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by

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1. INTRODUCTION

In this paper we are concerned with a few problems on the structure of invertible matrices. First, we describe an algorithm for constructing invertible matrices starting with a family of complex numbers. This algorithm resembles one of continued fraction type used by I. Schur in [20] in the analysis of the bounded analytic functions on the unit disc and the complex numbers we use for constructing an invertible matrix appear as a generalization of the so-called Schur parameters.

This Schur type algorithm is described in Section 2 and as a first consequence we obtain a formula for the determinant of the given matrix in terms of its Schur parameters. In Section 3 we show the connection of the Schur parameters with the extensions of band matrices treated in [11] and [3]. Moreover, connection with some other known results are presented.

In Section 4, these results are extended to partial matrices subjacent to chordal graphs. Here, the analysis is based on some graph theoretic results obtained in connection with the perfect Gaussian elimination. (see [15]). As the main result it is obtained that the graphs with property that for any subjacent partial matrix there exists a unique invertible completion whose inverse has elements zero on the unspecified positions of the given partial matrix are exactly the chordal graphs. The completion obtained in this way appears as a generalization of the band completion in [11].

In the next section, the induced subgraphs which inherite this completion are characterized. Thus, it is obtained another general result in connection with previous inheritance properties in [12], [13], [6], [17], [5], [2].

In the last section we return to determinantal formulae. We obtain extensions of some formulae derived in [2], computing the determinants of the completions of a partial matrix subjacent to a chordal graph.

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2. A SCHUR TYPE ALGORITHM

In this section we establish a certain structure of an arbitrary matrix (under suitable invertibility conditions) emphasizing the role played by some parameters uniquely determining the given matrix.

This procedure has its roots in a classical paper of I. Schur [20] and since then several variants and generalizations appeared - see, for instance, [1], [7], [14], [18]. Here we follow a line developed in [8] and [9] for positive matrices and, respectively, invertible hermitian matrices.

We begin with some simple remarks concerning the so-called Frobenius-Schur factorization. Let X be an invertible matrix, $L(X) = (XX^*)^{\frac{1}{4}}$ and $R(X) = (X^*X)^{\frac{1}{4}}$.

Defining $s(X) = X(X^*X)^{-\frac{1}{2}}$, a form of the polar decoposition of X is $X = L(X)s(X)R(X)$, $s(X)$ being a unitary matrix, i.e. $s(X)^*s(X) = s(X)s(X)^* = I$ (for a matrix X, X^* denotes its adjoint and I is the unit matrix). Consider W another invertible matrix and the block matrix

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

where Y and Z are matrices of appropriate dimensions.

The following result is a variant of the well-known Frobenius-Schur factorization.

2.1. LEMMA. Define the matrices G and H by the relations $Y = L(X)GR(W)$, $Z = L(W)HR(X)$ and $D = s(W) - Hs(X)^*G$. Then

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} L(X) & 0 \\ 0 & L(W) \end{bmatrix} \begin{bmatrix} s(X) & 0 \\ H & L(D) \end{bmatrix} \begin{bmatrix} s(X)^* & 0 \\ 0 & s(D) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} s(X) & G \\ 0 & R(D) \end{bmatrix} \begin{bmatrix} R(X) & 0 \\ 0 & R(W) \end{bmatrix}. \blacksquare$$

2.2. REMARKS. In view of the above factorization we can suppose that the elements on the main diagonal of the (block) matrices we will consider are unitary.

Note that a dual factorization also holds. Define $F = s(X) - Gs(W)^* H$, then

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} L(F) & G \\ 0 & W \end{bmatrix} \begin{bmatrix} s(F) & 0 \\ 0 & W^* \end{bmatrix} \begin{bmatrix} R(F) & 0 \\ H & W \end{bmatrix}$$

(we supposed that X and W are unitary matrices). The matrices $D = s(W) - Hs(X)^* G$ and $F = s(X) - Gs(W)^* H$ are the so-called Schur complements - see [19]. ∎

With the notation in Lemma 2.1. and Remark 2.2., we derive the following result.

