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STIINTIFICA SI TEHNICA

ISSN 0250 3638

CURVES IN ALGEBRAIC GROUPS AND ALGEBRAIC DIFFERENTIAL EQUATIONS

by

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PREPRINT SERIES IN MATHEMATICS
No. 35/1990

BUCURESTI

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May 1990

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Curves in algebraic groups and algebraic differential equations

A. Buium

1. Introduction

It was a conjecture of Lang [La_1] (which due to results of Raynaud [R_1] and Faltings [F] is now a theorem) that if A is an abelian $\mathbb{C}$-variety, $X \subset A$ is a smooth curve of genus $\geqslant 2$ and $\Gamma \subset A$ is a subgroup of finite rank then the set $X \cap \Gamma$ is finite.

The aim of this paper is to prove an analogue of Lang's conjecture in the theory of algebraic differential equations of Ritt [Ri] and Kolchin [K_1, K_2]. Our result easily implies Lang's conjecture for any X which does not descend to $\overline{\mathbb{Q}}$; it also easily implies the "geometric analogues" of Lang and Mordell conjectures. On the other hand our proofs are essentially elementary. The only prerequisites necessary are the language and general results of the Ritt-Kolchin theory plus some facts from [B_1]; the reader not familiar with this topic will find in section 2 a self-contained account of the necessary backgorund.

Our main result is:

(1.1) THEOREM. Let $\mathcal{F}$ be an ordinary constrainedly closed Δ-field of characteristic zero with field of constants $\mathcal{C}$ and assume $\mathcal{F}$ is Δ-algebraic over $\mathcal{C}$. Let G be an irreducible algebraic $\mathcal{F}$-group, $X \subset G$ a smooth curve of genus $\geqslant 2$ which does not descend birationally to $\mathcal{C}$ and $\Sigma \subset G$ a Δ-closed subgroup of Δ-type zero. Then the set $X \cap \Sigma$ is finite.

In the statement above X is assumed of course to be Zariski closed in G; by "X does not descend birationally to $\mathcal{C}$" we mean that X is not birationally equivalent to a curve which descends to $\mathcal{C}$.

We shall derive from (1.1) the following "non-differential" corollaries:

(1.2) COROLLARY. Let G be an irreducible commutative algebraic $\mathbb{C}$-group, $X \subset G$ a smooth curve of genus $\geqslant 2$ which does not descend birationally to $\overline{\mathbb{Q}}$ and $\Gamma \subset G$ a subgroup of finite rank. Then the set $X \cap \Gamma$ is finite.

Note that (1.2) reproves in particular the conjectures of Lang and Mordell for curves X over $\mathbb{C}$ which do not descend to $\overline{\mathbb{Q}}$ (Mordell says in this case that for any such X of genus $\geqslant 2$ and any field of definition $K \subset \mathbb{C}$ of X which is finitely generated over $\mathbb{Q}$ the set X_K is finite).

(1.3) COROLLARY. Let $k \subset K$ be an extension of algebraically closed fields of characteristic zero with $\text{tr.deg}.K/k < \infty$, let G be an irreducible commutative algebraic K-group, $H \subset G$ be an algebraic subgroup which descends to k, X a smooth curve in G of genus $\geqslant 2$ which does not descend birationally to k and $\Gamma \subset G_K$ a subgroup containing H_k such that Γ/H_k is of finite rank. Then the set $X \cap \Gamma$ is finite.

The above corollary reproves in particular the "geometric analogue" of Lang's conjecture (proved by Raynaud $[R_2]$). It also reproves the "geometric analogue" of Mordell's conjecture (proved by Manin and Grauert) saying that if L is a function field over on algebraically closed field k of characteristic zero and C is a smooth projective curve of genus $\geqslant 2$ over L which does not descend to k (over the algebraic closure K of L) then C_L is finite. Indeed let J be the Jacobian of C put $X = C \otimes_L K$, $G = J \otimes_L K$ and let H be the L/k-trace of J; then by Mordell-Weil $[La_2]$ p.71, J_L/H_k is finitely generated and we may apply (1.3) to $\Gamma = J_L$.

