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by

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$\bigvee$-dualities and $\perp$-dualities

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We introduce and study dualities $\Delta : \overline{R}^X \to \overline{R}^W$ (i.e., mappings $f \in \overline{R}^X \to f^\Delta \in \overline{R}^W$ such that $(\inf_{i \in I} f_i)^\Delta = \sup_{i \in I} f_i^\Delta$ for all $\{f_i\}_{i \in I} \subseteq \overline{R}^X$ and all index sets I), which satisfy the additional condition $(f \vee d)^\Delta = f^\Delta \wedge -d$ $(f \in \overline{R}^X, d \in \overline{R})$, and their duals, which are characterized as those dualities $\Delta^* : \overline{R}^W \to \overline{R}^X$, for which $(f \perp d)^{\Delta^*} = f^{\Delta^*} \top -d$ $(f \in \overline{R}^X, d \in \overline{R})$, where $\perp$ and $\top$ are two new binary operations on $\overline{R}$, which we introduce here. Furthermore, we give a characterization of those Δ which are also conjugations. Some applications are also mentioned.

§0. Introduction

We recall that if $E = (E, \leq)$ and $F = (F, \leq)$ are two complete lattices, a mapping $\Delta : E \to F$ is called (see e.g. [6], [24], [11]) a <u>duality</u> (or, a "polarity" [1], [15], [16]), if for each index set I (including the empty set $I = \emptyset$, with the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$, where $+\infty$ and $-\infty$ denote the largest and smallest elements, respectively), we have

$$\Delta(\inf_{i \in I} x_i) = \sup_{i \in I} \Delta(x_i) \qquad (\{x_i\}_{i \in I} \subseteq E). \tag{0.1}$$

We have studied dualities between (general and various concrete) lattices E and F, in [11] (see also [24]; for other recent results on dualities, see e.g. [3]-[5], and the references therein). An important particular class of dualities is that of conjugations, which has applications to duality in optimization. Let us recall that, if X and W are two sets and $E = (\overline{R}^X, \leq)$ (the complete lattice of all functions $f : X \to \overline{R} = [-\infty, +\infty]$, with the usual pointwise partial order), $F = (\overline{R}^W, \leq)$, a mapping $c : E \to F$ (or, we shall write, simply, $c : \overline{R}^X \to \overline{R}^W$) is called [21] <u>a conjugation</u>, if it is a duality, i.e., if for every index set I (including $I = \emptyset$) we have (denoting $c(f)$ by f^c)

$$(\inf_{i\in I} f_i)^c = \sup_{i\in I} f_i^c \qquad (\{f_i\}_{i\in I} \subseteq \overline{R}^X), \tag{0.2}$$

and if it satisfies the additional condition

$$(f \dotplus d)^c = f^c \underset{\cdot}{+} -d \qquad (f \in \overline{R}^X,\ d \in \overline{R}), \tag{0.3}$$

where we identify each $d \in \overline{R}$ with the constant function $h_d \in \overline{R}^X$ defined by $h_d(x) = d$ ($x \in X$), the operations $\dotplus, \underset{\cdot}{+}$ on $\overline{R}^X$ are defined pointwise (on X), and the operations $\dotplus, \underset{\cdot}{+}$ on $\overline{R}$ are the "upper addition" and "lower addition" defined ([13], [14]) by

$$a \dotplus b = a \underset{\cdot}{+} b = a + b \quad \text{if} \quad R \cap \{a,b\} \neq \emptyset \quad \text{or} \quad a = b = \pm\infty, \tag{0.4}$$

$$a \dotplus b = +\infty,\ a \underset{\cdot}{+} b = -\infty, \quad \text{if} \quad a = -b = \pm\infty. \tag{0.5}$$

For example, if X and W are two sets and $\varphi : X \times W \to \overline{R}$ is any function (called, following [13], [14], a coupling function), then the mapping $c(\varphi) : \overline{R}^X \to \overline{R}^W$ defined by

$$f^{c(\varphi)}(w) = \sup_{x\in X} \{\varphi(x,w) \underset{\cdot}{+} -f(x)\} \qquad (f \in \overline{R}^X,\ w \in W), \tag{0.6}$$

is a conjugation, called the (Fenchel-Moreau) conjugation associated to φ. — Moreover, the converse is also true. Indeed, by [21], theorem 3.1, for any conjugation $c : \overline{R}^X \to \overline{R}^W$ we have (at each $w \in W$)

$$f^c = \sup_{x\in X} \{(\chi_{\{x\}})^c \underset{\cdot}{+} -f(x)\} \qquad (f \in \overline{R}^X), \tag{0.7}$$

and hence there exists a uniquely determined coupling function $\varphi_c : X \times W \to \overline{R}$ — such that $c = c(\varphi_c)$, i.e., such that

$$f^c(w) = \sup_{x\in X} \{\varphi_c(x,w) \underset{\cdot}{+} -f(x)\} \qquad (f \in \overline{R}^X,\ w \in W); \tag{0.8}$$

namely, we have

$$\varphi_c(x,w) = (\chi_{\{x\}})^c(w) \qquad (x \in X,\ w \in W), \tag{0.9}$$

where $\chi_{\{x\}}$ denotes the indicator function of the singleton $\{x\}$. We recall that, for any subset M of a set X, the indicator function $\chi_M : X \to \overline{R}$ (of M) is defined by

$$\chi_M(x) = \begin{cases} 0 \text{ if } x \in M \\ +\infty \quad \text{if } x \in X \setminus M. \end{cases} \tag{0.10}$$

Thus, by the above, there is a one-to-one correspondece between conjugations $c : \overline{R}^X \to \overline{R}^W$ and coupling functions $\varphi : X \times W \to \overline{R}$; the function φ_c of (0.8), (0.9) is called [21] the coupling function associated to the conjugation c.

In the present paper we shall introduce and study dualities $\Delta : \bar{R}^X \to \bar{R}^W$, i.e., mappings Δ satisfying, for every index set I,

$$(\inf_{i \in I} f_i)^\Delta = \sup_{i \in I} f_i^\Delta \qquad (\{f_i\}_{i \in I} \subseteq \bar{R}^X), \tag{0.11}$$

which have another "second property", different from (0.3). First, we shall introduce and study "$\bigvee$-dualities" $\Delta : \bar{R}^X \to \bar{R}^W$, defined by a "second condition" on $(f \bigvee d)^\Delta$ (instead of $(f \dotplus d)^\Delta$), namely, condition (2.1) below, where $\bigvee$ and $\bigwedge$ stand for (pointwise) sup and inf, in $\bar{R}^X$ and $\bar{R}^W$ respectively. In order to determine the duals $\Delta^* : \bar{R}^W \to \bar{R}^X$ of $\bigvee$-dualities, we shall introduce two new binary operations on $\bar{R}$, denoted $\top$ and $\bot$, respectively, which may have interest for other applications as well (e.g., to the inversion of Boolean inequalities, as we shall show). Then, extending these operations (pointwise) to $\bar{R}^X$, we shall introduce and study "$\bot$-dualities" $\delta : \bar{R}^X \to \bar{R}^W$, defined by a "second condition" on $(f \bot d)^\delta$, namely, condition (3.1) below. It will turn out that $\bigvee$-dualities and $\bot$-dualities are dual to each other and that (Fenchel-Moreau) conjugations can be expressed with the aid of $\bigvee$-dualities or $\bot$-dualities. Examples of dualities which are simultaneously conjugations, $\bigvee$-dualities and $\bot$-dualities, are the dualities of type (2.3) below (containing, as a particular case, e.g. the Greenberg-Pierskalla [8] quasi-conjugations), and we shall show that they are the only dualities having simultaneously any two (and hence all three) of the above properties. Finally, we shall also give some examples of $\bot$-dualities related to lower subdifferentiability (in the sense of [17]) and mention some applications. For some further applications, to duality in optimization, see remark 4.2 b) and [12].

Let us also note that, while the indicator function χ_M of a set M (defined by (0.10)) has turned out to be useful for the study of conjugations, in the sequel we shall find it convenient to use, for the study of $\bigvee$-dualities and $\bot$-dualities, the following function $\rho_M : X \to \bar{R}$, introduced by Flachs and Pollatschek [7], which we shall call the representation function of M:

$$\rho_M(x) = \begin{cases} -\infty & \text{if } x \in M \\ +\infty & \text{if } x \in X \setminus M. \end{cases} \tag{0.12}$$

For any function $f : X \to \overline{R}$ and number $r \in R$, we shall use the strict lower r-level set $A_r(f)$ of f, defined by

$$A_r(f) = \{x \in X \mid f(x) < r\}. \tag{0.13}$$

We shall denote by min (respectively, max) an inf (respectively, sup) which is attained.

In the paper [11], instead of dualities $\Delta : \overline{R}^X \to \overline{R}^W$, we have also studied, more generally, dualities $\Delta : A^X \to B^W$, where $(A, \leq)$ and $(B, \leq)$ are subsets of $(\overline{R}, \leq)$, which are complete lattices, giving applications to various concrete sets A and B. Although the results of the present paper can be extended to that more general case, here we shall consider, for simplicity, only dualities $\Delta : \overline{R}^X \to \overline{R}^W$.

§ 1. The binary operations $\top$ and $\bot$

We shall now introduce two new binary operations on $\overline{R}$ (and extend them, pointwise, to $\overline{R}^X$), denoted $\top$ and $\bot$, respectively, which we shall need in order to determine the dual of a $\vee$-duality (see § 4).

Definition 1.1. For any $a, b \in \overline{R}$, let

$$a \top b = \begin{cases} a & \text{if } a > b \\ -\infty & \text{if } a \leq b, \end{cases} \tag{1.1}$$

$$a \bot b = \begin{cases} a & \text{if } a < b \\ +\infty & \text{if } a \geq b. \end{cases} \tag{1.2}$$

The binary operations $\top$ and $\bot$ are non-commutative and non-associative; for example, we have

$$(0 \top 0) \top 0 = -\infty \top 0 = -\infty, \tag{1.3}$$

$$0 \top (0 \top 0) = 0 \top -\infty = 0. \tag{1.4}$$

Nevertheless, these operations have some interesting properties. Indeed, first of all, they give an answer to the following questions on the "inversion" of Boolean inequalities: If $a \leq b \vee c$ or $a \geq b \wedge c$, how to move b or c to the left hand side?

Proposition 1.1. For any $a,b,c \in \overline{R}$, we have

$$a \leq b \vee c \Leftrightarrow a \top b \leq c \Leftrightarrow a \top c \leq b, \tag{1.5}$$

$$a \geq b \wedge c \Leftrightarrow a \perp b \geq c \Leftrightarrow a \perp c \geq b. \tag{1.6}$$

Proof. If $a > b$, then for any $c \in \overline{R}$ we have the equivalences

$$a \leq b \vee c \Leftrightarrow a \leq c \Leftrightarrow a \top b \leq c,$$

while if $a \leq b$, then for any $c \in \overline{R}$ we have $a \leq b \vee c$ and $a \top b = -\infty \leq c$. The second equivalence in (1.5) follows from the first one, since $b \vee c = c \vee b$. The proof of (1.6) is similar.

