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JORDAN STRUCTURES WITH APPLICATIONS. VIII
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Radu IORDANESCU

As is well known, JORDAN [64 a] stressed that the most fruitful attempt at generalizing of the standard Hilbert space structure of quantum mechanics would be to change the algebraic structures (see also the more recent opinion expressed by DIRAC in [23]). JORDAN [64 b] formulated a quantum mechanics in terms of commutative, but not associative (finite-dimensional) algebras of observables, now called (finite-dimensional) Jordan algebras. JORDAN, von NEUMANN and WIGNER [65] showed that this approach is equivalent to the realisation of the standard quantum mechanics in finite-dimensional subspaces of the physical Hilbert space with the single exception of $H_3(0)^+$. (The infinite-dimensional case was studied by von NEUMANN in [87 b]). A more recent tentative axiomatization was given by EMCH [33 a] (see § 1).

Exceptional quantum mechanics was reinvestigated by GÜNAYDIN, PIRON and RUEGG [55] and shown to be in accordance with the standard propositional formulation, with a unique probability function for the Moufang (non-Desarguesian) plane (see § 2).

TRUINI and BIEDENHARN [116 b] constructed a quantum mechanics for the complexified octonion plane by using the Jordan paire technique (see § 3).

In the Jordan technique, by contrast to the Dirac q-number approach, commutativity is replaced by non-commutativity and non-associativity by associativity. Section 4 deals with PEDROZA and VIANNA's results [90] concerning the dynamical variables for constrained and unconstrained systems described by the symmetric formulation of classical mechanics. These results can be connected with the recent researches on supersymmetriy and supermanifolds of BEREZIN [11].

ARAKI [5] improved the characterization of state spaces of JB-algebras given by ALFSEN and SHULTZ [2 b] to a form with more physical appeal (proposed by WITTSTOCK [127]) in the simplified case of a finite dimension. JB-algebras were fruitfully used by W.GUZ [60 c] in a tentative axiomatization for nonrelativistic quantum mechanics and KUMMER [71] gave recently a new approach (see § 5).

Results on Jordan (quantum) logics due to MOROZOVA and CHENTSOV [83 a,b], and on order unit spaces arising from sum logics due to ABBATI and MANIÀ [1 a,b] are sketched in § 6. BUNCE and WRIGHT's results [19 a,b] are also referred therein.

The important role hat Jordan algebras should play in string theories is examined in § 7.

1. Special Jordan algebras in traditional axioamtic quantum mechanics

It is well known that there are three different formulations of quantum mechanics, namely, the Hilbert space formulation, the Jordan algebra formulation and the propositional calculus approach of von Neumann.

In the Hilbert space formulation the state of a physical system is represented by a ray. The rays of the Hilbert space are represented by projection operators in the Jordan formulation, and these in turn correspond to realizations of propositions (yes - no - experiments) in the von Neumann formulation.

The Hilbert space formulation is equivalent to the Jordan algebra formulation except in the case of the exceptional Jordan algebra for which there exists no Hilbert space formulation.

In the present section we shall review briefly EMCH's (tentative) axiomatization of quantum mechanics [33 a], which emphasizes the role played by special Jordan algebras.

Note. EMCH [33 a] stated two "postulates" and ten "structure axioms" (see below for these postulates and structure axioms). The two postulates sumarize the structure of traditional quantum mechanics, as it emerged from the work of von Neumann (1927-1932). The axioms developed by Emch from what he thought would be a more intuitive approach, namely the systematic exploitation of the constructive interplay between the structure of two sets: the observables and the states of a physical system. In particular, Emch wanted to avoid the artificiality of introducing Hilbert spaces too early in the theory, a point of view which everyone learned to appreciate upon reading SEGAL [102].

Postulate 1. To each observable A of a given physical system corresponds a linear selfadjoint operator $\pi(A)$ acting on a Hilbert space $\mathcal{H}_{\pi}$, and conversely.

Remark. Immediately after the above postulate, Emch made the following remark: "We should notice that the 'converse' part of the postulate is now known to be untenable (existence of "superselection rules"); since, however, von Neumann made a rather mild use of the second part of the postulate, we shall keep it on a temporary basis and naturally exclude it from a more definitive axiomatization". (EMCH [33 a, p.34]). According to this remark, we confine ourselves only to coherent Hilbert spaces and to the observables realised by selfadjoint bounded operators.

Convention. The set of all observables of the physical system considered will be denoted by $\mathcal{A}$.

Remark 1. If $A, B \in \mathcal{A}$, then $\pi(A)\pi(B)$ does not, in general, belong to $\pi(\mathcal{A})$, whereas $\pi(A)\,\pi(B) + \pi(B)\,\pi(A)$ does. Therefore the symmetrized product

$$\pi(A) \circ \pi(B) := \frac{1}{2}\,(\pi(A)\,\pi(B) + \pi(B)\,\pi(A))$$

belongs to $\pi(\mathcal{A})$ for any two observables A and B of $\mathcal{A}$. This product is commutative and bilinear. Its definition does not require the knowledge of the ordinary product of two noncompatible observables (i.e. two observables such that the corresponding operators do not commute in the ordinary sense). Indeed, we have

$$\pi(A) \circ \pi(B) = \frac{1}{4}\left[(\pi(A) + \pi(B))^2 - (\pi(A) - \pi(B))^2\right].$$

Remark 2. The above symmetrized product is, in general, not associative, i.e. the operator

$$\left\{\pi(A), \pi(B), \pi(C)\right\} := (\pi(A) \circ \pi(B)) \circ \pi(C) - \pi(A) \circ (\pi(B) \circ \pi(C))$$

can be different from zero. The important concept of compatibility can be expressed by using only the weakened notion of symmetrized product and not the ordinary product between operators.

Remark 3. The set $\pi(\mathcal{A})$ has the property that $\left\{\pi(A)^2, \pi(B), \pi(A)\right\} = 0$ for all A,B of $\mathcal{A}$, which is the characteristic axiom of Jordan algebras. Then the set $\pi(\mathcal{A})$, with addition, multiplication by reals, and symmetrized product is a real Jordan algebra.

The state of a physical system is understood intuitively as a way to express the simultaneous knowledge of the expectation values of all observable of the physical system considered.

Postulate 2. To each state ϕ of the physical system considered corresponds a "density matrix", $\rho = \pi(\phi)$, acting on the Hilbert space $\mathcal{H}_\pi$ of Postulate 1, and such that the expectation values $\langle\phi;\, A\rangle$ can be computed by the rule $\langle\phi;\, A\rangle := \mathrm{Tr}\,(\pi(\phi)\,\pi(A))$.

Remark. From this second postulate follow the four properties of states that are listed below:

a) for any $A_i \in \mathcal{A}$ and any $\lambda_i \in \mathbb{R}$ we have $\mathrm{Tr}\,\pi(\phi) \sum \lambda_i \pi(A_i) = \sum \lambda_i \mathrm{Tr}\,\pi(\phi)\pi(A_i) = \sum \lambda_i \langle \phi ; A_i \rangle$;

b) if $A \in \mathcal{A}$ then $\mathrm{Tr}\,\pi(\phi)\,\pi(A)^2 \geqslant 0$;

c) $\mathrm{Tr}\,\pi(\phi)\, I = 1$;

d) for any sequence $\{\phi_i\}$ of states and any positive real numbers λ_i with $\sum \lambda_i = 1$, $\sum \lambda_i \langle \phi_i ; A \rangle$ defines a state with properties a)-c) above.

Structure Axiom 1. To each physical system, one can associate the triple $(\mathcal{A}, S, \langle ; \rangle)$ formed by the set $\mathcal{A}$ al all its observables, the set S of all its states, and a mapping $\langle ; \rangle : (\mathcal{A}, S) \to \mathbb{R}$ which associates with each pair (A, ϕ) in $(\mathcal{A}, S)$ a real number $\langle \phi ; A \rangle$. This number is interpreted as the expectation value of the observable A when the system is in the state ϕ.

Remark. The elements $\phi \in S$ can be considered as mappings from $\mathcal{A}$ to $\mathbb{R}$, and conversely, and we can regard each element $A \in \mathcal{A}$ as a mapping $\langle \phi ; A \rangle : S \to \mathbb{R}$. Accordingly, if T is a subset of S, the restriction of this mapping to T will be denoted by A_T.

Convention. One says that $A|_T \leq B|_T$ whenever $\langle \phi ; A \rangle \leq \langle \phi ; B \rangle$ for all ϕ in T. In particular, one writes $A \leq B$ if the above inequality holds for all states on $\mathcal{A}$, and $A \geqslant 0$ if and only if $\langle \phi ; A \rangle \geqslant 0$ for all $\phi \in S$ so that by definition states are positive functions on $\mathcal{A}$.

Definition. A subset $T \subseteq S$ is said to be full with respect to a subset $\mathcal{B} \subseteq \mathcal{A}$ if the inequality $A|_T \leq B|_T$ between any two elements A and B of $\mathcal{B}$ implies that $A \leq B$.

Structure Axiom 2. The relation $\leq$ is a partial order relation on $\mathcal{A}$. In particular $A \leq B$ and $B \leq A$ implies that $A \equiv B$.

Structure Axiom 3. (i) There exist in $\mathcal{A}$ two elements 0 and I such that, for all $\phi \in S$, we have $\langle \phi ; 0 \rangle = 0$ and $\langle \phi ; I \rangle = 1$.

(ii) For each observable A in $\mathcal{A}$ and any real number λ there exists an element (λA) in $\mathcal{A}$ such that $\langle \phi ; \lambda A \rangle = \lambda \langle \phi ; A \rangle$ for all $\phi \in S$.

(iii) For any pair of observables A and B in $\mathcal{A}$ there exists an element (A+B) in $\mathcal{A}$ such that $\langle \phi ; A+B \rangle = \langle \phi ; A \rangle + \langle \phi ; B \rangle$ for all ϕ in S.

Remark. The set $\mathcal{A}$ of observables becomes a real vector space, whereas the states are real linear functionals on $\mathcal{A}$.

The state of a system is a way to characterise the method used for its preparation. The state then manifests itself to the observer when for each observable A be performs a sequence (in principle infinite, in practice large enough to reach a reasonable degree of confidence) of independent trial measurements on systems prepared in an identical way.

What the observer receives is a distribution of real numbers, whose "proper" average he calls the expectation value $\langle \phi ; A \rangle$ of the observable A in the state ϕ.

Definition. A state is called dispersion-free on the observable A when the above distribution is concentrated on a single number, namely $\langle \phi ; A \rangle$.

Notation. Denote by S_A the set of all dispersion-free states on A, and by σ_A the set of all values assumed by A on its dispersion-free states.

Remark. The notion of dispersion-free states is closely related to that of simultaneous observability. Two observables A and B can be simultaneously measured with arbitrary precision whenever the system is in a state ϕ in $S_A \cap S_B$.

Definition. A subset $T \subseteq S$ is said to be complete if it is full with respect to the subset $\mathcal{A}_T \subset \mathcal{A}$ defined by $\mathcal{A}_T := \{ A \in \mathcal{A} \mid S_A \supseteq T \}$.
A complete subset $T \subseteq S$ is said to be deterministic for a subset

$\mathcal{B} \subseteq \mathcal{A}$ whenever $\mathcal{B} \subseteq \mathcal{A}_T$. A subset $\mathcal{B} \subseteq \mathcal{A}$ is said to be <u>compatible</u> if set $S_{\mathcal{B}} := \bigcap_{B \in \mathcal{B}} S_B$ is complete.

<u>Structure Axiom 4</u>. For every observable A the set S_A is deterministic for the one-dimensional subspace of $\mathcal{A}$ generated by A; for any two observables A and B, $S_{A+B} \supseteq S_A \cap S_B$ and $S_I = S$.

<u>Structure Axiom 5</u>. For any element A in $\mathcal{A}$ and any nonnegative integer n, there exists in $\mathcal{A}$ at least one element, denoted by A^n, such that

(i) $S_{A^n} \supseteq S_A$;

(ii) $\langle \phi ; A^n \rangle = \langle \phi ; A \rangle^n$ for all ϕ in S_A.

<u>Definition</u>. With each pair A,B of elements in $\mathcal{A}$ one associates an element $A \circ B$ of $\mathcal{A}$, called the <u>symmetrized product of A and</u> B, defined by $A \circ B := \frac{1}{2}\left[(A+B)^2 - A^2 - B^2\right]$.

<u>Notation</u>. Denote by $\{A,B,C\} := (A \circ B) \circ C - A \circ (B \circ C)$ the <u>associator</u> of the observables $A,B,C \in \mathcal{A}$.

<u>Structure Axiom 6</u>. For any triple A,B,C of observables in which A and C are compatible, the associator $\{A,B,C\}$ vanishes.

<u>Remark</u>. Note that since all powers (in the sense of the Structure Axiom 5) of an observable are compatible, we have $\{A^n,B,A^m\} = 0$ for all $A,B \in \mathcal{A}$ and non negative integers n and m. In particular, $A \circ (B \circ A^2) = (A \circ B) \circ A^2$.

<u>Theorem 1.1</u>. The set $\mathcal{A}$ of all observables on a physical system is a Jordan algebra.

<u>Note</u>. As was noticed by EMCH [33 a, p.47] , there are "two significant (and actually related) differences" between the work of JORDAN, von NEUMANN, WIGNER [65] and his approach [33 a]:

"First, the notion of state does not appear explicity in [65], although it was certainly lying in the background of their investigation, as can be guessed, for instance, from the papers by JORDAN [64 b,c] and von NEUMANN [87 a,b]; in particular, von Neumann's discussion contains the germs of the sysmmetric postulation we are presenting. We indeed introduced the postulates in an inductive sequence, devised to emphasise the complementary roles played by the observables and the states... The relative emphasis given to observable over states, or conversely, varies from one extreme to the other in the various axiomatic schemes to be found in the literature; ... so does the approach followed by MACKEY [76], who starts with an axiomatization of the probability measure p which associates with the triples (A, ϕ, M) (formed by an observable A, a state ϕ, and a Borel subset M of $\mathbb{R}$), the probability p(A, ϕ, M) that the observable A will take a value in M when the system is in the state ϕ. PIRON [91] is primarly concerned with the structure of a certain class of observables, whereas states appear later; so is SEGAL [102] who, however, considers from the outset a much larger class of observables; at the opposite extreme we mention a paper by EDWARDS [32], who starts his account with an axiom on the structure of the set of all states on a physical system".

"The second difference between the axioms of Jordan, von Neumann, Wigner, and ours is that we do not restrict $\mathcal{A}$ to have a <u>finite</u> linear basis; this restriction is obviously a severe one and had to be dropped from any general theory, since it excludes any ordinary quantum theory formulated on an infinite-dimensional Hilbert space (e.g., the description of a quantum particle moving on a real line!). The mathematical simplification introduced by this restriction is that it allows analysis of properties of $\mathcal{A}$ without having to use any explicit topological notion. In particular, Jordan, von Neumann, and Wigner were able to prove from their postulates two important results that already indicate the power of this line of approach. The first of these results is that a spectral theory can be completely worked out, and the second is that a complete classification of all the realizations of their axioms can be given".

In the following we shall say only a few words about the other four axioms, because they are not involved in the Joran algebra structure of $\mathcal{A}$.

Structure Axiom 7 (see EMCH [33 a, p.53]) endows $\mathcal{A}$ with a real Banach space structure relative to a natural norm, and the states ϕ in S are continuous (positive linear) functionals on $\mathcal{A}$ with respect to the topology induced by this norm.

Remark. The Structure Axioms 1 to 7 endow the set $\mathcal{A}$ of observables with the structure of a SEGAL algebra [102]. There are some differences between SEGAL's axiomatization and that of EMCH, namely: EMCH gives more attention to the concept of dispersion-free state at an earlier stage of the formulation (which leads to the earlier introduction of the concept of compatibility between observables), and, secondly, whereas the product $A \circ B$ is distributive, SEGAL does not postulate distributivity in general.

Structure Axiom 8 (see EMCH [33 a, p.65]) is a sufficient condition for a subset of $\mathcal{A}$ to be compatible.

Remark. The necessity of the condition formulated in Axiom 8 follows from the preceding axioms, whereas the sufficiency (Axiom 8) is imposed by physical reasons.

Structure Axiom 9. $\mathcal{A}$ can be identified with the set of all selfadjoint elements of a real or complex, associative, and involutive algebra $\mathcal{R}$ satisfying the following two conditions:

(i) for each R in $\mathcal{R}$ there exists an element A in $\mathcal{A}$ such that $R^*R = A^2$;

(ii) $R^*R = 0$ implies that $R = 0$.

Structure Axiom 10. To each pair of observables A and B in $\mathcal{A}$ there corresponds an observable C in $\mathcal{A}$ which provides the actual lower bound to the simultaneous observability of A and B in the sense that

$$\langle \phi, (A - \langle \phi; A \rangle)^2 \rangle \langle \phi; (B - \langle \phi; B \rangle)^2 \rangle \geqslant \langle \phi; C \rangle^2, \forall \phi \in S.$$

Comments. As was mentioned at the beginning of this chapter, the first axiomatization of the measuring process leading to Jordan algebras was done by JORDAN [64 c] and JORDAN, von NEUMANN, and WIGNER [65]. They associated with each observable A another observable A^n which returns the n^{th} power of the value returned by A on ← each measurement. The existence of a sum A+B of observable A and B satisfying

$$E_x (A + B) = E_x(A) + E_x(B)$$

for all states x, where $E_x(A)$ is the expectation value of A on state x, was also assumed. With the assumption that A^n is given by an algebra structure, i.e., that

$$A \cdot B := \frac{1}{2}\left[(A + B)^2 - A^2 - B^2\right]$$

is bilinear, one is led to a power-associative algebra which is formally real. Finite-dimensional formally real power associative algebras were shown to be Jordan algebras. FAULKNER [36 a,b] gave an axiomatization in which he reverts to taking a function of an observable. However, he assumes neither the existence of A + B, the quadratic nature of A^2, nor finite dimensionality. The basic new element which Faulkner introduced is a change in the measurement process due to a change in the counting observable. Roughly, he shows the existence of A + B, by changing the counting observable to make A and B compatible. Also, the quadratic nature of A^2 is a consequence of his axioms.

