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JORDAN STRUCTURES WITH APPLICATIONS. VII. ALGEBRAIC VARIETIES (OR MANIFOLDS) DEFINED BY JORDAN PAIRS

by

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JORDAN STRUCTURES WITH APPLICATIONS.VII.
ALGEBRAIC VARIETIES (OR MANIFOLDS) DEFINED BY JORDAN PAIRS

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The description of the algebraic varieties defined by simple finite-dimensional (quadratic) Jordan pairs over an algebraically closed field is given in Section 1. The next two Sections focus on local and global Jordan manifolds, a kind of manifold for which the underlying algebraic structure consists of (linear) Jordan pairs. Worth mentioning is that many familiar manifolds are Jordan manifolds.

§ 1. Homogeneous algebraic varieties defined by Jordan pairs

LOOS [24e] showed that every (quadratic) Jordan pair defines an affine algebraic group, the projective group of the Jordan pair. On the other hand, he observed [24d] that every Jordan pair defines an algebraic variety, related with the projective group of the Jordan pair in a natural manner.

We give a brief review of the above-mentioned constructions. The algebraic varieties defined by simple finite-dimensional Jordan pairs over an algebraically closed field are listed. Then comments on the results of PETERSSON [30] and JACOBSON [18] are made.

Let $\mathbb{K}$ be an arbitrary (commutative) ring of scalars and let G be a separated $\mathbb{K}$-group sheaf.

Definition. An action Ψ of the multiplicative group $\mathbb{K}_m$ of $\mathbb{K}$ on G by automorphisms is called elementary, and the pair (G, Ψ) is called an elementary system, if there exist subgroup sheaves H, U^+, U^- generating G, such that H is fixed under Ψ, U^+ and U^- are vector groups on which Ψ acts via $\Psi_t(u) := t^{\pm 1}u$, and $\Omega := U^-.H.U^+$ is open in G.

Remark. The conditions in the above Definition determine H, U^+, and U^- uniquely.

The Lie algebra $V^\pm$ of $U^\pm$ is a finitely generated and projective $\mathbb{K}$-module, and there exist unique isomorphisms $\exp: V_a^\pm \longrightarrow U^\pm$ with Lie (exp) = Id such that $\Psi_t(\exp(x)) = \exp(t^{\pm 1}, x)$ for all $t \in \mathbb{K}_m$ and $x \in V_a^\pm$ (see LOOS [24], 3.3).

There exists a unique Jordan pair structure on $V = (V^+, V^-)$ with the following property: for all $(x,y) \in V_R^+ \times V_R^-$, R being a (commutative associative unital) $\mathbb{K}$-algebra, we have $\exp(x)\exp(y) \in \Omega(R)$ if and only if (x,y) is quasi-invertible. In this case, $\exp(x)\exp(y) = \exp(y^x)b(x,y)\exp(x^y)$, with a morphism $b: W \longrightarrow H$, where H is the subscheme of quasi-invertible pairs. Also, H normalizes U^+ and U^- and acts on V by automorphisms via the adjoint representation (see LOOS [24], 4.1). V is called the Jordan pair associated with (G, Ψ).

In general, (G, Ψ) is not uniquely determined by V. However, given a Jordan pair V with $V^\pm$ finitely generated and projective, there exists an (essentially unique) $\mathbb{K}$-group sheaf $\overline{G} = PG(V)$, the projective group of V, and an elementary action $\overline{\Psi}$ on $\overline{G}$, whose associated Jordan pair is V, and such that the homomorphism $\overline{H} \longrightarrow \mathrm{Aut}(V)$ induced by the adjoint representation is an isomorphism. If $\overline{\gamma}: \mathbb{K}_m \longrightarrow \overline{H} \simeq \mathrm{Aut}(V)$ is the unique homomorphism such that $\mathrm{Ad}\,\overline{\gamma}(t).x = t^{\pm 1}.x$ for $t \in \mathbb{K}_m$, $x \in V_a^\pm$, then $\overline{\Psi}$ is given by conjugation with

$\overline{\gamma}$ (see LOOS [24d, 5.14 and 5.15]).

Let $V = (V^+, V^-)$ be a finite-dimensional Jordan pair over an algebraically closed field $\mathbb{F}$ and let $X = X(V)$ be the quotient of $V^+ \times V^-$ by the equivalence relation

$(x,y) \sim (x', y')$ if and only if $(x,y-y')$ is quasi-invertible and $x' = x^{y-y'}$. (As usual, x^y denotes the quasi-inverse in the Jordan pair V). LOOS [24c] proved that X is a quasi-projective variety containing V^+ as an open dense subset, and that X is projective if V is semisimple. Moreover, Loos showed that under the projective group of V the space X is homogeneous in a natural way, and that this projective group is isomorphic to the group of automorphisms of X if V is semisimple. This is essentially due to CHOW [9] in four classical cases. For a geometric characterization of the projective group see CHOW [9], FAULKNER [11a,b], FREUDENTHAL [12], HUA [16a,b], SPRINGER [37a], and, also, JSA.II.

Remark. The assumption that $\mathbb{F}$ be an algebraically closed field is not essential: in fact, $\mathbb{F}$ may be an arbitrary ring of scalars, provided V^+ and V^- are finitely generated and projective $\mathbb{F}$-modules. Then X is a smooth quasi-projective $\mathbb{F}$-scheme. Semisimplicity has to be replaced by separability (a finitely generated and projective Jordan pair V over a ring $\mathbb{F}$ is called separable if $V \otimes_{\mathbb{F}} k$ is semisimple for all fields k which are $\mathbb{F}$-algebras). If V is separable, then X is a projective $\mathbb{F}$-scheme and the projective group V is a semisimple $\mathbb{F}$-group scheme, isomorphic to the automorphism group scheme of X.

For each of the Jordan pairs from the classification of simple finite-dimensional Jordan pairs over an algebraically closed field $\mathbb{F}$ given by LOOS [24b, p.201] the variety X is as

follows:[1)]

Type $I_{p,q}$. $V^+ = V^- = M_{p,q}(\mathbb{F})$, $(p \times q)$-matrices over $\mathbb{F}$ $(p \leq q)$, with $Q(x)y := xy'x$, where y' denotes the transpose of y. In this case X is isomorphic to the Grassmannian $G_p(\mathbb{F}^{p+q})$ of p-dimensional subspaces of $\mathbb{F}^{p+q}$.

Type II_n. $V^+ = V^- = A_n(\mathbb{F})$, alternating $(n \times n)$-matrices with $Q(x)y := xy'x$. In this case X is isomorphic to the subvariety of $G_n(\mathbb{F}^{2n})$ consisting of all totally isotropic subspaces of $\mathbb{F}^{2n}$ of fixed parity with respect to the quadratic form $q(x_1, \ldots, x_n) :=$

$$:= \sum_{i=1}^{n} x_i x_{n+i}.$$

Type III_n. $V^+ = V^- = H_n(\mathbb{F})$, symmetric $(n \times n)$-matrices with $Q(x)y := xy'x$. In this case X is isomorphic to the subvariety of $G_n(\mathbb{F}^{2n})$ consisting of all maximal isotropic subspaces of $\mathbb{F}^{2n}$ with respect to the alternating form $\alpha(x,y) := \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$.

Type IV_n. $V^+ = V^- = \mathbb{F}^n$ with $Q(x)y := q(x,y)x - q(x)y$, where q is the standard quadratic form of $\mathbb{F}^n$, given by

$$q(x_1, \ldots, x_{2m}) := \sum_{i=1}^{m} x_i x_{m+i} \quad \text{if} \quad n = 2m,$$

$$q(x_o, \ldots, x_{2m}) := x_o^2 + \sum_{i=1}^{m} x_i x_{m+1} \quad \text{if} \quad n = 2m + 1$$

In this case X is isomorphic to the quadric of all isotropic lines through the origin in $\mathbb{F}^{2n}$.

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1) These Jordan pairs arise in a fairly obvious way from separable Jordan pairs over $\mathbb{Z}$ by reduction modulo char $\mathbb{F}$ and extending the prime field to $\mathbb{F}$. Thus, we are dealing with the fibres of a smooth projective $\mathbb{Z}$-scheme.

Type V. $V^+ = V^- = M_{1,2}(\mathcal{O})$, (1×2)-matrices over the octonion (Cayley) algebra $\mathcal{O}$ over $\mathbb{F}$. Here X is isomorphic to the projective octonion plane $\mathcal{P}(\mathcal{O})$ defined by the exceptional Jordan algebra $H_3(\mathcal{O})$ (see FAULKNER [11a] and, also, JSA.II, §§ 1,2).

Type VI. $V^+ = V^- = H_3(\mathcal{O})$, Hermitian (3×3)-matrices over the octonion (Cayley) algebra $\mathcal{O}$ over $\mathbb{F}$. In this case X is isomorphic to the space of lines $\mathbb{F}u$, where u is an element of rank one in the 56-dimensional space of all matrices $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{F}$, $a, b \in H_3(\mathcal{O}, \mathbb{F})$ (see FAULKNER [11b]).