2.3. LEMMA. *Suppose D and F invertible. Then, there exist uniquely determined matrices A and B such that*

$$\begin{bmatrix} G & L(F) \\ R(D) & A \end{bmatrix} \begin{bmatrix} W^* & 0 \\ 0 & s(F) \end{bmatrix} \begin{bmatrix} H & L(D) \\ R(F) & B \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & s(D)^* \end{bmatrix}$$

and

$$\begin{bmatrix} H & L(D) \\ R(F) & B \end{bmatrix} \begin{bmatrix} X^* & \\ 0 & s(D) \end{bmatrix} \begin{bmatrix} G & L(F) \\ R(D) & A \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & s(F)^* \end{bmatrix}.$$

PROOF. We have only the choose:

(2.1) $$A = -R(D)W^* HR(F)^{-1} s(F)^*$$

and

(2.2) $$B = -s(F)^* L(F)^{-1} GW^* L(D).$$ ∎

After the preliminaries we can introduce the objects we will be interested in throughout this section. Consider a family $(G_{ij})_{1 \leq i,j \leq N}$ of matrices and define

$S_{ii} = s(G_{ii}) = G_{ii}$ for $i = 1, \ldots, N$ and for $1 \leq i < j \leq N$,

(2.3) $$D_{ij} = S_{i,j-1} - G_{ij} S_{j,i+1} G_{ji}$$

(2.4) $$D_{ji} = S_{j,i+1} - G_{ji} S_{i,j-1} G_{ij}$$

where $S_{ij} = s(D_{ij})$ for all $1 \leq i,j \leq N$. This family will satisfy the following property:

(2.5) D_{ij} *and* D_{ji} *are invertible matrices.*

We define, according to Lemma 2.3, the matrices: for $1 \leq i \leq N$, $R_{ii} = I$ and for

$1 \le i < j \le N$,

(2.6) $$R_{ij} = \begin{bmatrix} G_{ij} & L(D_{ij}) \\ R(D_{ji}) & A_{ij} \end{bmatrix} \begin{bmatrix} S^*_{j,i+1} & 0 \\ 0 & S_{ij} \end{bmatrix}$$

(where A_{ij} is defined by formula (2.1)); for $1 \le i \le N$, $L_{ii} = I$ and

(2.7) $$L_{ji} = \begin{bmatrix} S^*_{j,i+1} & 0 \\ 0 & S_{ij} \end{bmatrix} \begin{bmatrix} G_{ji} & L(D_{ji}) \\ R(D_{ij}) & B_{ji} \end{bmatrix}$$

(where B_{ji} is defined by formula (2.2)). The next step is to consider for $1 \le i \le N$, $V(R_{ii}) = V(L_{ii}) = I$ and for $1 \le i < j \le N$,

(2.8) $$V(R_{ij}) = (R_{i,i+1} \oplus I_{j-i})(I \oplus R_{i,i+2} \oplus I_{j-i-1}) \dots (I_k \oplus R_{i,i+k} \oplus I_{j-i-k}) \dots (I_{j-i} \oplus R_{ij})$$

and

(2.9) $$V(L_{ji}) = (I \oplus L_{ji}) \dots (I_k \oplus L_{i+k,i} \oplus I_{j-i+k}) \dots (I \oplus L_{i+2,i} \oplus I_{j-i-1})(L_{i+1,i} \oplus I_{j-i}).$$

The following matrices will play the main role in the sequell: for $1 \le i \le N$, $U(R_{ii}) = U(L_{ii}) = I$ and for $1 \le i < j \le N$,

(2.10) $$U(R_{ij}) = V(R_{ij})(U(R_{i+1,j}) \oplus I)$$

and

(2.11) $$U(L_{ji}) = (U(L_{j,i+1}) \oplus I)V(L_{ji}).$$

Based on the matrices $U(R_{ij})$ and $U(L_{ji})$ we can construct triangular matrices which will produce a lower-upper triangular factorization of a given invertible matrix. That is, we define for $1 \le i < j \le N$,

(2.12) $$F_{ij} = \begin{bmatrix} F_{i,j-1} & U(R_{i,j-1})C_{ij} \\ 0 & R(D_{ji})R(D_{j,i+1}) \dots R(D_{j,j-1}) \end{bmatrix}$$

where $F_{ii} = G_{ii}$ and

(2.13) $$C_{ij} = (G_{j-1,j}, G_{j-2,j}R(D_{j,j-1}), \dots, G_{ij}R(D_{j,i+1}) \dots R(D_{j,j-1}))^t$$

