\},$$

$$\mathcal{G}_0 = \{(T,\hat{T}) \in \mathrm{End}_{\mathbb{R}} X \times \mathrm{End}_{\mathbb{C}} U \mid T \in \mathrm{Lie\ Aut}\ Y,$$
$$TS(d,w) = S(\hat{T} d,w) + S(d,\hat{T}w),\ d,w \in U\}.$$

From [18 d] it follows that

$$\mathcal{G}_{1/2} = \{(2S(u,\varphi(\bar{z})w),\ i\varphi(z)w + 2\varphi(S(u,w))u) \mid w \in P_{1/2}\},$$

$$\mathcal{G}_1 = \{(A_x(z)z,\ \varphi(z)\ \varphi(x)u \mid x \in P_1\}^{1)},$$

where $P_{1/2}$ is a complex subspace of U, P_1 is a real subspace of X (for a detailed description of $P_{1/2}$ and P_1, see DORFMEISTER [18 d]), while $A_x(z)$ and $\varphi(y)$ are defined as follows: let $((z_1,u_1),(z_2,u_2)) \longrightarrow \eta(-(i/2)(z_1-\bar{z}_2) - S(u_1,u_2))$ be the Bergmann kernel for D(Y,S), where $\eta : Y \longrightarrow \mathbb{R}^+$ is some positive function which can be holomorphically and continuously extended to $X \oplus iY$. Choose an element e of Y and keep it fixed. Denote by σ the

1) A detailed analysis of the representations of the formally real Jordan algebras that arise from the elements of $\mathcal{G}_1$ can be found in [39].

second differential $d_e^2 \log \eta$ at the point e; then σ is a positive defined bilinear form on X. The $\mathbb{C}$-bilinear extension of σ to $X^{\mathbb{C}}$ will also be denoted by σ. We now compare the third differential $d_e^3 \log \eta$ with σ,

$$\sigma(xy,a) := -\frac{1}{2} d_e^3 \log \eta(x,y,a), \qquad x,y,a \in X.$$

The product $(x,y) \longrightarrow xy := A(x)y$ imposes on X the structure of a <u>commutative algebra</u> $\mathcal{A} = \mathcal{A}(Y,S,e)$ (with identity e and left multiplications A(x)). In the complexified algebra $\mathcal{A}^{\mathbb{C}}$ of $\mathcal{A}$ we have $\sigma(x,y,z) = \sigma(x,yz)$ for $x,y,z \in \mathcal{A}^{\mathbb{C}}$. The parameter space P_1 is a subalgebra of $\mathcal{A}$ (see DORFMEISTER [18 d, Corollary 5.2]) and has a unit, which we denote by p.

$A_x(z)$ in the expression for $\mathcal{G}_1$ denotes a new product in the vector space X, defining the mutation $\mathcal{A}_x$ of $\mathcal{A}$, namely

$$A_x(a)b := (ax)b + a(xb) - x(ab).$$

If we set $\rho(u,w) := \sigma(e,S(u,w))$, with $u,w \in U$, then ρ is a positive definite Hermitian form on U. Finally, with $\rho(\varphi(z)u,w) :=$ $:= \sigma(z,S(u,w))$, where $z \in X^{\mathbb{C}}$ and $v,w \in U$, $\varphi : X^{\mathbb{C}} \longrightarrow \mathrm{End}_{\mathbb{C}} U$ is a $\mathbb{C}$-linear map.

Putting $P_{-1} := X$, $P_{-1/2} := U$, and $P_0 := \mathcal{G}_0$, we see that P_λ is a parameter domain, for $\mathcal{G}_\lambda$. The elements of $\mathcal{G}_\lambda$ are therefore often written as $\mathcal{X}_\lambda[w]$, $w \in P_\lambda$.

<u>Definition</u> (SATAKE [41]). A homogeneous Siegel domain D(Y,S) is called <u>quasisymmetric</u> if the algebra $\mathcal{A} = \mathcal{A}(Y,S,e)$ is a Jordan algebra.