In 1984 the fundamental paper [30] by PETERSSON appeared. It provides a uniform framework for generic splitting fields of associative algebras studied by HEUSER [15], KOVACS [22], SALTMAN [33], AMITSUR [2] and ROQUETTE [31,6], and generic zero fields of quadratic forms investigated by KNEBUSCH [20]. This is achieved by first observing that both, central simple associative algebras over a field $\mathbb{K}$ of arbitrary characteristic and absolutely nondegenerate quadratic forms over $\mathbb{K}$, are examples of absolutely simple Jordan pairs over $\mathbb{K}$, and then constructing generic reducing fields of Jordan pairs in a classification-free treatment.

Let $V = (V^+, V^-)$ be an absolutely simple Jordan pair over $\mathbb{K}$. A nonzero idempotent c of V is called <u>reduced</u>, if the Peirce space $V_2(c)^\varepsilon = \mathbb{K}\, c^\varepsilon$, $\varepsilon = \pm$, and V itself is called <u>reduced</u> if it contains a reduced idempotent.

<u>Remark</u>. For Jordan structures "split" is a more restrictive concept than "reduced", however "split" = "reduced" if V is the Jordan pair associated to a central simple associative algebra.

A field extension $\mathcal{K}/\mathbb{K}$ is called a <u>reducing field</u> of V in case the extended Jordan pair $V_{\mathcal{K}}$ is reduced, and such an extension is

called a generic reducing field if an arbitrary field extension $\mathcal{L}/\mathbb{K}$ reduces V if and only if there is a $\mathbb{K}$-place from $\mathcal{K}$ to $\mathcal{L}$.

Petersson has constructed two, in general not $\mathbb{K}$-isomorphic, generic reducing fields. Both are $\mathbb{K}$-rational function fields of irreducible projective $\mathbb{K}$-variables associated to V, A detailed discussion of examples, most natably Brauer-Severi varieties, and with an application to exceptional simple Jordan algebras arising from the first Tits construction is also given by PETERSSON [30].

In 1985, JACOBSON [18] considered two classes of projective varieties, namely norm hypersurfaces and varieties of reduced elements defined by finite-dimensional central simple Jordan algebras. JACOBSON's paper [18] overlaps substantially with the paper [30] by PETERSSON. However, the methods and points of view are different, supplementing each other.

§ 2. Local Jordan manifolds

WATSON [42a, Chapter II] defined a class of manifolds, called local Jordan manifolds, where the ground algebraic structure consists of (linear) Jordan pairs. For the sake of simplicity we shall omit the word "linear". As one can see from the examples given below, many familiar manifolds are local Jordan manifolds.

Note. All Jordan pairs considered in the following will be finite-dimensional vector spaces over $\mathbb{R}$ and the vector spaces V^+ and V^- of a Jordan pair V will be assumed to have the ordinary real topology.

Definition. A Jordan chart on a topological space M is a pair (ϕ, V) consisting of a Jordan pair $V = (V^+, V^-)$ and a map $\phi : U \longrightarrow V^+$, where $U \neq \emptyset$ is open in M, and ϕ is a homeomorphism of U onto an open subset of V^+.

Definition. A Jordan atlas on a topological space M is a set $\mathcal{A}$ of Jordan charts on M satisfying the following two conditions:

(a) the domains of the Jordan charts in $\mathcal{A}$ conver M;

(b) if $\phi_1: U_1 \longrightarrow V_1^+$ and $\phi_2: U_2 \longrightarrow V_2^+$ (where U_i is open in M and V_i is a Jordan pair) are elements of $\mathcal{A}$ and if $U_1 \cap U_2 \neq \emptyset$, then in a neighbourhood of each point in $\phi_1(U_1 \cap U_2)$, $\phi_2 \phi_1^{-1} : \phi_1 (U_1 \cap U_2) \subset V_1^+ \longrightarrow V_2^+$ is the restriction of a map from $L F_+ (V_1, V_2)$.

Let $\phi_i : U_i \longrightarrow V_i^+$, $i = 1,2$, be two Jordan charts in a Jordan atlas $\mathcal{A}$ on M. Suppose that $U_1 \cap U_2 \neq \emptyset$. Then for every $u \in U_1 \cap U_2$, in a neighbourhood of $\phi_1(u)$, $\phi_2 \phi_1^{-1}$ is the restriction of a unique linear fractional map, called the coordinate transition map and denoted by $C^u_{\phi_2 \phi_1}$.

Remark 1. Note that $u \longrightarrow C^u_{\phi_2 \phi_1}$ is locally constant.

Remark 2. Let $\phi_i: U_i \longrightarrow V_i^+$, $i = 1,2,3$ be an element of $\mathcal{A}$ and let $u \in U_1 \cap U_2 \cap U_3$. Then $C^u_{\phi_3 \phi_2} \circ C^u_{\phi_2 \phi_1} = C^u_{\phi_3 \phi_1}$, and, in particular, $C^u_{\phi_2 \phi_1} = \left(C^u_{\phi_1 \phi_2}\right)^{-1}$ and $C^u_{\phi_1 \phi_1} = \mathrm{Id}$.

Definition. Let $\mathcal{A}$ be a Jordan atlas on a topological space M. Then a set $\mathcal{B}$ of Jordan charts on M is said to be compatible with $\mathcal{A}$ if $\mathcal{A} \cup \mathcal{B}$ is a Jordan atlas on M.

Now let $\hat{\mathcal{A}}$ be the class of all Jordan charts on M compatible with $\mathcal{A}$. It can easily be checked that if $\mathcal{A}_1$ is a set, if $\mathcal{A}_1 \subset \hat{\mathcal{A}}$, and if the domains of the charts in $\mathcal{A}_1$ cover M,

then $\mathcal{A}_1$ is a Jordan atlas on M, and $\hat{\mathcal{A}}_1 = \hat{\mathcal{A}}$. The class $\hat{\mathcal{A}}$ is called the Jordan structure of M.

Definition. A local Jordan manifold is a pair $(M, \hat{\mathcal{A}})$ where M is a topological Hausdorff space and $\hat{\mathcal{A}}$ is a Jordan structure on M determined by a Jordan atlas $\mathcal{A}$ on M.

Remark. Note that M is not assumed to be connected, nor must all components of M have the same dimension. However, every component of M is a real analytic manifold.

Definition. Let $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ be local Jordan manifolds. A morphism from M_1 to M_2 is a continuous map $f: M_1 \longrightarrow M_2$ such that, for every pair of charts $\phi_1 : U_1 \longrightarrow V_1^+$ in $\mathcal{A}_1$ and $\phi_2: U_2 \longrightarrow V_2^+$ in $\hat{\mathcal{A}}_2$ with $U_1 \cap f^{-1}(U_2) \neq \emptyset$, in a neighbourhood of each point in its domain the map $\phi_2 f \phi_1^{-1}$: $: \phi_1(U_1 \cap f^{-1}(U_2)) \longrightarrow V_2^+$ is the restriction of a map from $L F_+ (V_1, V_2)$.

Remark. The composite of two morphisms (as maps) between local Jordan manifolds is again a morphism, and in this way the local Jordan manifolds and the morphisms between them form a category, denoted by LJM. Every morphism in LJM between connected local Jordan manifolds is a morphism in the category of real analytic manifolds.

Definition. Let V be a Jordan pair and let M be a topological space. A Jordan chart on M of the form (ϕ, V) is called a Jordan V-chart. A Jordan V-atlas on M is a Jordan atlas $\mathcal{A}$ on M consisting of Jordan V-charts.

Definition. Let V be a Jordan pair. A local Jordan V-manifold is a topological Hausdorff space M with a maximal Jordan V-atlas $\mathcal{A}$.

The standard local Jordan V-manifold is V^+.

Remark 1. Every local Jordan V-manifold is a local Jordan manifold.

Remark 2. Let $(M, \hat{\mathcal{A}})$ be a connected local Jordan manifold. One can see that on M we can specify a Jordan V-atlas contained in $\hat{\mathcal{A}}$, where V is any Jordan pair occurring in a Jordan chart in $\mathcal{A}$. The various Jordan pairs V generated by a Jordan atlas $\mathcal{A}$ on M need not be isomorphic.

Remark 3. A morphism $f : M_1 \longrightarrow M_2$ between two local Jordan manifolds M_1 and M_2 can be a diffeomorphism but can never be an isomorphism in LJM.

Theorem 2.1. Let $f : M_1 \longrightarrow M_2$ be a morphism between two local Jordan manifolds M_1 and M_2. Suppose that M_1 is a local Jordan V-manifold, where Rad $V^+ = 0$. Then f is an isomorphism in LJM if and only if it is a homeomorphism.

Let $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ be local Jordan manifolds. Since any $(f_1, f_2) \in LF_+(V_1, W_1) \times LF_+(V_2, W_2)$ can be identified in a natural way with an element of $LF_+(V_1 \oplus V_2, W_1 \oplus W_2)$, it is clear that $M_1 \times M_2$ possesses a natural Jordan structure. The (direct product) local Jordan manifold $M_1 \times M_2$ has the expected universal property requiring that whenever M is a local Jordan manifold and $p_i : M \longrightarrow M_i$, $i = 1,2$, are morphisms, there exists a unique morphism $p : M \longrightarrow M_1 \times M_2$ such that $\pi_i p = p_i$, where $\pi_i : M_1 \times M_2 \longrightarrow M_i$ is the projection morphism, for $i = 1,2$.