("t" denotes the matrix transpose). We also define for $1 \leq i < j \leq N$,

$$(2.14) \qquad H_{ji} = \begin{bmatrix} H_{j-1,i} & 0 \\ \\ C_{ji}U(L_{j-1,i}) & L(D_{j,j-1})\dots L(D_{ji}) \end{bmatrix}$$

where $H_{ii} = G_{ii}$ and

$$(2.15) \qquad C_{ji} = (G_{j,j-1}, L(D_{j,j-1})G_{j,j-2}, \dots, L(D_{j,j-1})\dots L(D_{j,i+1})G_{ji}).$$

We conclude this list of notation by defining for $1 \leq i \leq j \leq N$,

$$(2.16) \qquad E_{ji} = S^*_{ii} \oplus S_{i+1,i} \oplus \dots \oplus S_{ji}.$$

We use families of matrices $(G_{ij})_{1 \leq i,j \leq N}$ and their associated objects as above in order to describe a certain structure of the invertible matrices having all principal submatrices also invertible (and, according to a previous convention, all the elements on the main diagonal are unitary matrices). Denote such a matrix by $T = (T_{ij})_{1 \leq i,j \leq N}$ and the set of all these matrices by $\mathcal{T}$. Define $P_1 = (I, 0, \dots, 0)$ where the number of matrices 0 is required by the place where P_1 appears.

Now, we can state and prove the main result of this section.

2.4. THEOREM. Any matrix $T = (T_{ij})_{1 \leq i,j \leq N}$ belonging to $\mathcal{T}$ is uniquely determined by a family of matrices $(G_{ij})_{1 \leq i,j \leq N}$ satisfying (2.5).

This correspondence is realised by means of the formulae: for $1 \leq i \leq N$, $T_{ii} = G_{ii}$ and for $1 \leq i < j \leq N$,

$$T_{ij} = P_1 U(R_{i,j-1}) C_{ij}$$

and

$$T_{ji} = C_{ji} U(L_{j-1,i}) P_1.$$

Moreover, the following factorization holds:

$$T = H_{N1} E_{N1} F_{1N}.$$

PROOF. The proof can be performed by induction on the size N of T. The following assertions are verified in this vein: for $1 \leq i < j \leq N$

$(2.17)_{ij}$ $\quad (T_{ij}, T_{i+1,j}, \ldots, T_{j-1,j})^t = H_{j-1,i}E_{j-1,i}U(R_{i,j-1})C_{ij}$

$(2.18)_{ij}$ $\quad (T_{ji}, T_{j,i+1}, \ldots, T_{j,j-1}) = C_{ji}U(L_{j-1,i})E_{j-1,i}F_{i,j-1}$

$(2.19)_{ij}$ $\quad (T_{mp})_{i\leq m,p\leq j} = H_{ji}E_{ji}F_{ij}$

$(2.20)_{ij}$ $\quad T_{ij} = P_1U(R_{i,j-1})C_{ij}$

$(2.21)_{ji}$ $\quad T_{ji} = C_{ji}U(L_{j-1,i})P_1^*.$

As all these assertions follow by repeating (and slightly adapting) the computations made in the proof of Theorem 1.3 in [8] we illustrate here only the relevant case $N = 3$.

We take $G_{ii} = T_{ii}$, $i = 1, 2, 3$ and $G_{12} = T_{12}$, $G_{23} = T_{23}$, $G_{21} = T_{21}$, $G_{32} = T_{32}$ which satisfy the required property (2.5) by Lemma 2.1 and hypothesis. Further on, we define G_{13} and G_{31} by the formulae $T_{13} = P_1U(R_{12})C_{13}$ and, respectively, $T_{31} = C_{31}U(L_{21})P_1^*$ because we see that, once the other elements are fixed, the correspondence between T_{13} and G_{13} and, respectively T_{31} and G_{31} in these formulae is one-to-one. By direct computation using Lemma 2.3 and the definitions,

$$H_{21}E_{21}U(R_{12})C_{13} = (T_{13}, T_{23})^t.$$

Similarly

$$(T_{31}, T_{32}) = C_{31}U(L_{21})E_{21}F_{12}.$$

In order to verify that $T = H_{31}F_{31}F_{13}$, we note that

$$T = \begin{bmatrix} H_{21}E_{21}F_{12} & H_{21}E_{21}U(R_{12})C_{13} \\ & \\ C_{31}U(L_{21})E_{21}F_{12} & T_{33} \end{bmatrix}$$