<u>Remark</u>. From $\sigma(ab,c) = \sigma(a,bc)$, it follows that $\mathcal{A}$ is a <u>formally real</u> Jordan algebra.

From DORFMEISTER [18 a,d] the following theorem is immediate:

Theorem 4.1. For a homogeneous Siegel domain $D(Y,S)$ the following are equivalent:

1) $D(Y,S)$ is quasisymmetric;

2) $\mathcal{A}$ is a Jordan algebra;

3) Y is selfdual with respect to σ;

4) $\mathcal{G}_0$ is reductive;

5) $\mathcal{G}_0$ is selfadjoint with respect to $\sigma \oplus \varsigma$;

6) $(A(x), 1/2\, \varphi(x)) \in \mathcal{G}_0$ for $x \in X$.

For a quasisymmetric Siegel domain $D(Y,S)$ there exist subspaces $X_j \subset X$ and $U_j \subset U$ (uniquely determined up to the order) such that $X = \oplus X_j$, $U = \oplus U_j$, and $S(U_j, U_j) \subset X_j$ hold. If we denote by Y_j the projection of Y on X_j and by S_j the restriction of S to U_j, then $D(Y,S) = \oplus D(Y_j, S_j)$, where $D(Y_j, S_j)$ are irreducible. The cones Y_j are also irreducible and the formula $\eta(\sum x_j) = \prod \eta_j(x_j)$ holds, where η_j is defined via the Bergmann kernel of $D(Y_j, S_j)$. It follows that $\mathcal{A}(Y,S,e) = \oplus \mathcal{A}_j$, where $\mathcal{A}_j := \mathcal{A}(Y_j, S_j, e_j)$ and $e = \sum e_j$.

Proposition 4.2. (DORFMEISTER [18 d]). a) A Siegel domain is quasisymmetric if and only if all its irreducible components are quasisymmetric.

b) A quasisymmetric Siegel domain $D(Y,S)$ is irreducible if and only if $\mathcal{A}(Y,S,e)$ is simple.

Proposition 4.3. (SATAKE [41]). Let $D(Y,S)$ be an irreducible quasisymmetric Siegel domain. Then either $D(Y,S)$ is symmetric or $\mathcal{G}_{1/2} = \mathcal{G}_1 = 0$.

Remark. Proposition 4.3. is an immediate consequence of [18 d, Corollaries 5.2 and 5.9].

Notation. Let X be a finite-dimensional real vector space, and let U be a finite-dimensional complex vector space. We denote by $Q(X,U)$ the set of all triples (Y,S,e), where $e \in Y$, and $D(Y,S)$ is a quasisymmetric Siegel domain in $X^{\mathbb{C}} \times U$. We denote by $\mathcal{Q}(X,U)$ the set of all triples $(\mathcal{A}, \varphi, \varsigma)$ satisfying the following four

1) $\mathfrak{a}$ is a formally real Jordan algebra on X (with unit e);

2) ρ is a positive definite Hermitian form on U;

3) $\varphi : \mathfrak{a} \longrightarrow \mathrm{Sym}(U, \rho) := \{ X \in \mathrm{End}_{\mathbb{C}} U \mid X^{\rho} = X \}$ is a homomorphism of Jordan algebras;

4) $\varphi(e) = \mathrm{Id}$.

Theorem 4.4. (SATAKE [41]). There is a canonical bijection from $Q(X,U)$ onto $\mathcal{Q}(X,U)$.

Remark. DORFMEISTER gave [18 b] an ad hoc proof for Theorem 4.4. and explicitly stated the canonical bijection.

Notation. Consider the rational maps π and $\tilde{\pi}$ of $X^{\mathbb{C}} \times U$ into $X^{\mathbb{C}} \times U$ defined by

$$\pi(z,u) := ((z-ie)(z+ie)^{-1}, \alpha\varphi(z+ie)^{-1}u),$$

$$\tilde{\pi}(z,u) := (i(e+z)(e-z)^{-1}, \beta\varphi(e-z)^{-1}u),$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha\beta = 2i$.