As was pointed out by ZAITSEV in the introduction to the Russian edition of EMCH's book [33 b], algebraic methods (see also [16], [130]) offer necessary tools for the development of the modern quantum field theory and statistical physics (see the next section).

§ 2. The exceptional Jordan algebra and a generalization of quantum mechanics

In 1956, SHERMAN [105] studied the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$ within the framework of generalized quantum mechanics of SEGAL [102]. Then, the algebra $H_3(\mathbb{O})^{(+)}$ appeard in elementary particle physics through the work of GAMBA [42] on internal symmetries. GÜNAYDIN [52 a] proposed that $H_3(\mathbb{O})^{(+)}$ be used to represent the charge space of color quarks and strongly interacting particles. This proposal was later extended by GÜRSEY [58 b] to include leptons in the color singlet sector of the charge space and was used by him to motivate some grand unified theories based on exceptional groups (see GÜRSEY, RAMOND, SIKIVIE [59]).

GÜNAYDIN [52 b] gave exceptional realizations of Lorentz group *via* the algebra $H_3(\mathbb{O})^{(+)}$.

In this section we outline the "exceptional" quantum mechanics corresponding to the exceptional Jordan algebra considered by GÜRSEY [58 c,d] and more recently by GÜNAYDIN, PIRON and RUEGG [55].

The axioms of quantum mechanics as formulated by BIRKHOFF and von NEUMANN are equivalent to the axioms of projective geometry (see, for instance, VARADARAJAN [120, Vol.I]). Propositional calculus can then be interpreted as projective geometry with propositions corresponding to the points in the projective space. The pro ective geometry connected with the exceptional quantum mechanics is the non-Desarguesian projective plane first considered by MOUFANG [84] and the lattice of propositions is the orthocomplemented lattice whose existence was posed as an open problem by BIRKHOFF [14, p.124, Problem 56: "Which non-Desarguesian plane projective geometries admit orthocomplements ? "].

Note. The reader is referred to PIRON [91 b, Chapter 3] for an excellent treatment of the following problem: Given an irreducible proposition system, construct a projective geometry and canonically define an embedding of the lattice in the linear varieties. After answering this problem, Piron obtained the traditional Hilbert realisation of quantum mechanics.

A conventional quantum mechanical state is represented either by the ket $|\alpha\rangle \in \mathbb{C}^n$ or by the projection operator $P_\alpha := |\alpha\rangle\langle\alpha|$, where $\langle\alpha|$ is the transposed and complex conjugate of $|\alpha\rangle$, and n is the dimension of a complex Hilbert subspace of a Hilbert physical system.

Definition. The probability for a transition $\alpha \to \beta$ is denoted by $\Gamma_{\alpha\beta}$ and is given by

$$(2.1) \quad \Gamma_{\alpha\beta} := \mathrm{Tr}(|\alpha\rangle\langle\alpha||\beta\rangle\langle\beta|) = \mathrm{Tr}\,(P_\alpha P_\beta) = \mathrm{Tr}\,(P_\alpha \circ P_\beta),$$

where $P_\alpha \circ P_\beta := \frac{1}{2}(P_\alpha P_\beta + P_\beta P_\alpha)$.

For normalized states we have: $\mathrm{Tr}\, P_\alpha = 1$, $\mathrm{Tr}\, P_\beta = 1$, $P_\alpha^2 = P_\alpha$, $P_\beta^2 = P_\beta$.

Remark 1. Using (2.1) and the above relations, we can write

$$d^2_{\alpha\beta} := \frac{1}{2}\,\mathrm{Tr}\,(P_\alpha - P_\beta)^2 = 1 - \Gamma_{\alpha\beta} ,$$

and thus

$$\Gamma_{\alpha\beta} = 1 - d^2_{\alpha\beta} .$$

Remark 1. In a projective geometry in which the normalized idempotent P_α is represented by a point with n homogeneous coordinates and n-1 inhomogeneous coordinates, $d_{\alpha\beta}$ would be the invariant distance between the points α and β.

One can see that $d_{\alpha\beta}$ is related in a simple way to the transition probability $\Gamma_{\alpha\beta}$. When $\alpha = \beta$, their relative distance vanishes, while the transition probability equals one. Some subgroup H of U(n) (in this case $H = U(n-1) \times U(1)$) leaves the idempotent P_α invariant. This is called the stability group of the corresponding projective geometry leaving the point unchanged. The transformation of the coset $U(n)/_H$ will change the point, hence the state. The set of all transformed states is the complex projective space $P_{n-1}(\mathbb{C})$. The quantum mechanical (state) space $P_\infty(\mathbb{C})$ is obtained for $n \to \infty$.

Remark 1. In this quantum mechanical space, the transition probability can be determined from the invariant distance between two separate points.

Remark 2. As can be easily proved, in this algebraic formulation of quantum mechanics only Hermitian matrices and their Jordan product are involved (see also Theorem 1.1, § 1).

Important remark. It follows that if one can generalise Hermitian matrices, projections operators, and Jordan product to the case in which complex numbers are replaced by octonions, then a generalized quantum mechanics becomes possible. This is exactly what JORDAN, von NEUMANN, and WIGNER did by introducing the (3×3) - matrices, Hermitian with respect to octonionic conjugation [65].

A one-dimensional projection operator P of $H_3(\mathbb{O})^{(+)}$ is a matrix of $H_3(\mathbb{O})$ satisfying

$$P^2 := P \circ P = P, \qquad \mathrm{Tr}\, P = 1.$$

It is also referred to as an irreducible idempotent. According to a theorem of JORDAN [64 d], any irreducible idempotent P can be brought to the form

$$P = \begin{pmatrix} a \\ b \\ c \end{pmatrix} (\bar{a}\ \bar{b}\ \bar{c}) = \begin{pmatrix} a\bar{a} & a\bar{b} & a\bar{c} \\ b\bar{a} & b\bar{b} & b\bar{c} \\ c\bar{a} & c\bar{b} & c\bar{c} \end{pmatrix}.$$

where a,b,c are octonions, one of them being pure real, and satisfying

$$\mathrm{Tr}\ P = a\bar{a} + b\bar{b} + c\bar{c} = 1 .$$

The crucial properties needed in the sequel are the following:

Proposition 2.1. There exists a transformation belonging to the exceptional Lie group F_4 which brings an irreductible idempotent given by

$$P = \begin{pmatrix} a \\ b \\ c \end{pmatrix} (\bar{a}\ \bar{b}\ \bar{c}) = \begin{pmatrix} a\bar{a} & a\bar{b} & a\bar{c} \\ b\bar{a} & b\bar{b} & b\bar{c} \\ c\bar{a} & c\bar{b} & c\bar{c} \end{pmatrix}, \ \mathrm{Tr}\ P = 1,$$

to the form E_1[1].

Remark. The exceptional Lie group F_4, which is the automorphism group of $H_3(\mathbb{O})^{(+)}$, can also be uniquely characterized as the simultaneous invariance group of bilinear and trilinear forms of $H_3(\mathbb{O})^{(+)}$, i.e.

$$(J_1, J_2) := \mathrm{Tr}\ (J_1 \circ J_2)$$

and

$$(J_1, J_2, J_3) := \mathrm{Tr}\ (J_1 \times J_2) \circ J_3),$$

where

$$J_1 \times J_2 := J_1 \circ J_2 - \tfrac{1}{2} J_1\ \mathrm{Tr}\ J_2 - \tfrac{1}{2} J_2\ \mathrm{Tr}\ J_1 +$$

$$+\tfrac{1}{2}(\mathrm{Tr}\ J_1\ \mathrm{Tr}\ J_2 - \mathrm{Tr}\ (J_1 \circ J_2))\ \mathrm{Id}_3,$$

Id_3 being the (3×3)-identity matrix.

Proposition 2.2. There mutually orthogonal irreducible idempotents P_1, P_2, P_3 (i.e. $P_1 \circ P_2 = P_2 \circ P_3 = P_1 \circ P_3 = 0$), can

1) By E_i $(i=1,2,3)$, we denote the (3×3)-matrix with 1 on the i-th intersection of the row and i-k column, all other elements being zero.

be simultaneously brought to the form E_1, E_2, E_3 by a transformation of F_4.

Proposition 2.3. For any two irreducible idempotent P_1 and P_2 there exists a transformation of F_4 bringing them to real form.

Proposition 2.4. For any two irreducible idempotents P_1 and P_2, $Tr(P_1 \circ P_2) = 0$ implies that $P_1 \circ P_2 = 0$.

Proposition 2.5. Let P_1 and P_2 be two irreducible idempotents. An irreducible idempotent P_3 satisfies $P_1 \circ P_3 = P_2 \circ P_3 = 0$ if and only if P_3 is a multiple of $P_1 \times P_2$.

Proposition 2.6. Any element of $H_3(\mathbb{O})^{(+)}$ can be brought to diagonal form by a transformation of F_4.

As points of the projective plane we take equivalence classes of elements $P \in H_3(\mathbb{O})^{(+)}$ which satisfy the condition $P \times P = 0$. Recall that the condition $P \times P = 0$ implies that P is a scalar multiple of a primitive idempotent. If P and λP are real and nonzero, then they denote the same point. As a representative of an equivalence class one takes the P with $Tr\, P = 1$.

A line ℓ in the projective plane is represented by a two-dimensional projection, i.e.

$$\ell^2 := \ell \circ \ell = \ell \quad \text{and } Tr\, \ell = 2.$$

A point P is said to be contained in the line ℓ if and only if $P \circ \ell = P$.

Remark. The projective plane thus obtained is the Moufang plane, which is non-Desarguesian.

If two points P_1 and P_2 of the Moufang plane are given, then the point P corresponding to the superposition of P_1 and P_2 will all lie on the line passing through P_1 and P_2. The most general superposition P is given by the solution of the equation

$$P \circ (P_1 \times P_2) = 0.$$

As is known, the Moufang plane admits an orthocomplementation (a polarity which maps linear varieties into linear varieties and which reserves the order of inclusion). There exists a unique probability function, as was shown by GÜNAYDIN, PIRON and RUEGG [55].

GÜNAYDIN [52 c] generalized the octonionic quantum mechanics of Günaydin, Piron and Ruegg to the quadratic Jordan formulation, which extends without modification.

Remark 1. More general quantum mechanical spaces have been considered by GÜRSEY [58 d]. E.g., he considered the symmetric space $F_4/SO(9)$, which corresponds to the Moufang plane, and associates the distance function over the space $F_4/SO(9)$ with the probability function over the Moufang plane. So, he could immediately give examples of larger spaces, namely $E_6/SO(10) \times SO(2)$ and $E_7/E_6 \times SO(2)$, whose distance functions satisfy properties like that of $F_4/SO(9)$. The space $E_6/SO(10) \times SO(2)$, which is not a projective space (for instance, two lines can intersect at more than one point), is obtained from the Moufang plane by complexification (the exceptional Jordan algebra is considered over complex octonions). TRUINI, OLIVIERI and BIEDENHARN [117 b] studied the coset spaces $E_7/E_6 \otimes U(1)$ and $E_6/SO(10) \otimes U(1)$ by using the Jordan pairs of the corresponding Lie algebras. These coset spaces appeared as manifolds in the model of supergravity due to GÜNAYDIN, SIERRA and TOWNSEND [56]. (For further developments see the next section).

Remark 2. Another way to approach this problem has been suggested to GÜNAYDIN [52 c] by Faulkner. It consists of the use of an isotope of the Jordan algebra in which nilpotemt elements of the original Jordan algebra are represented by idempotent elements. In this connection, GÜNAYDIN [52 c] indicated how all measurable quantities of quantum mechanics (expectation values, transition probabilities) can be expressed in terms of a quadratic mapping. This holds for special as well as exceptional Jordan algebras.

Comments. If space-time degrees of freedom over the charge space corresponding to the Moufang plane are introduced, then we must go to a higher-dimensional projective geometry. On the other hand, it is known that Desargues'theorem holds in all projective spaces of dimension exceeding two. Therefore, the Moufang plane cannot be embedded in a higher-dimensional projective space. The method should be to embed this charge space into a higher-dimensional nonprojective geometry and then try to interpret this geometry in "projective" language. In these higher-dimensional geometries there will be new relationships between points, which may be connected with unusual properties of quarks. For the study of these higher-dimensional geometries, associated with the complex (quaternionic or octonionic) octonion plane, BIEDENHARN and HORWITZ [13] suggested the use of the structure group and Jordan pairs.

Finally, let us refer to other three papers of interest, as follows:

DOMOKOS and KÖVESI-DOMOKOS [29] outlined a quantum theory of quarks and gluons based on fields with values in a noncommutative Jordan algebra.

In [89 a], OKUBO starts from the remark that the so-called Dirac problem (which is concerned with quantization of classical mechanical system) for mechanics on the real line has no solution if we insist on a correspondence principle for all observables and on an associative algebra of quantum mechanical operations.

It has been shown, however, that the algebraic difficulties can be overcome if the associativity condition is relaxed, and that this leads to a flexible, Lie-admissible, noncommutative Jordan algebra A of operators. The algebraic structure of A is compatible with the Heisenberg equations of motion.

Recently, TILGNER [113 a] defined electromagnetic curvature structures (c.s.) as being bilinear in the two electromagnetic field matrices and electrovac c.s. by having the electromagnetic energy-momentum as Einstein tensor. It is shown that electromagnetic implies gravitational radiation, and conversely, that electromagnetic gravitational radiation is induced by electromagnetic radiation. A structure theory of c.s. is then described. More emphasis is laid on structures which are defined by semisimple Lie and Jordan algebras with respect to their standard bilinear forms.

Comments. As was noticed by TILGNER [113 b], it seems that the question of what a "physical" energy-momentum is, can be formulated in terms of the ω-domain or domain positivity of the Jordan algebra of Lorentz -selfadjoint (4×4)-matrices.

§ 3. Jordan pairs in quantum mechanics

In this section, we shall deal with the construction, due to TRUINI and BIEDENHARN [116 b], of a quantum mechanics for the complexified octonionic plane. This plane, as they showed (see [116 b, p.1337]), has automotphism group large enough to accomodate - as finite-dimensional quantum-mechanical charge spaces - a color-flavor structure which is not ruled out by current experimental evidence. The Truini-Biedenharn construction makes essential use of Jordan pairs. The construction of a quantum-mechanics over a complex octonion plane was begun by GÜRSEY [58 e,f] , without, however, using the concepts of linear ideals or Jordan pairs.

Let $V := (J,J)$ be the Jordan pair obtained by doubling the Jordan algebra J of (3×3) - Hermitian matrices over complex octonions. Hermiticity is considered only with respect to the octonionic conjugation, and the Jordan product is the symmetrized product

$$xy := \frac{1}{2}(x\cdot y + y\cdot x), \quad x,y\in J,$$

where the dot denotes the ordinary matrix product. Truini and Biedenharn considered these structures from the quadratic point of view, so that the quadratic and trilinear operators defining the pair structure are

$$U_{x^\sigma} y^{-\sigma} := Tr(x^\sigma, y^{-\sigma})x^\sigma - x^{\sigma\#} \times y^{-\sigma}$$

$$V_{x^\sigma, y^{-\sigma}} z^\sigma := (U_{x^\sigma + z^\sigma} - U_{x^\sigma} - U_{z^\sigma})y^{-\sigma},$$

where $\sigma = \pm$, and

$$Tr(x,y) := Tr(x\,y)$$

$$x^{\#} := x^2 - x\,Tr(x) - \frac{1}{2}\,Id_3(Tr(x^2) - (Tr x)^2)$$

$$x \times y := (x+y)^{\#} - x^{\#} - y^{\#},$$

Id_3 being the identity in J.

Note. In what follows, $V = (J,J)$ will be denoted as $V = (V^+, V^-)$.

Definition. An idempotent (x^+, x^-) of V is called a primitive normalized idempotent if $Tr(x^+, x^{+*}) = Tr(x^-, x^{-*}) = 1$, where x^* denotes the complex conjugate of x.

Convention. Putting in the Peirce decomposition of V (see JSA.1., § 4) $(V_i^+, V_i^-) := V_i$, $i=0,1/2,1$, we can formally write $V = V_0 \oplus V_1 \oplus V_{1/2}$.

Proposition 3.1. Let $V = V_o \oplus V_1 \oplus V_{1/2}$ be the Peirce decomposition of V with respect to a primitive normalized idempotent. Then V_1^σ and V_o^σ are principal inner ideals.

Consider now a primitive normalized idempotent $x = (x^+, x^-)$ in V. One can associate to it: (i) a point $x_{\star} := V_1(x)$, and (ii) a line $x^{\star} := V_o(x)$.