Remark that by Lemma 2.3,

$$U(L_{21})E_{21}U(R_{12}) = S_{22}^* \oplus S_{12}$$

and then

$$C_{31}(S_{22}^{*}\oplus S_{12})C_{13}+L(D_{32})L(D_{31})S_{31}R(D_{31})R(D_{32})=T_{33}$$

which finishes the proof in the considered case. ■

As a first consequence of Theorem 2.4 we obtain a formula for the determinant of T.

2.5. COROLLARY. $\det T=\prod_{j=1}^{N}\det S_{j1}\prod_{1\leq i<j\leq N}\det L(D_{ji})R(D_{ji})$. ■

2.6. REMARK. Without the assumption that T_{ii} are unitary matrices, the following modifications are necessary: $D_{ii}=G_{ii}=T_{ii}$, $S_{ii}=s(G_{ii})$, $i=1,\dots,N$. In these conditions, the formula in Corollary 2.5 becomes

$$\det T=\prod_{j=1}^{N}\det S_{j1}\prod_{1\leq i\leq j\leq N}\det L(D_{ji})R(D_{ji}).\ ■$$

3. COMPLETIONS OF BAND MATRICES

In this section we obtain some other applications of Theorem 2.4 to the completion of band matrices treated in [11] and [3].

Consider a partial block-matrix $T=(T_{ij}/1\leq i,j\leq N)$ where T_{ij} are specified matrices for $|i-j|\leq m$ and T_{ij} are unspecified matrices for $|i-j|>m$, where $m\geq 1$.

A completion of T will be a specification of the matrices T_{ij} for $|i-j|>m$. Our assumption here will be that all principal submatrices of T formed by specified matrices are invertible and we write in this case $T\in\mathcal{B}_m$.

By a result in [11] and [3], $T\in\mathcal{B}_m$ admits completions in the class $\mathcal{T}$ as defined in Section 2 (this is, of course, a consequence of Theorem 2.4 also). More than that, by Theorem 2.4, T is uniquely determined by a family of matrices $(G_{ij}/|i-j|\leq m)$ satisfying (2.5) and any completion $\tilde{T}$ of T in class $\mathcal{T}$ is uniquely determined by a family $(G_{ij}/|i-j|>m)$ of matrices satisfying (2.5) (with respect to the given family $(G_{ij}/|i-j|\leq m)$.

By Corollary 2.5, a formula for the determinant of $\tilde{T}$ is obtained in terms of the parameters $(G_{ij}/|i-j|\leq m)$ and $(G_{ij}/|i-j|>m)$, but, of course, it is quite desirable to avoid the use of $(G_{ij}/|i-j|\leq m)$ by the use of the known matrices $(T_{ij}/|i-j|\leq m)$. Here, the main role is played by a Fischer-Hadamard type formula. For a set of indices $A\subset\{1,2,\dots,N\}$ we denote by $T(A)$ the matrix $(T_{ij}|i,j\in A)$. Let us first explain the simplest (but generic) case $N=3$, $m=1$. By Remark 2.6, for any completion $\tilde T\in\mathcal{T}$ of T (and such a completion is determined by the parameters G_{13} and G_{31}, while T is determined by the parameters $(G_{ij}/|i-j|\leq 1,\ 1\leq i,j\leq 3)$,

$$\det\tilde T=\prod_{i=1}^{3}\det S_{i1}\prod_{1\leq i\leq j\leq 3}\det L(D_{ji})R(D_{ji})$$

$$\det T(1,2)=\prod_{i=1}^{2}\det S_{i1}\prod_{1\leq i\leq j\leq 2}\det L(D_{ji})R(D_{ji})$$

$$\det T(2,3)=\prod_{i=2}^{3}\det S_{i2}\prod_{2\leq i\leq j\leq 3}\det L(D_{ji})R(D_{ji}).$$