Moreover, any other local Jordan manifold M' having this property (for a pair of morphisms $\pi_i' : M' \longrightarrow M_i$, $i = 1,2$) is isomorphic in LMJ to $M_1 \times M_2$.

Remark . If M_i is a local Jordan V_i-manifold, $i = 1,2$, then $M_1 \times M_2$ is a local Jordan $V_1 \oplus V_2$-manifold.

Definition Let $(M, \hat{a})$ be a local Jordan manifold. A non--empty subset N of M is called a local Jordan submanifold of M if there exists a set $a_1 \subset \hat{a}$ such that for every $n \in N$ there exists a Jordan chart $\phi : U \longrightarrow V^+$ in a_1 with $n \in U$ and such that $\phi(U \cap N)$ is an open subset of a vector subspace W^+ of V^+.

Open subsets of local Jordan manifolds and affine subspaces of V^+, where $V = (V^+, V^-)$ is a Jordan pair, are simple examples of local Jordan submanifolds.

Proposition 2.2. Let N_1 and N_2 be local Jordan submanifolds of M_1 and M_2, respectively, and let $f : M_1 \longrightarrow M_2$ be a morphism. Then $f\big|_{N_1} : N_1 \longrightarrow M_2$ is a morphism. If $f(N_1) \subset N_2$ and $f : N_1 \longrightarrow N_2$ is an open mapping, then $f : N_1 \longrightarrow N_2$ is a morphism.

Definition. Let $f : M_1 \longrightarrow M_2$ be a morphism between local Jordan manifolds. Then f is called a local isomorphism if every point $p \in M_1$ has a neighbourhood U such that $f(U)$ is open in M_2 and $f : U \longrightarrow f(U)$ is an isomorphism in LJM (where U and f(U) are local Jordan submanifolds of M_1 and M_2, respectively).

Proposition 2.3. Let $(M, \hat{a})$ be a local Jordan manifold and let X be a topological Hausdorff space. Let $p : X \longrightarrow M$ be a

local homeomorphism. Then X has a unique Jordan structure such that p is a local isomorphism.

Let $\tilde{M}$ be a covering space of a local Jordan manifold M and let $p : \tilde{M} \longrightarrow M$ be the covering map. By Proposition 2.3 it follows that $\tilde{M}$ has a unique Jordan structure such that p is a local isomorphism. If M is a local Jordan V-manifold, then $\tilde{M}$ is also a local Jordan V-manifold. Let $\tilde{f} : \tilde{M} \longrightarrow \tilde{M}$ be a continuous map which lies over a continuous map $f : M \longrightarrow M$ ($p\tilde{f} = fp$); then $\tilde{f}$ is an LJM morphism if and only if f is an LJM morphism.

If $\tilde{M}$ is a connected local Jordan manifold and G is a group of LJM automorphisms acting freely and properly discontinuously on $\tilde{M}$, then the orbit space $M = \tilde{M}/G$ possesses a unique Jordan structure such that the natural projection $p : \tilde{M} \longrightarrow M$ is a local isomorphism. A Jordan atlas for M is constructed by taking charts of the form $\hat{\phi} = \phi\, p^{-1} : p(U) \longrightarrow V^+$, where $p : U \longrightarrow p(U)$ is a homeomorphism and $\phi : U \longrightarrow V^+$ is a Jordan chart on $\tilde{M}$. If U_i and ϕ_i, $i = 1,2$, determine two such charts on M and if $x \in p(U_1) \cap p(U_2)$, then there exist $x_1 \in U_1$, $x_2 \in U_2$ and $g \in G$ such that $p(x_1) = p(x_2) = x$ and $g(x_1) = x_2$. Hence $\hat{\phi}_2 \hat{\phi}_1^{-1} = \phi_2\, g\, \phi_1^{-1}$ in a neighbourhood of $\hat{\phi}_1(x)$, and hence locally it is the restriction of a linear fractional map.

As an example of an orbit space, let $\tilde{M} = V^+$, where $V = (V^+, V^-)$ is a Jordan pair, and let G be a group of translations of V^+ acting freely and properly discontinuously. Then $M = \tilde{M}/G$ is a local Jordan V-manifold. This shows that a torus, for instance, possesses many local Jordan manifold structures.

Examples of local Jordan manifolds. 1. Affine quadrics.

Let X be an n-dimensional real vector space equipped with a symmetric bilinear form $\langle x,y \rangle$. Consider the Jordan pair $\underline{X} = (X,X)$ with $Q(x)y := 2\langle x,y\rangle x - \langle x,x\rangle y$ (see Example (b) of § 4 of JSA.I). Suppose that $M := \left\{ x \in X \,\middle|\, \langle x,x\rangle = 1 \right\}$ is not empty. Then M is a local Jordan submanifold of $\underline{X}^+$.

2. Projective quadrics. Let X be a real vector space with the symmetric bilinear form $\langle x,y\rangle$ as in above example. For $x \in X$, let $x^\perp := \left\{ y \in X \,\middle|\, \langle x,y\rangle = 0 \right\}$. Let $P(X)$ be the manifold of one-dimensional subspaces of X. For $x \in X$, $x \neq 0$, let $x^\#$ denote the space $\mathbb{R}x$. Suppose that $M := \left\{ x^\# \in P(X) \,\middle|\, \langle x,x\rangle = 0 \text{ and } x^\perp \neq X \right\}$ is not empty. Let I be the set of all ordered pairs $(p,r) \in X \times X$, where $p^\#, r^\# \in M$ and $\langle p,r\rangle \neq 0$

For every $(p,r) \in I$ a Jordan chart is constructed on M. For $x^\# \in M$, $x^\# \neq r^\#$, the projective line through $r^\#$ and $x^\#$ meets $(p^\perp)^\#$ in $I(x^\#) = (\langle r,p\rangle x - \langle x,p\rangle r)^\#$. The projection of $I(x^\#)$ onto the vector space $r^\perp$ is

$$S_{p,r}(x^\#) = \frac{\langle p,r\rangle I(x^\#) - \langle I(x^\#), r\rangle p}{\langle I(x^\#), r\rangle} = \frac{\langle p,r\rangle x - \langle x,p\rangle r - \langle x,r\rangle p}{\langle x,r\rangle},$$

for $x^\# \in M$ with $\langle x,r\rangle \neq 0$. It is easy to establish that $S_{p,r}: \left\{ x^\# \in M \,\middle|\, \langle x,r\rangle \neq 0 \right\} \xrightarrow{\text{onto}} p^\perp \cap r^\perp$, and its inverse is

$S_{p,r}^{-1}(u) = (2\langle p,r\rangle (u+p) - \langle u,u\rangle r)^\#$ for $u \in p^\perp \cap r^\perp$. Hence $S_{p,r}$ is a kind of stereographic projection of M onto $p^\perp \cap r^\perp$. Let $\underline{X}_{p,r}$ be the subpair $(X^+_{p,r}, X^-_{p,r})$ of $\underline{X}$, where

$$X^+_{p,r} := p^\perp \cap r^\perp \quad \text{and} \quad X^-_{p,r} := \left\{ x \in X \;\middle|\; Q(X^+_{p,r})x \subset X^+_{p,r} \right\} .$$

Defining convenient isometries of X and performing some calculation, one can state that if (p_1,r), $(p_2,r) \in I$, then

$S_{p_2,r}\, S^{-1}_{p_1,r} \in L F_+ (X_{-p_1,r}, X_{-p_2,r})$ and if (p,r_1), $(p,r_2) \in I$, then

$$S_{p,r_2}\, S^{-1}_{p,r_1} \in L F_+ (X_{-p,r_1}, X_{-p,r_2}) .$$

Since $M \neq \emptyset$, one can split the argument into two cases:

(i) $\dim X^\perp + 2 < \dim X$

(ii) $\dim X^\perp + 2 = \dim X$

(i) In this case, M is connected and for all $p^\# \in M$, the set $\left\{ r^\# \in M \;\middle|\; (p,r) \in I \right\}$ is a dense Zariski open subset of M. Let $(p,r) \in I$. Then $\langle , \rangle$ is non-zero on $X^+_{p,r}$, and since $Q(u)x = 2\langle u,x\rangle u - \langle u,u\rangle x$, we can see that $X^-_{p,r} = X^+_{p,r}$. Now, an element $u \in X^+_{p,r}$ is invertible if and only if $Q(u)$ is invertible. But this holds if and only if $\langle u,u\rangle \neq 0$. Hence $X_{p,r}$ is a unital Jordan pair. The Jordan charts $S_{p,r}: \left\{ x^\# \in M \;\middle|\; \langle x,r\rangle \neq 0 \right\} \longrightarrow X^+_{p,r}$, for $(p,r) \in I$, form a Jordan atlas on M.

(ii) In this case M has two components (because of the assumption that $M \neq \emptyset$), M_1 and M_2, and $(p,r) \in I$ if and only if $p^\#$ and $r^\#$ belong to different components. Moreover,

$$S_{p,r}: \left\{ x^\# \in M \;\middle|\; \langle x,r\rangle \neq 0 \right\} = M_i \longrightarrow X^+_{p,r} ; \text{ where } p^\# \in M_i .$$

Note that $X^{-}_{p,r} = X$ and that $\underline{X}_{p,r}$ is not unital. The maps $S_{p,r}$, $(p,r) \in I$, form a Jordan atlas on M.