Remark 1.1. By proposition 1.1, the operation $\top$ (respectively, $\perp$) is, in a certain sense, the "inverse" of $\vee$ (respectively, $\wedge$). We think that this property alone could be already a sufficient algebraic motivation for introducing $\top$ and $\perp$, and that these operations may be useful for other applications as well.

b) Proposition 1.1 is similar to [14], formula (3.3), according to which

$$x \dotplus y \geq z \qquad x \geq z \,\underset{\cdot}{+}\, -y \qquad (x,y,z \in \overline{R}). \tag{1.7}$$

Corollary 1.1. For any $b,c \in \overline{R}$, we have

$$b \vee c = \max_{\substack{a \in \overline{R} \\ a \top c \leq b}} a = \max_{\substack{a \in \overline{R} \\ a \top b \leq c}} a, \tag{1.8}$$

$$b \wedge c = \min_{\substack{a \in \overline{R} \\ a \perp c \geq b}} a = \min_{\substack{a \in \overline{R} \\ a \perp b \geq c}} a, \tag{1.9}$$

$$b \perp c = \max_{\substack{a \in \overline{R} \\ a \wedge c \leq b}} a, \tag{1.10}$$

$$b \top c = \min_{\substack{a \in \overline{R} \\ a \vee c \geq b}} a. \tag{1.11}$$

Proof. We have $b \vee c = \max\{a \in \overline{R} \mid a \leq b \vee c\}$, whence, by (1.5), we obtain (1.8). The proofs of (1.9)-(1.11) are similar.

A connection between $\top$ and $\perp$ is given by

Proposition 1.2. For any $a,b \in \overline{R}$ we have

$$a \top b = -(-a \perp -b), \quad a \perp b = -(-a \top -b). \tag{1.12}$$

Proof. This follows from (1.1) and (1.2), using that $a > b$ if and only if $-a < -b$.

Remark 1.2. a) By proposition 1.2, each result on $\top$ is equivalent to a corresponding result on $\perp$ (for example, (1.5) and (1.6) are equivalent).

b) Proposition 1.2 is similar to [14], formula (2.2), according to which

$$x \dot{+} y = -(-x \,\dot{+}\, -y), \quad x \,\dot{+}\, y = -(-x \dot{+} -y) \qquad (x,y,z \in \overline{R}). \qquad (1.13)$$

Proposition 1.3. We have

$$a \top +\infty = -\infty = (-\infty) \top a \qquad (a \in \overline{R}), \qquad (1.14)$$

$$a \top -\infty = a = a \perp +\infty \qquad (a \in \overline{R}), \qquad (1.15)$$

$$a \perp -\infty = +\infty = +\infty \perp a \qquad (a \in \overline{R}), \qquad (1.16)$$

$$+\infty \top a = +\infty \qquad (a \in R \cup \{-\infty\}), \qquad (1.17)$$

$$(-\infty) \perp a = -\infty \qquad (a \in R \cup \{+\infty\}), \qquad (1.18)$$

$$(a \top b) \top c = a \top (b \vee c) = (a \top c) \top b \qquad (a,b,c \in \overline{R}). \qquad (1.19)$$

Proof. (1.14) - (1.18) are particular cases of (1.1) and (1.2), which we shall need in the sequel. Finally, by (1.1) and (1.5) we have, for any $a,b,c \in \overline{R}$,

$$(a \top b) \top c = \left\{ \begin{array}{ll} a \top b & \text{if } a \top b > c \\ -\infty & \text{if } a \top b \leq c \end{array} \right\} = \left\{ \begin{array}{ll} a \top b = a & \text{if } a > b \vee c \\ -\infty & \text{if } a \leq b \vee c \end{array} \right\} =$$

$$= a \top (b \vee c) = a \top (c \vee b) = (a \top c) \top b.$$

Proposition 1.4. For any set I, we have

$$(\sup_{i \in I} a_i) \top b = \sup_{i \in I} (a_i \top b) \qquad (\{a_i\}_{i \in I} \subseteq \overline{R},\ b \in \overline{R}), \qquad (1.20)$$

$$(\inf_{i \in I} a_i) \perp b = \inf_{i \in I} (a_i \perp b) \qquad (\{a_i\}_{i \in I} \subseteq \overline{R},\ b \in \overline{R}). \qquad (1.21)$$

Proof. If $\sup_{i \in I} a_i > b$, then $\{i \in I \mid a_i > b\} \neq \emptyset$, and, by (1.1),

$$(\sup_{i \in I} a_i) \top b = \sup_{i \in I} a_i = \sup_{i \in I, a_i > b} a_i = \sup_{i \in I, a_i > b} (a_i \top b) = \sup_{i \in I} (a_i \top b).$$

If $\sup_{i \in I} a_i \leq b$, then $(\sup_{i \in I} a_i) \top b = -\infty$ and, on the other hand, $a_i \leq b$ $(i \in I)$, whence $a_i \top b = -\infty$ $(i \in I)$, so $\sup_{i \in I} (a_i \top b) = -\infty$. The proof of (1.21) is similar (alternatively, (1.21) follows from (1.20) and (1.12)).

We shall use the above propositions in the sequel. Let us also mention, without proof, some further properties of the operations $\top$ and $\perp$. We have, for any $a,b \in \overline{R}$.

$$a \top b \in \{a, -\infty\}; \qquad a \perp b \in \{a, +\infty\}, \tag{1.22}$$

$$a \leq b \Leftrightarrow a \top c \leq b \quad (c \in \overline{R}) \Leftrightarrow a \leq b \perp c \quad (c \in \overline{R}), \tag{1.23}$$

$$a \leq b \Leftrightarrow a \top b = -\infty \Leftrightarrow b \perp a = +\infty, \tag{1.24}$$

$$a \top a = -\infty, \quad a \perp a = +\infty, \tag{1.25}$$

$$(a \top b) \perp b = \begin{cases} -\infty & \text{if } a \leq b \text{ and } b > -\infty \\ +\infty & \text{if either } a > b \text{ or } b = -\infty, \end{cases} \tag{1.26}$$

$$(a \vee b) \top b = a \top b, \tag{1.27}$$

$$a \top b \leq a \perp b; \tag{1.28}$$

moreover,

$$a \top b < a \perp b \qquad (a \in R,\ b \in \overline{R}). \tag{1.29}$$

Remark 1.3. Property (1.28) is similar to the obvious inequality (observed in [14], p.120)

$$x \dot{+} y \leq x \dot{+} y \qquad (x, y \in \overline{R}); \tag{1.30}$$

however, (1.29) contrasts with the obvious equality

$$x \dot{+} y = x \dot{+} y \qquad (x \in R,\ y \in \overline{R}). \tag{1.31}$$

One can also give further results on $\top$ and $\perp$, corresponding to those of [14] on $\dot{+}$ and $\dot{+}$, which we omit.

The binary operations $\top$ and $\perp$ can be extended to $\overline{R}^X$, where X is any set, as follows.

Definition 1.2. For any $f, h \in \overline{R}^X$, let

$$(f \top h)(x) = f(x) \top h(x) \qquad (x \in X), \tag{1.32}$$

$$(f \perp h)(x) = f(x) \perp h(x) \qquad (x \in X). \tag{1.33}$$

In the sequel we shall use (1.32) and (1.33) only in the particular case when either f or h is a constant function.

One can also define such binary operations $\top$ and $\perp$ for any partially ordered set having a greatest element $+\infty$ and a least element $-\infty$ (e.g., replacing $a > b$ by $a \not\leq b$ in (1.1) and $a < b$ by $a \not\geq b$ in (1.2)), but we shall not need this in the sequel.

§ 2. $\vee$-dualities

Definition 2.1. Let X and W be two sets. A duality $\Delta : \overline{R}^X \to \overline{R}^W$ will be called a $\vee$-duality (or, a max-duality), if

$$(f \vee d)^{\Delta} = f^{\Delta} \wedge -d \qquad (f \in \overline{R}^X,\ d \in \overline{R}). \qquad (2.1)$$

Remark 2.1. a) It is enough to assume (2.1) for all $d \in R$, since for $d = +\infty$ it reduces to $(+\infty)^{\Delta} = -\infty$, which holds for any duality, while for $d = -\infty$ it reduces to $f^{\Delta} = f^{\Delta}$ $(f \in \overline{R}^X)$.

b) In analogy with the above, conjugations might be called "$\dotplus$-dualities".

Example 2.1. For any coupling function $\psi: X \times W \to \overline{R}$, the mapping $\Delta(\psi): \overline{R}^X \to \overline{R}^W$ defined by

$$f^{\Delta(\psi)}(w) = \sup_{x \in X} \{\psi(x,w) \wedge - f(x)\} \qquad (f \in \overline{R}^X,\ w \in W), \qquad (2.2)$$

is a $\vee$-duality, which we shall call the $\vee$-duality associated to ψ. Indeed, by (2.2),

$$(\inf_{i \in I} f_i)^{\Delta(\psi)}(w) = \sup_{x \in X} \{\psi(x,w) \wedge -\inf_{i \in I} f_i(x)\} =$$

$$= \sup_{x \in X} \{\psi(x,w) \wedge \sup_{i \in I} (-f_i(x))\} = \sup_{x \in X, i \in I} \{\psi(x,w) \wedge - f_i(x)\} = \sup_{i \in I} f_i^{\Delta(\psi)}(w) \qquad (\{f_i\}_{i \in I} \subseteq \overline{R}^X,\ w \in W),$$

$$(f \vee d)^{\Delta(\psi)}(w) = \sup_{x \in X} \{\psi(x,w) \wedge - (f \vee d)(x)\} =$$

$$= \sup_{x \in X} \{\psi(x,w) \wedge - f(x) \wedge - d\} = f^{\Delta(\psi)}(w) \wedge - d \qquad (f \in \overline{R}^X,\ d \in \overline{R},\ w \in W).$$

Example 2.2. For any set $\Omega \subseteq X \times W$, the mapping $\Delta = \Delta_{\Omega}: \overline{R}^X \to \overline{R}^W$ defined by

$$f^{\Delta}(w) = - \inf_{\substack{x \in X \\ (x,w) \in \Omega}} f(x) \qquad (f \in \overline{R}^X,\ w \in W), \qquad (2.3)$$

is a $\vee$-duality, since it is a particular case of example 2.1, namely, with $\psi: X \times W \to \overline{R}$ defined by

$$\psi = -\rho_{\Omega}; \qquad (2.4)$$

indeed, for ψ of (2.4) we have, by (2.2), (0.12) and (2.3),

$$f^{\Delta(\psi)}(w) = \sup_{x \in X} \{ - \rho_{\Omega}(x,w) \wedge - f(x)\} = \sup_{\substack{x \in X \\ (x,w) \in \Omega}} (-f(x)) =$$

$$= -\inf_{\substack{x \in X \\ (x,w) \in \Omega}} f(x) = f^{\Delta}(w) \qquad (f \in \overline{R}^X,\ w \in W). \qquad (2.5)$$

Let us also recall (see e.g. [23], [10]) that Δ of (2.3) is a conjugation, namely, $\Delta = c(\varphi)$, where

$$\varphi = -\chi_{\Omega}. \qquad (2.6)$$

Remark 2.2. a) If X is a locally convex space and $W = X^* \times R$, where X^* is the conjugate space of X, then for

$$\Omega = \{(x,(\Phi,\lambda)) \in X \times W \mid \Phi(x) \geq \lambda\}, \tag{2.7}$$

the V-duality $\Delta = c(-\chi_\Omega) = \Delta(-\rho_\Omega)$ of (2.3) becomes

$$f^\Delta((\Phi,\lambda)) = -\inf_{\substack{x\in X \\ \Phi(x)\geq\lambda}} f(x) \qquad (f \in \overline{R}^X, \Phi \in X^*, \lambda \in R), \tag{2.8}$$

which is (modulo the inessential additive term $+\lambda$) the quasi-conjugation in the sense of Greenberg and Pierskalla [8]. Similarly, the semi-conjugation

$$f^\Delta((\Phi,\lambda)) = -\inf_{\substack{x\in X \\ \Phi(x)>\lambda-1}} f(x) \qquad (f \in \overline{R}^X, \Phi \in X^*, \lambda \in R), \tag{2.9}$$

introduced in [19] (modulo the additive term $+\lambda - 1$), is obtained by taking $W = X^* \times R$ and

$$\Omega = \{(x,(\Phi,\lambda)) \in X \times W \mid \Phi(x) > \lambda - 1\}; \tag{2.10}$$

note that the pseudo-conjugation, and the more general surrogate conjugations of [18], [20] can be also obtained in the above way, and hence they are V-dualities.

b) If $X = W$ is an arbitrary set, then, for the diagonal set

$$\Omega = \{(x,x) \in X \times X \mid x \in X\}, \tag{2.11}$$

the V-duality $\Delta = c(-\chi_\Omega) = \Delta(-\rho_\Omega)$ of (2.3) becomes

$$f^\Delta(w) = -f(w) \qquad (f \in \overline{R}^X, w \in X), \tag{2.12}$$

which has been considered in [21], example 2.4. Note that, in this case, ψ of (2.4) becomes

$$\psi(x,w) = -\rho_\Omega(x,w) = -\rho_{\{x\}}(w) \qquad (x,w \in X), \tag{2.13}$$

and a corresponding remark holds also for φ of (2.6).