Theorem 4.5. Let $D(Y,S)$ a quasisymmetric Siegel domain. Then π maps $D(Y,S)$ biholomorphically onto a homogeneous bounded domain.

Remark 1. LOOS [33 b] developed a theory of bounded symmetric domains using, in particular, (Hermitian) Jordan triple systems. He also mentioned the maps π and $\tilde{\pi}$.

Remark 2. Using the above-mentioned results, the theory of representations of Jordan algebras and the theory of Clifford algebras, it is possible to describe explicitly all quasisymmetric Siegel domains. This has been carried out by DORFMEISTER [18 b].

Definition. A Siegel domain $D(Y,S)$ is called symmetric if for each $x \in D(Y,S)$ there exists a biholomorphic map $g_x : D(Y,S) \longrightarrow D(Y,S)$ satisfying $g_x \circ g_x = \mathrm{Id}$ and having x as isolated fixed point[1].

Theorem 4.6. A Siegel domain $D(Y,S)$ is symmetric if and only if $\mathcal{G}(Y,S)$ is semisimple.

Proposition 4.7. A symmetric Siegel domain is quasisymmetric.

Remark. Proposition 4.7., stated in [41], also follows from the Theorem 4.6 above and [18 d, Theorem 3.12].

Comments. A characterization of symmetric Siegel domains in terms of quasisymmetric Siegel domains was given by DORFMEISTER [18 c, § 3].

Using a construction which associates Jordan pairs to certain Lie algebras[2], DORFMEISTER [18 c] constructed for every Siegel domain a Jordan pair. This Jordan pair induces another Jordan pair - denoted below by $\mathcal{V}$ - on $(X^{\mathbb{C}} \times U, P_1^{\mathbb{C}} \times P_{1/2})$. We shall recall here DORFMEISTER's results [18 c, § 5] concerning the Jordan pair $\mathcal{V}$.

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1) A Siegel domain $D(Y,S)$ is symmetric if and only if it is homogeneous and $\mathcal{G}(Y,S)$ is semisimple. Thus, by a result of VEY [49, Proposition 6.2], the assumption that $D(Y,S)$ is homogeneous may be omitted.

2) If W is a real or complex vector space and $\mathcal{P} = \mathcal{P}(W)$ is the set of polynomial maps from W to W, then we denote by $\mathcal{P}_\nu = \mathcal{P}_\nu(W)$ the subset of $\mathcal{P}$ of polynomials of degree $\nu + 1$. For $X, Y \in \mathcal{P}$, we define a product by $[X,Y](x) := d_xY(X(x)) - d_xX(Y(x))$. Thus $\mathcal{P}$ becomes a Lie algebra and we have $[\mathcal{P}_\lambda, \mathcal{P}_\mu] \subset \mathcal{P}_{\lambda+\mu}$. If $D(Y,S)$ is a Siegel domain, then $\mathcal{G}(Y,S)$ is a subalgebra of $\mathcal{P}(X^{\mathbb{C}} \times U)$. The following theorem holds: Let $\tilde{\mathcal{G}}_{-1} \subset \mathcal{P}_{-1}$, $\tilde{\mathcal{G}}_0 \subset \mathcal{P}_0$, $\tilde{\mathcal{G}}_1 \subset \mathcal{P}_1$ be subspaces, and assume that $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_{-1} \oplus \tilde{\mathcal{G}}_0 \oplus \tilde{\mathcal{G}}_1$ is a subalgebra of $\mathcal{P}$. Then $(\tilde{\mathcal{G}}_{-1}, \tilde{\mathcal{G}}_1)$ together with the trilinear maps

$$\{\ \}: \tilde{\mathcal{G}}_\varepsilon \times \tilde{\mathcal{G}}_{-\varepsilon} \times \tilde{\mathcal{G}}_\varepsilon \to \tilde{\mathcal{G}}_\varepsilon, \quad \varepsilon = \pm 1, \quad \{X_\varepsilon Y_{-\varepsilon} Z_\varepsilon\} := -[[X_\varepsilon, Y_{-\varepsilon}], Z_\varepsilon]$$

is a Jordan pair.