Definition. A Truini-Biedenharn plane $\mathcal{P}(J)$ consists of points $x_\star$ and lines $x^\star$ under the following relations

a) $x_\star | y^\star$, $x_\star$ incident to $y^\star$, if $V_1(x) \subset V_o(y)$;

b) $x_\star \cong y^\star$, $x_\star$ connected to $y^\star$, if $V_1(x) \subset V_o(x) \oplus V_{1/2}(y)$;

c) $x_\star \cong y_\star$, $x_\star$ connected to $y_\star$, if $V_1(x) \subset V_1(y) \oplus V_{1/2}(y)$;

d) $x^\star \cong y^\star$, $x^\star$ connected to $y^\star$, if $V_1(x) \subset V_1(y) \oplus V_{1/2}(y)$.

Proposition 3.2. Let $z = (z^+, z^-)$ and $x = (x^+, x^-)$ be two primitive normalized idempotents. Then $z^+ \in V_i^+(x)$ if and only if $z^- \in V_i^-(x)$.

Remark. By virtue of Proposition 3.2, one can choose as representative of points and lines just the elements of J which generate V_1^+ and V_o^+.

TRUINI and BIEDENHARN [116 b] considered the subgroup G of the structure group of J (which is isomorphic to the automorphism group of V) which maps primitive normalized idempotents into themselves.

Note. In this section we shall use TRUINI and BIEDENHARN's [116 b] notation (see also FREUDENTHAL [40]) for the real forms of a Lie algebra H of type G_2, F_4, E_6, E_7. When no further index, other than the one specifying the rank of the group is written for H, it is meant that H is complex. The compact real form of H is denoted by $H_{r,o}$ (i.e. $E_{6,o}$, $F_{4,o}$, ...). The group $H_{r,o}$ has signature (the Cartan index) equal to minus the number of generators of H. The Lie group associated with a certain Lie algebra will be denoted by the same, but script, letter.

The structure group of J has 79 generators, coming from the three-grading of the (complex) Lie algebra of $\mathcal{E}_7$. Among these 79 generators, 78 form the complex Lie algebra E_6 and the 79^{th} is just a change of scale. When we consider the subgroup G of the structure group of J we get from this last generator the (compact real form) U(1), and from $\mathcal{E}_6$ a subgroup H such that, for any x in J,

$$(g_+(x))^* = g_-(x^*) \tag{3.1}$$

for every g_+, $g_- \in H$, where $(g_+, g_-) \in \mathrm{Aut}\ (V)$.

Proposition 3.3. Equation (3.1) holds for every primitive normalized idempotent (x,x^*) if and only if

$$\mathrm{Tr}(x,x^*) = \mathrm{Tr}(g_+(x),\ g_+(x)^*). \tag{3.2}$$

Remark. From (3.2) it follows that H must be compact.

If we denote by G_+ and G_- the generators of g_+ and g_-, respectively, we get

$$\mathrm{Tr}(G_+(x),\ x^*) = -\ \mathrm{Tr}(x,(G_+(x))^*),$$

which implies that the Lie algebra of H is real.

Therefore $H = \mathcal{E}_{6,o}$ (see FREUDENTHAL [40]), the compact subgroup of $\mathcal{E}_6$ generated by the compact real form $E_{6,o}$ of E_6.

Remark (see [116 b, p.1336]). The compact real form $E_{6,o}$ must not be taken for the Lie algebra of the structure group of the real exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$, which is also the collineation group of the Moufang plane (see FREUDENTHAL [40]). The latter is the noncompact form E_{6,o^*} of signature - 26. $\mathcal{E}_{6,o^*}$ has only real representations.

Proposition 3.4. If $g \in \mathcal{E}_{6,o} \otimes U(1)$ then $x \in V_i(e)$ implies that $g(x) \in V_i(g(e))$, $i=0,1/2,1$, for x,e primitive normalized idempotents.

Definition. Define the natural action of $\mathcal{E}_{6,o} \otimes U(1)$ on the points of $\mathcal{P}(J)$ by $g(x_{\star}) := (g(x))_{\star}$.

Remark. From Proposition 3.3 and the previous considerations it follows that $\mathcal{E}_{6,o} \otimes U(1)$ preserves the relations in the geometry of $\mathcal{P}(J)$.

Definition. Two points $x_{\star}$ and $y_{\star}$ are called orthogonal if $x \in V_o(y)$.

Proposition 3.5. $\mathcal{E}_{6,o}$ acts transitively on points and on triples of mutually orthogonal points. The maximal subgroup of $\mathcal{E}_{6,o}$ leaving a point invariant is $SO(10) \otimes \widetilde{U}(1)$; therefore, the plane $\mathcal{P}(J)$ is the homogeneous space $\mathcal{E}_{6,o}/SO(10) \otimes U(1)$ (the tilde ever U(1) was used to distinguish it as a subgroup of $\mathcal{E}_{6,o}$ from the "overall" phase group U(1), which is obviously outside $\mathcal{E}_{6,o}$).

Note. The definitions of collineations, correlations, dualities, and polarities are the same as those given in JSA II, § 1. For the definition of an isotropic point, see JSA II., § 2.

Definition. A polarity with respect to which no point is isotropic is called an orthocomplementation.

Important remark. It can be immediately seen that the standard polarity $\pi_o : x_{\star} \longleftrightarrow x^{\star}$ is an orthocomplementation. This is a fundamental result, used by TRUINI and BIEDENHARN [116 b] in building a quantum theory on the geometry of $\mathcal{P}(J)$. The orthocomplementation is indeed needed in defining both the propositional system and the states of the quantum logic.

As was proved by FREUDENTHAL [40], an orthocomplementation is an "elliptic" polarity defining an "elliptic geometry".

SPRINGER and VELDKAMP [109] investigated planes, called Hjelmslev-Moufang planes, defined over an exceptional central simple Jordan algebra on a split Cayley algebra, and therefore including the complex algebra J considered by TRUINI and BIEDENHARN [116 b].

Remark 1. The Hjelmslev-Moufang plane, obtained by complexifying the real Jordan algebra $H_3(\mathbb{O})^{(+)}$, although similar to $\mathcal{P}(J)$, is defined over a hyperbolic polarity (i.e. a polarity admitting isotropic points). Therefore, it is difficult to give it a quantum mechanical interpretation.

Remark 2. The Truini-Biedenharn plane $\mathcal{P}(J)$ has a nonprojective geometry (tqo lines may intersect in more than one point) and, consequently, the propositions system is not a lattice.

Comments. There is a very close relationship between $\mathcal{P}(J)$ and the Hjelmslev-Moufang plane. The objects, points, and lines are essentially the same. However, a difference shows up when we consider transformations on points and lines. We have much more structure to preserve, namely, the pairing of a rank one element with its complex conjugate. This is reflected in the preservation of the standard polarity $\pi_0 : x_\star \longrightarrow x^\star$, and which is preserved by the group mapping points into points. In other words we can say that $\mathcal{P}(J)$ is a Hjelmslev-Moufang plane carrying a further structure to be preserved: the standard (elliptic) polarity π_0.

As is well known, the language of quantum mechanics has always been identified with the language of projective geometry, the points of the geometry being identified with the density matrices of the (pure) states, and the lines and hyperplanes with the propositions which are not atoms. The automorphism group of the geometry (that is, its collineation group) is, however, larger than the automorphism group of the quantum structure, because collineations need not preserve the traces (which are the canonical measures defining the quantum states) nor orthogonality, which has no projective meaning. In mathematical language we can say that the quantum logic requires an automorphism group which preserves an elliptic polarity. For instance, the automorphism group of the quantum system of GÜNAYDIN, PIRON and RUEGG [55] (see also § 2) for the Moufang plane is $\mathcal{F}_{4,o}$, whereas the collineation group of the plane itself is $\mathcal{E}_{6,o*}$, which contains $\mathcal{F}_{4,o}$ as maximal compact subgroup. For the same reason, TRUINI and BIEDENHARN [116 b] do not investigate the collineation group of $\mathcal{P}_{(J)}$; they have determine the group which is needed in describing automorphisms of the quantum system. This is the compact group $\mathcal{E}_{6,o} \otimes U(1)$,which preserves the trace Tr (x;x*) and the orthocomplementation (i.e. the elliptic polarity π_o).

TRUINI and BIEDENHARN [116 b] defined the propositional system as follows: the propositions are identified with the geometrical objects (points and lines correspond to the principal inner ideals ov V). They form a partially ordered set, with ordering given by the set inclusion of the inner ideals. The plane itself (i.e., the principal inner ideal generated by an invertible element) is the trivial proposition. We have an orthocomplementation $a \rightarrow a^{\perp}$, which is the standard polarity $a_{\star} \rightarrow a^{\star}$. Thus we can define orthogonality: $a \perp b$ if $a < b^{\perp}$, which is symmetric.

Comments (see [116 b, p.1338]). The only concepts of the standard theory which are weakened are the concepts of greatest lower bound ("meet") and least upper bound ("join"). They are not define here for every pair of propositions. Therefore we do not have a lattice structure. But the subsets consisting of nonconnected points and lines are sublattices of the partially ordered set.

Remark. As is well known, the lattice axiom is the axiom least justified experimentally since it is nonconstructive. It is the merit of the Truini-Biedenharn construction that it provides a model in which this axiom is denied in a natural way.

Because of the lack of a lattice structure, the definition of "state" given by TRUINI and BIEDENHARN [116 b] was suitable a "measure" with unusual properties thereby being defined. However, this measure coincides with the unique probability function (defined by GÜNAYDIN, PIRON and RUEGG [55] on the Moufang plane) when restricted to the <u>real</u> octonion case. Moreover, when restricted to the purely <u>complex</u> case, the measure coincides with the usual modulus (squared) of complex three-dimensional Hilbert space quantum mechanics.

TRUINI and BIEDENHARN [116 b] associated observables to the generators of the automorphism group $\mathcal{E}_{6,0} \otimes U(1)$ in exactly the same way as in the usual quantum theory, namely, by multiplying the skew-Hermitian generators by the imaginary unit to obtain Hermitian operators. Let us note, however, that the spectral theory of the observables thus defined is completely different from the ordinary one. The Hamiltonian of the system will be one of the Hermitian generators of $\mathcal{E}_{6,0} \otimes U(1)$. For details see [116 a,b].

<u>Open problem</u> (see [116 b, p.1329]). To obtain some kind of physical understanding of the role of the connected points which are responsible for all unusual features of Truini-Biedenharn quantum mechanics.

Finally, let us mention the opinion of TRUINI and BIEDENHARN [116 b, p.1328] that "It is our belief (noting the close relationship between geometries and quantum mechanics) that the concepts of quadratic Jordan algebras and inner ideals will be useful in physics".

§ 4. Jordan algebras in classical mechanics

A systematic study of the classical mechanics of systems described by usual c-number variables and by Grassmann variables was given by CASALBUONI [20 a,b]. Such a mechanics is the classical limit (in the sense $\hbar \to 0$) of a general quantum theory with Bose and Fermi operators.

Let us recall that in the standard exposition of the quantisation procedure of classical systems, the quantisation rules for unconstrained system are

$$\{,\}_- \longrightarrow -\frac{i}{\hbar}[\,,\,]_- \quad ,$$

while the quantisation rules for constrained system are

$$\{,\}^*_- \longrightarrow -\frac{i}{\hbar}[\,,\,]^*_- \quad ,$$

where $\{,\}_-$ is the minus Poisson bracket, $\{,\}^*_-$ is the minus Dirac bracket, $[\,,\,]_-$ and $[\,,\,]^*_-$ being commutators. They are valid only in the case of <u>integer-spin (Bose) systems</u>.

DROZ-VINCENT [30] showed that for unconstrained classical systems there exists another symmetric structure characterized by a new bracket, called <u>plus Poisson bracket</u> and denoted by $\{,\}_+$.

FRANKE and KÁLNAY [39] showed that for constrained classical systems there exists a dual symmetric partner of minus Dirac bracket, called <u>plus Dirac bracket</u> and denoted by $\{,\}^*_+$. RUGGERI [97] and KÁLNAY and RUGGERI [66] suggested that, in the case of <u>half-integer spin (Fermi) systems</u>, the quantisation rules for unconstrained system are

$$\{,\}_+ \to \xi[\,,\,]_+ \quad ,$$

while the quantisation rules for constrained systems are

$$\{,\}_+ \to \xi[\,,\,]^*_+$$

where ξ is a parameter in the theory, $[\,,\,]_+$ and $[\,,\,]^*_+$ being anti-commutators.

DROZ-VINCENT [30] observed that for systems described by plus Poisson brackets, the algebraic structure must be a Jordan algebra. However, it was not clear what the conditions are under which the classical dynamical variables constitute a Jordan algebra with respect to the plus Poisson or Dirac bracket. PEDROZA and VIANNA have solved this problem [90]. Their results are recalled in the sequel.

Notation. As was done in [85] and [90], we introduce in the 2N-dimensional phase space of a classical mechanical system with canonical variables $q^1,\dots,q^N$, $p_1,\dots,p_N$, the variables $\omega^1,\dots,\omega^{2N}$ as follows:

$$\omega^1 = q^1,\dots,\omega^N = q^N,\ \omega^{N+1} = p_1,\dots,\omega^{2N} = p_N\ ,$$

$$\omega = (\omega^1,\omega^2,\dots,\omega^{2N}) = (q,\ p).$$

With respect to the indices for coordinates in phase space, we shall use for q and p the indices r,s,t and for ω, we shall use the indices $\mu,\nu,\dots,\pi$. For the functions φ and θ we shall use the indices i,j,k,...,n and a,b,c, respectively. Local coordinates will be denoted by x^I, I = 1,2,...,2N. As usually, $\partial_I := \partial/\partial x^I$ $\partial_{IJ} := \partial^2/\partial x^I \partial x^J$, and $\partial_{IJK} := \partial^3/\partial x^I \partial x^J \partial x^K$.

Let f,g,...,h be real functions on a 2N-dimensional manifold V. According to [30], a plus Poisson bracket is defined by

$$\{f,M,g\}_+ := \{f,g\}_+ := \nabla_I(M^{IJ}f)\ \nabla_J\, g = \nabla^J f\ \nabla_J\, g,$$

where ∇ is the covariant derivative in the connection Γ and M is a second-rank symmetric tensor of contravariant type with vanishing covariant derivative. The functions f and g satisfy the conditions

(4.1) $$\nabla_K \nabla_L \nabla_J f = 0,\ \nabla_K \nabla_L \nabla_J g = 0.$$

The existence of the tensor M for the phase space is assured by virtue of the considerations given in [30].

Remark. The canonical rules for the minus Poisson bracket are also valid for the plus Poisson bracket, that is $\{q^r, q^l\}_+ = \{p_r, p_s\}_+ = 0$, $\{q^r, p_s\}_+ = \delta^r_s$.

Note. For simplicity, PEDROZA and VIANNA [90] considered only the case $\Gamma = 0$. However, their results can be extended to the general case when $\Gamma \neq 0$.

1. Let $F = \{f, g, \ldots, h\}$ be the set of all dynamical variables of an unconstrained classical system described by the symmetric formulation of classical mechanics. By (4.1), $f, g, \ldots, h$ must be such that

(4.2)

$$\partial_{IJL} f = 0, \quad \partial_{IJL} g = 0, \ldots, \partial_{IJL} h = 0,$$

with $1 \leq I, J, L \leq 2N$.

For $f, g \in F$ define

$$fg := \{f, M, g\}_+ := \{f, g\}_+ := M^{KI} \partial_I f \, \partial_K g. \tag{4.3}$$

As was shown by PEDROZA and VIANNA [90], the set of functions $f, g, \ldots, h$ which satisfy conditions (4.2), endowed with the product (4.3) defined by the plus Poisson bracket, is a real Jordan algebra.

2. Now we shall consider systems involving constraints. For such kind of systems the Hamiltonian equations of motion can be expressed in terms of the Dirac bracket in the same way in which equations of motion of unconstrained systems can be expressed in terms of the Poisson bracket. Before the introduction of the Dirac bracket, the constraints have to be separated in two classes: constraints of the first and second classes.

Notation. Following [90], we denote by $\xi_a(q,p)$ any one of the m constraints of a classical system. If

$$\{\xi_a, \xi_b\}_+ = 0 \text{ for all } b, \quad 1 \leq a, b \leq m,$$

then ξ_a is called a symmetric first-class constraint, while, if there exists a b such that

$$\{\xi_a, \xi_b\}_+ \neq 0, \qquad 1 \leq a,b \leq \mathcal{M}$$

then ξ_a is called a symmetric second-class constraint. Let $\Theta :=$ $:= \{\theta_1(\omega), \theta_2(\omega), \ldots, \theta_c(\omega), \ldots, \theta_{N_\Theta}(\omega)\}$ be the set of symmetric second-class constraint of a classical system. Let Φ be a set with $2N-N_\Theta$ independent functions $\varphi_1(\omega), \ldots, \varphi_m(\omega), \varphi_n(\omega), \ldots, \varphi_{2N-N_\Theta}(\omega)$, and such that $\Psi = \Phi \cup \Theta$ is a local coordinate system for the phase-space manifold. FRANKE and KÁLNAY [39] showed that

$$L^+_{\varrho\sigma}(\omega) := \varepsilon^+_{\mu\nu} \frac{\partial\omega^\mu}{\partial\psi^\varrho}\frac{\partial\omega^\nu}{\partial\psi^\sigma} \text{ with } \|\varepsilon^+_{\mu\nu}\| := \|\delta_{\mu,\nu+N} + \delta_{\mu+N,\nu}\|$$

is a second-rank symmetric tensor of covariant type, where $\psi_1, \psi_2, \ldots, \psi_\varrho, \psi_\sigma, \ldots, \psi_{2N}$ denote elements of Ψ. Also, they [39] showed that L^+_{mn} has an inverse tensor M^{mn}_+ with covariant derivatives restricted to the submanifold whose local coordinate system is Φ. It is also possible to define the plus Dirac bracket by

$$(4.4) \qquad \{f,g\}^*_+ (\theta,\varphi) := M^{mn}_+ (\theta,\varphi) \frac{\partial f}{\partial\varphi^m}\frac{\partial g}{\partial\varphi^n}.$$

Remark 1. In this approach, Franke and Kálnay have used the fact that any function of the variables ω^μ can be written as a function of the variables θ^a and φ^m. Partial differentiation with respect to a φ^m is carried out keeping the θ^a's and the other φ^m's constant.