Example 2.3. Fenchel-Moreau conjugations $c(\varphi): \overline{R}^X \to \overline{R}^W$ can be expressed with the aid of V-dualities $\Delta(\psi): \overline{R}^{X\times R} \to \overline{R}^W$, as follows: if $\varphi: X \times W \to \overline{R}$ is any coupling function, then

$$f^{c(\varphi)} = F^{\Delta(\psi)} \qquad (f \in \overline{R}^X), \tag{2.14}$$

where

$$\psi((x,r),w) = 2\varphi(x,w) - r \qquad (x \in X, r \in R, w \in W), \tag{2.15}$$

$$F(x,r) = 2f(x) - r \qquad (x \in X, r \in R). \tag{2.16}$$

Indeed, by the identity

$$\sup_{r\in R} (a - r)\wedge(r - b) = \frac{1}{2}(a \dot{+} - b) \qquad (a,b \in \overline{R}), \tag{2.17}$$

due, essentially, to Flachs and Pollatschek ([7], lemma 1) (see also [25], p.126, lemma), we obtain .

$$F^{\Delta(\psi)}(w) = \sup_{(x,r)\in X\times R} \{\psi((x,r),w)\wedge -F((x,r))\} = \sup_{x\in X}\sup_{r\in R} \{[2\varphi(x,w) - r]\wedge[r - 2f(x)]\} =$$
$$= \sup_{x\in X}\frac{1}{2}[2\varphi(x,w) \dot{+} - 2f(x)] = f^{c(\varphi)}(w) \qquad (w\in W).$$

Taking in (2.2) various coupling functions ψ, one obtains various V-dualities (see examples 2.2 and 2.3). Now we shall show that all V-dualities can be obtained in this way. To this end, we shall use

Lemma 2.1. For any set X, we have

$$f = \inf_{x\in X} \{\rho_{\{x\}} \vee f(x)\} \qquad (f\in \overline{R}^X). \tag{2.18}$$

Proof. By [21], lemma 3.1, we have

$$f = \inf_{x\in X} \{\chi_{\{x\}} \dot{+} f(x)\} \qquad (f\in \overline{R}^X), \tag{2.19}$$

so it is sufficient to observe that

$$\chi_M \dot{+} a = \rho_M \vee a \qquad (M\subseteq X,\ a\in \overline{R}). \tag{2.20}$$

Theorem 2.1. For any V-duality $\Delta : \overline{R}^X \to \overline{R}^W$ we have

$$f^{\Delta} = \sup_{x\in X} \{(\rho_{\{x\}})^{\Delta}\wedge - f(x)\} \qquad (f\in \overline{R}^X), \tag{2.21}$$

and hence there exists a uniquely determined coupling function $\psi_\Delta : X\times W \to \overline{R}$ such that $\Delta = \Delta(\psi_\Delta)$, i.e., such that

$$f^{\Delta}(w) = \sup_{x\in X} \{\psi_\Delta(x,w)\wedge - f(x)\} \qquad (f\in \overline{R}^X,\ w\in W); \tag{2.22}$$

namely, we have

$$\psi_\Delta(x,w) = (\rho_{\{x\}})^{\Delta}(w) \qquad (x\in X,\ w\in W). \tag{2.23}$$

Proof. By (2.18), (0.11) (with I = X) and (2.1) (with $d = f(x)\in \overline{R}$), we obtain, for any $f\in \overline{R}^X$,

$$f^{\Delta} = (\inf_{x\in X} \{\rho_{\{x\}} \vee f(x)\})^{\Delta} = \sup_{x\in X} (\rho_{\{x\}} \vee f(x))^{\Delta} = \sup_{x\in X} \{(\rho_{\{x\}})^{\Delta}\wedge - f(x)\},$$

which proves (2.21). Hence, for ψ_Δ defined by (2.23), we have (2.22). Finally, to prove the uniqueness of ψ_Δ, assume that $\Delta : \overline{R}^X \to \overline{R}^W$ and a function $\psi_\Delta : X\times W \to \overline{R}$ satisfy (2.22). Then, by (2.22) for $f = \rho_{\{x\}}$ and (0.12), we obtain

$$(\rho_{\{x\}})^{\Delta}(w) = \sup_{y\in X}\{\psi_{\Delta}(y,w)\wedge -\rho_{\{x\}}(y)\} = \psi_{\Delta}(x,w) \qquad (x\in X,\ w\in W),$$

so (2.23) holds.

Remark 2.3. a) The above proof is similar to that of [21], theorem 3.1. One can also deduce theorem 2.1 from the results of [11], as follows. By [11], part of theorem 3.2, for any duality $\Delta : \overline{R}^X \to \overline{R}^W$ we have

$$f^{\Delta}(w) = \sup_{x\in X} G_{\Delta}(x,w,f(x)) \qquad (f\in\overline{R}^X,\ w\in W), \tag{2.24}$$

where $G_{\Delta} : X\times W\times\overline{R} \to \overline{R}$ is the function defined by

$$G_{\Delta}(x,w,a) = (\chi_{\{x\}} \dotplus a)^{\Delta}(w) \qquad (x\in X,\ w\in W,\ a\in\overline{R}). \tag{2.25}$$

Now, if Δ is a $\bigvee$-duality, then, by (2.25), (2.20) and (2.1),

$$G_{\Delta}(x,w,a) = (\rho_{\{x\}}\vee a)^{\Delta}(w) = [(\rho_{\{x\}})^{\Delta}\wedge - a](w) = (\rho_{\{x\}})^{\Delta}(w)\wedge -a$$
$$(x\in X,\ w\in W,\ a\in\overline{R}), \tag{2.26}$$

whence, by (2.24), we obtain (2.21) (which, as above, implies (2.22) and the uniqueness of ψ_{Δ}). Note also that, by (2.23) and (2.26), we have

$$\psi_{\Delta}(x,w) = G_{\Delta}(x,w,-\infty) \qquad (x\in X,\ w\in W). \tag{2.27}$$

b) By example 2.1 and theorem 2.1, we have a <u>one-to-one correspondence between $\bigvee$-dualities</u> $\Delta : \overline{R}^X \to \overline{R}^W$ <u>and coupling functions</u> $\psi : X\times W \to \overline{R}$; we shall call ψ_{Δ} of theorem 2.1 <u>the coupling function associated to the $\bigvee$-duality</u> Δ. On the other hand (see §0), we have a one-to-one correspondence between conjugations $c : \overline{R}^X \to \overline{R}^W$ and coupling functions $\varphi : X\times W \to \overline{R}$. Hence, we obtain a <u>one-to-one correspondence between $\bigvee$-dualities</u> $\Delta : \overline{R}^X \to \overline{R}^W$ <u>and conjugations</u> $c : \overline{R}^X \to \overline{R}^W$, namely, $\Delta \to c(\psi_{\Delta})$ (with inverse $c \to \Delta(\varphi_c)$), satisfying

$$(\rho_{\{x\}})^{\Delta} = (\chi_{\{x\}})^{c(\psi_{\Delta})} \qquad (x\in X). \tag{2.28}$$

Similarly to [24], one can show that these one-to-one correspondences are complete lattice isomorphisms.

c) One can replace (2.18) of lemma 2.1 by

$$f = \inf_{(x,d)\in \mathrm{Epi}\, f}\{\chi_{\{x\}} + d\} = \inf_{(x,d)\in \mathrm{Epi}\, f}\{\rho_{\{x\}}\vee d\} \qquad (f\in\overline{R}^X), \tag{2.29}$$

where $\mathrm{Epi}\, f = \{(x,d)\in X\times R \mid f(x)\le d\}$, the epigraph of f. Then, by the above argument, using (2.29) and (0.11) with $I = \mathrm{Epi}\, f$ (or, using directly (2.21)), we obtain, for any $\bigvee$-duality $\Delta : \overline{R}^X \to \overline{R}^W$,

$$f^{\Delta}(w) = \sup_{(x,d)\in \mathrm{Epi}\, f} \{(\rho_{\{x\}})^{\Delta}(w) \wedge -d\} = \sup_{(x,d)\in \mathrm{Epi}\, f} \{\psi_{\Delta}(x,w) \wedge -d\} \qquad (f \in \overline{R}^X,\ w \in W). \tag{2.30}$$

Corollary 2.1. For an operator $\Delta : \overline{R}^X \to \overline{R}^W$ and a set $\Omega \subseteq X \times W$, the following statements are equivalent:

1°. We have (2.3).

2°. Δ is a $\bigvee$-duality, satisfying

$$(\rho_{\{x\}})^{\Delta}(w) = -\rho_{\Omega}(x,w) \qquad (x \in X,\ w \in W). \tag{2.31}$$

Proof. $1^\circ \Rightarrow 2^\circ$. If 1° holds, by example 2.2 and theorem 2.1, Δ is a $\bigvee$-duality, with

$$(\rho_{\{x\}})^{\Delta}(w) = \psi_{\Delta}(x,w) = -\rho_{\Omega}(x,w) \qquad (x \in X,\ w \in W).$$

$2^\circ \Rightarrow 1^\circ$. If 2° holds, then, by (2.21), we obtain

$$f^{\Delta}(w) = \sup_{x\in X} \{(\rho_{\{x\}})^{\Delta}(w) \wedge - f(x)\} = \sup_{x\in X} \{-\rho_{\Omega}(x,w) \wedge - f(x)\} = -\inf_{\substack{x\in X \\ (x,w)\in\Omega}} f(x) \quad (f \in \overline{R}^X,\ w \in W).$$

Remark 2.4. Corollary 2.1 remains valid, with a similar proof, if we replace 2° by

$2'$. Δ is a conjugation, satisfying

$$(\chi_{\{x\}} \dotplus d)^{\Delta}(w) = -\chi_{\Omega}(x,w) \dotplus -d \qquad (x \in X,\ w \in W,\ d \in R). \tag{2.32}$$

Corollary 2.2. For $\Delta : \overline{R}^X \to \overline{R}^W$ and ψ_{Δ} as in theorem 2.1, we have

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ -d\top - \psi_{\Delta}(\cdot,w) \leq f}} d \qquad (f \in \overline{R}^X,\ w \in W). \tag{2.33}$$

Proof. By (2.22), (1.6) and (1.12), for any $f \in \overline{R}^X$ and $w \in W$ we have

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ f^{\Delta}(w)\leq d}} d = \min_{\substack{d\in\overline{R} \\ \psi_{\Delta}(\cdot,w)\wedge - f \leq d}} d = \min_{\substack{d\in\overline{R} \\ -d\top - \psi(\cdot,w)\leq f}} d. \tag{2.34}$$

One can express f^{Δ} with the aid of the level sets of f, as follows.