Therefore M is a local Jordan manifold.

3. Projective space. Let X be an (n+1)-dimensional ($n \geqslant 1$) real vector space and let X^* be its dual space. The action of X^* on X and that of X on X^* will be denoted by $\langle y \mid x \rangle = \langle x \mid y \rangle$ for $x \in X$ and $y \in X^*$. For $x \in X$ we set $x^{\perp} := \{ w \in X^* \mid \langle x \mid w \rangle = 0 \}$, and, similarly for $y \in X^*$, we set $y^{\perp} := \{ v \in X \mid \langle y \mid v \rangle = 0 \}$.

Let P(X) and $x^{\#}$ be as in Example 2 above. Let I be the set of all $(v,w) \in X \times X^*$ such that $\langle w \mid v \rangle \neq 0$. Given $(v,w) \in I$, construct a Jordan chart on P(X) as follows. Define

$$S_{v,w}: \{ x^{\#} \in P(X) \mid x \notin w^{\perp} \} \xrightarrow{\text{onto}} w^{\perp} \quad \text{by}$$

$$S_{v,w}(x^{\#}) := \frac{\langle w \mid v \rangle x - \langle w \mid x \rangle v}{\langle w \mid x \rangle} .$$

The inverse of $S_{v,w}$ is $S^{-1}_{v,w}(u) = (u+v)^{\#}$, $u \in w^{\perp}$. Consider now the Jordan pair $\underline{X}_{v,w} := (w^{\perp}, v^{\perp})$ with $Q(x)\, y := \langle x \mid y \rangle x$. Hence, $\underline{X}_{v,w}$ is isomorphic to $\underline{\mathbb{R}}_{1,n}(\mathbb{R})$ (see Example (c) in § 4 of JSA.I). The Jordan charts $S_{v,w}$, $(v,w) \in I$, form a Jordan atlas for P(X). Indeed, one can prove that: if (v,w_1), $(v,w_2) \in I$, then $S_{v,w_2} S^{-1}_{v,w_1} \in L\,F\,+\,(\underline{X}_{v,w_1}, \underline{X}_{v,w_2})$; if (v_1,w), $(v_2,w) \in I$, then $S_{v_2,w} S^{-1}_{v_1,w} \in L\,F\,+\,(\underline{X}_{v_1,w}, \underline{X}_{v_2,w})$; if (v_1,w_1), $(v_2,w_2) \in I$, then $S_{v_2,w_2} S^{-1}_{v_1,w_1} \in L\,F\,+\,(\underline{X}_{v_1,w_1}, \underline{X}_{v_2,w_2})$ and $S_{v,w}(v^{\#}) = 0$. Clearly it follows that P(X) is a local Jordan $\underline{\mathbb{R}}_{1,n}(\mathbb{R})$-manifold.

Comments. It would be interesting to reconsider MARTINELLI's results [27 a,b] on quaternionic projective planes in this more general setting of Jordan pairs. An open problem is to find an algebraic characterization of integrability of a geometric structure.

4. One-dimensional local Jordan manifolds: a) $\underset{\sim}{\mathbb{R}}_0^+$ ($\mathbb{R}$ is the + space of the Jordan pair $\underset{\sim}{\mathbb{R}}_0 := (\mathbb{R},0)$, and as such it is a local Jordan $\underset{\sim}{\mathbb{R}}_0$-manifold, denoted by $\underset{\sim}{\mathbb{R}}_0^+$) ;

b) $\underset{\sim}{\mathbb{R}}_1^+$ ($\mathbb{R}$ is also the + space of the unital Jordan pair $\underset{\sim}{\mathbb{R}}_1 := (\mathbb{R}, \mathbb{R})$ with $Q(x)y := x^2y$. Hence it is a local Jordan $\underset{\sim}{\mathbb{R}}_1$-manifold, denoted by $\underset{\sim}{\mathbb{R}}_1^+$);

c) S^1 (by Example 1, $S^1 := \left\{ (x_1,x_2) \in \mathbb{R}^2 \;\middle|\; x_1^2 + x_2^2 = 1 \right\}$ is a local Jordan $\underset{\sim}{\mathbb{R}}_1$-manifold);

d) $\overline{S^1}$ (Consider the covering map $p:\mathbb{R} \longrightarrow S^1$ defined by $p(t) = (\cos t, \sin t)$. Then by virtue of the considerations made on page 11, $\mathbb{R}$ possesses a unique Jordan structure such that p is a local isomorphism. This local Jordan $\underset{\sim}{\mathbb{R}}_1$-manifold will be denote by $\overline{S^1}$) ;

e) Let G_n be the group of rotations of S^1 generated by rotation over $\frac{2\pi}{n}$, (n= 1,2, ...). Then S^1/G_n is a local Jordan $\underset{\sim}{\mathbb{R}}_1$-manifold, topologically isomorphic to S^1.

5. Two-dimensional local Jordan manifolds: a) the torus $S^1 \times S^1$ is a local Jordan $\underset{\sim}{\mathbb{R}}_1 \oplus \underset{\sim}{\mathbb{R}}_1$-manifold ;

b) the sphere $S^2 := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1 \right\}$ is a local Jordan $\underset{\sim}{\mathbb{R}}_1(\mathbb{C})$-manifold (see Example 1 above and the examples in § 4 of JSA.I);

c) the real projective plane $\mathcal{P}(\mathbb{R}^3)$ is a local Jordan $\underset{\sim}{\mathbb{R}}_{1,2}(\mathbb{R})$-manifold (see Example 3 above and the examples in § 4 of JSA.I);

d) by Example 1 the cylinder $C := \left\{ (x_1,x_2,x_3) \in \mathbb{R}^3 \middle| x_1^2+x_2^2=1 \right\}$ is a local Jordan $\underset{\sim}{\mathbb{R}}_1 \otimes \underset{\sim}{\mathbb{R}}(\varepsilon)$-manifold, $\mathbb{R}(\varepsilon)$, $\varepsilon^2 = 0$, being the algebra of dual numbers.

By constructing the universal covering manifolds of the examples a), c) and d), the following local Jordan manifolds are obtained:

a') the plane $\mathbb{R}^2$ has a local Jordan $\underset{\sim}{\mathbb{R}}_1 \oplus \underset{\sim}{\mathbb{R}}_1$-manifold structure derived from the covering $p:\mathbb{R}^2 \longrightarrow S^1 \times S^1$, $p(s,t) := ((\cos s, \sin s), (\cos t, \sin t))$;

c') the sphere S^2 has a local Jordan $\underset{\sim}{\mathbb{R}}_{1,2}(\mathbb{R})$-manifold structure derived from the covering $p: S^2 \longrightarrow \mathcal{P}(\mathbb{R}^3)$, $p(x) := \bar{x}$;

d') the plane $\mathbb{R}^2$ has a local Jordan $\underset{\sim}{\mathbb{R}}_1 \otimes \mathbb{R}(\varepsilon)$-manifold structure derived from the covering $p:\mathbb{R}^2 \longrightarrow C$, $p(s,t) = (\cos s, \sin s, t)$.

By taking orbit spaces, the following local Jordan manifolds are obtained:

a'') the torus $S^1 \times S^1$ is a local Jordan V-manifold, V being any Jordan pair with two-dimensional V^+, since it can be identified with the orbits of a group of translations of V^+ (see page 11);

c'') the real projective plane is a local Jordan $\underset{\sim}{\mathbb{R}}_1(\mathbb{C})$--manifold, since it is the orbit space of the group $\left\{ Id, -Id \right\}$ acting on S^2 ;

d'') the cylinder C is a local Jordan V-manifol, V being a Jordan pair with two-dimensional V^+, since, like the torus, it is the orbit space of a group of translations acting on V^+.

Definition. Let V be a unital Jordan pair, and let $\eta = (\eta_+, \eta_-) \in \Gamma(V, V^{op})$ be an involution of V (i.e. $\eta_{-\sigma} \circ \eta_\sigma = Id_V$, $\sigma = \pm$). The set $\bigcirc_\sigma(V, \eta)$ of orthogonal elements in V^σ

is defined by $O_\sigma(V, \eta) := \left\{ x \in V^\sigma \,\middle|\, i_\sigma(x) = \eta_\sigma(x) \right\}$.

Remark. Note that i_σ is defined at x if and only if x is invertible.

Proposition 2.4. Let V and η be as in the definition above. Then $r \in O_\sigma(V, \eta)$ if and only if $r^{-1} \in O_{-\sigma}(V, \eta)$.

Definition. Let V be a unital Jordan pair, and let η be an involution of V. The tangent space of $O_\sigma(V, \eta)$ at the point $p \in O_\sigma(V, \eta)$ is $T_p O_\sigma(V, \eta) := T_p O_\sigma := \left\{ u \in V^\sigma \,\middle|\, di_\sigma(p;u) = \eta_\sigma(u) \right\}$.

Theorem 2.5. Let V be a unital Jordan pair and let η be an involution of V. Then $O_\sigma(V, \eta)$ is a local Jordan submanifold of V^σ, $\sigma = \pm$.