If case $G = G_a^N$ (1.1) implies more than (1.3) namely:

(1.4) COROLLARY. Let $k \subset K$ be as in (1.3) and let X be a smooth curve of genus ≥ 2 over K which does not descend birationally to k and is embedded into the N-affine space K^N. Let V be finite dimensional k-linear subspace of K^N. Then the set $X \cap V$ is finite.

The link between our Theorem (1.1) and Corollaries (1.2) and (1.3) is made by the following easy consequence of results from [C] and [B_1](in which a subgroup of an algebraic $\mathcal{F}$-group is called of Δ-type zero if its Δ-closure has Δ-type zero):

(1.5) PROPOSITION. Let G be an irreducible commutative algebraic $\mathcal{F}$-group and $\Gamma' \subset \Gamma \subset G$ subgroups such that Γ/Γ' has finite rank. Assume Γ' has Δ-type zero; then the same holds for Γ. In particular any subgroup of finite rank in G has Δ-type zero.

It worths noting that the groups Γ appearing in Lang's conjecture are at most countable while the groups Σ in (1.1) are generally uncountable if $\mathcal{C}$ is so (see (3.4)). So the finiteness of $X \cap \Sigma$ in (1.1) is much stronger than finiteness of $X \cap \Gamma$ in Lang's conjecture. Note also that unlike in Lang's conjecture we allow in (1.1) G to be non-commutative and that there are many interesting examples of Δ-closed subgroups of Δ-type zero of non-commutative algebraic $\mathcal{F}$-groups [B_3]. Note finally that we expect (1.1) to hold without the assumption that $\mathcal{F}$ is Δ-algebraic over $\mathcal{C}$ and in the partial differential rather than ordinary case.

Our paper is organized as follows. In section 2 we review the basic concepts of Ritt-Kolchin theory involved in (1.1) and we recall some facts from [B_1]; the basic references for section 2 are [B_1] and [K_2]. In section 3 we prove Proposition (1.5). In section 4 we prove Theorem (1.1). In section 5 we prove Corollaries (1.2), (1.3), (1.4). In Section 6 we describe the effect of Theorem (1.1) on the program (initiated in [B_1, B_2]) of study of Δ-polynomial functions on projective varieties; the present paper is implicitely part of this program.

It is a pleasure to aknowledge our debt to E. Kolchin and P. Cassidy for their kind encouragement.

2. Review of Δ-polynomial functions

For details on facts presented in this section we send to [K_1, K_2, B_1].

(2.1) Start with a field $\mathcal{F}$ of characteristic zero on which we have fixed a derivation δ; such an $\mathcal{F}$ is called an (ordinary) Δ-field. The Ritt-Kolchin theory is an analogue of algebraic geometry in which polynomial equation are replaced by algebraic differential equations "over $\mathcal{F}$". The standard hypothesis in algebraic geometry about the ground field is that it is algebraically closed. In the Ritt-Kolchin theory the most natural hypothesis seems to be that $\mathcal{F}$ is "constrainedly closed" (cf. [K_2] p.79); it is irrelevant to explain here the definition of this concept for which we send to loc.cit. In any case constrainedly closed Δ-fields are in particular algebraically closed. For any smooth $\mathcal{F}$-variety X (in the usual sense of algebraic geometry) we often identify X with its set $X_{\mathcal{F}}$ of $\mathcal{F}$-points. A function $f:X\to\mathcal{F}$ will be called Δ-polynomial if, locally in the Zariski topology of X, f is defined by a polynomial in the coordinates and their derivatives (e.g. if $X=\mathbb{A}^2=\mathcal{F}^2$ then $f(y_1,y_2)=(\delta^2 y_1)^3-y_1^5(\delta^3 y_2)^4$ is Δ-polynomial). A subset Σ of X is called Δ-closed if, locally in the Zariski topology on X, Σ is the set of common zeroes of finitely many Δ-polynomial functions; by a basic result of Ritt, Δ-closed subsets are the closed sets of a Noetherian topology on X. Let Σ be a Δ-closed subset of X, U an affine Zariski open subset of X meeting Σ and consider the ring $\mathcal{F}\{\Sigma\cap U\}$ obtained by dividing the ring of all Δ-polynomial functions on U by the ideal of those vanishing on $\Sigma\cap U$; Σ is called of Δ-type zero if for any U as above the residue fields of $\mathcal{F}\{\Sigma\cap U\}$ at the minimal primes have finite transcendence degree over $\mathcal{F}$ (Intuitively Σ has Δ-type zero if the "general solution" of the system of algebraic differential equations defining it around each point depends on finitely many "integration constants" rather than on "arbitrary functions").