Notation . Given a homogeneous Siegel domain D(Y,S), the explicit expressions for $\mathcal{X}_\lambda \in \mathcal{G}_\lambda$, $\lambda = \pm 1/2, \pm 1$, show that $\mathcal{G}^{\mathbb{C}} = (\mathcal{G}^{\mathbb{C}} \cap \mathcal{P}_{-1}) \oplus (\mathcal{G}^{\mathbb{C}} \cap \mathcal{P}_o) \oplus (\mathcal{G}^{\mathbb{C}} \cap \mathcal{P}_1)$. We put $\mathcal{J} = \mathcal{J}(Y,S) :=$ $:= (J^+, J^-)$, where $J^\varepsilon := \mathcal{G}^{\mathbb{C}} \cap \mathcal{P}_{\varepsilon 1}$.

From the theorem recalled in the footnote it follows that $\mathcal{J}$, together with $\{X_\varepsilon, Y_{-\varepsilon}, Z_\varepsilon\} := - [[X_\varepsilon, Y_{-\varepsilon}], Z_\varepsilon]$, is a Jordan pair over $\mathbb{C}$. If $\mathcal{X} \in \mathcal{G}$, then we denote by $\mathcal{X}^k$ (resp., $\mathcal{X}^l$, $\mathcal{X}^q$) the constant (resp., linear, quadratic) component of $\mathcal{X}$ in $\mathcal{G}^{\mathbb{C}}$. It is obvious that $\mathcal{X} = \mathcal{X}^k$ for $\mathcal{X} \in \mathcal{G}_{-1}$, $\mathcal{X} = \mathcal{X}^l$ for $\mathcal{X} \in \mathcal{G}_o$, and $\mathcal{X} = \mathcal{X}^q$ for $\mathcal{X} \in \mathcal{G}_1$.

We set $V^+ := X^{\mathbb{C}} \oplus U$, $V := P_1^{\mathbb{C}} \oplus P_{1/2}$, and impose on V^+ the canonical $\mathbb{C}$-structure, while on V^- we impose the dual complex structure (i.e. $\alpha(z+w) := \bar{\alpha} z \oplus \bar{\alpha} w$, $\alpha \in \mathbb{C}$, $z \in P_1^{\mathbb{C}}$, $w \in P_{1/2}$, where $\bar{\alpha}$ is the complex conjugate of α and $\bar{\alpha} z$, resp. $\bar{\alpha} w$, is scalar multiplication in $P_1^{\mathbb{C}}$, resp. $P_{1/2}$). Let $\mathcal{V} := \mathcal{V}(Y,S,e) :=$ $:= (V^+, V^-)$, and

$$j_\varepsilon : V^\varepsilon \longrightarrow J^\varepsilon \quad , \qquad \varepsilon = \pm \quad ,$$

$$j_+(a \oplus d) := \mathcal{X}_{-1}[a]^k + \mathcal{X}_{-1/2}[d]^k \quad ,$$

$$j_-(x \oplus w) := \mathcal{X}_1[\bar{x}]^q + \mathcal{X}_{1/2}[w]^q \; .$$

Proposition 4.8. The map $j_\varepsilon : V^\varepsilon \longrightarrow J^\varepsilon$, $\varepsilon = \pm$, is an isomorphism of complex vector spaces.

Corollary. Putting $\{\ \} : V^\varepsilon \times V^{-\varepsilon} \times V^\varepsilon \longrightarrow V^\varepsilon$,

$\{v_1^\varepsilon\, v_2^\varepsilon\, v_3^\varepsilon\} := j_\varepsilon^{-1} \{ j_\varepsilon(v_1^\varepsilon), j_{-\varepsilon}(v_2^{-\varepsilon}), j_\varepsilon(v_3^\varepsilon)\}$ the structure of a Jordan pairs is defined on $\mathcal{V} = (V^+, V^-)$.