Examples. a) Affine quadrics. Let X, $\underline{X}$, and M be as in Example 1 above. The Jordan pair $\underline{X}$ is unital. Let $\eta_\sigma = Id_X$. Then $\eta = (\eta_+, \eta_-)$ is trivially an involution of $\underline{X}$. Obviously, $M = O_+(\underline{X}, \eta)$;

b) Orthogonal, unitary, and symplectic groups. Let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let X be a right vector space over $\mathbb{F}$ with a positive definite Hermitian form $\langle x, y \rangle$. Let $V^+ = V^-$ be the real associative algebra of $\mathbb{F}$-linear endomorphisms of X. Let $V = (V^+, V^-)$ be the real Jordan pair with $Q(u)v := uvu$, for $u \in V^\sigma$ and $v \in V^{-\sigma}$. Then V is isomorphic to $\underline{R}_n(\mathbb{F})$, where n is the $\mathbb{F}$-dimension of X (for $\underline{R}_n(\mathbb{F})$ and other standard Jordan pairs, see § 4, Examples of JSA.I). Moreover, V is unital and $i_\sigma(u) = u^{-1}$, where u^{-1} is the inverse of u in the associative algebra $V^\sigma = V^{-\sigma}$. We now define an involution $\eta = (\eta_+, \eta_-)$ of V by $\eta_\sigma(u) := u^*$, where u^* is the adjoint of u with respect to $\langle\ ,\ \rangle$. Then for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, $O_+(V, \eta)$ is the orthogonal, unitary, and symplectic group, respectively. For $n \geqslant 2$, $O_+(V, \eta)$ is a local Jordan $\underline{SH}_n(\mathbb{F})$-manifold.

c) Grassmann manifolds. Let $\mathbb{F}$, X be as in Example b) above. Let H be the real vector space of Hermitian $\mathbb{F}$-linear endomorphisms of X. Thus $u \in H$ if and only if $u = u^*$, where u^* is the adjoint of u. Let $\underline{H} := (H,H)$ be the Jordan pair with $Q(u)\,v := uvu$. Then $\underline{H}$ is a unital Jordan pair with $i_\sigma(u) := u^{-1}$, where u^{-1} is the inverse of u in the associative algebra of $\mathbb{F}$-linear endomorphisms of X. Note that $\underline{H}$ is isomorphic to $\underline{H}_n(\mathbb{F})$, where n is the $\mathbb{F}$-dimension of X. Define now an involution $\eta = (\eta_+, \eta_-)$ of $\underline{H}$ by $\eta_\sigma(x) := x$, $x \in H$. Given an $\mathbb{F}$-subspace Y of X, define $p(Y) = p \in \mathcal{O}_+(\underline{H}, \eta)$ by $p(x) := x$ for $x \in Y$ and $p(x) := -x$ for $x \in Y^\perp$. Then the map $Y \longrightarrow p(Y)$ defines a bijection between the $\mathbb{F}$-subspaces of X and $\mathcal{O}_+(\underline{H}, \eta)$. The components of $\mathcal{O}_+(\underline{H}, \eta)$ are the Grassmann manifolds of $\mathbb{F}$-subspaces of X of a fixed dimension.

Comments. At the OAGR Conference, Neptun-Romania, September 1980, Gel'fand gave a lecture on "Integral geometry", mentioning the results obtained by himself and some of his coworkers on transformations between Grassmann manifolds. In a discussion with Gel'fand after that lecture, IORDANESCU [17c] suggested him to reconsider these results in a Jordan algebra (pair) setting.

Remark. Let V be a unital Jordan pair, and let η be an involution of V. Then each component of the set of invertible elements in V^+ becomes a symmetric space in the sense of LOOS with the symmetry around the point p given by $p.x := Q_+(p)x^{-1}$ (see LOOS [24a] and, also [17b, Chapter II, § 2]). Furthermore, each component of $\mathcal{O}_+(V, \eta)$ is a symmetric space. The symmetric spaces arising in this manner have been studied by HELWIG (see [14], in which it is shown, in particular, that many symmetric spaces have this form ; see, also, [17b, Chapter II § 2]). Similarly, each component of $\mathcal{O}_-(V, \eta)$ is a symmetric space with $p.x := Q_-(p)x^{-1}$.

Comments 1. IORDANESCU's results on quaternionic Grassmann manifolds [17 a,e] can be developed in this more general setting. An open problem is to study the map Φ (see Proposition 3.25 in [17 e]) in terms of Jordan pairs.

Comments 2. Results reported by GEL'FAND and MacPHERSON on polyhedra in real Grassmann manifolds [13] could be reconsidered in a Jordan pair setting, as IORDANESCU [17 d] suggested to MacPherson in 1980.

Proposition 2.6. If V is a unital Jordan pair, and η is an involution of V, then the map $i_+ : \mathcal{O}_+(V,\eta) \longrightarrow \mathcal{O}_-(V,\eta)$ (see Proposition 2.4) induces LJM isomorphisms and symmetric space isomorphisms between corresponding components.

Definition. Let $(M,\hat{\mathcal{A}})$ be a local Jordan manifold and let $p \in M$. Denote by $F_p(M)$ (or F_p) the real algebra of germs at p of real C^∞ functions defined in a neighbourhood of p. The vector space of derivations of F_p is called the tangent space of M at p and is denoted by $T_p(M)$.

Remark. Let $\hat{\mathcal{A}}_p$ be the class of all Jordan charts $\varphi \in \hat{\mathcal{A}}$ with $p \in \mathrm{Dom}\,\varphi$. If $\varphi \in \hat{\mathcal{A}}_p$, $\varphi : U \longrightarrow V^+$, then φ induces a linear isomorphism $\varphi^*(p) : T_p(M) \longrightarrow V^+$, abbreviated as φ^*, and defined by $v(f) = d(f\varphi^{-1})(\varphi(p); \varphi^*(v))$, for $f \in F_p$ and $v \in T_p(M)$. If $\varphi_i \in \hat{\mathcal{A}}_p$, $\varphi_i : U_i \longrightarrow V_i^+$, $i = 1,2$, then $\varphi_2^* \varphi_1^{*-1} = d\mathcal{C}^\flat_{\varphi_2\varphi_1}(\varphi_1(p)) \in \Gamma_+(V_1,V_2)$. So $T_p(M)$ is a local Jordan manifold and $\{\varphi^* \mid \varphi \in \hat{\mathcal{A}}_p\}$ is contained in its Jordan structure. Moreover, $\varphi^* : T_p(M) \longrightarrow V^+$, for $\varphi \in \hat{\mathcal{A}}_p$, is a LJM isomorphism.

Definition. A map $\alpha : U \longrightarrow T_p(M)$, where U is open in the local Jordan manifold M, is called a tangent Jordan chart if $\alpha(U)$ is open in $T_p(M)$ and $\alpha : U \longrightarrow \alpha(U)$ is an LJM isomorphism. (Here U and $\alpha(U)$ are local Jordan submanifolds of M and $T_p(M)$, respectively.)

Remark. Let $\hat{a}$ be the Jordan structure of M. If $\varphi \in \hat{a}_p$, $\varphi : U \longrightarrow V^+$, then $\varphi^{*-1}\varphi : U \longrightarrow T_p(M)$ is a tangent Jordan chart.

Definition. Let $(M, \hat{a})$ be a local Jordan manifold. The set of tangent Jordan charts on M is called the tangent Jordan structure of M and is denoted by $\mathcal{T}(M, \hat{a})$ (or simply by $\mathcal{T}$).

Let $\alpha : U \longrightarrow T_p(M)$ be a tangent Jordan chart, and let $\varphi \in \hat{a}_p$. Then $\varphi^*(p)\alpha$ is a Jordan chart. Now let $\alpha_i : U_i \longrightarrow T_{p_i}(M)$ $i = 1,2$, be tangent Jordan charts with $U_1 \cap U_2 \neq \emptyset$. Then for any $u \in U_1 \cap U_2$, in a neighbourhood of $\alpha_1(u)$, $\alpha_2\alpha_1^{-1}$, the restriction of a unique rational map $C^u_{\alpha_2\alpha_1} : T_{p_1}(M) \longrightarrow T_{p_2}(M)$. The rational map $C^u_{\alpha_2\alpha_1}$ is called a coordinate transition map. The $C^u_{\alpha_2\alpha_1}$ have the same properties as those given in Remarks 1 and 2 on page 7 for the coordinate transition maps belonging to the Jordan structure.

Definition. A subset $\mathcal{S}$ of $\mathcal{T}$ such that the domains of the charts in $\mathcal{S}$ cover M is called a tangent Jordan atlas on M.

Let $f : M_1 \longrightarrow M_2$ be a morphism between local Jordan manifolds $(M_1, \hat{a}_1)$ and $(M_2, \hat{a}_2)$. Let $p_1 \in M_1$ and set $p_2 = f(p_1)$. Then f induces a map $df(p_1) : T_{p_1}(M_1) \longrightarrow T_{p_2}(M_2)$ defined by $\left[df(p_1)v\right](h) := v(hf)$, for $v \in T_{p_1}(M_1)$ and $h \in T_{p_2}(M_2)$.