(2.2) Now one should note that in $[B_1, B_2]$ we worked with the above concepts over a universal Δ-field $\mathcal{U}$ ($[K_1]$ p.133) rather than over a constrainedly closed Δ-field $\mathcal{F}$; but everything which was said in $[B_1, B_2]$ holds with $\mathcal{F}$ instead of $\mathcal{U}$! The reason for which we shift here from $\mathcal{U}$ to $\mathcal{F}$ is that we sometimes need our $\mathcal{F}$ be Δ-algebraic over its field of constants $\mathcal{C}$ (which means that for any $x \in \mathcal{F}$ the family $x, \delta x, \delta^2 x, \ldots$ is algebraically dependent over $\mathcal{C}$) while $\mathcal{U}$ is never Δ-algebraic over its constant field $\mathcal{K}$. It is therefore of interest to give examples of constrainedly closed Δ-fields $\mathcal{F}$ which are Δ-algebraic over $\mathcal{C}$. One such example is the field $\mathcal{F} = \mathcal{U}_0$ of elements in a universal Δ-field $\mathcal{U}$ which are Δ-algebraic over $\mathcal{K}$; in this case we have $\mathcal{C} = \mathcal{K}$. Another example is the following: start with any Δ-field $\mathcal{F}_1$ which is Δ-algebraic over its field of constants $\mathcal{C}_1$ and let $\mathcal{F}$ be a constrained closure of $\mathcal{F}_1$, cf $[K_2]$ p.79. Then $\mathcal{C}$ is the algebraic closure of $\mathcal{C}_1$ ($[K_1]$ p.143).

(2.3) Let us explain one of our main results in $[B_1]$. For any irreducible algebraic $\mathcal{F}$-group G we constructed a Δ-closed subset of some affine space $\mathcal{F}^N$ (call it here $\bar{G} \subset \mathcal{F}^N$) which has a structure of group whose multiplication $\bar{G} \times \bar{G} \to \bar{G}$ and inverse $\bar{G} \to \bar{G}$ extend to Δ-polynomial maps $\mathcal{F}^N \times \mathcal{F}^N \to \mathcal{F}^N$, $\mathcal{F}^N \to \mathcal{F}^N$ (this is what Cassidy calls in [C] and affine differential algebraic group with Δ-polynomial law) and we constructed a surjective Δ-polynomial homomorphism $G \to \bar{G}$ (i.e. a homomorphism whose components are Δ-polynomial) whose kernel $G^{\#}$ has Δ-type zero (see $[B_1]$ (5.1)). It follows that a Δ-closed subgroup of G has Δ-type zero iff its image in $\bar{G}$ has Δ-type zero.

(2.4) It will be also useful to recall from $[B_1]$, section 3, a dictionary relating Δ-closed sets to "D-schemes". Let $D = \mathcal{F}[\delta]$ be the ring of differential operators generated by $\mathcal{F}$ and δ. Then by a D-scheme we understand an $\mathcal{F}$-scheme V such that we are given an extension of δ from $\mathcal{F}$ to a derivation (still denoted by δ) of $\mathcal{O}_V$; D-schemes form a category in the obvious way. Group objects in this category are called D-group schemes. The forgetful functor from {D-schemes} to {$\mathcal{F}$-schemes} has a right adjoint $X \mapsto X^{\infty}$. The latter functor takes closed immersions into closed immersions and group objects into group objects.

Finally, given a Δ-closed subset Σ of an $\mathcal{F}$-variety X one can construct [B_1] (3.9) a reduced closed D-subscheme Σ^{∞} of X^{∞} such that Σ identifies with $\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, \Sigma^{\infty})$. If Σ is a Δ-closed subgroup (of Δ-type zero) of an algebraic $\mathcal{F}$-group G then Σ^{∞} is a D-subgroup scheme of G^{∞} (respectively an "algebraic D-group" i.e. a D-group scheme whose underlying $\mathcal{F}$- group scheme is an algebraic $\mathcal{F}$-group) [B_1] (3.12).