Corollary 2.3. We have

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ \sup_{x\in A_{-d}(f)} \psi_{\Delta}(x,w)\leq d}} d \qquad (f \in \overline{R}^X,\ w \in W). \tag{2.35}$$

Proof. Since $\psi_{\Delta}(x,w) \wedge -f(x) \leq -f(x) \leq d$ $(x \in X \setminus A_{-d}(f))$ and $-f(x) > d$ $(x \in A_{-d}(f))$, we have

$$\{d \in \overline{R} \mid \psi_{\Delta}(x,w) \wedge -f(x) \leq d \ (x \in X)\} = \{d \in \overline{R} \mid \psi_{\Delta}(x,w) \leq d \ (x \in A_{-d}(f))\},$$

whence, by (2.34), we obtain (2.35).

Remark 2.5. Formula (2.35) is equivalent to

$$f^{\Delta}(w) = \min\{d \in \overline{R} \mid \inf_{\substack{x \in X \\ \psi_{\Delta}(x,w) > d}} f(x) \geq -d\} \qquad (f \in \overline{R}^X, w \in W). \tag{2.36}$$

Finally, let us also mention another expression for $f^{\Delta(\psi)}$.

Proposition 2.1. We have

$$f^{\Delta(\psi)}(w) = \sup_{x \in X} \min_{\substack{d \in \overline{R} \\ d \perp \psi(x,w) \geq -f(x)}} d \qquad (f \in \overline{R}^X, w \in W). \tag{2.37}$$

Proof. This follows from (2.2) and (1.9).

For the $\bigvee$-duality $\Delta = \Delta_{\Omega}$ of example 2.2, there holds

Proposition 2.2. If $\Omega \subseteq X \times W$, then, for the $\bigvee$-duality $\Delta = \Delta_{\Omega} : \overline{R}^X \to \overline{R}^W$ defined by (2.3) we have

$$f^{\Delta}(w) = \min_{\substack{d \in \overline{R} \\ (x,w) \notin \Omega \ (x \in A_{-d}(f))}} d \qquad (f \in \overline{R}^X, w \in W). \tag{2.38}$$

Proof. For any $f \in \overline{R}^X$ and $w \in W$ we have

$$f^{\Delta}(w) = -\inf_{\substack{x \in X \\ (x,w) \in \Omega}} f(x) = \sup_{\substack{x \in X \\ (x,w) \in \Omega}} -f(x) = \min_{\substack{d \in \overline{R} \\ d \geq -f(x) \ (x \in X, (x,w) \in \Omega)}} d =$$

$$= \min_{\substack{d \in \overline{R} \\ (x,w) \notin \Omega \ (x \in X, d < -f(x))}} d = \min_{\substack{d \in \overline{R} \\ (x,w) \notin \Omega \ (x \in A_{-d}(f))}} d.$$

Remark 2.6. For a locally convex space X, $W = X^* \times R$ and Ω of (2.7), i.e., for the Greenberg-Pierskalla quasi-conjugation (2.8), from (2.38) we obtain

$$f^{\Delta}((\Phi, \lambda)) = \min_{\substack{d \in \overline{R} \\ \Phi(x) < \lambda \ (x \in A_{-d}(f))}} d \qquad (f \in \overline{R}^X, \Phi \in X^*, \lambda \in R), \tag{2.39}$$

which is, essentially (namely, modulo the additive term $+\lambda$ in the definition of f^{Δ}), theorem 2, formula (8), of [22]. Similarly, for Δ of (2.2), with $W = X^* \times R$ and Ω of

(2.10), from (2.35) we obtain theorem 2, formula (10), of [22].

We recall that if X and W are two sets, the <u>dual</u> of an operator $\Theta : \overline{R}^X \longrightarrow \overline{R}^W$ is the operator $\Theta^* : \overline{R}^W \longrightarrow \overline{R}^X$ defined (see [21], defintion 4.1) by

$$g^{\Theta^*} = \inf_{\substack{h \in \overline{R}^X \\ h^\Theta \leq g}} h \qquad (g \in \overline{R}^W). \tag{2.40}$$

Example 2.4. If $\Delta = \Delta_\Omega : \overline{R}^X \longrightarrow \overline{R}^W$ is the $\bigvee$-duality (2.3), for some set $\Omega \subseteq X \times W$, then

$$g^{\Delta^*}(x) = -\inf_{\substack{w \in W \\ (x,w) \in \Omega}} g(w) \qquad (g \in \overline{R}^W,\ x \in X). \tag{2.41}$$

Indeed, this is well-known, by $\Delta = c(-\chi_\Omega)$ (see example 2.2) and e.g. [21], corollary 4.5 (see [10], the formula after (5.1)).

The dual of a duality is again a duality (see e.g. [16]) and the dual of a conjugation is a conjugation ([21], theorem 4.1), but the dual of a $\bigvee$-duality need not be a $\bigvee$-duality, nor a "$\bigwedge$-duality". In § 4 we shall determine what kind of duality is the dual of a $\bigvee$-duality, i.e., we shall characterize it by a "second condition" (besides (0.11)). To this end, in § 3 we shall introduce and study $\perp$-dualities.

§ 3. $\perp$-dualities

Definition 3.1. Let X and W be two sets. A duality $\delta : \overline{R}^X \longrightarrow \overline{R}^W$ will be called a <u>$\perp$-duality</u>, if

$$(f \perp d)^\delta = f^\delta \top -d \qquad (f \in \overline{R}^X,\ d \in \overline{R}). \tag{3.1}$$

Remark 3.1. It is enough to assume (3.1) for all $d \in R$, since for $d = -\infty$ it reduces (by (1.16) and (1.14)) to $(+\infty)^\delta = -\infty$, while for $d = +\infty$ it reduces (by (1.15)) to $f^\delta = f^\delta$ $(f \in \overline{R}^X)$.

Example 3.1. For any coupling function $\sigma : X \times W \longrightarrow \overline{R}$, the mapping $\delta(\sigma) : \overline{R}^X \longrightarrow \overline{R}^W$ defined by

$$f^{\delta(\sigma)}(w) = \sup_{x \in X} \{-f(x) \top -\sigma(x,w)\} =$$

$$= \sup_{\substack{x \in X \\ f(x) < \sigma(x,w)}} -f(x) = -\inf_{\substack{x \in X \\ f(x) < \sigma(x,w)}} f(x) \qquad (f \in \overline{R}^X,\ w \in W), \tag{3.2}$$

is a $\perp$-duality, which we shall call the $\perp$-duality associated to σ (the second equality in (3.2) holds by (1.1)). Indeed, by (3.2), (1.20), (1.12) and (1.19),

$$(\inf_{i \in I} f_i)^{\delta(\sigma)}(w) = \sup_{x \in X} \{[-\inf_{i \in I} f_i(x)] \top - \sigma(x,w)\} = \sup_{x \in X} \{[\sup_{i \in I}(-f_i(x))] \top - \sigma(x,w)\} =$$

$$= \sup_{x \in X, i \in I} \{-f_i(x) \top - \sigma(x,w)\} = \sup_{i \in I} f_i^{\delta(\sigma)}(w) \qquad (\{f_i\}_{i \in I} \subseteq \overline{R}^X,\ w \in W),$$

$$(f \perp d)^{\delta(\sigma)}(w) = \sup_{x \in X} \{-(f \perp d)(x) \top - \sigma(x,w)\} =$$

$$= \sup_{x \in X} \{(-f(x) \top - d) \top - \sigma(x,w)\} = f^{\delta(\sigma)}(w) \top - d \qquad (f \in \overline{R}^X,\ d \in \overline{R},\ w \in W).$$

Example 3.2. For any set $\Omega \subseteq X \times W$, the mapping $\delta = \delta_\Omega = \Delta : \overline{R}^X \to \overline{R}^W$ defined by (2.3) is a $\perp$-duality, since it is a particular case of example 3.1, namely, with $\sigma : X \times W \to \overline{R}$ defined by

$$\sigma = -\rho_\Omega ; \tag{3.3}$$

indeed, for σ of (3.3) we have, by (3.2), (0.12) and (2.3),

$$f^{\delta(\sigma)}(w) = -\inf_{\substack{x \in X \\ f(x) < -\rho_\Omega(x,w)}} f(x) = -\inf_{\substack{x \in X \\ (x,w) \in \Omega}} f(x) = f^{\delta}(w) \qquad (f \in \overline{R}^X,\ w \in W). \tag{3.4}$$

Remark 3.2. From example 3.2 and remark 2.2 it follows that the mappings $\delta = \Delta$ of (2.8), (2.9) and (2.12), as well as the pseudo-conjugations and surrogate conjugations of [18] and [20], are $\perp$-dualities.

Example 3.3. Fenchel-Moreau conjugations $c(\varphi) : \overline{R}^X \to \overline{R}^W$ can be also expressed with the aid of $\perp$-dualities $\delta(\sigma) : \overline{R}^{X \times R} \to \overline{R}^W$, as follows: if $\varphi : X \times W \to \overline{R}$ is any coupling function, then

$$f^{c(\varphi)} = F^{\delta(\sigma)} \qquad (f \in \overline{R}^X), \tag{3.5}$$

where

$$\sigma((x,r),w) = \varphi(x,w) - 2r \qquad (x \in X,\ r \in R,\ w \in W), \tag{3.6}$$

$$F(x,r) = \frac{1}{2} f(x) - r \qquad (x \in X,\ w \in W). \tag{3.7}$$

Indeed, we have

$$F^{\delta(\sigma)}(w) = \sup_{\substack{(x,r)\in X\times R \\ F(x,r)<\sigma((x,r),w)}} -F(x,r) = \sup_{\substack{(x,r)\in X\times R \\ (1/2)f(x)-r<\varphi(x,w)-2r}} \{-\frac{1}{2}f(x) + r\} =$$

$$= \sup_{x\in X}\{-\frac{1}{2}f(x) \dot{+} \sup_{\substack{r\in R \\ (1/2)f(x)<\varphi(x,w)-r}} r\} = \sup_{x\in X}\{-\frac{1}{2}f(x) + [\varphi(x,w) \dot{+} -\frac{1}{2}f(x)]\} =$$

$$= \sup_{x\in X}\{\varphi(x,w) \dot{+} - f(x)\} = f^{c(\varphi)}(w) \qquad (w \in W).$$

Taking in (3.2) various coupling functions σ, one obtains various $\perp$-dualities. Now we shall show that all $\perp$-dualities can be obtained in this way.