Remark 2. Compare (4.4) with (4.3).

Let $\mathcal{F} = \{f(\theta,\varphi), g(\theta,\varphi), \ldots, h(\theta,\varphi)\}$ be the set of dynamical variables of a constrained classical system described by the symmetric formulation of classical mechanics, so that

$$(4.5) \qquad \partial_{\ell mn} f = 0, \; \partial_{\ell mn} g = 0, \ldots, \partial_{\ell mn} h = 0,$$

with $1 \leq \ell,m,n \leq 2N-N_\Theta$, where $\partial_{\ell mn} := \partial^3/\partial\varphi^\ell\partial\varphi^m\partial\varphi^n$.

For $f,g \in \mathcal{F}$ define

$$(4.6)\quad \{fg\}(\theta,\varphi) := \{f(\theta,\varphi)\ g(\theta,\varphi)\}^{*}_{+} := M^{mn}_{+}(\theta,\varphi)\,\partial_m f \partial_n q := \\ := M^{mn}_{+}\,\partial_m f \partial_n g,$$

where $\partial_m f := \partial f/\partial \varphi^m$.

As was noticed by PEDROZA and VIANNA [90], the set of functions $f(\theta,\varphi)$, $g(\theta,\varphi),\ldots,h(\theta,\varphi)$ which satisfy conditions (4.5), endowed with the product (4.6) defined by the plus Dirac bracket, is a real Jordan algebra if the following conditions are satisfied:

$$(4.7)\qquad M^{jk}_{+}M^{mn}_{+}\partial_{kn}M^{\ell i}_{+} + M^{jk}_{+}\partial_k M^{mn}_{+}\partial_n M^{\ell i}_{+} - M^{kn}_{+}\partial_k M^{jm}_{+}\partial_n M^{\ell i}_{+} = 0,$$

$$(4.8)\qquad 2M^{jk}_{+}M^{mn}_{+}\partial_k M^{\ell i}_{+} + 2M^{jn}_{+}M^{mk}_{+}\partial_k M^{\ell i}_{+} - M^{im}_{+}\,M^{nk}_{+}\partial_k\,M^{\ell j}_{+} + \\ + 2M^{jk}_{+}M^{\ell i}_{+}\partial_k M^{mn}_{+} - 2M^{\ell i}_{+}\,M^{kn}_{+}\partial_k\,M^{jm}_{+} = 0$$

$$(4.9)\qquad (\partial_{ijk}M^{mn}_{+}(\theta,\varphi) + \partial_{ij}M^{mn}_{+}(\theta,\varphi)\partial_k + \partial_{ik}M^{mn}_{+}(\theta,\varphi)\partial_j + \\ + \partial_{kj}M^{mn}_{+}(\theta,\varphi)\partial_i + \partial_i M^{mn}_{+}(\theta,\varphi)\partial_{kj} + \partial_j M^{mn}_{+}(\theta,\varphi)\partial_{ki} + \\ + \partial_k\,M^{mn}_{+}(\theta,\varphi)\partial_{ij})\partial_m f \partial_n g = 0.$$

Remark. The relation (4.8) is satisfied if $M_{+}(\theta,\varphi)$ is independent of φ^m ($m=1,2,\ldots,2N-N_\theta$).

Hence, for unconstrained classical systems described by the symmetric formulation of classical mechanics, the set of all dynamical variables is a real Jordan algebra with respect to ordinary addition and plus Poisson bracket while for constrained systems, the set of dynamical variables is a real Jordan algebra under ordinary addition and the plus Dirac bracket (4.4) if conditions (4.7), (4.8) and (4.9) are satisfied.

In this respect, let us mention PEDROZA and VIANNA's comments from [90] : "We note an important difference between the Lie and Jordan algebraic structure for classical systems. The Lie algebraic structure appears in classical mechanics in a natural way, but for Jordan algebra it is different. In fact, Droz-Vincent's symmetric brackets are only defined for dynamical variables f such that ($\Gamma = 0$) $\partial_{IJK} f = 0$ for unconstrained systems ($1 \leq I,J,K \leq 2N$) and $\partial_{\ell mn} f = 0$ for constrained systems ($1 \leq \ell, m, n \leq 2N - N_\theta$). Consequently, in this symmetric formulation of classical mechanics, if f and g are dynamical variables, the quantity fg = gf is not necessarily a dynamical variable. This result restricts the set of admissible dynamical variables in the present theory. However, our result is not worse than the usual theory with minus Poisson brackets. Indeed, as STREATER [110] has shown, the Dirac quantisation procedure for minus Poisson brackets is also only possible for a restricted set of dynamical variables". (See [90, p.830]).

Comments. The results recalled in this section can be extended to symplectic manifolds (generalized phase spaces).

Remark. In the case of simplectic manifolds the variables $\omega^1, \ldots, \omega^{2N}$ are local canonical coordinates.

Open problem. (suggested by GHEORGHE [44]). To classify and construct smooth manifolds on which $\{,\}_+^*$ is *globally* defined.

Comments. Solving the above mentioned open problem is important for the construction of supergroups and supermanifolds describing constrained quantum systems with spin.

Finally, let us mention a recent paper [95] by ROCHA FILHO and VIANNA, where it is shown that the set of observable functionals associated with a constrained field theory satisfying two given assumptions is a Jordan algebra under the symmetric Dirac bracket composition law.

§ 5. JB-algebras in quantum mechanics

In order to improve actually existing axiom systems for non-relativistic quantum mechanics, GUZ [60 c] developed the general axiomatic scheme given by Axioms A,B,C,D (see below) in two directions: one (see Axioms I1-I6 below) which is very close to the well known quantum logic approach (originated in [15] and developed and improved by many others) and another which is the outcome of Guz's general axiomatics (see Axioms III1-III7 below), and is based on introducing the structure of a partially ordered real vector space in the set $\mathcal{A}_b$ of bounded observables and then establishing the Jordan-Banach structure in $\mathcal{A}_b$, the latter being deduced from a set of physically plausible postulates. In this way Guz developed of his general axiomatics based on Axioms A,B, C,D along the lines of the algebraic axiomatic scheme initiated as far back as JORDAN [64 a,b,c], JORDAN, von NEUMANN and WIGNER [65], von NEUMANN [87 b] and latter modified by SEGAL [102], EMCH [33 a] (see also § 2).

For an improvement in the finite-dimensional case of the characterisation of state spaces of JB-algebras to a form with more physical appeal see ARAKI [5].

Here we recall the basic facts from the above-mentioned developments given by GUZ (see [60 a-d]) and then briefly refer on KUMMER's recent approach [71].

<u>Notation</u>. As in § 1 let $\mathcal{A}$ and S be the set of all observables and all states, respectively, of a given physical system. Denote by $\mathbb{R}_+$ the non-negative part of $\mathbb{R}$, and by $B(\mathbb{R})$ the σ-algebra of all Borel subsets of $\mathbb{R}$.

Following MACKEY [76] (see also [60 c, p.66]) we assume:

<u>Axiom A</u>. There exists a function $p : \mathcal{A} \times S \times B(\mathbb{R}) \to \mathbb{R}_+$ which, for fixed $A \in \mathcal{A}$ and $\phi \in S$, is a probability measure on $B(\mathbb{R})$.

<u>Axiom B.</u> If $p(A_1, \phi, E) = p(A_2, \phi, E)$ for all $\phi \in S$ and $E \in B(\mathbb{R})$, then $A_1 = A_2$.

<u>Axiom C</u>. If $p(A, \phi_1, E) = p(A, \phi_2, E)$ for all $A \in \mathcal{A}$ and $E \in B(\mathbb{R})$, then $\phi_1 = \phi_2$.

Axiom D. For each sequence $\phi_1, \phi_2, \ldots$ of states and each sequence $t_1, t_2, \ldots$ of positive real numbers with $\sum_{i=1}^{\infty} t_i = 1$, there exists a state $\phi \in S$ such that $p(A,\phi,E) = \sum_{i=1}^{\infty} t_i\, p(A,\phi_i, E)$ all $A \in \mathcal{A}$ and $E \in B(\mathbb{R})$.

Remark. For the physical interpretation of Axioms A-D see GUZ [60 c, pp.66-67].

Convention and notation. An ordered pair $(A,E) \in \mathcal{A} \times B(\mathbb{R})$ is identified with the experimentally verifiable proposition stating that "a measurement of an observable A yields to a value in a Borel set E", and the number $p(A,\phi,E)$ is then interpreted as the probability that the proposition (A,E) is true for the system in the state ϕ (MACKEY [76], MACZYNSKI [77], GUZ [60 c]). In the set $\mathcal{A} \times B(\mathbb{R})$ one can define two operators, called implication and negation, respectively,

$$(A,E) \rightarrow (B,F) \text{ if and only if } p(A,\phi,E) \leq p(B,\phi,E) \text{ for all } \phi \in S;$$

$$\neg(A,E) := (A, \mathbb{R} \setminus E).$$

Two propositions (A,E) and (B,F) are called equivalent, written as $(A,E) \sim (B,F)$, if $(A,E) \rightarrow (B,F)$ and $(B,F) \rightarrow (A,E)$, i.e. if $p(A,\phi,E) = p(B,\phi,F)$ for every $\phi \in S$. The set $L := (\mathcal{A} \times B(\mathbb{R}))/\sim$, which is called the logic of a physical system (or the logic of propositions, see MACKEY [76], MACZYNSKY [77]) has been shown to be a partially ordered set with involution, provided we define

$$|(A,E)| \leq |(B,F)| \text{ if and only if } (A,E) \rightarrow (B,F);$$

$$|(A,E)|' := |\neg(A,E)|,$$

where $|(A,E)|$ denotes the equivalence class of the proposition (A,E). The equivalence classes $|(A,E)|$ will also be called propositions. Two propositions $a = |(A,E)|$ and $b = |(B,F)|$ are called orthogonal, written as $a \perp b$, if $a \leq b'$.

Definitions. Let A be from $\mathcal{A}$. The smallest closed set $F \subseteq \mathbb{R}$ satisfying $p(A, \phi, F) = 1$ for all $\phi \in S$ is called the spectrum of A; it is denoted by sp A. An observable $A \in \mathcal{A}$ for which sp A is a bounded set is called bounded. In the latter case, the number $\sup \{ |t| \mid t \in \text{sp } A \}$ is called the spectral norm of A; it is denoted by $\|A\|$ sp. The set of all bounded observables will be denoted by $\mathcal{A}_b$.

Remark. $\text{sp } A \subseteq [-\|A\| \text{sp}, \|A\| \text{sp}]$.

Definition. Consider $A \in \mathcal{A}$ and $\phi \in S$. If the integral $\int_{-\infty}^{\infty} t \, p(A, \phi, dt)$ exists and is finite, then it is called the expectation value (or mean value) of the observable A for the system in the state ϕ , and is denoted by $\langle A, \phi \rangle$.

Remark 1. Compare the above definition with Postulate 2 from § 2.

Remark 2. If $A \in \mathcal{A}_b$, then $\langle A, \phi \rangle \leq \|A\|$ sp, so that every bounded observable has finite expectation value in all states.

Notation. Let M_p be the convex set of probability measures on $B(\mathbb{R})$. Identify S with the family of maps $p_\phi : \mathcal{A} \to M_p$, and consider the space $V := \left\{ \sum_{i=1}^{n} s_i \, p_{\phi_i} \,\middle|\, s_i \in \mathbb{R}, \phi_i \in S, n = 1, 2, \ldots \right\}$.
We can associate with each $A \in \mathcal{A}_b$ a linear functional on V as follows: $L_A(x) := \int_{-\infty}^{\infty} t(x(A))dt$, $x \in V$. It is obvious that $L_A(p_\phi) = \langle A, \phi \rangle$. The notation $\langle A, x \rangle$ will be extended to all $x \in V$, i.e. $\langle A, x \rangle := L_A(x)$.

An important assumption of the quantum logic approach to the foundations of quantum mechanics is the so-called "orthogonality postulate" (see, for instance, MACKEY [76]), which asserts

Axiom I 1. If $a_i = |(A_i, E_i)|$, i=1,2,..., is a sequence of pairwise orthogonal propositions from L, then there exists a proposition $a = |(A,E)|$ such that $p(A,\phi,E) = \sum_{i=1}^{\infty} p(A_i,\phi, E_i)$ for all $\phi \in S$.

From Axioms A,B,C and I 1, it follows that the propositional logic $(L, \leq, ')$ becomes an orthomodular σ-orthocomplete orthocomplemented partially ordered set with 0 and 1 (the least upper bound for an orthogonal sequence $\{a_i\}_{i=1,2,\ldots,\infty} \subseteq L$ is given by the proposition $a \in L$ defined above in Axiom I 1).

Convention. Any state $\phi \in S$ can be identified (see MACZINSKI [77]) with the probability measure μ_ϕ on L defined by $\mu_\phi(|(A,E)|) := p(A,\phi,E)$, and every observable $A \in \mathcal{A}$ - with the L-valued measure x_A (that is, x_A is a σ-homomorphism from $B(\mathbb{R})$ to L) defined by $x_A(E) := |(A,E)|$.

We have $p(A,\phi, E) = \mu_\phi(x_A(E))$, and the family $\{\mu_\phi \mid \phi \in S\}$ of all the probability measures associated with states of a physical system is easily seen to be order determining.

Note. The proportional logic L appears now as a primary object of the theory, while the sets of states and observables become secondary, as they arise here as some constructions on L (the probability measures on L and the L-valued measures, respectively).

Convention. After the identification of the states with the corresponding probability measures on L, $\phi(a)$ will be written instead of $\mu_\phi(a)$, $a \in L$.

Assume now (GUZ [60 b,c]) :

Axiom I 2. There exists a subset $P \subseteq S$ whose members, called pure states, are assumed to satisfy the following requirements:

(i) for every non-zero proportion $a \in L$ there exists a pure state $p \in P$ such that $p(a) = 1$;

(ii) if for every pure state $p \in P$ satisfying $p(a) = 1$ we also have $p(b) = 1$, where $a, b \in L$, then $a \leq b$;

(iii) for each pure state $p \in P$, there exists a proposition $a \in L$ such that $p(a) = 1$ and $q(a) < 1$ for all pure state q distinct from p.

GUZ [60 c] showed that, assuming Axioms A,B,C and I 1, Axiom I 2 above is equivalent to the following statement: The propositional logic L is atomistic (i.e. L is atomic and each $a \in L$ is the least upper bound of the atoms contained in it), and there exists a bijection $s : P \rightarrow A(L)$ of the set P of all pure states onto the set A(L) of all atoms in L such that, for every $p \in P$,

(1) $p(s(p)) = L$;

(2) $p(a) = 1$, where $a \in L$, implies that $a \geqslant s(p)$.

<u>Definition</u>. Two states ϕ_1 and ϕ_2 from S are called <u>orthogonal</u>, written as $\phi_1 \perp \phi_2$, if for some proposition $a \in L$ we have $\phi_1(a) = 1$ and $\phi_2(a) = 0$.

<u>Notation and definitions</u>. For any subset $M \subseteq P$ we denote by $M^{\perp}$ the set of all pure states $p \in P$ such that $p \perp q$ for all $q \in M$, and write M^{-} instead of $M^{\perp\perp}$. If $M = M^{-}$, then the set M is called <u>closed</u>. The family $C(P, \perp)$ of all closed subsets of P is called the <u>phase geometry</u> associated with a physical system.

GUZ proved [60 b] the following theorem.

<u>Theorem 5.1.</u> For every $a \in L$, the set $a^1 := \{p \mid p \in P, p(a) = 1\}$ is an element of $C(P, \perp)$, and the correspondence $a \rightarrow a^1$ defines an orthoinjection of the propositional logic L into the phase geometry $C(P, \perp)$, the latter being an atomistic complete orthocomplemented lattice.

<u>Notation</u>. If ϕ_1 and ϕ_2 are two arbitrary states of a phy-

sical system, then the number $\inf\{\phi_1(a) \mid a \in L, \phi_2(a) = 1\}$ is called the degree of dependence of ϕ_1 on ϕ_2, and is denoted by $(\phi_1 : \phi_2)$ (see GUZ [60 c]) [1].

Suppose that the initial state of a physical system is described by the density operator ϕ , and, that after a measurement performed on the system, the proposition described by the projection operator $\mathcal{P}$ is verified to be true, then the subsequent state of the system is described by the density operator

$$\phi_{\mathcal{P}} := \mathcal{P}\phi\mathcal{P} / Tr(\mathcal{P}\phi).$$

Ignoring normalization of the state we obtain a linear map $\phi \to \mathcal{P}\phi\mathcal{P}$, called the conditional probability map from the Banach space of selfadjoint operator of trace class (acting on the Hilbert space corresponding to the quantum-mechanical system under study) into itself.