(2.5) The facts reviewed in (2.1)-(2.4) are sufficient to understand the proofs of (1.1)-(1.5); they are not sufficient however to understand our final section 6 for which a deeper familiarity with [B_1, B_2] is required.

3. Finite rank groups and groups of Δ-type zero

(3.1) Let $G_a^N = \mathcal{F}^N$ be an algebraic vector group. Then by [C] p.911 it follows that the Δ-closed subgroups of G_a^N of Δ-type zero are precisely the finite dimensional $\mathcal{C}$-linear subspaces of $\mathcal{F}^N$. This implies (1.5) in case $G=G_a^N$.

(3.2) Let $G_m^N=(\mathcal{F}^*)^N$. Then by [C] p.937 the Δ-closed subgroups of G_m^N of Δ-type zero containing $(\mathcal{C}^*)^N$ are precisely those of the form $(\ell\delta)^{-1}(V)$ where $\ell\delta : (\mathcal{F}^*)^N \to \mathcal{F}^N$ is the logarithmic derivative $(x_1,\dots,x_N) \longmapsto (x_1^{-1}\delta x_1,\dots,x_N^{-1}\delta x_N)$ and V is a finite dimensional $\mathcal{C}$-linear subspace of $\mathcal{F}^N$. This implies that (1.5) holds for any irreducible commutative linear G.

(3.3) Let's prove (1.5) for arbitrary irreducible commutative G. Consider the Δ-polynomial homomorphism $f:G \to \bar{G}$ from (2.3). Since $G^{\#}$ has Δ-type zero we may assume Γ' is Δ-closed and contains $G^{\#}$. By [C] p.914 (and with terminology from loc.cit.) there exists a Δ-rational injective homomorphism $u:\bar{G} \to GL_{\mathcal{F}}(n)$. The Zariski closure H of the image of u is then a commutative linear algebraic $\mathcal{F}$-group. We conclude by applying (1.5) to H

(cf. (3.2) and (2.3)).

(3.4) Assume $\mathcal{U}$ is uncountable and $\Sigma \neq 0$ is a Δ-closed subgroup of Δ-type zero of an irreducible commutative algebraic $\mathcal{F}$-group G. If G is linear then Σ must be uncountable by (3.2). We conjecture that Σ is uncountable for G non necessarily linear. This is clearly so if $\Sigma \not\subset G^{\#}$ (which in its turn holds for Σ the Δ-closure of a "sufficiently general" subgroup Γ of G of finite rank!).

4. Proof of Theorem (1.1)

Clearly we may assume for the proof of (1.1) that $\mathcal{U} = \mathbb{C}$. We need some analytic ingredients:

(4.1) First we will use a classical theorem of Picard [Pic, GK] which we recall for convenience: if C, X are two smooth algebraic curves over $\mathbb{C}$ with C affine and X complete of genus ≥ 2 then any holomorphic map from C to X is rational.

(4.2) Another analytic ingredient will be an elementary yet remarkable result of Hamm [Ha], which we now recall and put it in a form suitable for our purpose. Let $\Sigma_B \to B$ be a local submersion of analytic manifolds where B is a disk in $\mathbb{C}$ and let $\theta, \tilde{\theta}$ be non-vanishing vector fields on B and Σ_B respectively with $\tilde{\theta}$ lifting θ. Assume we are given analytic maps $\mu: \Sigma_B \times_B \Sigma_B \to \Sigma_B$, $\Sigma_B \to \Sigma_B$, $B \to \Sigma_B$ over B satisfying the "usual" axioms of multiplication, inverse and unit (analogue to the axioms of a group scheme) and assume moreover that these maps are equivariant (with respect to the vector fields $\theta, \tilde{\theta}$ and $(\tilde{\theta}, \tilde{\theta})$ on B, Σ_B and $\Sigma_B \times_B \Sigma_B$ respectively). Then there exists an analytic B-isomorphism $\sigma: \Sigma_B \to \Sigma_o \times B$ where Σ_o is a Lie group such that σ transports the map μ into the map $\mu_o \times 1_B$ (where $\mu_o: \Sigma_o \times \Sigma_o \to \Sigma_o$ is the multiplication on Σ_o) and such that σ transports $\tilde{\theta}$ into $\frac{\partial}{\partial b}$ (where b is a coordinate on B). See also [B_3], (11.1) for an exposition of Hamm's result.