Remark. Using the description of $\mathcal{G}$ as recalled at the beginning of this section, DORFMEISTER [18 c] gave an explicit formulation for the triple product $\{\ \ \}$ of $\mathcal{V}$ in terms of $\mathcal{A}$, As an application of the decomposition of D(Y,S) into irreducible components, recalled before as Proposition 4.2, we give the following proposition.

Proposition 4.9. Let $D(Y,S) = \bigoplus D(Y_j, S_j)$ be the decomposition of the Siegel domain D(Y,S) into irreducible components. Then $\mathcal{V} = \bigoplus \mathcal{V}_j$ holds, where $\mathcal{V}_j = \mathcal{V}_j(Y_j, S_j, e_j)$ is the Jordan pair constructed for $D(Y_j, S_j)$ and e_j.

Theorem 4.10. a) With Rad $\mathcal{V}$ defined as in JSA.I, § 4, we have Rad $\mathcal{V} = (P_1^{\perp} \oplus P_{1/2}^{\perp}, 0)$, where $P_i^{\perp}$ is the orthogonal component of P_i, $i = 1, 1/2$.

b) Set $\mathcal{V}_s^+ := P_1^{\mathbb{C}} \oplus P_{1/2}$ and endow $\mathcal{V}_s^+$ with the canonical complex structure; put $V_s^+ := V^-$. Then $\mathcal{V}_s := (V_s^+, V_s^-)$ is a semisimple subpair of $\mathcal{V}$.

c) The Jordan pair of a symmetric Siegel domain is semisimple.

For a given homogeneous Siegel domain D(Y,S), DORFMEISTER [18 c] applied the above-mentioned results to prove explicit expressions for 1-parameter groups of elements of $\mathcal{G}_\lambda$. This yields a description of the connected component of Aut D(Y,S) that contains the identity (see [18 c, Theorems 6.3, 6.4, and 6.7]).

§ 5. Theta functions for Jordan pairs

RESNIKOFF [37 b] defined theta functions associated with complex finite-dimensional Jordan pairs admitting a positive Hermitian involution. Using an abstract of his talk (see [37 b]), below we sketch this construction.

Let (V^+, V^-) be a complex finite-dimensional Jordan pair admitting a positive Hermitian involution $* : V^- \longrightarrow V^+$.

Convention. The triple product $V^+ \times V^- \times V^+ \longrightarrow V^+$ will be denoted by $(u^*, v, w^*) \longrightarrow \{uvw\}$, where $u,v,w \in V^-$, and we write u,w in the product instead of u^*, w^*.

Let $e_1, e_2 \in V^-$ be tripotents such that $e_1 + e_2$ is maximal, and let $V^- = \bigoplus V^-_{ij}$ be the corresponding Peirce decomposition. We have $V^-_{11} \oplus V^-_{12} \oplus V^-_{22} = \mathcal{A} \oplus \mathbb{C}$, where $\mathcal{A}$ is a formally real Jordan algebra.

Convention. Write $V^-_{ij} := \mathcal{A}_{ij} \otimes \mathbb{C}$ for $1 \leq i \leq j \leq 2$.

Choose lattices $\mathcal{L}_{12} \subset \mathcal{A}_{12}$ and $\mathcal{L}_{10} \subset V^-_{10}$, an element h such that $0 < h \in \mathcal{A}_{11}$, and let $z = z_{22} \oplus z_{20} \in V^-_{22} \oplus V^-_{20}$ belong to the Siegel domain of type 2

$$D := \left\{ z_{22} \oplus z_{20} \in V^-_{22} \oplus V^-_{20} \;\middle|\; \frac{z_{22} - z^*_{22}}{2i} - \frac{1}{2} \{ z_{20}, z_{20}, e_2 \}^* > 0 \right\}.$$

Let $U \in V^-_{12} \oplus V^-_{10}$. Set $\sigma(u,v) := c \operatorname{Tr} D(u,v)$, where $D(u,v)w := \{uvw\}$ and the constant c is determined so that $\sigma|_{\mathcal{A} \times \mathcal{A}}$ concides with the reduced trace. Finally, put $Q(x)y := \frac{1}{2}\{xyx\}$.