Theorem 3.1. For any $\perp$-duality $\delta : \overline{R}^X \longrightarrow \overline{R}^W$ there exists a uniquely determined coupling function $\sigma_\delta : X\times W \longrightarrow \overline{R}$ such that $\delta = \delta(\sigma_\delta)$, i.e., such that

$$f^{\delta}(w) = \sup_{x\in X}\{-f(x) \top - \sigma_\delta(x,w)\} \qquad (f \in \overline{R}^X, w \in W); \tag{3.8}$$

namely, we have

$$\sigma_\delta(x,w) = \sup_{\substack{a\in R \\ (\chi_{\{x\}}\dot{+}a)^{\delta}(w) = -a}} a = \min_{\substack{a\in \overline{R} \\ (\chi_{\{x\}}\dot{+}a)^{\delta}(w) = -\infty}} a \qquad (x \in X, w \in W). \tag{3.9}$$

Proof. Similarly to remark 2.3 a), let

$$G_\delta(x,w,a) = (\chi_{\{x\}} \dot{+} a)^{\delta}(w) \qquad (x \in X, w \in W, a \in \overline{R}). \tag{3.10}$$

Then, since for any $a' > a$ we have $\chi_{\{x\}} \dot{+} a = (\chi_{\{x\}} \dot{+} a)\perp a'$ (by (0.10) and (1.2)), from (3.10), (3.1) and (1.1) we obtain, for any $a' > a$,

$$G_\delta(x,w,a) = (\chi_{\{x\}} \dot{+} a)\perp a']^{\delta}(w) = [(\chi_{\{x\}} \dot{+} a)^{\delta} \top - a'](w) =$$

$$= -\infty \text{ if } G_\delta(x,w,a) \leq -a'. \tag{3.11}$$

Hence, either $G_\delta(x,w,a) = -\infty$, or $G_\delta(x,w,a) > -a'$ for all $a' > a$, that is, $G_\delta(x,w,a) \geq -a$. Furthermore, since $(\chi_{\{x\}} \dot{+} a)\perp a = +\infty$ (by (0.10) and (1.2)), from (3.1), (1.1) and (3.10) we obtain

$$-\infty = (+\infty)^{\delta} = [(\chi_{\{x\}} \dot{+} a)\perp a]^{\delta}(w) = [(\chi_{\{x\}} \dot{+} a)^{\delta} \top - a](w) =$$

$$= G_\delta(x,w,a) \text{ if } G_\delta(x,w,a) > -a, \tag{3.12}$$

whence $G_\delta(x,w,a) \leq -a$. Hence, if $G_\delta(x,w,a) > -\infty$, then, by the above, it follows that $G_\delta(x,w,a) = -a$. Thus,

$$G_\delta(x,w,a) \in \{-a, -\infty\} \qquad (a \in \overline{R}). \tag{3.13}$$

Hence, since $G_\delta(x,w,\cdot): a \to G_\delta(x,w,a)$ is non-increasing and lower semi-continuous (by [11], theorem 3.2), there exists $\sigma_\delta(x,w) \in \overline{R}$ such that

$$G_\delta(x,w,a) = \begin{cases} -a & \text{if } a < \sigma_\delta(x,w) \\ -\infty & \text{if } a \geq \sigma_\delta(x,w) \end{cases} = -a \top -\sigma_\delta(x,w). \tag{3.14}$$

Consequently, by [11], theorem 3.2, we obtain

$$f^\delta(w) = \sup_{x \in X} G_\delta(x,w,f(x)) = \sup_{x\in X}\{-f(x) \wedge -\sigma_\delta(x,w)\} \qquad (f \in \overline{R}^X,\ w \in W). \tag{3.15}$$

Furthermore, according to [11], theorem 3.2, $G_\delta : X \times W \times \overline{R} \to \overline{R}$ is uniquely determined by δ. Hence, since (by (3.14))

$$\sigma_\delta(x,w) = \sup_{\substack{a \in R \\ G_\delta(x,w,a)=-a}} a = \min_{\substack{a \in \overline{R} \\ G_\delta(x,w,a)=-\infty}} a, \tag{3.16}$$

σ_δ is also uniquely determined by δ. Finally, by (3.16) and (3.10), we have (3.9).

Remark 3.3. a) For theorem 3.1 we do not have a proof similar to the above proof of theorem 2.1, since the only result (corresponding to lemma 2.1 and to [21], lemma 3.1) expressing $f \in \overline{R}^X$ with the aid of the operation $\perp$, is the formula

$$f = \inf_{x \in X} \{f(x) \perp -\rho_{\{x\}}\} \qquad (f \in \overline{R}^X), \tag{3.17}$$

to which we cannot apply (3.1). In order to show (3.17), it is enough to observe that for any $x,y \in X$ we have, by (1.2) and (0.12),

$$f(x) \perp -\rho_{\{x\}}(y) = \begin{cases} f(x) & \text{if } f(x) < -\rho_{\{x\}}(y) \\ +\infty & \text{if } f(x) \geq -\rho_{\{x\}}(y) \end{cases} = \begin{cases} f(x) & \text{if } x = y \\ +\infty & \text{if } x \neq y; \end{cases}$$

indeed, if $x \neq y$, then $f(x) \geq -\infty = -\rho_{\{x\}}(y)$, while if $x = y$, then $f(x) \geq -\rho_{\{x\}}(y) = +\infty$ — is possible if and only if $f(x) = +\infty$.

b) By example 3.1 and theorem 3.1, we have a one-to-one correspondence between $\perp$-dualities $\delta : \overline{R}^X \to \overline{R}^W$ and coupling functions $\sigma : X \times W \to \overline{R}$; we shall call σ_δ of theorem 3.1 <u>the coupling function associated to the $\perp$-duality</u> δ. One can compose this one-to-one correspondence with those of remark 2.3 b).

Corollary 3.1. <u>For</u> $\delta : \overline{R}^X \to \overline{R}^W$ <u>and</u> σ_δ <u>as in theorem</u> 3.1, <u>we have</u>

$$f^{\delta}(w) = \min_{\substack{d\in\overline{R} \\ (-d)\wedge\sigma_{\delta}(\cdot,w)\leq f}} d \qquad (f\in\overline{R}^{X},\ w\in W). \tag{3.18}$$

Proof. By (3.8) and (1.12) we have, for any $f\in\overline{R}^{X}$ and $w\in W$,

$$f^{\delta}(w) = \min_{\substack{d\in\overline{R} \\ f^{\delta}(w)\leq d}} d = \min_{\substack{d\in\overline{R} \\ -f\top-\sigma_{\delta}(\cdot,w)\leq d}} d = \min_{\substack{d\in\overline{R} \\ f\perp\sigma_{\delta}(\cdot,w)\geq -d}} d, \tag{3.19}$$

whence, by (1.6), we obtain (3.18).

One can express f^{δ} with the aid of the level sets of f, as follows.

Corollary 3.2. We have

$$f^{\delta}(w) = \min_{\substack{d\in\overline{R} \\ \sigma_{\delta}(\cdot,w)|_{A_{-d}(f)}\leq f|_{A_{-d}(f)}}} d \qquad (f\in\overline{R}^{X},\ w\in W). \tag{3.20}$$

Proof. This follows from (3.18), since

$$(-d)\wedge\sigma_{\delta}(\cdot,w)\leq f \Leftrightarrow \sigma_{\delta}(x,w)\leq f(x) \qquad (x\in A_{-d}(f)).$$

Remark 3.4. Formula (3.20) is equivalent to

$$f^{\delta}(w) = \min\{d\in\overline{R}\,|\inf_{\substack{x\in X \\ f(x)<\sigma_{\delta}(x,w)}} f(x)\geq -d\} \qquad (f\in\overline{R}^{X},\ w\in W). \tag{3.21}$$

Finally, let us also mention another expression for $f^{\delta(\sigma)}$.

Proposition 3.1. We have

$$f^{\delta(\sigma)}(w) = \sup_{x\in X}\ \min_{\substack{d\in\overline{R} \\ (-d)\wedge\sigma(x,w)\leq f(x)}} d \qquad (f\in\overline{R}^{X},\ w\in W). \tag{3.22}$$

Proof. By (3.2) and (1.11), we have, for any $f\in\overline{R}^{X}$ and $w\in W$,

$$f^{\delta(\sigma)}(w) = \sup_{x\in X}\ \min_{\substack{d\in\overline{R} \\ d\vee-\sigma(x,w)\geq -f(x)}} d = \sup_{x\in X}\ \min_{\substack{d\in\overline{R} \\ (-d)\wedge\sigma(x,w)\leq f(x)}} d.$$

§ 4. The duals of $\vee$-dualities

Theorem 4.1. If $\Delta:\overline{R}^{X}\to\overline{R}^{W}$ is a $\vee$-duality, then its dual $\Delta^{*}:\overline{R}^{W}\to\overline{R}^{X}$ is a $\perp$-duality, and

$$\sigma_{\Delta^{*}}(w,x) = \psi_{\Delta}(x,w) \qquad (w\in W,\ x\in X). \tag{4.1}$$

Proof. By theorem 3.1, we have to prove that, for any $\vee$-duality $\Delta:\overline{R}^{X}\to\overline{R}^{W}$, there holds

$$g^{\Delta^*} = \sup_{w\in W}\{-g(w)\top - \psi_\Delta(\cdot,w)\} \qquad (g\in \bar{R}^W). \tag{4.2}$$

Now, by (2.40), (2.22), (1.6) and (1.12), we have

$$g^{\Delta^*} = \inf_{\substack{h\in\bar{R}^X \\ h^\Delta\leq g}} h = \inf_{\substack{h\in\bar{R}^X \\ \psi_\Delta(\cdot,\cdot)\wedge -h\leq g}} h = \inf_{\substack{h\in\bar{R}^X \\ -h\leq g\perp\psi_\Delta(\cdot,\cdot)}} h = \inf_{\substack{h\in\bar{R}^X \\ h\geq -g\top-\psi_\Delta(\cdot,\cdot)}} h \qquad (g\in\bar{R}^W) \tag{4.3}$$

(where $\psi_\Delta(\cdot,\cdot)\wedge -h \leq g$ means that $\psi_\Delta(x,w)\wedge -h(x)\leq g(w)$ for all $x\in X$, $w\in W$), whence

$$g^{\Delta^*} \geq \sup_{w\in W}\{-g(w)\top-\psi_\Delta(\cdot,w)\} \qquad (g\in\bar{R}^W). \tag{4.4}$$

On the other hand, the function h_o defined by

$$h_o(x) = \sup_{w\in W}\{-g(w)\top - \psi_\Delta(x,w)\} \qquad (x\in X), \tag{4.5}$$

belongs to the set $\{h\in\bar{R}^X \mid h\geq -g\top-\psi_\Delta(\cdot,\cdot)\}$, whence, by (4.3), we obtain $g^{\Delta^*}\leq h_o$, which, together with (4.4), yields (4.2).