For $\varphi_i \in \hat{a}_{i,p_i}$, $\varphi_i : U_i \longrightarrow V_i^+$, $i = 1,2$ we have

$$\varphi_2^*(p_2)\, df(p_1)\, \varphi_1^*(p_1)^{-1} = d(\varphi_2 f \varphi_1^{-1})(\varphi_1(p_1)) \in \Gamma_+(V_1, V_2).$$

Hence $df(p_1) : T_{p_1}(M_1) \longrightarrow T_{p_2}(M_2)$ is a morphism. The linear map $df(p_1)$ is called the differential of f at p_1.

§ 3. Jordan manifolds

In this section we shall focus on the important subcategory JM of LJM consisting of the global Jordan manifolds (or simply, the Jordan manifolds) (see WATSON [42 a, Chapter III]). Every + space of a Jordan pair V^+ can be embedded in a Jordan manifold $\Delta(V^+)$ such that every $f \in LF_+(V_1, V_2)$ admits an extension to a morphism $\Delta(f) : \Delta(V_1^+) \longrightarrow \Delta(V_2^+)$. The Jordan manifold $\Delta(V^+)$ is called the completion of V^+, and plays a central role in the theory of Jordan manifolds. In particular, the completion is used to examine the image of a morphism between Jordan manifolds. The completion has been constructed for various special cases by de BRUIJN [7], MAKAREVICH [25], ROZENFELD et al. [32], and for more general Jordan pairs by LOOS [24b]. For a detailed study of $\Delta(V^+)$ and linear fractional group $\sum_+(V)$ the reader is referred to WATSON [42a].

Notation and construction. Let M be a connected local Jordan V-manifold and let $\mathcal{A}$ be the maximal Jordan V-atlas on M. For $\alpha \in \mathcal{A}$, let $\mathrm{Dom}\,\alpha = U_\alpha$, and thus $\alpha : U_\alpha \longrightarrow V^+$. Let G(V) (or, briefly, G) be the group of automorphisms of V^+ in the quasi-category LF_+ (see JSA.V, § 1). Endow G with the discrete topology. For $\alpha, \beta \in \mathcal{A}$, there is a map $C_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow G$, defined by $u \longrightarrow C^u_{\beta\alpha}$ (note that $U_\alpha \cap U_\beta$ may be empty). If $\alpha, \beta, \gamma \in \mathcal{A}$, then $C^u_{\gamma\beta} \circ C^u_{\beta\alpha} = C^u_{\gamma\alpha}$, for all $u \in U_\alpha \cap U_\beta \cap U_\gamma$. The maps $C_{\beta\alpha}$, for $\alpha, \beta \in \mathcal{A}$, can therefore be used to define a principal fibre bundle with structure group G and base space M in which they are the bundle coordinate transformations (see STEENROD [38, § 3]).

Below we present WATSON's adapted version (see [42 a, p.817]) of STEENROD's construction [38, 3.2].

Let X be the set $M \times G \times \mathcal{A}$. Endow $\mathcal{A}$ with the discrete topology and X with the product topology. Consider $X' := \{(u,g,\alpha) \in X \mid u \in U_\alpha\}$ and endow X' with the subspace topology. We define an equivalence relation $\sim$ on X' by

$(u_1,g_1,\alpha_1) \sim (u_2,g_2,\alpha_2)$ if and only if $u_1 = u_2$ and

$$C^{u_2}_{\alpha_2\alpha_1} \circ g_1 = g_2 .$$

Let the equivalence class of $(u,g,\alpha) \in X'$ be denoted by $q(u,g,\alpha)$ and let $E(M,\mathcal{A})$ (or, briefly, E) be the set of equivalence classes. Thus, $q : X' \longrightarrow E$. The topology of E is determined by the requirement that $W \subset E$ is open if and only if $q^{-1}(W)$ is open in X'. Therefore, q is continuous. Now, define $p : E \longrightarrow M$ by $pq(u,g,\alpha) := u$.

Lemma 3.1. The map $p : E \longrightarrow M$ is a continuous surjection.

Lemma 3.2. Let $u \in U_\alpha, \alpha \in \mathcal{A}$. Then the fibre over u is

$$p^{-1}(u) = \{ q(u,g\alpha) \mid g \in G \}.$$

Lemma 3.3. The sets $q(U \times g \times \alpha)$, for $U \times g \times \alpha \subset X'$ and U open in M, form a basis for the topology of E. (Here $U \times g \times \alpha$ was written instead of $U \times \{g\} \times \{\alpha\}$.)

Lemma 3.4. E is a Hausdorff space.

Lemma 3.5. For $U \times g \times \alpha \subset X'$ and U open in M, $p : q(U \times g \times \alpha) \longrightarrow U$ is a homeomorphism. The open sets $q(U \times g \times \alpha)$, $g \in G$, are pairwise disjoint.

Lemma 3.6. For $h \in G$, define $R_h : E \longrightarrow E$ by $R_h q(u,g,\alpha) := q(u,g \circ h,\alpha)$. Then R_h is a homeomorphism of E and $p R_h = p$.

The Jordan structure of E. Since p is a local homeomorphism, E has a unique Jordan structure such that p is a local isomorphism (see Proposition 2.3., § 2). Since M is a local Jordan V-manifold, E is also a local Jordan V-manifold.

Lifting of curves. Let $\omega : [a,b] \longrightarrow M$ be a curve and let $x = q(\omega(a), g, \alpha)$ belong to the fibre over $\omega(a)$. Let $a=t_0 < t_1 < \ldots < t_n = b$ an increassing sequence in $[a,b]$, and $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ a sequence in $\mathcal{A}$, such that $\omega [t_i, t_{i+1}] \subset U_{\alpha_i}$ for $i = 0,1, \ldots, n-1$. Set $u_i = \omega(t_i)$, $i = 1, \ldots, n$. Define $\overline{\omega} : [a,b] \longrightarrow E$ by $\overline{\omega}(t) := q(\omega(t), g, \alpha_0)$ for $t \in [t_0, t_1]$ and $\overline{\omega}(t) := q(\omega(t), C^{u_i}_{\alpha_i \alpha_{i-1}} \circ \ldots \circ C^{u_1}_{\alpha_1 \alpha_0} \circ g, \alpha_i)$ for $t \in [t_i, t_{i+1}]$, $i = 1, \ldots, n-1$. Then $\overline{\omega}$ is well defined and continuous. It is the unique curve lying over ω ($p\,\overline{\omega} = \omega$) with initial point x.

Proposition 3.7. Let $x \in E$, and let E_x be the component of x in E. Then $p : E_x \longrightarrow M$ is a regular convering of M.

Remark. The Jordan structure of E_x arising from the covering $p : E_x \longrightarrow M$ is the same as that given to E_x as a local submanifold of E.

Definition. Let M be a connected local Jordan V-manifold. If the covering map $p : E_x \longrightarrow M$ is an injection, then M is called a global Jordan V-manifold, or simply a Jordan V-manifold.

Remark. By Lemmas 3.2. and 3.6, it follows that this concept is independent of the component of E chosen. Since p is a local isomorphism, M is a Jordan V-manifold if and only if $p : E_x \longrightarrow M$ is an isomoprhism. An arbitrary local Jordan V-manifold M is called a Jordan V-manifold if each component of M is a Jordan V-manifold.

Definition. Let $\mathcal{A}$ be a Jordan atlas on a topological space M. Then $\mathcal{A}$ is said to be transitive if whenever $\omega : [a,b] \longrightarrow M$

is a closed curve, $a = t_0 < t_1 < \dots < t_n = b$ is an increasing sequence in $[a,b]$, and $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ is a sequence of charts in $\mathcal{A}$, such that $\omega[t_i, t_{i+1}] \subset U_{\alpha_i}$, $i = 0,1, \dots, n-1$, we have $C^{u_n}_{\alpha_0\alpha_{n-1}} \circ C^{u_{n-1}}_{\alpha_{n-1}\alpha_{n-2}} \circ \dots \circ C^{u_1}_{\alpha_1\alpha_0} = \mathrm{Id}_{V^+_{\alpha_0}}$, where $u_i = \omega(t_i)$, $i = 1, \dots, n$.

Proposition 3.8. Let M be a connected local Jordan V-manifold with maximal Jordan V-atlas $\mathcal{A}$. Let $\mathcal{A}' \subset \hat{\mathcal{A}}$ be a Jordan atlas on M. Then M is a Jordan V-manifold if and only if $\mathcal{A}'$ is transitive.

Definition. If $(M, \hat{\mathcal{A}})$ is a connected local Jordan manifold then, by virtue of the considerations made on pp. 9-10, § 2, M is a local Jordan V-manifold for any Jordan pair V used in $\hat{\mathcal{A}}$. Now M is called a global Jordan manifold, or simply, a Jordan manifold, if M is a Jordan V-manifold.

Remark 1. By Proposition 3.8, M is a Jordan manifold if and only if $\mathcal{A}$ is transitive, in which case all Jordan atlases $\mathcal{A}_1 \subset \hat{\mathcal{A}}$ on M are transitive. An arbitrary local Jordan manifold M is called a Jordan manifold if every component of M is a Jordan manifold. It is easy to see that this is equivalent to M having a transitive Jordan atlas (contained in this Jordan structure).

Remark 2. Jordan manifolds possess many properties which local Jordan manifolds need not possess.