It is also convenient to formulate the following

(4.3) LEMMA. Let $f: X_Y \to Y$ be a smooth projective morphism of smooth $\mathbb{C}$-varieties whose fibres are connected curves of genus ≥ 2. Let moreover θ be an algebraic vector field on Y without zeroes such that the field of constants

$\mathbb{C}(Y)^{\theta}$ of the Δ-field $(\mathbb{C}(Y),\theta)$ equals $\mathbb{C}$. Assume moreover that for any analytic disk B embedded into Y which is an integral subvariety for θ we have that all fibres of f above points in B are isomorphic. Then the curve $X_Y \times_Y \mathrm{Spec}\, \overline{\mathbb{C}(Y)}/\overline{\mathbb{C}(Y)}$ descends to $\mathbb{C}$.

Proof. Let $m: Y \to (\mathcal{M}_g)_{\mathbb{C}}$ be the morphism to the moduli space of smooth projective curves of genus g over $\mathbb{C}$ induced by f and assume the closure Z of its image has dimension $\geqslant 1$. We claim that the image of $\mathbb{C}(Z)$ in $\mathbb{C}(Y)$ via m^* is contained in $\mathbb{C}(Y)^{\theta}$ and this will be a contradiction. Indeed let $\varphi \in \mathbb{C}(Z)$ and $y \in Y$ be any point at which the rational function $\tilde{\varphi}: Y \xrightarrow{m} Z \overset{\varphi}{\dashrightarrow} \mathbb{C}$ is defined. Take a disk B which is an integral subvariety of Y for θ through y. By hypothesis B is contained in the fibre of $\tilde{\varphi}$ through y. Consequently the latter fibre is tangent to θ, hence $\tilde{\varphi} \in \mathbb{C}(Y)^{\theta}$ and we are done.

(4.4) Finally we recall for convenience another classical result due to Severi which will be needed: if C is an algebraic curve over $\mathbb{C}$ then the set of isomorphism classes of smooth projective curves X over $\mathbb{C}$ of genus $\geqslant 2$ dominated (as algebraic varieties) by C is finite. What will be used in fact is a weak form of this saying that C cannot dominate a "non-isotrivial family" of X's of genus $\geqslant 2$ (this is an "easy exercise" with the Hilbert scheme of divisors on XxX!).

(4.5) The rest of this section is devoted to the proof of (1.1).

Let $X \to G$ be our closed immersion and let $X^{\infty} \to G^{\infty}$ be the induced closed immersion of D-schemes (2.4). Moreover let Σ^{∞} be the D-subgroup scheme of G^{∞} corresponding to Σ; it is an algebraic D-group (2.4). Using (2.4) once again we see that $X \cap \Sigma$ identifies with $\mathrm{Hom}_{D\text{-}sch}(\mathrm{Spec}\,\mathcal{F}, X^{\infty} \cap \Sigma^{\infty})$ so what we must prove is that $X^{\infty} \cap \Sigma^{\infty}$ is either empty or zero dimensional. Assume it has dimension > 0 and look for a contradiction. If $\pi: G^{\infty} \to G$ is the morphism of schemes arrising from adjunction then $\pi(X^{\infty} \cap \Sigma^{\infty}) \subset X$. We claim that any positive dimensional component W of $X^{\infty} \cap \Sigma^{\infty}$ dominates X. Indeed if $\pi(W)=\{x\}$ we could find two morphisms $\alpha_1, \alpha_2 \in \mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, W)$, $\alpha_1 \neq \alpha_2$ (this is possible since by [K$_2$] p.84 $\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, W)$ is Zariski dense in $\mathrm{Hom}_{\mathcal{F}\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, W)$) hence by composing them with the inclusion $W \subset G^{\infty}$ we would get two morphisms $\alpha'_1, \alpha'_2 \in \mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, G^{\infty})$, $\alpha'_1 \neq \alpha'_2$ such that $\pi \circ \alpha'_1 = \pi \circ \alpha'_2$ which contradicts the adjunction property!