Consequently,

$$\det\tilde T=(\det T(1,2)\det T(2,3)/\det T(2))(\det S_{31}/\det S_{32})\det L(D_{31})R(D_{31}). \tag{3.1}$$

Using the same Remark 2.6, we can obtain the following extension of formula (3.1) as a Fischer-Hadamard type formula. Define the index sets $Y_k=\{k,k+1,\dots,k+m\}$, $k=1,2,\dots,N-m$.

3.1. PROPOSITION. Let $T\in\mathcal{B}_m$ and $\tilde T\in\mathcal{T}$ be a completion of T, determined by the parameters $(G_{ij}/|i-j|>m)$. Then

$$\det\tilde T=\Big(\prod_{k=1}^{N-m}\det T(Y_k)/\prod_{k=1}^{N-m-1}\det T(Y_k\cap Y_{k+1})\Big)K(\tilde T),$$

where

$$K(\tilde T)=\Big(\prod_{j-i>m}\det S_{ji}/\prod_{j-i>m}\det S_{j,i+1}\Big)\prod_{j-i>m}\det L(D_{ji})R(D_{ji}).\ \blacksquare$$

It is important to know when the residual factor $K(\tilde T)$ disappears. An obvious sufficient condition is that

$$G_{ji}S_{i,j-1}G_{ij}=0\qquad\text{for } j-i>m \tag{3.2}$$

but this is not necessary.

However, the condition $G_{ij} = 0$ for $|i - j| > m$ plays a role in connection with the so-called band completion in [11] (and also in [3]). We have the following result.

3.2. PROPOSITION. Let $T \in \mathcal{B}_m$, then $(\tilde{T}^{-1})_{ij} = 0$ for $|i - j| > m$ if and only if $G_{ij} = 0$ for $|i - j| > m$.

PROOF. Once again it is sufficient to illustrate the case $N = 3$, $m = 1$. Moreover, without loss of generality, T_{ii} are supposed unitary matrices.

Consider $\tilde{T} \in \mathcal{T}$ a completion of T. By (2.19),

$$\tilde{T} = H_{31} E_{31} F_{13}$$

and then

$$\tilde{T}^{-1} = F_{13}^{-1} E_{31} H_{31}^{-1}.$$

In view of the triangularity of F_{13} and H_{31}, $(\tilde{T}^{-1})_{13} = 0$ if and only if $(F_{13}^{-1})_{13} = 0$ and $(\tilde{T}^{-1})_{31} = 0$ if and only if $(H_{31}^{-1})_{31} = 0$. But, in view of the definition of F_{13},

$(F_{13}^{-1})_{13} = -P_1 F_{12}^{-1} U(R_{12}) C_{13} R^{-1}(D_{32}) R^{-1}(D_{31})$ and by direct computations, using Lemma 2.3, we get

$$(F_{13}^{-1})_{13} = -R^{-1}(D_{12}) G_{13} L(D_{32}) R^{-1}(D_{32}) R^{-1}(D_{31})$$

so that, $(F_{13}^{-1})_{13} = 0$ if and only if $G_{13} = 0$. Similarly, $(H_{31}^{-1})_{31} = 0$ if and only if $G_{31} = 0$. The general case is essentially reduced to this one. ■

3.3.REMARK. The problem of finding necessary and sufficient conditions on T for the existence of a completion $\tilde{T}$ with $(\tilde{T}^{-1})_{ij} = 0$ for $|i - j| > m$ (this $\tilde{T}$, denoted from now on by T° is called the band completion of T) is solved in [11]. Our conditions are slightly restrictive than those in [11], but it is quite simple to see that we can develop a formalism similar to that in Theorem 2.3 (i.e. a Schur analysis) also in conditions of [11] (and which will be lesser explicit than Theorem 2.3). So that, we preffer to remain in class $\mathcal{T}$.