Definition. Any collection $\{E_a\}$ of transformations of the set P of pure states into itself, indexed by nonzero propositions from L and satisfying:

(1) the domain $D(E_a)$ of E_a consists of those $\phi \in P$ for which $\phi(a) > 0$, and for every $\phi \in D(E_a)$ we have $(E_a\phi)(a) = 1$;

(2) if $\phi(a) = 1$, then $E_a\phi = \phi$,

is called the family of (pure) filters (or pure conditional probability maps) associated to the propositional logic L.

Convention. In order to study the properties of the maps E_a, the transition probability function (:) is extended onto the set $P_0 := P \cup \{0\}$, where 0 denotes the improper "pure" state, called the zero state, adjoined to P and defined as the zero function on L: For all $p \in P_0$, $(0:p) := (p:0) := 0$ (see [60 c, p.79]).

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1) The number $(\phi_1 : \phi_2)$ was introduced in 1969 independently by MIELNIK [82] under the name transition probability between ϕ_1 and ϕ_2.

<u>Definition</u>. A map E_a ($E_a : p \rightarrow p_a$) of the set P_o into itself is called a (<u>pure</u>) <u>filter associated with the proposition</u> $a \in L$ if it satisfies the following conditions:

(i) $(p:p_a) = p(a)$ for all $p \in P_o$;

(ii) E_a is an idempotent map;

(iii) $(p:p_a) = 0$ implies that $p_a = 0$.

<u>Remark</u>. Any filter E_a possesses the property

$$(p : p_a) \geqslant (p : q_a) \tag{5.1}$$

fo all $p, q \in P_o$.

<u>Definition</u>. A filter E_a is called <u>proper</u> if the inequality (5.1) becomes strict whenever $p_a \neq 0$ and $p_a \neq q_a$.

<u>Remark</u>. Each proper filter E_a can be identified with the corresponding <u>Sasaki projection</u> $s_a : A(L) \cup \{0\} \rightarrow A(L) \cup \{0\}$ defined by $s_a(e) := a' \vee e - a' = (a' \vee e) \wedge a$, where $a \in L$, $e \in A(L) \cup \{0\}$, and $\wedge$ stands for the greatest lower bound in L.

<u>Axiom I 3</u>. With every nonzero proposition $a \in L$ a proper pure filter $E_s : P_o \rightarrow P_o$ is associated.

<u>Notations</u>. GUZ [60 c] considered, after GUNSON [57], the vector space (L) defined as the linear span of the image of the propositional logic L under the canonical embedding $L \rightarrow V'$ defined by $|(A,E)| \rightarrow q_{(A,E)}$, where V denotes the complete base-norm space spanned by states of a physical system and $q_{(A,E)} : V \rightarrow \mathbb{R}$ is given by $q_{(A,E)}(x) := (x(A))(E)$, $x \in V$ (see [60 c, p.71]). Similarly, (L_f) denotes the linear span of the set $L_f \subseteq L$ of all finite elements (i.e. which are the join of a finite number of atoms) of L. Since every finite proposition $a \in L_f$ can be written as a (finite) join of pairwise orthogonal atoms, we have $(L_f) = (A(L))$ = the linear span of the set of all atoms in L.

Define, after GUNSON [57], the following pseudo product for atomic propositions:

$$e \circ f := \frac{1}{2}(Q_e - Q_{e'} + Id)f \in (L_f),$$

where Q_a, $a \in L$, is given by

$$Q_a e := p^e(a)s_a(e), \qquad e \in A(L),$$

with p^e defined as $p^e := a^{-1}(e)$ and Id is the identity map.

Axiom I 4. Each Q_a, $a \in L$, can be extended to an affine map $\hat{Q}_a : (L_f)_+ \rightarrow (L_f)_+$, where $(L_f)_+$ is the generating cone in (L_f) defined by $(L_f)_+ := \left\{ \sum_{i=1}^{n} t_i e_i \,\middle|\, t_i \geqslant 0,\ e_i \in A(L),\ n=1,2,\dots \right\}$.

Remark. $\hat{Q}_a$ can be extended to a linear map $T_a : (L_f) \rightarrow (L_f)$ by setting $T_a u := \hat{Q}_a u_1 - \hat{Q}_a u_2$ whenever $u = u_1 - u_2$ with $u_1, u_2 \in (L_f)_+$. The maps T_a are called dual filters associated with propositions from L.

Axiom I 5. For any pair p,q of pure states we have $(p:q) = (q:p)$.

Theorem 5.2. The pseudoproduct $\circ$ can be extended to a commutative product on U (= the norm closure of $(L_f) \oplus \mathbb{R}\,1$ in the order unit space $(V',1)$) such that $(U, \circ, 1)$ becomes a distributive Segal algebra with 1 acting on it as unit element.

Axiom I 6. If $a \leqslant b$ and $p(b) > 0$, $a, b \in L$ and $p \in P$, then $P_b(a) = p(a)/p(b)$.

Theorem 5.3. If we assume the validity of Axiom I 6, then the space U endowed with the product $\circ$ becomes a real Jordan algebra.

As a consequence of Theorems 5.2 and 5.3 we have

Theorem 5.4. The space U endowed with the product $\circ$ and with the order unit norm inherited from V', where V is the base norm space spanned by states of a physical system, becomes a JB-algebra.

Comments. The quantum logic approach, as modified above (see Axioms I 1 - I 6), is intimately connected with the Jordan-Banach algebraic scheme, and at this point GUZ [60 c] followed the pioneering work [57] of GUNSON.

The first three axioms (see below) of the second development of the general axiomatic scheme (Axioms A-D above) given by GUZ [60 c] are connected with the introduction of a linear structure on $\mathcal{A}_b$ (see also MACKEY [76] and EMCH [33 a]).

Axiom II 1. If $A, B \in \mathcal{A}_b$ and $\langle A, \phi \rangle = \langle B, \phi \rangle$ for all states $\phi \in S$, then $A = B$.

Axiom II 2. i) For each pair $A, B \in \mathcal{A}_b$ there exists an observable $A + B \in \mathcal{A}_b$ such that

$$\langle A + B, \phi \rangle = \langle A, \phi \rangle + \langle B, \phi \rangle$$

for all $\phi \in S$,

ii) for every bounded observable $A \in \mathcal{A}_b$ and every $t \in \mathbb{R}$ there exists an observable $t\,A \in \mathcal{A}_b$ satisfying

$$\langle t\,A, \phi \rangle = t \langle A, \phi \rangle ,$$

for all $\phi \in S$,

iii) there exist observables $0, I \in \mathcal{A}_b$ such that

$$\langle 0, \phi \rangle = 0, \ \langle I, \phi \rangle = 1,$$

for all $\phi \in S$.

Remark. By Axiom II 1, every $A \in \mathcal{A}_b$ can be identified with the corresponding mean value functional L_A. Axiom II 2 introduces in the set $\mathcal{A}_b$ a real vector space structure. Moreover, after identifying each $A \in \mathcal{A}_b$ with L_A, one obtains in $\mathcal{A}_b$ the structure of a partially ordered vector space inherited from V' (= the Banach dual of the space V), and $\mathcal{A}_b$ becomes in fact an order unit space.

Note. From this point on, GUZ [60 c] closely follows the path of the first development (Axioms I 1 - I 6) by introducing the concept of the conditional probability map, and then by establishing, step by step, the Jordan-Banach algebra structure in $\mathcal{A}_b$, so that finally he is in a position to appeal to GNS representation theorem proved for JB-algebras by ALFSEN, SHULTZ and STØRMER [3].

Following POOL [92], GUZ [60 c] assumed

Axiom II 3. With every nonzero proposition $a \in L$ is associated a map E_a of the set S into itself whose domain is $D(E_a) = \{\phi \in S \mid a(\phi) > 0\}$ [1)] such that

(1) $a(E_a\phi) = 1$ for all $\phi \in D(E_a)$;

(2) $E_a\phi = \phi$, whenever $a(\phi) = 1$.

The physical interpretation of E_a is as follows: if, after a measurement performed on a physical system being initially in the state ϕ, the proposition $a \in L$ is verified to be true, then the subsequent state of the system is $E_a\phi$. Hence in other words, $E_a\phi$ describes the state of the system conditioned by the fact of occurence of an "event" $a \in L$ (see GUZ [60 c, pp.98-99]).

According to the above interpretation, GUZ [60 c] calls E_a the conditional probability map associated with the (nonzero) proposition $a \in L$.

Now it is convenient to pass (similarly ad above, see Convention on page 37) from E_a to the transformation $P_a : V_+ \to V_+$, where $V_+ := \mathbb{R}_+ \hat{S}$, $\hat{S} := \{p_\phi \mid \phi \in S\}$, is defined by

$$P_a x := \begin{cases} a(x)E_a(x/\|x\|) & \text{when } a(x) > 0, \\ 0 & \text{when } a(x) = 0, \end{cases}$$

where $x \in V_+$ and $\|x\| := \inf\{t > 0 \mid x \in t[-1,1]\}$. (Clearly, if $a = 0$ then $P_o = 0$).

1) Here, following GUZ [60 c], we prefer the notation $a(\phi)$ in place of the more conventional $\phi(a)$. This is in accordance with the fact that L is considered here as a subset of the Banach dual V!

<u>Convention</u>. Identifying the set S with its canonical image $\hat{S}$, we shall write $P_a \phi$ instead of $P_a p_\phi$, where $\phi \in S$. P_a is called the <u>filter</u> associated with the proposition $a \in L$.

<u>Axiom II 4</u>. For each $a \in L$ and each $A \in \mathcal{A}_b$ there exists a bounded observable $B \in \mathcal{A}_b$ such that

$$\langle B, \phi \rangle = \langle A, P_a \phi \rangle \quad ,$$

for all $\phi \in S$.

<u>Remark 1</u>. B is necessarily unique, by Axiom II 1. It will be denoted by Q_a A.

<u>Remark 2</u>. P_a can be uniquely extended to a linear map acting on the whole space V and the extension will be denoted by the same letter P_a. Moreover, it can be shown that all maps $P_a : V \rightarrow V$, where $a \in L$ are continuous with respect to the weak topology $\sigma(V, \mathcal{A}_b)$ in V, given the duality $\langle , \rangle$. Hence $Q_a = P_a^*$, where P_a^* denotes the linear operator in $\mathcal{A}_b$ weakly dual to P_a.

<u>Definition</u>. Two filters P_a, P_b are called <u>compatible</u>, written as $P_a \leftrightarrow P_b$, if for each state $\phi \in S$ we have $\| P_b (P_a + P_{a'}) \phi \| = \| P_b \phi \|$. (see [60 c, p.102]).

Passing to dual filters $Q_a = P_a^*$, GUZ [60 c, p.102] defines

$$Q_a \leftrightarrow Q_b \text{ if and only if } P_a \leftrightarrow P_b .$$

Moreover, Guz defines compatibility of Q_a with a bounded observable $A \in \mathcal{A}_b$ in two steps, as follows: 1) if $A \geq 0$, then $Q_a \leftrightarrow A$ if and only if $Q_a A \leq A$ and 2) if A is arbitrary, then $Q_a \leftrightarrow A$ if and only if there exists a decomposition $A = A_1 - A_2$, where $A_1, A_2 \in \mathcal{A}_b$ such that $Q_a \leftrightarrow A_i$, $i=1,2$,.

<u>Axiom II 5</u>. If Q_a is compatible with $A \in \mathcal{A}_b$, then $Q_a \leftrightarrow Q_{|(A,E)|}$ for all Borel subsets $E \subseteq \mathbb{R}$.

<u>Axiom II 6</u>. The space $\mathcal{A}_b$ of bounded observables is pointwise monotone σ-complete, that is, for every increasing sequence

$\{A_i\} \subseteq \mathcal{A}_b$ bounded above there exists an $A \in \mathcal{A}_b$ such that $\langle A, \phi \rangle =$ $= \sup \langle A_i, \phi \rangle$ for all $\phi \in S$.

The next axiom, introduced for the first time by ALFSEN and SHULTZ [2], formulates the key physical property needed for obtaining a Jordan algebra structure on $\mathcal{A}_b$.

Axiom II 7. Let P_1, P_2 be weakly continuous positive projections on either A or V with at most 1 and admitting a complement with norm at most 1. Then, for each state $\phi \in S$, the probability of the exclusive disjunction of P_1 and P_2, defined by

$$\text{Prob}\ ((P_1\ \&\ P_2')\ \text{or}\ (P_1'\ \&\ P_2))_\phi := \|P_2' P_1 \phi\| + \|P_2 P_1' \phi\|,$$

is independent of the order of P_1 and P_2, that is,

$$\text{Prob}\ ((P_1\ \&\ P_2')\ \text{or}\ (P_1'\ \&\ P_2))_\phi = \text{Prob}\ ((P_2'\ \&\ P_1)\ \text{or}\ (P_2'\ \&\ P_1))_\phi .$$

Remark. ALFSEN and SHULTZ [2] showed that the property expressed by Axiom II 7 is sufficient and necessary for the space $\mathcal{A}_b$, which is a pointwise monotone σ-complete order unit space in spectral duality with the base-norm space V, to be a JB-algebra with a Jordan product defined by $AB := \frac{1}{2}\,((A+B)^2 - A^2 - B^2)$. So, one can apply the GNS representation theorem for JB-algebras (see [3]) to obtain the Hilbert space representation for $\mathcal{A}_b$.

Comments. LOUPIAS [75] proved that a system of observables of a quantum system, closed under linear combinations a convenient squaring operation, and complete with respect to an appropiate norm topology, possesses a Jordan-Banach structure.

Recently KUMMER [71] formulated an axiomatic theory which describes a class of "yes-no" experiments, involving a fixed basic source, a fixed basic detector, and various filters. It is assumed that all filters considered can be constructed from a set of primitive filters by composition and stochastic selection. Two physically plausible axioms are formulated which allow Kummer to define the concept of a system in the present context. To each system he can attach an order unit module ($\hat{^oV}$, $\hat{^oV}_+$, $|1\rangle$, $\langle 1|$) whereby ($\hat{^oV}$, $\hat{^oV}_+$, $|1\rangle$) is a complete, separable order unit space. Two additional axioms are proposed which have the effect that the space ($\hat{^oV}$, $\hat{^oV}_+$, $|1\rangle$) becomes isomorphic to the order unit space underlying a JB-algebra, at least in the case where $\hat{^oV}$ is finite-dimensional.

Open problem. (see KUMMER [71, p.52]). Search if KUMMER's additional axioms (see [71, p.43 and p.49]) have the same effect in the infinite-dimensional case.

KUMMER's work [71] has been inspired, as himself asserts, "by a deep and beautiful theorem of KOECHER [70] which characterizes finite-dimensional JB-algebras within the category of all partially ordered finite-dimensional vector spaces" (see [71, p.1]). With the help of this theorem, Kummer deduced the JB-algebra structure of quantum mechanics from four physically transparent axioms, at least in the finite-dimensional case. This work bears some recemblance to the paper [5] by ARAKI who, likewise confining himself to finite-dimensional case, is able to recover the JB-algebra structure of quantum mechanics from a few axioms. However, the physical content of Asaki's axioms remains rather obscure since he uses, as is castomary in this field of inquiry, primitive concepts of a highly idealized nature such as pure states, idealized filters, and the like.

By contrast, the work of KUMMER [71] takes a constructive approach to the same field of investigation; that is, Kummer's main primitive concept, the concept of a filter term, is quite a direct concept. The more idealized concepts, such as pure states, etc., appeared later within the theory as derived concepts.

Kummer's approach owes much to the work of GILES [46 a,b], who a long time ago proposed that an ideally formulated physical theory should have the form of an axiomatic theory, supplemented by a set of rules of interpretation of the logically primitive concept.

§ 6. Jordan (quantum) logics

In 1975 MOROZOVA and CHENTSOV [83 a] considered logics and quasilogics of subspaces of a finite-dimensional unitary space $\mathcal{H}$ with vector addition, orthogonal subtraction, isolation of contracts (noncommutative generalized meet) ot two subspaces, and coherent combination of isocline subspaces as logical operations (see the definitions below). Such lattices of events appear instead of Boolean algebras and rings in describing the logic of quantum phenomena. [83 b] gives a classification of $\mathbb{R}$-quasi-logics, similar to that of finite-dimensional special Jordan algebras and establishes a one-to-one correspondence between Jordan algebras of selfadjoint operators on $\mathcal{H}$ and $\mathbb{R}$-quasi-logics.

In what follows, we shall briefly recall MOROZOVA and CHENTSOV's results [83 a,b], and comment on the results due to ABBATI and MANIÀ [1 a,b], and BUNCE and WRIGHT [19].

Let $\mathcal{H}$ be a finite-dimensional unitary vector space over $\mathbb{C}$ endowed with a scalar product $\langle x|y\rangle = \overline{\langle y|x\rangle}$ which is linear in the second argument and antilinear in the first argument.

<u>Convention</u>. Linear subspaces of $\mathcal{H}$ will be denoted by italics, while orthoprojections on them are denoted by the corresponding printed letters.

For any subspaces $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{H}$ define the following operations

(6.1) $\mathcal{F} \to \mathcal{F}^{\perp}$, where $\mathcal{F}^{\perp} := \{x \mid x \in \mathcal{F}, \langle y|x\rangle = 0, \forall y \in \mathcal{F}\}$;

(6.2) $(\mathcal{F},\mathcal{G}) \to \mathcal{F}+\mathcal{G}$, where $\mathcal{F}+\mathcal{G} := \{ z | z = x+y,\ x\in\mathcal{F},\ y\in\mathcal{G}\}$;

(6.3) $(\mathcal{F},\mathcal{G}) \to \mathcal{F}\ominus\mathcal{G}$, where $\mathcal{F}\ominus\mathcal{G} := \{x | x\in\mathcal{F},\ \langle y|x\rangle = 0, \forall y\in\mathcal{G}\}$.