Proposition 3.9. Let M be a local Jordan manifold. If every component of M is simply connected, then M is a Jordan manifold.

Proposition 3.10. Let M be a Jordan manifold. If N is a local Jordan submanifold of M, then N is a Jordan manifold.

Definition. In the above situation, N is called, a Jordan submanifold of M.

Proposition 3.11. Let $(M, \hat{\mathcal{A}})$ be a local Jordan manifold. If the Jordan atlas $\mathcal{A}$ has the properties:

a) for every three charts $\alpha_1, \alpha_2, \alpha_3$ in $\mathcal{A}$, $\bigcap_{i=1}^{3} U_{\alpha_i} \neq \emptyset$;

b) the map $u \longrightarrow C^u_{\alpha_2 \alpha_1}$ is constant on $U_{\alpha_1} \cap U_{\alpha_2}$ for all $\alpha_1, \alpha_2 \in \mathcal{A}$,

then M is a Jordan manifold.

Proposition 3.12. If M_i, $i = 1,2$, are local Jordan manifolds, then $M_1 \times M_2$ is a Jordan manifold if and only if M_i, $i = 1,2$, are Jordan manifolds.

Notation. Let $\xi : X \longrightarrow M$ be a local isomorphism between local Jordan manifolds X and M. Let $\mathcal{A}$ be a Jordan atlas on X such that, for every $\varphi : U_\varphi \longrightarrow V^+_\varphi$ in $\mathcal{A}$, the map $\xi : U_\varphi \longrightarrow \xi(U_\varphi)$ is an isomorphism. For $\varphi \in \mathcal{A}$, construct the Jordan chart $\overline{\varphi} = \varphi \xi^{-1}$: $\xi(U_\varphi) \longrightarrow V^+_\varphi$ in the Jordan structure of M. Select a Jordan atlas $\mathcal{A}_1$ in M such that $\overline{\varphi} \in \mathcal{A}_1$ for all $\varphi \in \mathcal{A}$. Note that if $\varphi_1, \varphi_2 \in \mathcal{A}$, then $C^u_{\varphi_2 \varphi_1} = C^{\xi(u)}_{\overline{\varphi}_2 \overline{\varphi}_1}$ for all $u \in U_{\varphi_1} \cap U_{\varphi_2}$.

Proposition 3.13. Let M be a Jordan manifold, X a local Jordan manifold and $\xi : X \longrightarrow M$ a local isomorphism. Then X is a Jordan manifold.

Examples of Jordan manifolds. a) V^+, the + space of a Jordan pair V, evidently has a transitive atlas (and is also simply connected);

b) affine quadrics (see Example 1, § 2); since affine quadrics are local Jordan submanifolds of + spaces of Jordan pairs, they are Jordan manifolds;

c) projective quadrics (see Example 2, § 2); every component of a projective quadric has a Jordan atlas satisfying the conditions of Proposition 3.11, hence is a Jordan manifold;

d) projective space (see Example 3, § 2); as in c), real projective space has a Jordan atlas satisfying the conditions of Proposition 3.11, hence is a Jordan manifold;

e) one-dimensional Jordan manifolds; the local Jordan manifolds $\underset{\sim}{\mathbb{R}}^+_0$, $\underset{\sim}{\mathbb{R}}^+_1$, S^1, and $\overline{S^1}$ (see Example 4, §2) are all Jordan manifolds;

f) two-dimensional Jordan manifolds; the torus $S^1 \times S^1$, the sphere S^2, the projective plane $\mathcal{P}(\mathbb{R}^3)$, and the cylinder C (see Example 5 a), b), c) and d), respectively, § 2) are all Jordan manifolds, as are their universal covering manifolds;

g) sets of orthogonal elements; the set $O_\sigma(V, \eta)$ of orthogonal elements in V^σ (see the Definition on pp. 16-17, § 2) is a Jordan submanifold of V.

Comments. Taking into account Example d) above, it would be interesting to reconsider MARTINELLI's results [27 a, b] on quaternionic projective planes in this more general setting of Jordan pairs. An open problem would be to find an algebraic characterization of integrability of a geometric structure.

Definition. Let M be a connected local Jordan manifold. Then a transitive closure of M is a pair (C,p), where C is a connected Jordan manifold and $p : C \longrightarrow M$ is a local isomorphism, such that, whenever $\xi : X \longrightarrow M$ is a local isomorphism of a connected Jordan manifold X, there exists a local isomorphism $\mu : X \longrightarrow C$ such that $p\mu = \xi$.

Definition. Two transitive closures (C_i, p_i), $i = 1,2$, of M are called isomorphic if there exists an isomorphism $\varphi : C_1 \longrightarrow C_2$ such that $p_2 \varphi = p_1$.

Theorem 3.14. Every connected local Jordan manifold M has a unique (up to isomorphism) transitive closure.

Corollary. If (C,p) is the transitive closure of a connected local Jordan manifold M, then M is a Jordan manifold if and only if p is injective, in which case p is an isomorphism.

Examples. a) The sphere S^2 is a Jordan $\underset{\sim}{\mathbb{R}}_1(\mathbb{C})$-manifold (see Example 5 b, § 2 and Example f above), and the orbit space of the group $\{Id, -Id\}$ acting on S^2 leads to a covering $p : S^2 \longrightarrow \mathcal{P}(\mathbb{R}^3)$ (see example c'', § 2); then $\mathcal{P}(\mathbb{R}^3)$ is a local Jordan $\underset{\sim}{\mathbb{R}}_1(\mathbb{C})$-manifold and (S^2, p) is its transitive closure;

b) four examples of transitive closures of local Jordan manifolds homeomorphic to a torus are listed below:

i) the torus $S^1 \times S^1$ (see example 5 a, § 2) is a Jordan $\underset{\sim}{\mathbb{R}}_1 \oplus \underset{\sim}{\mathbb{R}}_1$-manifold, hence its own transitive closure;

ii) the cylinder C is a Jordan $\underset{\sim}{\mathbb{R}}_1 \otimes \mathbb{R}(\varepsilon)$-manifold (see example 5 d, § 2 and example f above), and the orbit space of the group of automorphisms of C generated by $T(x_1, x_2, x_3) := (x_1, x_2, x_3+1)$ leads to a covering $p : C \longrightarrow M$, where M is a local Jordan $\underset{\sim}{\mathbb{R}}_1 \otimes \mathbb{R}(\varepsilon)$-manifold homeomorphic to the torus; the pair (C, p) is the transitive closure of M;

iii) consider the Jordan $\underset{\sim}{\mathbb{R}}_1 \oplus \underset{\sim}{\mathbb{R}}_0$-manifold $S^1 \times \underset{\sim}{\mathbb{R}}_0^+$, and take the orbit space of the group of automorphisms of $S^1 \times \underset{\sim}{\mathbb{R}}_0^+$ generated by $T(x,y) := (x, y+1)$, $x \in S^1$, $y \in \underset{\sim}{\mathbb{R}}_0^+$; this produces a covering $p : S^1 \times \underset{\sim}{\mathbb{R}}_0^+ \longrightarrow M$, where M is a local Jordan $\underset{\sim}{\mathbb{R}}_1 \oplus \underset{\sim}{\mathbb{R}}_0$-manifold homeomorphic to the torus; the pair $(S^1 \times \underset{\sim}{\mathbb{R}}_0^+, p)$ is the transitive closure of M;

iv) let V be the radical Jordan pair $(\mathbb{R}^2, 0)$ and take the orbit space of the group of automorphisms of V^+ generated by $T_1(x_1, x_2) := (x_1+1, x_2)$ and $T_2(x_1, x_2) := (x_1, x_2+1)$; we then obtain a covering $p : V^+ \longrightarrow M$, where M is a local Jordan V-manifold homeomorphic to the torus; the pair (V^+, p) is the transitive closure of M.

Important remark. The Jordan manifolds, and the morphisms (in LJM) between them, form a subcategory of LJM, denoted by JM. The category JM is closed with respect to direct products, covering spaces and local Jordan submanifolds. Moreover, a local Jordan manifold is

covered by a Jordan manifold.

Definition. Let $V = (V^+, V^-)$ be a Jordan pair and define on $V^+ \times G(V)$ an equivalence relation $\sim$ by

$(x_1, \varphi_1) \sim (x_2, \varphi_2)$ if and only if $x_1 \in \mathrm{Dom}(\varphi_2 \circ \varphi_1^{-1})$ and $(\varphi_2 \circ \varphi_1^{-1})(x_1) = x_2$.

The equivalence class of $(x, \varphi) \in V^+ \times G(V)$ will be denoted by $[x|\varphi]$, the set of equivalence classes of $\sim$ will be called the completion of V^+ and will be denoted by $\Delta(V^+)$.

For $\varphi \in G(V)$, let $U_\varphi := \{ [x|\varphi] \mid x \in V^+ \}$ and define $\underline{\varphi} : U_\varphi \longrightarrow V^+$ by $\underline{\varphi}[x|\varphi] = x$. For $x \in V^+$ and $y \in V^-$, let $[x \mid y] := [x \mid T_+^-(y)]$. Thus, $[x_1 \mid y_1] = [x_2 \mid y_2]$ if and only if $(x, y_1 - y_2)$ is quasi-invertible and $x_1^{y_1 - y_2} = x$. Let $U_y := \{ [x \mid y] \mid x \in V^+ \}$.