In Proposition 3.2 we have explained at the level of Schur analysis the band completion phenomenon. Moreover, in this way, Proposition 3.1 appears as a generalization of a determinantal formula in [3] (which is obtained exactly for the band completion case, when $K(T^{\circ}) = 1$ by Proposition 3.2). On the other hand, it is easily seen that the condition $K(\tilde{T}) = 1$ does not imply $\tilde{T}$ is the band completion of T (as is the case for positive matrices). ▨

Now, we can state an inheritance principle for the band completion proved for positive matrices in [12] and for hermitian invertible matrices in [13].

3.4. PROPOSITION. Let T° be the band completion of T. Then $T^{\circ}(Y)$ is the band completion of $T(Y)$ for any $Y = \{k,\ldots,m\}$, $1 \leq k \leq m \leq N$.

PROOF. It is only a consequence of Proposition 3.2. ▨

4. SCHUR ANALYSIS AND GAUSSIAN ELIMINATION

An expression of the connection between the Schur analysis and Gaussian elimination already appeared in Proposition 3.2. Our goal in this section is to obtain a general result in this direction, extending also the band completion principle in [11] and [3].

Consider a partial matrix $T = (t_{ij})_{1\leq i,j\leq N}$, $t_{ij} \in \mathbb{C}$, in the sense that some of elements t_{ij} are specified and some of them are not specified. The assumptions are that the main diagonal is specified (and its elements are of modulus 1) and all the principal submatrices formed by specified elements are invertible. We say in this case that T belongs to $\mathcal{PT}$. Moreover, we say that T has property (P) if it admits a unique completion $T^{\circ} \in \mathcal{T}$ such that $(T^{\circ})^{-1}_{ij} = 0$ for the unspecified positions (i,j).

With a partial matrix $T = (t_{ij})$, an undirected graph $G = (V,E)$ is associated in the following way: $V = \{1,2,\ldots,N\}$ and an edge between i and j exists ($i \neq j$) if both t_{ij} and t_{ji} are specified elements. We call G the associated graph of T and we say that T is subjacent to G. A graph G has property (Q) if any subjacent matrix of G in $\mathcal{PT}$ has the property (P).

Some more terminology on graphs is needed (we follow here [15]). $A \subset V$ is a <u>clique</u> if $(x,y) \in E$ for all distinct $x,y \in A$. The <u>subgraph</u> of G <u>induced</u> by A is $G_A = (A,E_A)$, where $E_A = \{(x,y) \in E/x,y \in A\}$.

G is <u>chordal</u> if every cycle of length strictly greater then 3 has a chord, i.e. an edge joining two nonconsecutive vertices of the cycle.

We can obtain now the main result of this section.

4.1. THEOREM. G <u>has property</u> (Q) <u>if and only if it is a chordal graph</u>.

PROOF. Consider $T \in \mathcal{PT}$ sujacent to the chordal graph $G = (V,E)$. First, we obtain a parametrization of all completions in $\mathcal{T}$ of T. For this purpose, fix a sequence of chordal graphs $G = G_o, G_1, \dots, G_t = K_N$ (K_N is the complete graph based on the vertices of G, N being the cardinality $|V|$ of V) such that each $G_j = (V,E_j)$ is obtained from G_{j-1} by adding exactly one newledge (u_j,v_j). The existence of such a sequence follows from the fact that G is chordal (see for instance [16]). By [16], there exists a unique maximal clique V_j in G_j which is not a clique in G_{j-1}. In this way, after a reordering of V_j if necessary,

$$T(V_j) = \begin{bmatrix} A & B & t_{u_j,v_j} \\ C & D & E \\ t_{v_j,u_j} & F & G \end{bmatrix}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} D & E \\ F & G \end{bmatrix}$ are in class $\mathcal{T}$. Moreover, at least one of t_{u_j,v_j} and t_{v_j,u_j} is unspecified. Using Theorem 2.4, t_{u_j,v_j} is uniquely determined by a complex number $g(u_j,v_j)$ and t_{v_j,u_j} is uniquely determined by a complex number $g(v_j,u_j)$ such that

$$\varepsilon_1^{(j)} - g(u_j,v_j)\varepsilon_2^{(j)} g(v_j,u_j) \neq 0, \quad |\varepsilon_1^{(j)}| = |\varepsilon_2^{(j)}| = 1.$$