Remark 4.1. a) Theorem 4.1 can be also deduced from the results of [11], as follows. By part of [11], theorem 3.5, for any duality $\Delta : \bar{R}^X \to \bar{R}^W$ we have

$$g^{\Delta^*}(x) = \sup_{w\in W} G_{\Delta^*}(w,x,g(w)) \qquad (g\in\bar{R}^W,\ x\in X), \tag{4.6}$$

where

$$G_{\Delta^*}(w,x,b) = \min_{\substack{a\in\bar{R} \\ G_\Delta(x,w,a)\leq b}} a \qquad (w\in W,\ x\in X,\ b\in\bar{R}), \tag{4.7}$$

with G_Δ of (2.25). Now, by (2.26) and (2.23), there holds

$$G_\Delta(x,w,a) = \psi_\Delta(x,w)\wedge -a \qquad (x\in X,\ w\in W,\ a\in\bar{R}), \tag{4.8}$$

whence, by (4.7) and (1.11),

$$G_{\Delta^*}(w,x,b) = \min_{\substack{a\in\bar{R} \\ \psi_\Delta(x,w)\wedge -a\leq b}} a = (-b)\top-\psi_\Delta(x,w) \qquad (w\in W,\ x\in X,\ b\in\bar{R}), \tag{4.9}$$

which, together with (4.6), yields (4.2).

b) Let us also mention the following direct proof of the first part of theorem 4.1. If $\Delta : \bar{R}^X \to \bar{R}^W$ is a $\vee$-duality, then Δ^* is a duality and, by (2.40), (1.6), (2.1) and (1.5), we have

$$(g\perp d)^{\Delta^*} = \inf_{\substack{h\in\bar{R}^X \\ h^\Delta\leq g\perp d}} h = \inf_{\substack{h\in\bar{R}^X \\ h^\Delta\wedge d\leq g}} h = \inf_{\substack{h\in\bar{R}^X \\ (h\vee -d)^\Delta\leq g}} h =$$

$$= \inf_{\substack{h\in\overline{R}^X \\ h \vee -d \geq g^{\Delta^*}}} h = \inf_{\substack{h\in\overline{R}^X \\ h\geq g^{\Delta^*} \top -d}} h = g^{\Delta^*} \top - d \qquad (g \in \overline{R}^W, d \in \overline{R}),$$

so Δ^* is a $\perp$-duality.

c) By theorems 4.1, 3.1 and 2.1, for any $\vee$-duality $\Delta : \overline{R}^X \to \overline{R}^W$ we have

$$\sup_{\substack{b\in R \\ (\chi_{\{x\}} \dot{+} b)^{\Delta^*}(x) = -b}} b = \min_{\substack{b\in\overline{R} \\ (\chi_{\{x\}} \dot{+} b)^{\Delta^*}(x) = -\infty}} b = (\rho_{\{x\}})^{\Delta}(w) \qquad (x \in X, w \in W). \tag{4.10}$$

d) It will be useful to also express theorem 4.1, in the following equivalent form (by theorems 2.1 and 3.1): If $\Delta(\psi) : \overline{R}^X \to \overline{R}^W$ is a $\vee$-duality (2.2), then its dual $\Delta(\psi)^* : \overline{R}^W \to \overline{R}^X$ is the $\perp$-duality $\delta(\sigma_\psi)$ (in the sense (3.2)), where

$$\sigma_\psi(w,x) = \psi(x,w) \qquad (w \in W, x \in X). \tag{4.11}$$

Let us consider now, for a $\vee$-duality $\Delta : \overline{R}^X \to \overline{R}^W$, the "second dual" (called also the $\Delta^*\Delta$-hull) $f^{\Delta\Delta^*} = (f^\Delta)^{\Delta^*} \in \overline{R}^X$ of a function $f \in \overline{R}^X$. Some results on $f^{\Delta\Delta^*}$ can be obtained from those on g^{Δ^*} $(g \in \overline{R}^W)$, applied to $g = f^\Delta$. For example, for $\Delta = \Delta_\Omega$ of (2.3), applying (2.41) to $g = f^\Delta$, we obtain (see [10], formula (5.2))

$$f^{\Delta\Delta^*}(x) = \sup_{\substack{w\in W \\ (x,w)\in\Omega}} \inf_{\substack{y\in X \\ (y,w)\in\Omega}} f(y) \qquad (f \in \overline{R}^X, x \in X). \tag{4.12}$$

Theorem 4.2. For any $\vee$-duality $\Delta(\psi) : \overline{R}^X \to \overline{R}^W$, we have

$$f^{\Delta(\psi)\Delta(\psi)^*}(x) = \sup_{w\in W} \{-f^{\Delta(\psi)}(w) \top -\psi(x,w)\} = -\inf_{\substack{w\in W \\ f^{\Delta(\psi)}(w)<\psi(x,w)}} f^{\Delta(\psi)}(w) \qquad (f \in \overline{R}^X, x \in X). \tag{4.13}$$

Proof. The first equality follows from (4.2) applied to $\Delta = \Delta(\psi)$ and $g = f^{\Delta(\psi)} \in \overline{R}^W$. The second equality holds by (1.1) (similarly to (3.2)).

Remark 4.2. a) Theorem 4.2 can be also deduced from the results of [11]. Indeed, by (4.6) and (4.9) applied to $\Delta = \Delta(\psi)$ and $g = f^{\Delta(\psi)}$, we have

$$f^{\Delta(\psi)\Delta(\psi)^*}(x) = \sup_{w\in W} G_{\Delta(\psi)^*}(w,x,f^{\Delta(\psi)}(w)) = \sup_{w\in W} \{-f^{\Delta(\psi)}(w) \top - \psi(x,w)\} \qquad (f \in \overline{R}^X, x \in X).$$

b) If F and X are two sets, $x_0 \in X$, $p : F \times X \to \overline{R}$, $f(x) = \inf_{y\in F} p(y,x)$ $(x \in X)$, and

$\psi: X \times W \to \overline{R}$, then, for the "primal" infimization problem

(P) $$\alpha = \inf_{y \in F} p(y,x_o) = f(x_o), \tag{4.14}$$

formula (4.13) suggests to construct a "Lagrangian duality theory, using $\vee$-dualities", by introducing the "dual" problem

(Q) $$\beta = \sup_{w \in W} \{-f^{\Delta(\psi)}(w) \top - \psi(x_o,w)\} = f^{\Delta(\psi)\Delta(\psi)^*}(x_o). \tag{4.15}$$

A similar remark can be also made for $\perp$-dualities, using, e.g., formula (5.9) below. We shall not consider these duality theories in the present paper.

Corollary 4.1. For any $\vee$-duality $\Delta(\psi): \overline{R}^X \to \overline{R}^W$, we have

$$f^{\Delta(\psi)\Delta(\psi)^*}(x) = \sup_{w \in W} \min_{\substack{d \in \overline{R} \\ \psi(x,w) \wedge -d \leq f^{\Delta(\psi)}(w)}} d \qquad (f \in \overline{R}^X,\ x \in X). \tag{4.16}$$

Proof. By (4.13) and (1.11), we have

$$f^{\Delta(\psi)\Delta(\psi)^*}(x) = \sup_{w \in W} \min_{\substack{d \in \overline{R} \\ -f^{\Delta(\psi)}(w) \leq -\psi(x,w) \vee d}} d \qquad (f \in \overline{R}^X,\ x \in X),$$

whence (4.16).

Theorem 4.3. For any $\vee$-duality $\Delta(\psi): \overline{R}^X \to \overline{R}^W$, we have

$$f^{\Delta(\psi)\Delta(\psi)^*} = \sup_{\substack{w \in W,\ b \in \overline{R} \\ b \top - \psi(\cdot,w) \leq f}} \{b \top - \psi(\cdot,w)\} \qquad (f \in \overline{R}^X). \tag{4.17}$$

Proof. By [11], theorem 3.6, for any duality $\Delta(\psi)$ we have

$$f^{\Delta(\psi)\Delta(\psi)^*} = \sup_{\substack{w \in W, b \in \overline{R} \\ G_{\Delta(\psi)^*}(w,\cdot,b) \leq f}} G_{\Delta(\psi)^*}(w,\cdot,b) \qquad (f \in \overline{R}^X,\ w \in W), \tag{4.18}$$

with $G_{\Delta(\psi)^*}$ of (4.7), where $G_{\Delta(\psi)}$ is that of (2.25). Hence, if $\Delta(\psi)$ is a $\vee$-duality, then, by (4.9), we obtain (4.17).

Remark 4.3. Theorem 4.3 shows that, for any $\vee$-duality $\Delta(\psi): \overline{R}^X \to \overline{R}^W$, the $\Delta(\psi)^* \Delta(\psi)$-hull of f coincides with the "V-convex hull" of f, in the sense of [2], where

$$V = \{b \top - \psi(\cdot,w) \mid w \in W,\ b \in \overline{R}\}, \tag{4.19}$$

or, in other words, that, for any $\vee$-duality $\Delta(\psi): \overline{R}^X \to \overline{R}^W$, the "elementary functions", in a sense similar to that of [14], are the functions $\gamma_{w,b} \in \overline{R}^X$ defined by

$$\gamma_{w,b} = b\top - \psi(\cdot,w) = -\chi_{\{x\in X \mid b>-\psi(x,w)\}} \dotplus b = -\chi_{A_b(-\psi(\cdot,w))} + b \qquad (w\in W,\ b\in\overline{R}). \quad (4.20)$$

Corollary 4.2. For any $\bigvee$-duality $\Delta(\psi) : \overline{R}^X \to \overline{R}^W$, we have

$$f^{\Delta(\psi)\Delta(\psi)^*} = \sup_{\substack{w\in W, b\in\overline{R} \\ \sup_{x\in A_b(f)} \psi(x,w)\le -b}} \{b\top - \psi(\cdot,w)\} \qquad (f\in\overline{R}^X,\ w\in W). \quad (4.21)$$

Proof. This follows from (4.17) and the equivalences

$$b\top-\psi(\cdot,w) \le f \Longleftrightarrow b \le f \vee - \psi(\cdot,w) \Longleftrightarrow \psi(\cdot,w)\wedge -f \le -b \Longleftrightarrow \psi(x,w)\le -b \qquad (x\in X,\ -f(x)>-b).$$

Example 2.3. above, which expresses $f^{c(\varphi)}$ as $F^{\Delta(\psi)}$ (with $F : X\times R \to \overline{R}$ of (2.16) and $\psi : (X\times R)\times W \to \overline{R}$ of (2.15)) cannot be used to express $f^{c(\varphi)c(\varphi)^*}$ with the aid of $F^{\Delta(\psi)\Delta(\psi)^*}$. Nevertheless, this aim can be also achieved, with a different method, as shown by

Example 4.1. If $\varphi: X\times W \to \overline{R}$ is any coupling function and $f\in\overline{R}^X$, $g\in\overline{R}^W$, define $\psi : (X\times R)\times(W\times R) \to \overline{R}$, $F : (X\times R) \to \overline{R}$ and $G : (W\times R) \to \overline{R}$ by

$$\psi((x,r),(w,s)) = 2\varphi(x,w) - r - 2s \qquad (x\in X,\ w\in W,\ r,s\in R), \quad (4.22)$$

$$F(x,r) = 2f(x) - r \qquad (x\in X,\ r\in R), \quad (4.23)$$

$$G(w,s) = g(w) - s \qquad (w\in W,\ s\in R). \quad (4.24)$$

Then, by (2.2), (2.17) and (3.2),

$$F^{\Delta(\psi)}(w,s) = \sup_{x\in X, r\in R} \{\psi((x,r),(w,s)) \wedge - F(x,r)\} = \sup_{x\in X, r\in R} \{(2\varphi(x,w) - r - 2s)\wedge(r - 2f(x))\} =$$
$$= \sup_{x\in X} \{\tfrac{1}{2}[(2\varphi(x,w) - 2s) \dotplus - 2f(x)]\} = f^{c(\varphi)}(w) - s \qquad (w\in W,\ s\in R), \quad (4.25)$$

$$G^{\Delta(\psi)^*}(x,r) = -\inf_{\substack{w\in W, s\in R \\ G(w,s)<\psi((x,r),(w,s))}} G(w,s) = -\inf_{\substack{w\in W, s\in R \\ g(w)-s<2\varphi(x,w)-r-2s}} \{g(w) - s\} =$$

$$= -\inf_{\substack{w\in W, s\in R \\ -s>g(w)\dotplus-2\varphi(x,w)+r}} \{g(w) - s\} = -\inf_{w\in W} \{g(w) \dotplus (g(w) \dotplus - 2\varphi(x,w) + r)\} =$$

$$= 2\sup_{w\in W} \{\varphi(x,w) \dotplus - g(w)\} - r = 2g^{c(\varphi)^*}(x) - r \qquad (x\in X,\ r\in R). \quad (4.26)$$

In particular, if $g = f^{c(\varphi)}$, then, by (4.24) and (4.25),

$$G(w,s) = f^{c(\varphi)}(w) - s = F^{\Delta(\psi)}(w,s) \qquad (w \in W,\ s \in R), \tag{4.27}$$

whence, by (4.26) (with $g = f^{c(\varphi)}$), we obtain

$$F^{\Delta(\psi)\Delta(\psi)^*}(x,r) = 2f^{c(\varphi)c(\varphi)^*}(x) - r \qquad (x \in X,\ r \in R). \tag{4.28}$$