So pick a positive dimensional component W of $X^{\infty} \cap \Sigma^{\infty}$ and let $\bar{X}$ be a smooth projective model of X. We may find a Δ-subfield $\mathcal{F}'$ of $\mathcal{F}$ containing $\mathcal{C} = \mathbb{C}$ and Δ-finitely generated over $\mathbb{C}$ such that the diagram below is defined over $\mathcal{F}'$:

$$(*) \qquad \bar{X} \leftarrow X \leftarrow W \rightarrow \Sigma^{\infty}$$

and moreover the structure of algebraic D-group of Σ^{∞} descends to $\mathcal{F}'$. So we dispose of a diagram of $\mathcal{F}'$-varieties:

$$(x)' \qquad \bar{X}' \leftarrow X' \leftarrow W' \rightarrow \Sigma'$$

from which $(*)$ is deduced by base change and we dispose of a structure of algebraic D'-group on Σ' (where $D' = \mathcal{F}'[\hat{\delta}]$) inducing that of Σ^{∞} such that W' is a D'-subscheme of Σ'. Since $\mathcal{F}/\mathcal{C}$ is Δ-algebraic, $\mathcal{F}'/\mathcal{C}$ is finitely generated as a non-differential field extension [K$_1$] p.112. So we may find an affine smooth $\mathbb{C}$-variety Y with $\mathbb{C}(Y) = \mathcal{F}'$ and a diagram of integral Y-schemes:

$$(*)_Y \qquad \bar{X}_Y \leftarrow X_Y \leftarrow W_Y \rightarrow \Sigma_Y$$

from which $(*)'$ is deduced by base change such that $X_Y \rightarrow \bar{X}_Y$ is an open immersion, $W_Y \rightarrow \Sigma_Y$ is a closed immersion, the fibres of $W_Y \rightarrow Y$ dominate the fibres of $X_Y \rightarrow Y$, $X_Y \rightarrow Y$ is smooth, $\Sigma_Y \rightarrow Y$ is a smooth group scheme and the derivation δ induces a non-vanishing vector field θ on Y and a vector field $\tilde{\theta}$ on Σ_Y. Then W_Y must be an integral subvariety of Σ_Y for $\tilde{\theta}$. Let B be an analytic disk embedded into Y which is an integral subvariety for θ and let

$(*)_B$. $\bar{X}_B \leftarrow X_B \leftarrow W_B \rightarrow \Sigma_B$

be deduced from $(*)_Y$ via base change $B \rightarrow Y$. Then W_B is an integral subvariety of Σ_B for $\tilde{\theta}$. By Hamm's result we dispose of an analytic isomorphism $\sigma: \Sigma_B \rightarrow \Sigma_o xB$ as in (4.2). Let's make the obvious remark that any closed analytic subset of $\Sigma_o xB$ which is an integral subvariety for $\frac{\partial}{\partial b}$ (notations as in (4.2)) is necessarily of the form $\Sigma_1 xB$ where Σ_1 is a closed analytic subset of Σ_o. Hence $\sigma(W_B) = \Sigma_1 xB$ with Σ_1 as above. For any $b \in B$ let $\bar{X}_b, X_b, W_b, \Sigma_b$ be the fibres of the spaces in $(*)_B$ at b. Not fix a point $b_o \in B$; since W_{b_o} dominates X_{b_o} one can find a smooth Zariski locally closed curve C_o in W_{b_o} dominating X_{b_o}. Now consider the analytic map of analytic surfaces:

$$u: C_o xB \xrightarrow{v} W_B \rightarrow \bar{X}_B$$

where $v(c,b) = \sigma^{-1}(p_1(\sigma(c)),b)$, $c \in C_o$, $b \in B$, $p_1: \Sigma_o xB \rightarrow \Sigma_o$ being the first projection. Since the map $u_{b_o}: C_o \rightarrow \bar{X}_{b_o}$ defined by $c \mapsto u(c,b_o)$ is non-constant there exists an open subset B_o of B such that for any $b \in B_o$ the analytic map $u_b: C_o \rightarrow \bar{X}_b$ defined by $c \mapsto u(c,b)$ is non-constant. By Picard's theorem (4.1) u_b is rational .. By Severi's theorem (4.4) all X_b's are isomorphic for $b \in B_o$ hence for $b \in B$. By Lemma (4.3) $\bar{X}$ descends to $\mathbb{C}$, contradiction. Our Theorem is proved.

5. Deducing "non-differential" statements from "differential" ones

(5.1) LEMMA. Let $K \subset \mathbb{C}$ be a subfield finitely generated over $\mathbb{Q}$ and let $\delta \in$ Der K. Then δ extends to a derivation of $\mathbb{C}$ (still denoted by δ) such that $\mathcal{F} := (\mathbb{C}, \delta)$ is constrainedly closed and Δ-algebraic over constants.

Proof. By Seidenberg's "Lefschetz principle" [S] we may embed the Δ-field K into the field M of meromorphic functions is some region of $\mathbb{C}$. Consider the Δ-subfield L of M generated by the functions $f_\alpha(z) = \exp(\alpha z)$, $\alpha \in \mathbb{C}$. The compositum KL in M is then Δ-algebraic over its constant field and has the cardinality equal to that of $\mathbb{C}$. Then the constrained closure $\mathcal{F}$ of KL will have the same cardinality hence is abstractly isomorphic to $\mathbb{C}$ over K and we are done by (2.2).

(5.2) Proof of (1.2). Let x be the $\mathbb{C}$-point corresponding to the smooth complete model of X on the moduli space of curves $(\mathcal{M}_g)_{\mathbb{C}}$ and let $K=\mathbb{Q}(x)$ be the field generated by the coordinates of x. We can choose a derivation $\delta\in \operatorname{Der} K$, $\delta\neq 0$ and extend it to a derivation δ of $\mathbb{C}$ as in (5.1). Now X does not descend birationally to the constant field $\mathcal{C}$ of $\mathcal{F}=(\mathbb{C},\delta)$ because K is not contained in $\mathcal{C}$. So by (1.1) and (1.5) $X\cap\Gamma$ is finite.

(5.3) LEMMA. Let $k\subset K$ be as in (1.3). Then there exists $\delta\in \operatorname{Der}_k K$ such that the constant field of (K,δ) equals k.

Proof. Let $x_1,\dots,x_n$ be a transcendence basis for K/k. Then the derivations $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ of $k(x_1,\dots,x_n)$ lift to a K-basis of $\operatorname{Der}_k K$. One easily checks that if $a_1,\dots a_n\in k$ are linearly independent over $\mathbb{Q}$ then the derivation $\delta=\sum_{i=1}^{n} a_i x_i \frac{\partial}{\partial x_i}$ has the property that the field of constants of $(k(x_1,\dots,x_n),\delta)$ equals k. Since the field of constants of (K,δ) is algebraic over that of $(k(x_1,\dots,x_n),\delta)$ we are done.

(5.4) Proof of (1.3) and (1.4). By (5.3) there exists $\delta\in \operatorname{Der}_k K$ such that the field of constants of (K,δ) equals k. Let $\hat{\mathcal{F}}$ be a constrained closure of (K,δ). By (2.2) $\hat{\mathcal{F}}$ is Δ-algebraic over its constant field $\mathcal{C}=k$. By hypothesis $X_{\hat{\mathcal{F}}}$ does not descend birationally to $\mathcal{C}$. In the situation of (1.4) V is a Δ-closed subgroup of $\hat{\mathcal{F}}^N$ of Δ-type zero (3.1) hence by (1.1) $X_{\hat{\mathcal{F}}}\cap V$ a finite hence so is $X_K\cap V$ (which is contained in $X_{\hat{\mathcal{F}}}\cap V$). In order to prove (1.3) note that $H_K=H_{\mathcal{C}}$ is a Δ-closed subgroup of $G_{\hat{\mathcal{F}}}$ of Δ-type zero. ⟵⟶

⟵⟶ By (1.5) Γ is a subgroup of Δ-type zero in $G_{\hat{\mathcal{F}}}$ and we conclude by (1.1) once again.