Definition. The theta function of order h associated with the lattice $\mathcal{L} := \mathcal{L}_{12} \oplus \mathcal{L}_{10}$ and the Siegel domain D is

$$\theta_{\mathcal{L}}(z, U : h) :=$$

$$:= \sum_{\substack{\lambda \in \mathcal{L} \\ \lambda = \lambda_{12} \oplus \lambda_{10}}} \exp i\pi\sigma(Q(\lambda)z + i\{e_1 \lambda_{10} \lambda_{10}\} + \{e_1 U \lambda\}, h).$$

Comments. This series satisfies analogues of all usual functional and differential equations. It reduces to the theta function associated with a Jordan algebra when $V^-_{10} = V^-_{20} = 0$. (see RESNIKOFF [37 a] and also JSA.V, § 7).

§ 6. Differential equations in Jordan triple systems and Jordan pairs

Let T be a (quadratic) finite-dimensional Jordan triple system over a field $\mathbb{K}$ of characteristic $\neq 2$. Define, as usually, the trilinear composition by $\{xyz\} := P(x+z)y - P(x)y - P(z)y$ (see JSA.I, § 3). Let $B(x,y)z := z - \{xyz\} + P(x)P(y)z$. Let $\boxed{\mapsto}(T)$ be the subgroup of the group of birational transformations of $T \times T$ generated by the structure group and all $(t_a, \tilde{t}_a)$, $(\tilde{t}_a, t_a)$ with $a \in T$, $t_a(x) := x+a$, and $\tilde{t}_a(x) := x^{-a}$, The group $\boxed{\mapsto}(T)$ has been studied by KOECHER [29 a] (see also [29 b]). This group is the group of $\mathbb{K}$-rational points of an affine algebraic $\mathbb{K}$-group.

KÜHN [30] defined the group $\circleddash(T)$ of all (f,g), where f and g are birational maps of T with the property that

$$B(f(x),\ g(y)) = \frac{\partial f(x)}{\partial x} B(x,y)\ H_{f,g}(y),$$

and

$$B(g(x),\ f(y)) = \frac{\partial g(x)}{\partial x} B(x,y)\ H'_{f,g}(y),$$

for suitable rational maps $H_{f,g}$ and $H'_{f,g}$.

In general we have $\boxed{\mapsto}(T) \subset \circleddash(T)$. She proved that $\boxed{\mapsto}(T) = \circleddash(T)$ in case char $\mathbb{K} = 0$ or T has no extreme radical. For characterizations of these groups involving a norm on T the reader is referred to KÜHN [30].

Concerning the Riccati differential equation in Jordan pairs, BRAUN [7] proved a result recalled below (see Theorem 6.1). Linearization of the matrix Riccati differential equation derived from $(m \times n)$-matrices (see LEVIN [32]) and the Riccati differential equation for operators in a Banach space (see TARTAR [46]) are assumed to be known to the reader.

Let V be a Jordan pair with V^σ, $\sigma = \pm$, Banach spaces, and let D and Q be the derivation and the quadratic representation defined as usually (see JSA.I, § 4). Let I be an $\mathbb{R}$-interval, let η be an initial point in I, and let k be a given initial value, $k \in V^+$. Let $v(\xi)$, $w(\xi)$ be given continuous functions, $v : I \longrightarrow V^-$, $w : I \longrightarrow V^+$, and let D and Q be continuous. The Riccati differential equation (without linear term) is defined by

$$\frac{\partial x}{\partial \xi} = Q(x)v + w.$$

The solution $x : I \times I \longrightarrow V^+$ with initial value k at the point η will be denoted by $x(\xi, \eta)$.

<u>Notation</u>. $B(u,t) := Id - D(u,t) + Q(u)Q(t)$, $u^t := B(u,t)^{-1}(u - Q(u)t)$, for $u \in V^+$, $t \in V^-$, if the inverse of $B(u,t)$ exists.