The cosine $\rho(\mathcal{F},\mathcal{G})$ of the minimal angle between $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\sup_{x\in\mathcal{F}} \frac{\langle x|G|x\rangle}{\langle x|x\rangle} = \rho(\mathcal{F},\mathcal{G}) = \sup_{y\in\mathcal{G}} \frac{\langle y|F|y\rangle}{\langle y|y\rangle}$$

and we have

$$\rho^2(\mathcal{F},\mathcal{G}) = \sup_{x\in\mathcal{F},\, y\in\mathcal{G}} \frac{\langle x|y\rangle\langle y|x\rangle}{\langle x|x\rangle\langle y|y\rangle}$$

where it is supposed that x and y are nonzero.

Note that $\rho(\mathcal{F},\mathcal{G}) = 0$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are orthogonal.

Definition. Subspaces which are not orthogonal to one another are called subspaces in contact.

We now define the following operation

(6.4) $$(\mathcal{F},\mathcal{G}) \to \mathcal{F} \,\mathrm{in}\, \mathcal{G},$$

where $\mathcal{F} \,\mathrm{in}\, \mathcal{G} := \left\{ x | x\in\mathcal{F},\ \sup_{0\neq y\in\mathcal{G}} \frac{\langle x|y\rangle\langle y|x\rangle}{\langle y|y\rangle} = \rho^2(\mathcal{F},\mathcal{G})\langle x|x\rangle \right\}$

in the case when $\mathcal{F}$ and $\mathcal{G}$ are in contact, and $\mathcal{F} \,\mathrm{in}\, \mathcal{G} = 0$ in the case when $\mathcal{F}$ and $\mathcal{G}$ are orthogonal.

Proposition 6.1. If $\mathcal{F}$ and $\mathcal{G}$ are subspaces in contact, then the operations $\rho^{-1}F$ and ^{-1}G give an isometry $I: \mathcal{F} \,\mathrm{in}\, \mathcal{G} \to \mathcal{F} \,\mathrm{in}\, \mathcal{G}$ as follows :

$$x \xrightarrow{I} \rho^{-1}G\,x = y \to \rho^{-1}F\,y = x.$$

Definition. Two subspaces $\mathcal{F}$ and $\mathcal{G}$ which are in contact are called isocline if $x|G|x = \rho(\mathcal{F},\mathcal{G})\langle x|x\rangle$, $\forall x\in\mathcal{F}$ and $\langle y|F|y\rangle = \rho(\mathcal{F},\mathcal{G})\langle y|y\rangle$, $\forall y\in\mathcal{G}$.

Proposition 6.2. If $\mathcal{F}$ and $\mathcal{G}$ are two isocline subspaces, then $\mathcal{F} = \mathcal{F} \,\mathrm{in}\, \mathcal{G}$ and $\mathcal{G} = \mathcal{G} \,\mathrm{in}\, \mathcal{F}$, and conversely.

For two isocline subspaces $\mathcal{F}$ and $\mathcal{G}$ define the operation

(6.5) $(\mathcal{F},\mathcal{G}) \to \mathcal{K}_{a:b}(\mathcal{F},\mathcal{G})$, where $\mathcal{K}_{a:b} := \{w = ax+b \updownarrow Ix \mid x \in \mathcal{F}\}$,

I being the canonical isometry given by Proposition 6.1, and $a,b \in \mathbb{R}$.

Definition. A system L of subspaces of $\mathcal{H}$ which is stable with respect to the operations (6.1), (6.2), (6.4), and (6.5) is called an $\mathbb{R}$-logic.

Note. If L is stable with respect to a proper operation (6.3) (i.e. defined only for pairs $\mathcal{F} \supseteq \mathcal{G}$) and to the operations (6.2), (6.4), and (6.5), then it is called an $\mathbb{R}$-quasilogic.

Definition. A system of subspaces of a unitary vector space is called a $\mathbb{C}$-logic (resp., a $\mathbb{C}$-quasilogic) if it is stable with respect to the operations (6.1) (resp., a proper operation (6.3)), (6.2), (6.4), and (6.5), $a,b \in \mathbb{C}$.

Remark. $\mathbb{C}$-logics were later used by MOROZOVA and CHENTSOV [83 c] in the study of the structure of the family of stationary stated of a quantum Markov chain.

Theorem 6.3. Let J be the Jordan algebra of selfadjoint operators acting on $\mathcal{H}$. Suppose that J is closed under the operator topology. Then the idempotents of J are esentially orthoprojections on the elements of an $\mathbb{R}$-quasilogic L_J consisting of subspaces of $\mathcal{H}$.

Remark. L_J from Theorem 6.3 is an $\mathbb{R}$-logic if and only if it contains the identity operator.

MOROZOVA and CHENTSOV [83 a] gave a nonclassical equality connecting the values of an operator-valued measure on a pencil of isocline subspace. They have also shown that if a linear space S of such measures contains the positive and negative parts of each $\mu \in S$, then the carriers of S-measure also form an $\mathbb{R}$-quasilogic. The difference between $\mathbb{C}$-logics and $\mathbb{R}$-logics (i.e. quantum logics of von Neumann and Jordan, respectively) are also discussed [83 a]

MOROZOVA and CHENTSOV [83 b] proved:

Theorem 6.4. For every $\mathbb{R}$-quasilogic L of subspaces of a finite-dimensional unitary vector space $\mathcal{H}$, there exists a corresponding Jordan algebra J_L of selfadjoint operators on $\mathcal{H}$. Conversely, for every Jordan algebra J of selfadjoint operators, there exists a corresponding $\mathbb{R}$-quasilogic of subspaces L_J, and we have

$$J_{L_J} = J, \quad L_{J_L} = L.$$

ABBATI and MANIÀ [1 b] developed a spectral theory for a particular class of μ-complete order unit spaces in terms of decision effects. These order unit spaces are associated to sum logics admitting a μ-complete set of expectation value functions.

Remark. The concrete representation of sum logics is an open problem.

If some conditions on the "spectral" order unit spaces arising from sum logics are added, one obtains JB-algebras and Alfsen 's representation theory can be used.

Thus, a spectral theory for order unit spaces may be of interest also in the representation theory for sum logics. However, the μ-completeness requirement on sum logics is, in general, not satisfied, and the duality for quantum logics does not completely correspond to duality for order unit spaces. Consequently, ABBATI and MANIÀ [1 a] developed a spectral theory for not necessary complete order unit spaces. In this theory they do not assume any duality, as is done in spectral theories, in terms of decision effect or projective units.

BUNCE and WRIGHT extended in [19 a] the Gleason-Christensen-Yeadon theorem (see [24, 128]) from von Neumann algebras to JBW-algebras, while in [19 b] they showed that a very large class of quantum logics (i.e. complete orthomodular lattices) may be identified with lattices of "projections" arising as natural geometric objects in certain convex sets. As an application, they gave a geometric characterisation of those logics which are isomorphic to the lattice of all projections in a von Neumann algebra on a JBW-algebra.

§ 7. Jordan structures and string theories

We shall present in the first part of this Section recent results that leads us to belive that octonions or exceptional Jordan algebra should play an important role in recent fundamental physical theories, namely, in the theory of superstrings. In the second part we shall point out how Jordan structures could be related to string theories via the infinite-dimensional Grassmann manifold method of SATO [99].

1. Let us briefly recall that the exceptional Jordan algebra ma e a dramatic appearence within the framework of supergravity theories through the work of GÜNAYDIN, SIERRA and TOWNSEN [56 a,b,c,d]. In their work on the construction and classification of N = 2 Maxwell-Einstein supergravity theories, they showed that there exist four remarkable theories of this type that are uniquely determined by simple Jordan algebras of degree three. These are the Jordan algebras of (3×3)-Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Their symmetry groups in five, four and three space-time dimensions give the famous magic square. From this largest one, namely the exceptional N = 2 Maxwell-Einstein supergravity defined by $H_3(\mathbb{O})^{(+)}$ emerge all the remarkable features of the maximal N = 8 supergravity theory in the respective space-time dimensions. In refs. [52 d, 56 a,b,d,e] it was speculated that a larger theor that includes the exceptional N = 2 theory and the N = 8 theory may provide us with a unique framework for a realistic unificatior of all known interactions. Such a theory, if it exists, may well

turn out to be a string theory (see GÜNAYDIN and HYUN [54, p.498]).

Remark. The work by Günaydin, Sierra and Townsend was reviewed by TRUINI in [115] which aimed at indicating the usefulness and naturalness of implementing the Jordan pair language in such a theory.

A crucial question in superstring theory is the following: What mathematical structures have a large degree of uniqueness and can also be associated with strings? FOOT and JOSHI suggested in [41a] that the exceptional Jordan algebra may be such a structure. This algebra is indeed unique as it is the only formally real Jordan algebra whose elements cannot be expressed in terms of real matrices. Although quantum mechanically superstring theories appear to be consistent only in ten space-time dimensions, classically superstring theorie are consistent in space-time dimensions of 3,4,6 and 10. These dimensions are suggestive of the sequence of division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ whose respective dimensions correspond to the number of transverse degree of freedom in d = 3,4,6 and 10. These remarks prompted FOOT and JOSHI [41 a] to look for mathematical structures which automatically single out d = 3, 4, 6 and 10 with d = 10 perhaps appearing special. They investigated the sequence of Jordan algebras $H_3(\mathbb{K})^{(+)}$ consisting of (3×3)-Hermitian matrices over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and showed that variables of the superstring can be interpreted as elements of the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$. The other algebras in this sequence correspond to classical superstring theories. One of the motivations for introducing the sequence of algebras $H_3(\mathbb{K})^{(+)}$ is that it is naturally supersymmetric: for $H_3(\mathbb{R})^{(+)}$, the spinor corresponds to a Majorana spinor of SO(2,1), for $H_3(\mathbb{C})^{(+)}$, the spinor corresponds to a Weyl spinor of SO(3,1), for $H_3(\mathbb{H})^{(+)}$, the spinor corresponds to a Weyl spinor of SO(5,1) and for $H_3(\mathbb{O})^{(+)}$, the spinor corresponds to a Majorana-Weyl spinor of SO(9,1). In each case the number of spinor degrees of freedom agrees with the number of vector degrees of freedom. Thus the sequence automatically incorporates equal Bose and Fermi degrees of freedom.

In FOOT and JOSHI's approach [41 a], transverse Lorentz rotations are contained in the automorphism group of the algebra $H_3(\mathbb{K})^{(+)}$.

In conclusion, the classical superstring theories can be expressed in a unified way using sequence $H_3(\mathbb{K})^{(+)}$. Furthermore the d = 10 case is especially interesting as it corresponds to the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$.

As FOOT and JOSHI pointed out [41 a], the GREEN-SCHWARZ [50 a,b] superstring is not the only mathematically consistent candidate for a unified theory of all interactions. Nevertheless, Foot and Joshi analysed the superstring because of its central role in the other string theories. Of particular interest is the heterotic string (see GROSS, HARVEY, MARTINEC and ROHM [51 a,b]), which can incorporate the exceptional gauge group $E_8 \otimes E_8$. The appearence of the exceptional group $E_8 \otimes E_8$ is interesting because E_8, like F_4 can be related to octonions.

In 1986 WITTEN [124 a] made some interesting remarks concerning a new approach to string field theory. Witten attempted to interpret the interactions of the open bosonic string in terms of noncommutative differential geometry. Furthermore he suggested that closed bosonic strings may be connected with some kind of commutative but nonassociative algebra.

Motivated by Witten's ideas, FOOT and JOSHI investigated in [41 b] the incorporation of Jordan algebras, to obtain a manifestly <u>commutative</u> but <u>nonassociative</u> string theory. Namely, they showed that the free bosonic string theory can be reformulated using the special Jordan algebra. Then they proceded to incorporate the exceptional Jordan algebra into the bosonic string. This leads to an exceptional group structure at the level of first quantization, which they interpreted as the appearence of the gauge group.

However, as Foot and Joshi pointed out, they are unable to construct the general N-point scattering probability, and thus a conclusive proof of the consistency of their model, at the interacting level is lacking. So, further work is required and it emerges the following

Open problem. Establish whether in fact Foot-Joshi string is consistent. In particular the construction of the N-point scattering probability warrants attention.

Comments. The above-mentioned open problem appears to be a difficult one, possibly requiring a new type of field theory based on the nonassociative Jordan formulation of quantum mechanics

Remark. The appearence of the transformation group SO(8) in Foot-Joshi's approach [41 b] suggests that a matrix of the exceptional Jordan algebra with fixed eigenvalues may be related to d = 10. It may thus be possible to incorporate this work into the heterotic string, which consists of closed bosonic strings in d = 26, and d = 10 fermionic strings [51 a,b].

Let us mention now the work of LI, PESCHANSKI, and SAVOY [74] by which a generalization of no-scale supergravity models is presented, where scale transformations and axion-like classical symmetries of the superstrings in four-dimensions are explicitly realized as dilatations and translations of the scalar fields in the Kähler manifold. A sufficient condition is that the (dimension one) dilaton field matrices can be arranged in matrices of a Jordan algebra. This determines four possible classes of irreducible manifolds which are symmetric spaces. An arbitrary number of matter (dimension one half) fields can be included in the Kähler potential in such a way to preserve the algebra of isometries. This inclusion defines a generalization of flat potential models with zero cosmological constant and scalar-fermion degeneracy except for massive fermions along the flat directions of scalar potential. For two classes of manifolds and trilinear superpotential, a SU (1,1) $\times$ U (1) subgroup can be promoted to an exact symmetry of the effective Lagrangean.

GODDARD, NAHM, OLIVE, RUEGG and SCHWIMMER [47] analysed the algebraic structure of dependent fermions, namely ones interrelated by the vertex operator construction. They are associated with special sorts of lattice systems which are introduced and discussed. The explicit evaluation of the relevant cocycles leads to the results that the operator product expansion of the fermions is related in a precise way to one or other of the division algebras given by $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. In ref.[96] RUEGG showed that from the fermionic operator product expansion one can define a product with the same algebraic properties as the Jordan product.

The Goddard-Nahm-Olive-Ruegg-Schwimmer octonion result has an important physical application in the formulation of the superstring theory of particle interactions. The fermionic vertex operators related to octonions are associated with short roots of F_4 and fall into three orbits under the action of the Weyl group of D_4, the subalgebra of F_4 defined by its long roots D_4= so(8) is the residue of the Lorentz invariance group of the superstring in the light cone gauge. In superstring theory the fermionic vertex operators are familiar and important constructions. For points of the orbit constituting vector weights of D_4 they are Ramond/

Neveu-Schwarz fields. For one of the other two orbits, comprising spinor or conjugate spinor weights, they are the fermion emission-absorbtion vertices (see [49]). Thus the algebra of these quantities which is essential to the evaluation of superstring scattering amplitudes appears to be related to the algebra of octonions or to the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$.

Let us mention also the works of FAIRLIE and MANONGE [35], SIERRA [107 b], CHAPLINE and GÜNAYDIN [23], and GÜRSEY [58 g] who spectulated on the possible role that the exceptional Jordan algebra may play in the framework of string theories.

FERREIRA, GOMEZ and ZIMERMAN [38 a] discussed the contruction of Lie algebra in terms of Jordan algebra generatora. A generalisation to Kac-Moody algebras in terms of vertex operators is proposed and may provide a clue for a construction of new representations of Kac-Moody algebras in terms of Jordan fields (For Jordan fields as a generalizatioh of Fermi fields see [38 b]).

GÜNAYDIN and HYUN [54] gave a stringly construction of the exceptional Jordan algebra $H_3(\mathbb{O})^{(+)}$. Specifically, they constructed $H_3(\mathbb{O})^{(+)}$ using Fubini-Vene iano vertex operators. This is a very special application of a géneral vertex operator construction of nonassociative algebras and their affine extensions developed recently by GÜNAYDIN [52 e]. This construction gives not only $H_3(\mathbb{O})^{(+)}$ but also its natural affine extension in terms of the vertex operators.

GÜRSEY [58 h] considered the discrete Jordan algebras of (1×1)— (2×2)- and (3×3)-Hermitian matrices over integer elements of the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They are transformed under discrete subgroups of groups associated with the magic square. Points corresponding to a discrete Jordan matrix belong to a lattice generated by Weyl reflections that are expressed by means of Jacobson's triple product. Special cases include the $O(32)$, $E_8 \times E_8$ and E_{10} lattices that occur in superstring theories.

2. As we presented in Section 8 of JSA.V., SATO and SATO [100] have an outstanding contribution to the study of the so-called soliton equations.

In is well known that a soliton is a nonlinear wave whose properties are characterized as follows:

A. a localized wave propagates without changing its properties (shape, velocity, etc.);

B. localized waves are stable against mutual collisions and each wave conserves its individuality.

The first property has been known in hydrodynamics since the midle of the last century as a solitary wave condition. The second means that the localized wave behaves like a particle. In modern physics, a suffix"on" implies the particle property, for instance, phonon and photon. In 1965, emphasizing the particle-like behaviour of the solitary wave, Zabusky and Kruskal called waves with the properties A and B "soliton". (For more hystorical details, see, for instance, WADATI and AKUTSU [121]).