Theorem 3.15. If $V = (V^+, V^-)$ is a Jordan pair, then the completion $\Delta(V^+)$ of V^+ has a unique Jordan structure such that all $\underline{\varphi} : U_\varphi \longrightarrow V^+$, $\varphi \in G(V)$, form a Jordan V-atlas on $\Delta(V^+)$ (contained in that Jordan structure). Moreover $\Delta(V^+)$ with this Jordan structure is a connected Jordan V-manifold.

The map $I_{V^+} : V^+ \longrightarrow \Delta(V^+)$ defined by $I_{V^+}(x) := [x \mid o]$ is an injective morphism, called the canonical embedding of V^+ into $\Delta(V^+)$.

Remark. Note that $I_{V^+} : V^+ \longrightarrow U_o$ is an isomorphism of V^+ onto the Jordan submanifold U_o of $\Delta(V^+)$. The inverse morphism is $\underline{Id}_{V^+}$.

The most important property of the completion is expressed by

Theorem 3.16. Let V and W be Jordan pairs and let $f \in LF_+(V, W)$. Then there exists a unique continuous map $\Delta(f)$:

: $\Delta(V^+) \longrightarrow \Delta(W^+)$ such that $\Delta(f)I_{V^+} = I_{W^+}f$ on Dom f.. Moreover, $\Delta(f)$ is a JM morphism.

An extension of Theorem 3.16 is

Theorem 3.17. Let V_1 and V_2 be Jordan pairs. Let U be a nonempty connected open subset of $\Delta(V_1^+)$ and let $f:U \longrightarrow \Delta(V_2^+)$ be a morphism (U is a Jordan submanifold of $\Delta(V_1^+)$). Then there exist a unique morphism $F : \Delta(V_1^+) \longrightarrow \Delta(V_2^+)$ such that $F\big|_U = f$.

The next theorem gives a relationship between Jordan manifolds and completions.

Theorem 3.18 .Let M be a connected local Jordan manifold. Then M is a Jordan manifold if and only if there exists a local morphism $\varsigma : M \longrightarrow \Delta(V^+)$.

By extending Theorem 3.17, the following fundamental properties of Jordan manifolds are obtained.

Theorem 3.19. Let M_i be Jordan manifolds and let $\varsigma_i : M_i \longrightarrow \Delta(V_i^+)$ be local isomorphisms, i= 1,2. Suppose that U is a nonempty connected open subset of M_1, and $f : U \longrightarrow M_2$ a morphism. Then there exists a unique morphism $f' : \Delta(V_1^+) \longrightarrow \Delta(V_2^+)$ such that $f'\varsigma_1 = \varsigma_2 f$ on U.

Theorem 3.20. Let M be a connected Jordan manifold and let U be a nonempty connected subset of M. If $f : U \longrightarrow \Delta(V^+)$ is a morphism, then there exists a unique morphism $F : M \longrightarrow \Delta(V^+)$ such that $F\big|_U = f$.

Proposition 3.21. Let M be a local Jordan manifold and let $p : \Delta(V^+) \longrightarrow M$ be a surjective local isomorphism. If ξ : : $X \longrightarrow M$ is a morphism on the connected Jordan manifold X, then there exists a morphism $\mu : X \longrightarrow \Delta(V^+)$ such that $p\mu = \xi$.

Corollary. Let M be a local Jordan manifold and let $p : \Delta(V^+) \longrightarrow M$ be a surjective local isomorphism. Then $(\Delta(V^+), p$

is the transitive closure of M and p is a regular covering map.

The main theorem about images of morphisms between Jordan manifolds is

Theorem 3.22. Let M_1 be a connected Jordan manifold and let M_2 be a Jordan manifold. If $f : M_1 \longrightarrow M_2$ is a morphism, then $N_2 := f(M_1)$ is a Jordan submanifold of M_2 and $f : M_1 \longrightarrow N_2$ is an open morphism.

Results concerning morphisms between local Jordan manifolds are

Proposition 3.23. Let M_1 and M_2 be local Jordan manifolds. If M_1 is connected and $f : M_1 \longrightarrow M_2$ is a morphism, then the rank of $df(p)$, $p \in M_1$, is constant.

Theorem 3.24. (Open mapping theorem.) Let M_1 and M_2 be local Jordan manifolds. If M_1 is connected and $f: M_1 \longrightarrow M_2$ is a surjective morphism, then f is open.

Three applications of the preceeding statements are

Theorem 3.25. Let M_1 be a connected local Jordan V-manifold where Rad $V^+ = 0$. If $f : M_1 \longrightarrow M_2$ is a bijective morphism onto a local Jordan manifold M_2, then f is an isomorphism.

Theorem 3.26. Let M_1 be a connected local Jordan V-manifold, where V is simple. If $f : M_1 \longrightarrow M_2$ is a surjective nonconstant morphism onto the local Jordan manifold M_2, then f is a local isomorphism.

Theorem 3.27. Let M_1 be a connected open Jordan submanifold of $\triangle(V^+)$, where V is a simple Jordan pair. If $f:M_1 \longrightarrow M_2$ is a nonconstant morphism into a Jordan manifold M_2, then $N_2 := f(M_1)$ is a Jordan submanifold of M_2, and $f:M_1 \longrightarrow M_2$ is an isomorphism.

A result concerning the fibres of a morphism is given by

Proposition 3.28. Let $f : M_1 \longrightarrow M_2$ be a morphism between local Jordan manifolds M_1 and M_2. Then $f^{-1}(u_2)$, $u_2 \in M_2$, is either

empty or a local Jordan submanifold of M_1.

Remark 1. The results presented above indicate that properties of the completion $\Delta(V^+)$ to a large extent determine properties of Jordan V-manifolds. For this reason WATSON undertook a detailed study of the completion $\Delta(V^+)$ and the linear fractional group $\sum_+(V)$, for which the reader is referred to WATSON's [42 a]. E.g., the interplay between the idempotents of a Jordan pair V, the topology of the completion $\Delta(V^+)$, and the one-dimensional Jordan submanifolds of $\Delta(V^+)$ is examined in [42 a, Chapter V].

Remark 2. For every nilpotent algebra A and local Jordan manifold M, WATSON [42 a, Chapter IV] has constructed a local Jordan manifold A(M), called the A-scalar bundle of M. The various jet bundles on M are realized as special cases of scalar bundles on M.

Remark 3. The properties of linear fractional maps needed for the theory of Jordan manifolds are easily derived by considering the group $\Delta\sum_+(V)$, acting on the completion of a quasi-invertible dense Jordan pair V (as recently defined and studied by WATSON [42 b]).

It is obvious that the Jordan manifolds defined by Watson have much in common with the theory of differentiable manifolds. (The main examples are differentiable manifolds.)

Open problem. To find an exact relationship between the category of Jordan manifolds and that of differentiable manifolds.

In 1951 VAGNER began (see [41 b,c]) a series of studies which led to a mathematical tool necessary to formulate and solve the following problem: To find the geometrical properties of differentiable manifolds which are derived only from algebraic properties of the pseudogroup of local homeomorphisms and the atlas.

By means of $[a,b,c] := c.b^{-1}.a$, where a,b,c are elements of an atlas, a ternary operation is defined and this led to the algebraic notion of heap (heath), a structure defined by a weakened set of a group postulates (see CLIFFORD [10], CERTAIN [8], SUSHKEVICH [39]).

This algebraic notion was used in differential geometry by VAGNER (see [41 b,d,e]). In [41 e] the following question has been raised: Do there exist properties of differentiable manifolds which depend only on the atlas ?

BÂNZEI [5 a] has answered this question affirmatively and has defined a kind of generalized manifolds, called $\mathcal{H}$-manifolds, which he studied in detail. A large number of examples have been given by him.

Open problem. To compare the category of Jordan manifolds with the category of $\mathcal{H}$-manifolds.

Remark. It would be of interest to re-consider the results on the above-mentioned ternary operation, as well as the results of BÂNZEI [5 b], in the theory of Jordan manifolds.

Let us mention now other categories whose objects are constructed in a similar way as differentiable manifolds. Among them we note topological manifolds, VAGNER's compound manifolds [41 a], ABRAHAM's [1] (C^r, C^s)-manifolds, SIKORSKI's [34] differentiable spaces, SMITH's [35] differentiable spaces, ARONSZAJN's [3], and MARSHALL's [26] C^∞-subcartesian spaces (for the differential topology of these spaces see [29 b]), and SPALIĖK's [36] N-differentiable spaces.

MOTREANU [29 a] (see also [29 c]) has constructed a category of so-called preringed manifolds, which contains as particular cases all categories mentioned above. Roughly speaking, a preringed manifold is a topological space M which is locally determined by a triple $(E, \mathcal{F}, V)$ where E is a topological space,

$\mathcal{F}$ is a presheaf of real-valued functions on E, and V is a vector space, two such triples being compatible with respect to change of charts. Here E describes locally the topology of M and V plays the role of tangent space to M. By a suitable choice of E, $\mathcal{F}$ and V one obtains the particular theories recalled above. So, the following problem is immediate:

Open problem. To describe the Jordan manifolds in terms of pre-ringed manifolds.

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