<u>Theorem 6.1.</u> Let x_o be the solution of the Riccati equation with initial value $k = 0$ at $\eta = 0$. Put

$$x(\xi, \eta) := x_o + h_+(k)^{h_-(z)}$$

$h_\sigma : I \times I \longrightarrow \mathrm{Aut}\, V^\sigma$, $z : I \times I \longrightarrow V^-$. Solve the linear system

$$\frac{\partial h_+}{\partial \xi} = D(x_o, v)h_+ , \quad \frac{\partial h_-}{\partial \xi} = - D(v, x_o)h_- ,$$

so that $\frac{\partial z}{\partial \xi} = h_-^{-1}(v)$ with $h_\sigma(\eta, \eta) = Id$; $z(\eta, \eta) = 0$. Then $x(\xi, \eta)$ is the solution with initial value $x(\eta, \eta) = k$ (in a neighborhood of η).

Recently, WALCHER [51 a] gave a characterization of regular Jordan pairs and its application to the Riccati differential equation as follows. Let V be a finite-dimensional vector space over $\mathbb{R}$, $P : V \longrightarrow \mathrm{Hom}(V,V)$ a quadratic map, $G \subset V$ open $(G \neq \emptyset)$, and

$\varphi \in C^1(G,V)$. Suppose that for all $a \in V$ one has $(d/dt)\varphi(z(t)) = -a$ whenever $z(t) \in G$ is a solution of the Riccati differential equation $\dot{x} = P(x)\, a$. By differentiation $D\varphi(x).\, P(x) = -Id$. Moreover, Walcher showed that the identity $P(x, P(x)z)y = P(x,P(x)y)z$ is satisfied for all $x,y,z \in V$. Thus there exists a Jordan pair structure (P,Q_-) on $\mathcal{V} = (V,V)$ and by Theorem 6.1 the following is true: Let $a : I \longrightarrow V^-$, $c : I \longrightarrow V^+$, $(B_+, B_-) : I \longrightarrow \mathrm{Der}\,\mathcal{V}$ be continuous. If $z(t)$ solves

$$\dot{x} = P(x)a + B_+x + c$$

and $P(z(t))$ is invertible, then $P(z(t))^{-1}\, z(t)$ solves

$$\dot{x} = -Q_-(x)\, c + B_-x - a.$$

Let us recall that a system of ordinary differential equations $\ddot{x} = F(t,x)$ is said to have a fundamental system of solutions if there exist finitely many solutions that determine (almost) all other solutions; it is called a system of <u>polynomial differential equations</u> if, for all values of t, $F(t,x)$ is a polynomial in x. A theorem of Lie implies that a system of polynomial differential equations has a fundamental system of solutions if $F(t,x) = \sum \lambda_i(t) f_i(x)$ and the polynomials $f_i(x)$ generate a finite-dimensional subalgebra of the Lie algebra Pol V, where V is the vector space on which the system is defined.
Recently, WALCHER [51 b] determined these subalgebras in the case dim V = 1 and showed to they correspond to the Riccati (including linear) and the Bernoulli equation.
For dim V > 1, Walcher investigated the finite-dimensional, graded subalgebras L of Pol V. Denoting by $\mathrm{Pol}_i V$ the subspace of all polynomials of degree i+1, it is shown that the semisimplicity of

$$L = L_{-1} \oplus L_0 \oplus \ldots \oplus L_m$$

with $L_i \subseteq Pol_i V$, implies $m = 1$.

L is said to be transitive if $L_{-1} = V$. By a result of Kantor, it is known that a finite-dimensional,graded, transitive subalgebra with $m > 1$ is reducible; that is, there exists a subspace U of L_{-1} with $0 \neq U \neq V$ such that for all k with $0 \leq k \leq m$ and all $p \in L_k$, $p(V, \ldots, V,U) \subseteq U$. This allows one to reduce the discussion of transitive subalgebras to those whose degree equals 1. The latter are shown to arise from finite-dimensional Jordan pairs. In case dim V = 2, this permits a complete ennumeration of all finite-dimensional, maximal, transitive subalgebras of Pol V of fixed degree m. Walcher also discussed how these results can be used to find all solutions of certain types of systems of polynomial differential equations.

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