For an elementary introduction to SATO theory we refer the reader to the paper [88] by OHTA, SATSUMA, TAKAHASHI and TOKIHIRO. Starting with an ordinary differential equation, introducing an infinite number of time variables, and imposing a certain time dependence on the solutions, they obtained the Sato equation which governs the time development of the variable coefficients. It is shown that the generalized Lax equation, the Zakharov-Shabat equation and the inverse scattering transform scheme are generalized from the Sato equation. It is also revealed that the τ-function becomes the key function to express the solution of the Sato equation. By using the results of the representation theory of groups, they showed that the τ-function is governed by the partial differential equations in the bilinear forms which are closely related to the Plücker relations.

TAKASAKI, inspired by Sato's theory for soliton equations, gave in [112 a,b,c] a new approach to the self-dual Yang-Mills equations, which is an alternative method also based on the viewpoint of a complet integrability. It is remarkable, that the self-dual Yang-Mills equations admit such an approach parallel to Sato's approach to soliton equations.

Remark. A close relationship with MULASE's method [86] can be pointed out.

An application of the above-mentioned Takasaki's approach would be expected to higher dimensional generalizations of gauge field equations.

Another application in eight dimensions was solved by SUZUKI [111 a,b] using Grassmann manifold method. Witten's gauge fields are interpreted by SUZUKI [111 c] as motions on an infinite-dimensional Grassmann manifold. Unlike the case of self-dual Yang-Mills equations in TAKASAKI's work [112 a,b],the initial data must satisfy a system of differential equations since Witten' equations comprise a pair of spectral parameters. Solutions corresponding to (anti-) self-dual Yang-Mills fields are characterized in the space of initial data and in application, some Yang-Mills fields which are not self-dual, anti-self-dual nor abelian can be constructed.

Let us also mention the JIMBO and MIWA's approach to the theory of soliton equations [63]. They considered an infinite-dimensional Lie algebra and its representation on a function space. The group orbit of the highest weight vector is an infinite-dimensional Grassmann manifold. Its defining equations on the function space, expressed in the form of differential equations, are then exactly the soliton equations. To put it the other way, there is a transitive action of an infinite-dimensional group on the manifold of solutions.

MANIN and RADUL [78] gave a supersymmetric extension of the one-component KP hierarchy as the Lax equations. The finite-dimensional version of the KP hierarchy was called by Ueno the Grassmann hierarchy. In the theory of Grassmann hierarchy the fundamental role is played by a linear algebraic equation which is called the Grassmann equation. UENO and YAMADA gave in [118 a,b] a supersymmetric extension of one-component hierarchies from the viewpoint of the Grassmann equation. Their approach is slightly different from that of MANIN and RADUL [78]. YAMADA generalized in [126] the results of [118 a,b] to the multicomponent case. In ref. [118 c], UENO and YAMADA revealed that the super KP hierarchy is equivalently transformed to the super Grassmann equation that connects a point in the universal super Grassmann manifold with an initial data of a solution.

As TAKASAKI pointed out [112 d], recently, physicists have come to recognize the relevance of the theory of universal Grassmann manifold (sketched by SATO and SATO [100]) to physical new topics, such as conformal field theories and strings (see ISHIBASHI, MATSUO and OOGURI [62], VAFA [119], ALVAREZ-GAUMÉ, GOMEZ and REINA [4], WITTEN [124 b], KAWAMOTO, NAMIKAWA, TSUCHIYA and YAMADA [68], MICKELSSON [80 a], see also ARBARELLO, DE CONTINI, KATS and PROCESI [6] for an application to the moduli geometry of algebraic curves which has a close relation to string) and anomalies (see MICKELSSON [80 b,c] and MICKELSSON and RAJEEV [81]). Almost all of them are based on the framework developed by SEGAL and WILSON [101] and PRESSLEY and SEGAL [93]. Their functionalanalytical formulation have a number of advantages, and is now widley recognized as a standard framework. Admitting this fact, TAKASAKI have rewritten everything in the spirit of SATO and SATO [100]. Their highly abstract and algebraic standpoint is fairly distinct from common sense of most physicists, who are much more familiar with the use of Hilbert spaces rather than abstract vector spaces. As TAKASAKI remarked [112 d,pp.3-4],"the algebraic method however has several advantages despite of these unfamiliar features, the most important being the fact that one can develope a theory not only on the basis of real and complex numbers but also within a more abstract world such as that of p-adic numbers".

A particular choice of affine coordinates on Grassmann manifolds, for both the finite - and infinite - dimensional case, made by TAKASAKI [112 d] turns onto to be very useful for the understanding of geometric structures therein. The so-called "Kac-Peterson cocycle", which is physically a kind of "commutator anomaly", then arises as a cocycle of a Lie-algebra of infinitesimal transformations on the universal Grassmann manifold. These ideas are extended in [112 d] to a multi-component theory. A simple application to a nonlinear realization of current and Virasoro algebras is also presented for illustration in [112 d].

SAITO [98 b] (see also [98 c]) showed that the vertex operator of the three-bosonic-string interaction of Della Selva and Sa to (see [27]) is an element of the universal Grassmann manifold. The correspondence between string theories and soliton theories is made explicite through the transformation of evolution parameters of solitons to string coordinates, the same transformation which relates Fay's trisecant formula (see [37]) to Hirota's bilinear difference equation (see [61]).

GILBERT [45], based on the approach to infinite Grassmannians as the space of solutions of KP equations (see [106], [86 a,b],

[31]), described in simple terms the infinite sequence of non-linear partial differential equations (the KP equations) and gave possible applications to a fundamental description of interacting strings. Gilbert also indicated in [45] lines of research likely to prove useful in formulating a description of non-perturbative string configurations.

An interesting connection between Witten's string field theory and the infinite Grassmannian, and the possible characterization of the group orbit on the Grassmannian by the bilinear identity are examined by GAO [43].

AWADA and CHAMSEDDINE introduced [8 a] the infinite-dimensional graded Grassmann manifolds in terms of free field operators and studied their properties. They showed the embedding of the graded Diff S^1/S^1 manifold in the graded Grassmannians, and commented on the possible supersymmetric KP hierarchy.

Let us recall at this point that there are two attractive views of string theory, both based on holomorphic geometry. The first is the formulation of quantum string theory as integrable analytic geometry on the universal moduli space of Riemann surfaces. The second is based on the concept of loop space and formulated as a holomorphic vector bundle over the manifold Diff S^1/S^1. In both cases, there exists an one-to-one embedding of the base manifold into the infinite-dimensional Grassmannians. As AWADA and CHAMSEDDINE pointed out [8 a]",there are various advantages of working with the Grassmannians, mainly that most computations become algebraic as well as having the promise of providing a non-perturbative treatment for moduli spaces of all Riemann surfaces, including the infinite genus one".

Recently, AWADA and CHAMSEDDINE [8 b] formulated the closed string theory as Hermitian geometry on Grassmannians.

Open problem (see [8 b]). Generalise the Awada-Chamseddine approach [8 b] to the closed superstring and heterotic string.

As we already mentioned, SEGAL and WILSON [101], and PRESSLEY and SEGAL [93] developed a framework which is a different approach to infinite Grassmannians. It consists of the space of choices of fermion boundary conditions for the free fermion field theory on a disc. In the ref. [123] is described how the modified KdV equations fit into the Grassmannian framework, topic not tuched in ref. [101]. Recently, WITTEN [124 b] clarified some aspects of the relation between quantum field theory and infinite-dimensional Grassmannians. More precisely, he described in physical terminology some aspects of relation, surveyed by SEGAL and WILSON [101] between Riemann surfaces and infinite-dimensional Grassmannians. This relation has been essential in recent studies of the Schottky problem (see MULASE [86 b], SHIOTA [106]), and its relation with quantum field theory and string theory have been subject of recent discussion from a physical point of view (see ISHIBASHI, MATSUO, OOGURI [62], ALVAREZ-GAUMÉ, GOMEZ, REINA [4], VAFA [119]).

MICKELSSON and RAJEEV [81] extended the methods of PRESSLEY and SEGAL [93] for constructing cocycle representations of the restricted general linear group in infinite dimensions to the case of a larger linear group modeled by Schatten classes of rank $1 \leq p < \infty$ (see SIMON [108]). An essential ingredient is the generalization of the determinant line bundle over an infinite-dimensional Grassmannian to the case of an arbitrary Schatten rank $p \geqslant 1$. The results are used to obtain highest weight representations of current algebras in d+1 dimensions when the space dimension d is any odd number.

Conjecture (see SEMENOFF [103]). Similar problems to that of MICKELSSON and RAJEEV [81] must afflict the electric field operators constructed by SEMENOFF in [103].

Recently, YAMAGISHI [127] pointed out an interesting relation between the KP hierarchy and the extended Virasoro algebra, namely, he showed that the simply extended KP equation has enough information to determine the extended Virasoro algebra. LEVI and WINTERNITZ [73] showed that a class of integrable nonlinear differential equations in 2+1 dimensions, including the physically important cylindrical KP equation, has a symmetry algebra with a specific Kac-Moody-Virasoro structure. KODAMA [69] presented a systematic method to produce a class of exact solutions of the dispersionless KP equation, using the conservation equations derived from the semi-classical limit of the KP theory. These exact solutions include rarefaction waves (global solutions) and shock waves (breaking solutions in finite time). ZABRODIN [129] proved that the scattering matrix for free massless fermions on a Riemann surface of finite genus generates the quasiperiodic solutions of the KP equation. The operator changing the genus of the solution is constructed and the composition law of such operators is discussed. Zabrodin's construction extends the well-known operator approach in the case of soliton solutions to the general case of the quasiperiodic τ-functions. DAVID, LEVI and WINTERNITZ [26] constructed a general class of fourth order scalar partial differential equations, invariant under the same group of local point transformations as the KP equation.

Finally, let us refer on other papers of interest, as follows:

EVANS [34] established an explicit correspondence between simple super Yang-Mills and classical superstrings in dimensions 3,4,6,10 and the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. A gamma matrix identity necessary and sufficient for their existence is shown to yield trialities, objects which are equivalent to division algebras. Evans interpreted then the identities necessary for supersymmetry from a more mathematical point of view related to the work [107 b] by SIERRA.

Various aspects of the connection between Kähler manifolds and string theories are examined by RAJEEV [94 a] (see also BOWICK and RAJEEV [18]), ZANON [131], CECOTTI, FERRARA, GIRARDELLO and PORRATI [21].

Comments. As we already mentioned in § 8 of JSA.V., it would be very interesting to find an algebraic (may be Jordan) description for the infinite-dimensional Grassmann manifold appearing in Sato's approach, corresponding to the Jordan structure description of finite-dimensional Grassmann manifolds presented in § 2 of JSA.III. and § 3 of JSA.VI. At any case, it would be fruitful to make use of the Jordan algebra description of finite-dimensional Grassmann manifolds which correspond (see § 8 of JSA.V) to rational solutions of soliton equations.

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JORDAN STRUCTURES WITH APPLICATIONS. IX.

JORDAN ALGEBRAS IN MATHEMATICAL BIOLOGY.

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This paper describes the applications of Jordan algebras to population genetics and color perception, as well as their possible use in bioenergetics.

§ 1. Jordan algebras in population genetics

Note. For a comprehensive account on algebras in genetics up to 1980, the reader is referred to WÖRZ-BUSEKROS' monograph [32 a].

ETHERINGTON [6 a, b] showed how a nonassociative algebra can be made to correspond to a given genetic system. The fact that many of these algebras have common properties has prompted their study from a purely abstract standpoint. Furthermore, these algebraic studies gave new ways of tackling problems in genetics.

In a study of nonassociative algebras arising in genetics, SCHAFER [23] proved that the so-called gametic and zygotic algebras (see ETHERINGTON [6 b]) for a single diploid locus are Jordan algebras.

HOLGATE [12 a] proved Schafer's results by methods which do not make use of transformation algebras (employed by SCHAFER [23]), which therefore accommodate the multi-allelic case more easily, and in which the main object is to maximise the interplay between the algebraic formalism and the genetic situation to which it corresponds.

The first part of the present section deals with these results as treated by HOLGATE [12 a], while the second deals with the results due to PIACENTINI CATTANEO [20], WÖRZ-BUSEKROS [32 c,d], and WALCHER [31].

We consider the gametic algebra $\mathcal{G}$ of a single locus with n+1 alleles, i.e., the algebra $\mathcal{G}$ over $\mathbb{R}$ with basis $\{a_o, \ldots, a_n\}$, whose elements correspond to the actual allelic forms, its multiplication table being

$$a_i a_j := \frac{1}{2}(a_i + a_j).$$

For an element $x = \sum_{i=o}^{n} x_i a_i$, the weight w is defined by

$$w(x) := \sum_{i=o}^{n} x_i .$$

Remark. It is easily seen that $x^2 = w(x)x$.

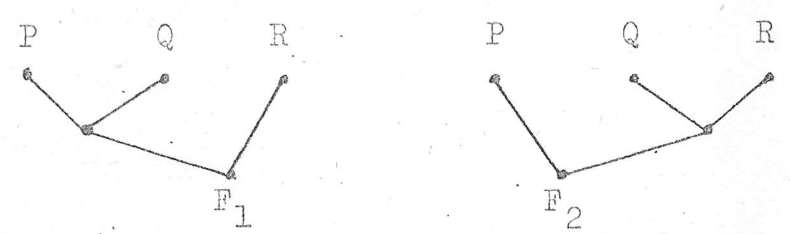
Proposition 1.1 (Algebraic). Every element of unit weight in $\mathcal{G}$ is idempotent.

(Genetic). In the absence of selection, the gametic proportions remain constant from one generation to another.

Remark 1. The algebraic result is more comprehensive, since only those elements of unit weight for which all the x_i are non-negative correspond to populations.

Remark 2. The nonassociativity of genetic algebras corresponds to the fact that if P, Q, and R are populations and if P and Q mate and the offspring mates with R, the final result is, in general, different from that arising from mating between P, and the offspring of mating between Q and R. The two situations are shown in the diagram below:

(1.1)



Proposition 1.2 (Algebraic). The algebra $\mathcal{G}$ is a Jordan algebra.

(Genetic). In the mating schemes shown in (1.1), the populations F_1 and F_2 have the same genetic proportions if P is the offspring of mating of R with itself.

Notation. The algebra $\mathcal{Z}$, corresponding to proportions of zygotic types, is formed by duplicating $\mathcal{G}$ (see ETHERINGTON [6 a,c]): its basis elements are pairs (x,y) of basis elements of $\mathcal{G}$ with the multiplication rule $(x,y)(u,v):=(xy,uv)$. A canonical basis may be taken in $\mathcal{G}$ by setting: $c_o:=a_o$, $c_i:=a_o-a_i$ $(i\neq 0)$, for which the multiplication table is

$$c_o^2 = c_o, \quad c_o c_i = \frac{1}{2} c_i, \quad c_i c_j = 0 \qquad (i,j \neq 0).$$

Then on writing $d_{ij} := (c_i, c_j)$, the multiplication table for the duplicate $\mathcal{Z}$ can be written as

$$d_{oo}^2 = d_{oo}, \; d_{oo}d_{oi} = \frac{1}{2} d_{oi}, \; d_{oi}d_{oj} = \frac{1}{4} d_{ij} ,$$

other products being zero $(i,j \neq 0)$.

Remark. The weight of an element $x = \sum_{i,j=o}^{n} x_{ij}d_{ij}$, $x_{ij}=x_{ji}$ is $w(x) = d_{oo}$.

Proposition 1.3 (Algebraic). Every element of the form $y := x^2 - w(x)x$ annihilates $\mathcal{Z}$.

(Genetic). The extent to which the zygotic proportions in a population differ from the Hardy-Weinberg equilibrium state has no effect on the offspring distribution produced by mating between this population and any other.

Proposition 1.4. The algebra $\mathcal{Z}$ is a Jordan algebra.

Remark. Let $\mathcal{A}$ be the algebra over $\mathbb{C}$ with basis $\{a_o,\ldots,a_n\}$ whose multiplication table is

(1.2) $$a_i \circ a_j := a_i \ .$$

Obviously, $\mathcal{A}$ is associative. Consider the special Jordan algebra $\mathcal{A}^{(+)}$ obtained from the vector space $\mathcal{A}$ by means of the product $xy := \frac{1}{2}(x.y + y.x)$. It can easily be seen that $\mathcal{A}^{(+)}$ is isomorphic to $\mathcal{G}$.

If it were possible to know in advance that the genes of only one of two given populations mating together are transmitted to the offspring, these could be written first in the product, and the system would correspond to the multiplication table (1.2). The fact that $\mathcal{G}$ is a special Jordan algebra appears as a consequence of inheritance being symmetric in the parents.

Recall from GONSHOR [8 b, I] the following

Definition. A special train algebra is a commutative algebra over $\mathbb{C}$ for which there exists a basis $\{a_o, \ldots, a_n\}$ with a multiplication table of the following kind: $a_i a_j := \sum x_{ijk} a_k$, where

(1.3) $$x_{ooo} = 1$$

(1.4) $$\text{for } k < j, \qquad x_{ojk} = 0,$$

(1.5) $$\text{for } i,j > 0, \ \ k \leq \max(i,j), \quad x_{ijk} = 0,$$

and all powers of the ideal $(a_1, a_2, \ldots, a_n)$ are ideals. (The powers I^r of an ideal I are defined by $I^r := I^{r-1} I$.)

Remark. A commutative algebra over $\mathbb{C}$ for which only conditions (1.3), (1.4) and (1.5) are required was called by GONSHOR genetic algebra (see [8 a]). Schafer's concept of genetic algebra coincides with that of Gonshor (see GONSHOR [8 a, Theorem 2.1]) [1]. WÖRZ-BUSEKROS defined [32 b] three kinds of noncommutative Gonshor genetic algebras and characterized them in terms of matrices.