Finally, taking $f = g^{c(\varphi)^*}$, from (4.26) and (4.23) we obtain

$$G^{\Delta(\psi)^*}(x,r) = 2g^{c(\varphi)^*}(x) - r = 2f(x) - r = F(x,r) \qquad (x \in X,\ r \in R), \tag{4.29}$$

whence, by (4.25) with $f = g^{c(\varphi)^*}$,

$$G^{\Delta(\psi)^*\Delta(\psi)}(w,s) = F^{\Delta(\psi)}(w,s) = g^{c(\varphi)^*c(\varphi)}(w) - s \qquad (w \in W,\ s \in R). \tag{4.30}$$

§ 5. The duals of $\perp$-dualities

Theorem 5.1. *If $\delta : \overline{R}^X \to \overline{R}^W$ is a $\perp$-duality, then its dual $\delta^* : \overline{R}^W \to \overline{R}^X$ is a $\vee$-duality, and*

$$\psi_{\delta^*}(w,x) = \sigma_\delta(x,w) \qquad (w \in W,\ x \in X). \tag{5.1}$$

Proof. By theorem 2.1, we have to prove that for any $\perp$-duality $\delta : \overline{R}^W \to \overline{R}^X$, there holds

$$g^{\delta^*} = \sup_{w \in W} \{\sigma_\delta(\cdot,w) \wedge - g(w)\} \qquad (g \in \overline{R}^W). \tag{5.2}$$

Now, by (2.40), (3.8) and (1.5), we have

$$g^{\delta^*} = \inf_{\substack{h \in \overline{R}^X \\ h^\delta \leq g}} h = \inf_{\substack{h \in \overline{R}^X \\ -h \top -\sigma_\delta(\cdot,\cdot) \leq g}} h = \inf_{\substack{h \in \overline{R}^X \\ -h \leq g \vee -\sigma_\delta(\cdot,\cdot)}} h = \inf_{\substack{h \in \overline{R}^X \\ h \geq \sigma_\delta(\cdot,\cdot) \wedge -g}} h \qquad (g \in \overline{R}^W), \tag{5.3}$$

whence

$$g^{\delta^*} \geq \sup_{w \in W} \{\sigma_\delta(\cdot,w) \wedge - g(w)\} \qquad (g \in \overline{R}^W). \tag{5.4}$$

On the other hand, the function h_o defined by

$$h_o(x) = \sup_{w \in W} \{\sigma_\delta(x,w) \wedge - g(w)\} \qquad (x \in X), \tag{5.5}$$

belongs to the set $\{h \in \overline{R}^X \mid h \geq \sigma_\delta(\cdot,\cdot) \wedge - g\}$, whence, by (5.3), we obtain $g^{\delta^*} \leq h_o$, which, together with (5.4), yields (5.2).

Remark 5.1. a) Theorem 5.1 can be also deduced from (4.6), (4.7) and (3.14) (with Δ replaced by δ), as follows. By (4.7), (3.14) and (1.5), we have

$$G_{\delta^*}(w,x,b) = \min_{\substack{a \in \overline{R} \\ -a \top -\sigma_\delta(x,w) \leq b}} a = \min_{\substack{a \in \overline{R} \\ b \vee -\sigma_\delta(x,w) \geq -a}} a =$$

$$= \min_{\substack{a \in \overline{R} \\ \sigma_\delta(x,w) \wedge -b \leq a}} a = \sigma_\delta(x,w) \wedge -b \qquad (w \in W,\ x \in X,\ b \in \overline{R}), \tag{5.6}$$

whence, by (4.6) (with Δ replaced by δ), we obtain (5.2).

b) Let us also mention the following direct proof of the first part of theorem 5.1. If $\delta : \overline{R}^X \to \overline{R}^W$ is a $\perp$-duality, then, by (2.40), (1.5), (3.1) and (1.6), we have

$$(g \vee d)^{\delta *} = \inf_{\substack{h \in \overline{R}^X \\ h^\delta \leq g \vee d}} h = \inf_{\substack{h \in \overline{R}^X \\ g \geq h^\delta \top d}} h = \inf_{\substack{h \in \overline{R}^X \\ g \geq (h \perp (-d))^\delta}} h =$$

$$= \inf_{\substack{h \in \overline{R}^X \\ h \perp -d \geq g^{\delta *}}} h = \inf_{\substack{h \in \overline{R}^X \\ g^{\delta *} \wedge -d \leq h}} h = g^{\delta *} \wedge -d \qquad (g \in \overline{R}^W, d \in \overline{R}),$$

so δ^* is a $\vee$-duality.

c) By theorems 5.1, 2.1 and 3.1, for any $\perp$-duality $\delta : \overline{R}^X \to \overline{R}^W$, we have

$$(\rho_{\{w\}})^{\delta *}(x) = \sup_{\substack{a \in R \\ (\chi_{\{x\}} \dot{+} a)^\delta (w) = -a}} a = \min_{\substack{a \in \overline{R} \\ (\chi_{\{x\}} \dot{+} a)^\delta (w) = -\infty}} a \qquad (x \in X, w \in W). \tag{5.7}$$

d) Theorem 5.1 can be also expressed in the following equivalent form: If $\delta(\sigma) : \overline{R}^X \to \overline{R}^W$ is a $\perp$-duality, then $\delta(\sigma)^*$ is the $\vee$-duality $\Delta(\psi_\sigma)$, where

$$\psi_\sigma(w,x) = \sigma(x,w) \qquad (x \in X, w \in W). \tag{5.8}$$

Corollary 5.1. a) Every $\vee$-duality is the dual of a $\perp$-duality.

b) Every $\perp$-duality is the dual of a $\vee$-duality.

Proof. It is well known that for any duality Δ we have $\Delta^{**} = (\Delta^*)^* = \Delta$. Hence, if Δ is a $\vee$-duality ($\perp$-duality), then it is the dual of Δ^*, which, by theorem 4.1 (theorem 5.1), is a $\perp$-duality (respectively, a $\vee$-duality).

Corollary 5.2. a) An operator $\Delta : \overline{R}^X \to \overline{R}^W$ is a $\vee$-duality if and only if Δ^* is a $\perp$-duality.

b) Δ is a $\perp$-duality if and only if Δ^* is a $\vee$-duality.

Let us consider now, for a $\perp$-duality $\delta : \overline{R}^X \to \overline{R}^W$, the second dual (the $\delta^* \delta$-hull) $f^{\delta \delta *} \in \overline{R}^X$ of a function $f \in \overline{R}^X$.

Theorem 5.2. For any $\perp$-duality $\delta(\sigma) : \overline{R}^X \to \overline{R}^W$, we have

$$f^{\delta(\sigma)\delta(\sigma)^*}(x) = \sup_{w \in W} \{\sigma(x,w) \wedge - f^{\delta(\sigma)}(w)\} \qquad (f \in \overline{R}^X, x \in X). \tag{5.9}$$

Proof. Apply (5.2) to $\delta = \delta(\sigma)$ and $g = f^{\delta(\sigma)} \in \overline{R}^W$.

Remark 5.2. Alternatively, (5.9) also follows from (4.6) (with Δ replaced by δ) and (5.6).

Corollary 5.3. For any $\perp$-duality $\delta(\sigma) : \overline{R}^X \to \overline{R}^W$, we have

$$f^{\delta(\sigma)\delta(\sigma)^*}(x) = \sup_{w\in W} \min_{\substack{d\in\overline{R} \\ -d \top -\sigma(x,w) \le f^{\delta(\sigma)}(w)}} d \qquad (f \in \overline{R}^X,\ x \in X). \tag{5.10}$$

Proof. By (5.9) and (1.9), we have

$$f^{\delta(\sigma)\delta(\sigma)^*}(x) = \sup_{w\in W} \min_{\substack{d\in\overline{R} \\ -f^{\delta(\sigma)}(w) \le d \perp \sigma(x,w)}} d \qquad (f \in \overline{R}^X,\ x \in X),$$

whence, using (1.12), we obtain (5.10).

Theorem 5.3. For any $\perp$-duality $\delta(\sigma) : \overline{R}^X \to \overline{R}^W$, we have

$$f^{\delta(\sigma)\delta(\sigma)^*} = \sup_{\substack{w\in W, b\in\overline{R} \\ \sigma(\cdot,w)\wedge b \le f}} \{\sigma(\cdot,w)\wedge b\} \qquad (f \in \overline{R}^X). \tag{5.11}$$

Proof. This follows from (4.18) (with $\Delta(c)$ replaced by $\delta(\sigma)$) and (5.6).

Remark 5.3. Theorem 5.3 shows that, for any $\perp$-duality $\delta(\sigma) : \overline{R}^X \to \overline{R}^W$, the $\delta(\sigma)^* \delta(\sigma)$ – hull of f coincides with the V'-convex hull of f, in the sense of [2], where

$$V' = \{\sigma(\cdot,w)\wedge b \mid w \in W,\ b \in \overline{R}\}, \tag{5.12}$$

and that the "elementary functions" for the $\perp$-duality $\delta(\sigma)$ (in a sense corresponding to that of [14], § 4, for conjugations $c(\varphi)$) are the functions

$$\gamma'_{w,b} = \sigma(\cdot,w)\wedge b \qquad (w \in W,\ b \in \overline{R}). \tag{5.13}$$

Corollary 5.4. For any $\perp$-duality $\delta(\psi) : \overline{R}^X \to \overline{R}^W$, we have

$$f^{\delta(\sigma)\delta(\sigma)^*} = \sup_{\substack{w\in W, b\in\overline{R} \\ \sigma(\cdot,w)|_{A_b(f)} \le f|_{A_b(f)}}} \{\sigma(\cdot,w)\wedge b\} \qquad (f \in \overline{R}^X). \tag{5.14}$$

Proof. This follows from (5.11) and the equivalence

$$\sigma(\cdot,w)\wedge b \le f \Leftrightarrow \sigma(x,w) \le f(x) \qquad (x \in X,\ f(x) < b,\ w \in W).$$

Example 3.3 above, which expresses $f^{c(\varphi)}$ as $F^{\delta(\sigma)}$ (with $F : X \times R \to \overline{R}$ of (3.7) and

$\sigma: (X \times R) \times W \to \overline{R}$ of (3.6)) cannot be used to express $f^{c(\varphi)c(\varphi)^*}$ with the aid of $F^{\delta(\sigma)\delta(\sigma)^*}$. Nevertheless, this aim can be also achieved, with a different method, as shown by

Example 5.1. If $\varphi: X \times W \to \overline{R}$ is any coupling function and $f \in \overline{R}^X$, $g \in \overline{R}^W$, define $\sigma(X \times R) \times (W \times R) \to \overline{R}$, $F: X \times R \to \overline{R}$ and $G: W \times R \to \overline{R}$ by

$$\sigma((x,r),(w,s)) = 2\varphi(x,w) - 2r - s \qquad (x \in X, w \in W, r, s \in R), \tag{5.15}$$

$$F(x,r) = f(x) - r \qquad (x \in X, r \in R), \tag{5.16}$$

$$G(w,s) = 2g(w) - s \qquad (w \in W, s \in R). \tag{5.17}$$

Then, similarly to example 4.1, we obtain

$$F^{\delta(\sigma)}(w,s) = 2f^{c(\varphi)}(w) - s \qquad (w \in W, s \in R), \tag{5.18}$$

$$G^{\delta(\sigma)^*}(x,r) = g^{c(\varphi)^*}(x) - r \qquad (x \in X, r \in R), \tag{5.19}$$

$$F^{\delta(\sigma)\delta(\sigma)^*}(x,r) = f^{c(\varphi)c(\varphi)^*}(x) - r \qquad (x \in X, r \in R), \tag{5.21}$$

$$G^{\delta(\sigma)^*\delta(\sigma)}(w,s) = 2g^{c(\varphi)^*c(\varphi)}(w) - s \qquad (w \in W, s \in R). \tag{5.21}$$

Finally, let us give some characterizations of the operators $\Delta : \overline{R}^X \to \overline{R}^W$ of the form (2.3).