6. Δ-character map

In what follows the reader is assumed to be familiar with $[B_1]$ and $[B_2]$. Theorem (1.1) has a nice interpretation in terms of Δ-polynomial functions on curves in the spirit of $[B_2]$. We would like to give this interpretation here; we shall be lead then to a conjecture on the restriction map $\mathcal{O}^{\Delta}(J) \longrightarrow \mathcal{O}^{\Delta}(X)$ where X is a curve and J is its Jacobian.

Assume $\mathcal{F}$ is constrainedly closed and Δ-algebraic over $\mathcal{C}$ and let X be a smooth projective curve over $\mathcal{F}$ of genus $g \geq 2$ which does not descend to $\mathcal{C}$. Then we dispose of two sequences of Δ-polynomial maps

$$\varphi_d : X \to \mathcal{F}^{N_d}, \quad d \geq 1$$
$$\psi_r : X \to \mathcal{F}^{M_r}, \quad r \geq 1$$

whose definition will be given below; the φ's are the analogues of pluricanonical maps (φ_d was called in $[B_2]$ the Δ-pluricanonical map of degree d) while the ψ's will be the analogues of the Albanese map (ψ_r will be called the Δ-character map of order r). Recall from $[B_2]$ that the components of φ_d are a basis of the space of all Δ- polynomial functions on X of order ≤ 1 and degree $\leq d$; by $[B_2]$ φ_d is a Δ-closed embedding for $d \gg 0$ (and in fact for $d \geq 3$ if X is non-hyperelliptic with $\mathrm{rank}_{\Delta}(X)=g$; in this case $N_3=8g-8$).

Now to define the ψ's consider for each $r \geq 1$ the $\mathcal{F}$-space of all Δ-polynomial characters of order $\leq r$ on the Jacobian J of X. This space is finite dimensional, say of dimension M_r; pick a basis of this space and consider the Δ-polynomial homomorphism $J \to \mathcal{F}^{M_r}$ defined by this basis. Finally embed X into J via the Albanese map (this depends of course upon fixing a point in X). Then ψ_r is by definition the composition $X \to J \to \mathcal{F}^{M_r}$. The main consequence of Theorem (1.1) in our situation here is that ψ_r has finite fibres for $r \gg 0$. Indeed by $[B_1]$ (6.1) $\mathrm{Ker}(J \to \mathcal{F}^{M_r})=J^{\#}$ for $r \gg 0$ and we apply (2.3) and Theorem (1.1) to the various translates of X in J.

Note also that by $[B_1]$ (6.1) if X is non-hyperelliptic with $\mathrm{rank}_{\Delta}(X)=g$ then $\mathrm{Ker}\,\psi_2=J^{\#}$ and $M_2 = g$.

So for X "Δ-generic" in the moduli space $\mathcal{M}_g$ the Δ-polynomial maps

$$\varphi_3 : X \to \mathcal{F}^{8g-8}$$

$$\varphi_2 : X \to \mathcal{F}^{g}$$

already give a Δ-closed embedding respectively a finite-to-one map.

It is tempting to make the following:

Conjecture 1) If X does not descend to $\mathcal{C}$ the restriction map $\mathcal{O}^{\Delta}(J) \longrightarrow \mathcal{O}^{\Delta}(X)$ is finite.

2) For X "Δ-generic" in $\mathcal{M}_g$ the restriction map $\mathcal{O}^{\Delta}(J) \longrightarrow \mathcal{O}^{\Delta}(X)$ is surjective.

Statement 1) would imply that $\varphi_r(X)$ is Δ-closed in $\mathcal{F}^{M_r}$ for $r \gg 0$. On the other hand 2) would imply that for X "Δ-generic" in $\mathcal{M}_g$ the map φ_r is a Δ-closed embedding for $r \geq 2$. The latter conjecture is somewhat supported by the fact that the "sufficiently general" complex curves contain no non-zero torsion point of their Jacobians [Sz].

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