Comments. Let us mention in this respect that in the mathematical theory of algebras in genetics, whose origins are in several papers by Etherington, fundamental contributions have been made by Schafer, Gonshor, Holgate, Reiersøl, Heuch and Abraham (for a detailed account see [32 a]).

Definition. The x_{ojj} are called the train roots of the algebra. (They are the characteristic roots of the operator which is multiplication by a_o.)

Remark. From SCHAFER [23, Theorem 5], it follows that a special train algebra can only be a Jordan algebra if its train roots all have values among 1, $\frac{1}{2}$, 0. This excludes the genetic algebras corresponding to polyploidy of several loci. Therefore, the appearance of Jordan algebra seems to be bound up with the property of attaining equilibrium after a single generation of mating.

PIACENTINI CATTANEO considered [20] the gametic algebra $\mathcal{G}$ (see the beginning of this section) of simple Mendelian inheritance. Suppose that mutation occurs in the chromosomes, i.e. suppose that a rate of alleles a_i mutate into the alleles a_j, $j \neq i$. If we denote this rate by r_{ij} (setting $r_{kk} = 0$), we can construct a new algebra, denoted by $\mathcal{G}_m$, called a gametic algebra of mutation (see [20, p.180]). The new multiplication table then is

1) Concerning genetic algebras, the fundamental idea has been to define a basis $\{G_1, \ldots, G_n\}$ with a one-to-one correspondence to the genotypes $g_1, \ldots, g_n$ considered, and then give a multiplication table so that the product G_iG_j of two basis elements be equal to a linear combination $\sum p_{ijk} G_k$, where p_{ijk} is the probability of getting genotype g_k in a cross between g_i and g_j individuals.

$$a_j^2 = (1 - \sum_{i=o}^{n} r_{ji})a_j + \sum_{i=1}^{n} r_{ji}a_i ;$$

$$a_j a_k = \frac{1}{2}(1 - \sum_{i=o}^{n} r_{ji})a_j + \frac{1}{2}\sum_{i=o}^{n} r_{ji}a_i + \frac{1}{2}(1 - \sum_{i=o}^{n} r_{ki})a_k + \frac{1}{2}\sum_{i=o}^{n} r_{ki}a_i ,$$

$$(j \neq k).$$

Proposition 1.5. Let $\mathcal{G}_m$ be a gametic algebra of mutation, with mutation rates r_{ij}. For $\mathcal{G}_m$ to be a Jordan algebra, it is necessary and sufficient that the following system of n identities in x_k(k=o,1, ..., n) holds:

$$(1.6) \quad \sum_{j=1}^{n} \beta_j (r_{oi} - r_{ji}) + \beta_i (1 + \sum_{k=o}^{n} r_{ik}) = 0, \quad i=1,\dots, n,$$

where

$$\alpha_t = \alpha_t(x_o, \dots, x_n) = (\sum_{k=o}^{n} x_k) r_{ot} - \sum_{k=1}^{n} x_k r_{kt} + \sum_{k=o}^{n} x_t r_{tk} ,$$

$$\beta_j = \beta_j(x_o, \dots, x_n) = (1 - \sum_{k=o}^{n} r_{jk})\alpha_j - \sum_{t=1}^{n} \alpha_t (r_{oj} - r_{tj}).$$

In the same paper [20], PIACENTINI CATTANEO used conditions (1.6) to determine the restrictions of the r_{ij}'s for $\mathcal{G}$ to be a Jordan algebra in specific cases.

Recently, PERESI [19 a] proved that if A is a nonassociative algebra that verifies $A^2 = A$ and has an idempotent, then A and its duplicate have isomorphic automorphism groups and isomorphic derivation algebras. This result is then applied by Peresi to the gametic algebra for polyploidy with multiple alleles.

Definitions. An algebra A, not necessarily associative, over a commutative field $\mathbb{K}$ of characteristic different from two, that admits a nontrivial homomorphism $w : A \longrightarrow \mathbb{K}$ is said to be baric.

A baric algebra A that satisfies the identity $(a^2)^2 = w^2(a)\, a^2$ for all $a \in A$ is called, a Bernstein algebra.

Remark. SINGH and SINGH [27] showed that Lie and Clifford algebras are never baric. On the other hand, starting with a baric algebra, it is possible to derive new algebras which are Lie, Jordan, alternative or associative.

Every Bernstein algebra A possesses at least one idempotent e. It can be decomposed into the direct sum of subspaces $A = E \oplus U \oplus Z$ with $E := Ke$, $U := \{ ey \mid y \in \operatorname{Ker} w \}$, $Z := \{ z \in A \mid ez = o \}$.

Is A has finite dimension, wich is at least 1, dim A=1+n, then one can associate to A a pair of integers (r+1,s), called type of A, whereby

$$r := \dim U \quad , \quad s := \dim Z ,$$

hence $r + s = n$.

Recently, WÖRZ-BUSEKROS [32 d] showed that for each decomposition $n = r + s$ there exists a Bernstein algebra of type $(r + 1, s)$. Thereby the so-called trivial Bernstein algebra of type $(r + 1, s)$ has been introduced as Bernstein algebra of the corresponding type where $(\operatorname{Ker} w)^2 = \{o\}$.

WÖRZ-BUSEKROS [32 c] showed that the well-konown decomposition of a Bernstein algebra with respect to an idempotent is nothing else but the Peirce decomposition known for finite-dimensional, power-associative algebras with idempotent, especially for Jordan algebras with idempotent.

Note. Bernstein algebras are not in general power-associative.

In terms of Peirce theory, WÖRZ-BUSEKROS [32 c] showed that in a Bernstein algebra all idempotents are principal and thus primitive. Hence, the Peirce decomposition cannot be further decomposed. She deduced a necessary and sufficient condition for a Bernstein algebra to be Jordan, and obtained a number of special results from it (the principal two being Proposition 1.6 and Theorem 1.7 below).

Proposition 1.6 (see WÖRZ-BUSEKROS [32 c, p.396]. A trivial Bernstein algebra of type (r+1, s) is a special Jordan algebra.

Remark. Proposition 1.6 is a generalization of HOLGATE's result [12 a] (see Proposition 1.4 above and Remark which follows), who proved that all gametic algebras for simple Mendelian inheritance are special Jordan algebras. Thereby the gametic algebra for simple Mendelian inheritance with n+1 alleles is a trivial Bernstein algebra of type (n+1,o), cf. WÖRZ-BUSEKROS [32 d].

Definition. Let A be an algebra over $\mathbb{K}$ with weight homomorphism $w: A \longrightarrow \mathbb{K}$. Then A is called a normal algebra, if the identity $x^2y = w(x)xy$ is satisfied in A.

Theorem 1.7 (see WÖRZ-BUSEKROS [32 c, p.397]). Every normal algebra is a Jordan algebra.

Recently, WALCHER [31] gave a characterization of Bernstein algebras which are Jordan algebras (called by him Jordan Bernstein algebras) over a field of characteristic different from 2 or 3, and listed some of their properties.

Theorem 1.8 (see WALCHER [31, p.219]). Let A be a baric algebra over a field of characteristic different from 2 or 3, and w the nontrivial homomorphism from the definition of A. The following statements are equivalent:

(i) A is a Jordan Bernstein algebra

(ii) A is a power-associative Bernstein algebra

(iii) $x^3 - w(x)x^2 = 0$ for all $x \in A$.

As a corollary of Proposition 1 from WALCHER [31], it follows that every Jordan Bernstein algebra is genetic. Thus, by WÖRZ-BUSEKROS [32 a, Theorem 3.18], for dim A=m+1, we have a chain of ideals of A

$$N := \operatorname{Ker} w \supset N_1 \supset N_2 \supset \dots \supset N_m \supset \{0\}$$

such that $\dim N_i = m + 1 - i$ and $N_i N_j \subset N_{k+1}$, where $k := \max \{i, j\}$, for all i and j.

Notation. Let c be an idempotent of A from Theorem 1.8 above, and let L(c) denotes, as usual, the left multiplication by c.

Proposition 1.9 (see WALCHER [31, p.221]). Let A be a Jordan Bernstein algebra of dimension m + 1. Then there exists a basis $\{v_1, \dots, v_m\}$ of N such that v_i is an eigenvector of L(c) for $1 \leq i \leq m$ and N_i is spanned by $v_i, \dots, v_m$ $(1 \leq i \leq m)$.

Comments. The WALCHER's results [31] should at least make the construction of Jordan Bernstein algebras a manageable task: Start with a basis of eigenvectors in N (the eigenvalues preassigned), take into account the composition rules for the eigenspaces and note that the only thing to be checked besides this is the identity $x^3 = 0$ in N.

Recently, HOLGATE [12 e] examined conditions under which the entropic law is satisfied in genetic algebras, and the consequences of imposing it when it is not. It appears that, as with the Jordan identity (see HOLGATE [12 a], and MICALI and QUATTARA [17]), the entropic law only interacts incisively with the properties of genetic algebras for small rank or dimension.

§ 2. Jordan algebras and color perception

This section deals with some of the results given by RESNIKOFF [21].

In order to endow the set $\mathcal{C}$ of perceived colors with a geometrical structure, various standard experimental results are taken as axioms. One can show that there exists a real vector space $\mathcal{V}$ spanned by the set $\mathcal{C}$ in which $\mathcal{C}$ is a cone of perceived colors. Denote by GL($\mathcal{C}$) the group of orientation-preserving linear transformations of $\mathcal{V}$ which preserve the cone $\mathcal{C}$. GL($\mathcal{C}$) is a subgroup of GL($\mathcal{V}$), and therefore a Lie group.

Making use of standard results in the theory of homogeneous spaces, $\mathcal{C}$ can be identified with the homogeneous space GL($\mathcal{C}$)/K, where K is isomorphic to the subgroup of GL($\mathcal{C}$) which leaves some point of $\mathcal{C}$ fixed, hence to a closed subgroup of the orthogonal group, and consequently to a compact subgroup of GL($\mathcal{C}$).

Finally it follows that $\mathcal{C}$ is a homogeneous space equivalent either to $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ or to $\mathbb{R}^+ \times SL(2,\mathbb{R})/SO(2)$, $\mathbb{R}^+$ denoting the positive real numbers.

The GL($\mathcal{C}$)-invariant metric (see (2.7) below) yield in the first case STILES' generalization [28] of HELMHOLTZ' color metric [10], and in the second a new color metric with respect to which $\mathcal{C}$ is not isometric to a Euclidean space.

RESNIKOFF [21 a] showed how the concept of Jordan algebra provides an unification of both cases.

Namely, let $\mathcal{A}$ be a (finite-dimensional) formally real Jordan algebra and consider $\exp \mathcal{A} := \{ \exp a \mid a \in \mathcal{A} \}$ (see JSA.I, § 1). Consider on $\mathcal{A}$ the form μ (see JSA.III, § 1) given by

$$\mu(a) := \frac{s}{n} \operatorname{Tr} L(a), \qquad a \in \mathcal{A}.$$

If $\mathcal{A} = \mathbb{R}$ $(= M_1(\mathbb{R})^{(+)})$, then $\mu(a) = a$, while if $\mathcal{A} = M_r(\mathbb{R})^{(+)}$, $\mu(a) = \operatorname{Tr} a$.

It can easily be seen that for $\alpha > 0$ the map $a \longrightarrow a/\alpha$ is an isomorphism of $\mathbb{R}$ onto a Jordan algebra $\mathcal{A}_{(\alpha)}$ with unit element $1/\alpha$ and that

$$\exp \mathcal{A}_{(\alpha)} = \{ \exp \alpha a \mid a \in \mathcal{A} \} = \{ (\exp a)^{\alpha} \mid a \in \mathcal{A} \} = \\ = \{ x^{\alpha} \mid x \in \exp \mathcal{A} \}.$$

Writing $\mathcal{A}_{(\alpha_1 \alpha_2 \alpha_3)} := \mathcal{A}_{(\alpha_1)} \oplus \mathcal{A}_{(\alpha_2)} \oplus \mathcal{A}_{(\alpha_3)}$,

it follows that

$$\exp \mathcal{A}_{(\alpha_1 \alpha_2 \alpha_3)} = \left\{ (x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}) \;\middle|\; x_i \in \mathbb{R}^+ \right\} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$$

and

$$\exp M_2(\mathbb{R})^{(+)} = \left\{ \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} = x \;\middle|\; x \quad \text{is positive definite} \right\}.$$

Thus,

$$\exp \mathcal{A} = \mathcal{C} = \text{space of perceived colors}$$

$$\text{if } \mathcal{A} = \mathcal{A}_{(\alpha_1 \alpha_2 \alpha_3)} \quad \text{or} \quad \mathcal{A} = M_2(\mathbb{R})^{(+)}.$$

The group GL(exp $\mathcal{A}$) is generated by the map P(a) for $a \in \mathcal{A}$ (P(a) being the quadratic representation of $\mathcal{A}$, see JSA.I, § 1); exp $\mathcal{A}$ is a homogeneous space of GL(exp $\mathcal{A}$), and a GL(exp$\mathcal{A}$)-invariant metric on exp$\mathcal{A}$ is given by

(2.7) $$ds^2 := \mu((P^{-1}(x)dx)dx).$$

Remark. With the unification provided by the concept of a Jordan algebra, the arguments concerning brightness can be conceptually reversed (see RESNIKOFF [21 a, pp. 122-123]).

§ 3. Jordan algebras and bioenergetics

As is widely accepted nowadays, proteins are the principal workhorses of the cell. They are the major organizers and manipulators of biological energy and enzymes that catalyze and maintain the life process. The proteins are responsible for the active transport of ions into and out of the cell, as well as for cellular and intracellular movement. That is why the discipline of bioenergetics, which is the study of how cells generate and transfer their energy supply, is primarily the investigation of how proteins work.

At present, the composition and three-dimensional structure of about two-hundred proteins are known. However, there is no generally accepted model of how proteins operate dynemically.

The idea that the energy released in the hydrolysis of adenosine triphosphate (ATP) molecules transforms into that of soliton excitation and is transferred with great efficiency along protein molecules was used by DAVYDOV as early as 1973 (see [5 a]) to explain the contraction mechanism of transversely striated muscles of animals at the molecular level. Davydov et al.considered, in addition, the idea that α-helical proteins may facilitate electron transport through a soliton mechanism. In this case, an extra electron causes a lattice distortion in the protein that stabilizes the electron's motion.

Thus it may be reasonable to consider charge transfer across membranes, energy coupling across membranes, and energy transport along filamentous cytoskeletal proteins in terms of a soliton mechanism, since proteins that carry out these functions contain structural units with significant α-helical character (see DAVYDOV [5 b]). The Davydov model leads to a nonlinear Schrödinger equation which has soliton solutions (see [14 p.13]).

Note. As was observed by LOMDAHL, LAYNE and BIGIO [14, p.6] the soliton model is one among several concepts for protein dynamics which should attract the careful attention of biologists. Clearly, it cannot explain every aspect of protein dynamics, but it is motivating exciting questions and new experiments.

LAYNE [13] presented a simplified theoretical model for anesthesia activity, taking advantage of the fact that the α helix is an important structure in membrane and cytoskeletal proteins.

More precisely, LAYNE [13, p.24] formulated the following question: How does the binding of an anesthetic molecule to a protein modify normal protein behavior? He answered this question using the soliton model as a paradigm for normal protein functioning . The soliton model proposes that α-helical proteins effect the transport of ATP hydrolysis energy through a coupling of vibrational excitations to displacements along the spines of the helix. This coupling leads to a self-focusing of vibrational energy that has remarkably stable qualities. LAYNE [13, p.24] suggests that the binding of an anesthetic molecule to a protein interferes with soliton propagation. He suggests further that this type of interference is most important in two separate regions of a cell where soliton propagation is an attractive candidate: first, in the α-helical proteins of the inner mitochondrial membrane, which appear to participate in ATP synthesis and electron transport and secondly, in the membrane proteins of neurons, which are responsible for chemical reception and signal transduction.

Remark [13, p.26]. If the Davydov soliton finds experimental support in biology, then such a model may help to explain some of the molecular mechanisms behind general anesthesia.

Let us mention that TAKENO [30 a] studied vibron (i.e. vibrational exciton) solitons in one-dimensional molecular crystals by employing a coupled oscillator-lattice model. Takeno showed that although vibron solitons in his theory and those in the Davydov theory are both described by the non-linear Schrödinger equation, their nature is fairly different from each other. The nonlinear Schrödinger equation arises in the Takeno theory from modulations of vibrons by nonlinear coupling with acoustic phonons propagating along helics of the α-proteins, while that in the Davydov theory follows immediately from the quantal Schrödinger equation for the exciton probability. Recently, TAKENO [30 c] presented an exactly tractable model of an oscillator-lattice system which is capable of incorporating both of the pictures of FRÖHLICH (see [7], [3]) and that of Davydov in a unified way and to make a more detailed study of vibron solitons by giving a significant improvement of the theory developed in [30 a].

Comments. As was already pointed out (see JSA.V, § 8), an open problem is to find an algebraic description of the GM (as well as of $\widetilde{GM}$ and UGM) appearing in SATO's approach [22 a,b] to soliton equations,resembling the Jordan algebra description of finite-dimensional Grassmann manifolds given by HELWIG in [11] and recalled in JSA.III, § 2 Taking into account the previous considerations, solving this open problem could be useful in bioenergetics.

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