Theorem 5.4. For any operator $\Delta : \overline{R}^X \to \overline{R}^W$, the following statements are equivalent:

1°. There exists a (unique) set $\Omega \subseteq X \times W$ such that we have (2.3).

2°. Δ is both a conjugation and a $\vee$-duality.

3°. Δ is both a conjugation and a $\perp$-duality.

4°. Δ is both a $\vee$-duality and a $\perp$-duality.

5°. Both Δ and Δ^* are $\vee$-dualities.

6°. Both Δ and Δ^* are $\perp$-dualities.

Moreover, in these cases, we have

$$\varphi_\Delta(x,w) = \varphi_{\Delta^*}(w,x) = -\chi_\Omega(x,w) \qquad (x \in X, w \in W), \tag{5.22}$$

$$\psi_\Delta(x,w) = \sigma_\Delta(x,w) = \psi_{\Delta^*}(w,x) = \sigma_{\Delta^*}(w,x) = -\rho_\Omega(x,w) \qquad (x \in X, w \in W). \tag{5.23}$$

Proof. The implications $1° \Rightarrow 2°, 3°, 4°$ and the equalities

$$\varphi_\Delta = -\chi_\Omega, \ \psi_\Delta = \sigma_\Delta = -\rho_\Omega \tag{5.24}$$

follow from examples 2.2 and 3.2 and the uniqueness of φ_Δ, ψ_Δ and σ_Δ. Furthermore, the equivalences $4^\circ \Leftrightarrow 5^\circ \Leftrightarrow 6^\circ$ and the other equalities of (5.22), (5.23) follow from corollary 5.2 and (2.41), (5.1) and (4.1) respectively.

$2^\circ \Rightarrow 1^\circ$. Assume 2° and let $(x,w) \in X \times W$ be such that $\varphi_\Delta(x,w) > -\infty$. Then, for any $d \in R$ satisfying $\varphi_\Delta(x,w) \geq d$, we have, by (0.9), (0.3), (2.20) and (2.1),

$$0 \leq \varphi_\Delta(x,w) - d = (\chi_{\{x\}})^\Delta(w) - d = (\chi_{\{x\}} + d)^\Delta(w) = (\chi_{\{x\}} \vee d)^\Delta(w) =$$
$$= (\chi_{\{x\}})^\Delta(w) \wedge -d = \varphi_\Delta(x,w) \wedge -d.$$

Thus, $0 \leq \varphi_\Delta(x,w)$ and $d \leq 0$ for any $d \in R$ with $\varphi_\Delta(x,w) > d$, whence $\varphi_\Delta(x,w) \leq d$ for all $d > 0$; therefore, $\varphi_\Delta(x,w)=0$. This proves that $\varphi_\Delta(x,w) \in \{0,-\infty\}$ for all $(x,w) \in X \times W$, whence $\varphi_\Delta = -\chi_\Omega$ (so $\Delta = c(-\chi_\Omega)$ of (2.3)), with

$$\Omega = \{(x,w) \in X \times W \mid \varphi_\Delta(x,w) = 0\}; \tag{5.25}$$

moreover, since φ_Δ is uniquely determined by the conjugation Δ, so is Ω.

$3^\circ \Rightarrow 1^\circ$. If 3° holds, then, by [21] and theorem 5.1, $\Delta^* : \overline{R}^W \to \overline{R}^X$ is both a conjugation and a $\vee$-duality. Hence, by the implication $2^\circ \Rightarrow 1^\circ$ (proved above), there exists a (unique) set $\Omega' \subseteq W \times X$ such that $\varphi_{\Delta^*} = -\chi_{\Omega'}$. Hence, since (by [21])

$$\varphi_\Delta(x,w) = \varphi_{\Delta^*}(w,x) \qquad (x \in X, w \in W), \tag{5.26}$$

we obtain

$$\varphi_\Delta(x,w) = -\chi_{\Omega'}(w,x) = -\chi_\Omega(x,w) \qquad ((x,w) \in X \times W), \tag{5.27}$$

where Ω is the (uniquely determined) set

$$\Omega = \{(x,w) \in X \times W \mid (w,x) \in \Omega'\}; \tag{5.28}$$

therefore, $\Delta = c(-\chi_\Omega)$ of (2.3).

$4^\circ \Rightarrow 1^\circ$. Assume 4° and let $(x,w) \in X \times W$ be such that $\psi_\Delta(x,w) > -\infty$. Then, for any $d \in R$ we have $\rho_{\{x\}} = \rho_{\{x\}} \perp d$ (by (0.12), (1.16) and (1.18)), whence, by (2.23) and (3.1) (with $\delta = \Delta$),

$$-\infty < \psi_\Delta(x,w) = (\rho_{\{x\}})^\Delta(w) = (\rho_{\{x\}} \perp d)^\Delta(w) = (\rho_{\{x\}})^\Delta(w) \top -d = \psi_\Delta(x,w) \top -d,$$

and hence, by (1.1), $\psi_\Delta(x,w) > -d$ $(d \in R)$, i.e., $\psi_\Delta(x,w) = +\infty$. This proves that $\psi_\Delta(x,w) \in \{+\infty, -\infty\}$ for all $(x,w) \in X \times W$, whence $\psi_\Delta = -\rho_\Omega$ (so $\Delta = \Delta(-\rho_\Omega)$ of (2.3)), with the (unique) set

$$\Omega = \{(x,w) \in X \times W \mid \psi_\Delta(x,w) = +\infty\}. \tag{5.29}$$

§6. Appendix : Some ⊥-dualities related to lower subdifferentiability

We shall give now some examples of ⊥-dualities and mention, briefly, some of their applications.

Example 6.1. Let X be a locally convex space and $W = X^* \times R$, and define a coupling function $\sigma : X \times W \to \overline{R}$ by

$$\sigma(x,(\Phi,\lambda)) = \Phi(x) + \lambda \qquad (x \in X,\ \Phi \in X^*,\ \lambda \in R). \tag{6.1}$$

Then, by theorem 5.3, for the ⊥-duality $\delta(\sigma) : \overline{R}^X \to \overline{R}^W$ we have

$$f^{\delta(\sigma)\delta(\sigma)^*} = \sup_{\substack{\Phi \in X^*, \lambda \in R, b \in \overline{R} \\ (\Phi+\lambda)\wedge b \leq f}} \{(\Phi + \lambda)\wedge b\} = \sup_{\substack{\Phi \in X^*, \lambda, b \in R \\ (\Phi+\lambda)\wedge b \leq f}} \{(\Phi + \lambda)\wedge b\} \qquad (f \in \overline{R}^X). \tag{6.2}$$

Indeed, to see the last equality in (6.2), it is enough to observe that the inequality $\leq$ holds true, but this follows from

$$\sup_{\substack{\Phi \in X^*, \lambda \in R \\ (\Phi+\lambda)\wedge(-\infty) \leq f}} \{(\Phi + \lambda)\wedge(-\infty)\} = -\infty,$$

$$\sup_{\substack{\Phi \in X^*, \lambda \in R \\ (\Phi+\lambda)\wedge(+\infty) \leq f}} \{(\Phi + \lambda)\wedge(+\infty)\} = \sup_{\substack{\Phi \in X^*, \lambda \in R \\ \Phi+\lambda \leq f}} \{\Phi + \lambda\},$$

which are $\leq$ than the last term of (6.2) (since for any $\Phi \in X^*$, $\lambda \in R$ with $\Phi + \lambda \leq f$ and any $x_o \in X$, the number $b_o = \Phi(x_o) + \lambda \in R$ satisfies $(\Phi + \lambda)\wedge b_o \leq \Phi + \lambda \leq f$ and $\Phi(x_o) + + \lambda = (\Phi(x_o) + \lambda)\wedge b_o$).

Following [9], §5, let us consider the conjugation $c(\ell) : \overline{R}^X \to \overline{R}^W$, ______ where $\ell : X \times W \to \overline{R}$ (with the same $W = X^* \times R$) is the coupling function defined by

$$\ell(x,(\Phi,\lambda)) = \Phi(x)\wedge\lambda \qquad (x \in X,\ \Phi \in X^*,\ \lambda \in R). \tag{6.3}$$

Then since,

$$\ell(\cdot,(\Phi,\lambda)) + b = (\Phi\wedge\lambda) + b = (\Phi + b)\wedge(\lambda + b) \qquad (\Phi \in X^*,\ \lambda, b \in R), \tag{6.4}$$

from (6.2) and [14], §4, it follows that

$$f^{\delta(\sigma)\delta(\sigma)^*} = f^{c(\ell)c(\ell)^*} \qquad (f \in \overline{R}^X). \tag{6.5}$$

Hence, combining the theory developed in the preceding Sections with the results of

[9] on $c(\ell)$, one obtains new formulas for various functional hulls, characterizations of lower subdifferentiability, lower subgradients, etc. (For example, from (6.5) and [9], corollary 5.3, it follows that $f^{\delta(\sigma)\delta(\sigma)^*} = \min\{f_{\bar{q}}, \lambda_f\}$, where $f_{\bar{q}}$ denotes the lower semi-continuous quasi-convex hull of f and λ_f denotes the supremum of those $\lambda \in R$ for which there exists a non-constant continuous affine function minorizing f on $A_\lambda(f)$). We omit the details.

Finally, let us consider the case when X is a normed linear space, with norm $\|\cdot\|$, say. We recall that $f : X \to R$ is said to be Lipschitz with constant N, or N-Lipschitz, if

$$|f(x_1) - f(x_2)| \leq N\|x_1 - x_2\| \qquad (x_1, x_2 \in X). \tag{6.6}$$

Example 6.2. Let X be a normed linear space and $W_N = B^*(0,N) \times R$, where $B^*(0,N) = \{\Phi \in X^* \mid \|\Phi\| \leq N\}$, the ball in X^* with center at the origin and radius N (with $\|\cdot\|$ being, as usual, the norm $\|\Phi\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\Phi(x)|$ on X^*), and define a coupling function $\sigma_N : X \times W_N \to \bar{R}$ by

$$\sigma_N(x,(\Phi, \lambda)) = \Phi(x) + \lambda \qquad (x \in X,\ \Phi \in B^*(0,N),\ \lambda \in R). \tag{6.7}$$

Then, similarly to example 6.1, we obatin

$$f^{\delta(\sigma_N)\delta(\sigma_N)^*} = f^{c(\ell_N)c(\ell_N)^*} \qquad (f \in \bar{R}^X), \tag{6.8}$$

where, following [9], § 5, $\ell_N : X \times W_N \to \bar{R}$ is the coupling function defined by

$$\ell_N(x,(\Phi, \lambda)) = \Phi(x) \wedge \lambda \qquad (x \in X,\ \Phi \in B^*(0,N),\ \lambda \in R). \tag{6.9}$$

Hence, combining the preceding results with those of [9] on $c(\ell_N)$, one obtains new results on quasi-convex Lipschitz functions with constant N, lower subgradients of norm $\leq N$, etc. (for example, from (6.8) and [9], theorem 5.12, it follows that $f^{\delta(\sigma)\delta(\sigma)^*}$ is the greatest quasi-convex N-Lipschitz minorant of f). We omit the details.

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