If t_{u_j,v_j} is unspecified, $g(u_j,v_j)$ is choosen as parameter. Similarly for t_{v_j,u_j}. This process can be used for $j = 1,2,\dots,t$ and finally get a one-to-one correspondence between the completions of T in class $\mathcal{T}$ and the families of complex numbers

$$\left\{ g(u_i,v_i),\ g(v_i,u_i) / \ i = 1,\dots,t,\ t_{u_i,v_i},\ t_{v_i,u_i} \text{ unspecified} \right\}$$

such that

$$\varepsilon_1^{(i)} - g(u_i,v_i)\varepsilon_2^{(i)} g(v_i,u_i) \neq 0 \tag{4.1}$$

where $|\varepsilon_1^{(i)}| = |\varepsilon_2^{(i)}| = 1$.

More than that, T^o is obtained exactly for $g(u_i,v_i) = 0 = g(v_i,u_i)$, for $i = 1,\dots,t$ and t_{u_i,v_i}, t_{v_i,u_i} unspecified, when the condition (4.1) holds.

Indeed, by Proposition 3.2, the choice $g(u_i,v_i) = 0$ (or $g(v_i,u_i) = 0$) makes $(T^o(V_i))^{-1}_{u_i,v_i} = 0$ (or respectively, $(T^o(V_i))^{-1}_{v_i,u_i} = 0$) (or both of them, if it is the case).

We have to show that in the above mentioned process of parametrizing the completions of T these zero entries are not changed (there is no fill-in). For this, we have only to use Lemma 6.2 in [5] and Theorem 2.3 in [6] as in the proof of Theorem 6.1 in [5].

In conclusion, we obtained that if G is a chordal graph, then T, a partial matrix in $\mathcal{PT}$ subjacent to G, has a unique completion $T^o \in \mathcal{T}$ such that $(T^o)^{-1}_{ij} = 0$ for the unspecified positions (i.j) of T, and it is given by the parameters zero with respect to the above parametrization of all completions of T in class $\mathcal{T}$.

Conversely, suppose G is not chordal. It contains a cycle of length strictly greater than 3. First, consider the case of $G = C_4$, the cycle of length 4.

Define the hermitian partial matrix $T = (t_{ij})_{1\le i,j\le 4}$ with $t_{ii} = 1$, $1 \le i \le 4$, $t_{12} = t_{14} = t_{34} = 2$ and $t_{23} = 0$. Then T has two (also hermitian) completions in $\mathcal{T}$, $T^{(1)}$, $T^{(2)}$ given by $t_{13}^{(k)} = t_{24}^{(k)} = a_k$, $k = 1,2$, where $a_1 = \frac{1}{2}(-1+\sqrt{17})$ and $a_2 = \frac{1}{2}(-1-\sqrt{17})$, with the property that their inverses have 0 on the unspecified positions of T.

Further on, consider a cycle C_N of length $N > 4$ and the (hermitian) partial matrix subjacent to C_N, $T = (t_{ij})_{1\le i,j\le N}$ given by $t_{ii} = 1$, $1 \le i \le N$, $t_{12} = t_{1N} = t_{N-1,N} = 2$ and $t_{i,i+1} = 0$ for $1 < i < N-1$. We consider the following two (hermitian) completions of T in $\mathcal{T}$: $T^{(k)} = (t_{ij}^{(k)})_{1\le i,j\le N}$, $k = 1,2$, where $t_{1,N-1}^{(k)} = t_{2,N}^{(k)} = b_k$, $t_{2,N-1}^{(k)} = c_k$ and $t_{ij}^{(k)} = 0$ elsewhere (of course, where the positions are unspecified in T), and $b_1 = -7$, $c_1 = 134/3$, $b_2 = -4$, $c_2 = 56/3$. The inverses of $T^{(1)}$ and $T^{(2)}$ have also the property that they have 0 on the unspecified positions of T.

Now, these constructions can be used in an obvious way for concluding that if G is not chordal, then it has not the property (Q). ∎

5. INHERITANCE PRINCIPLES

Chordal graphs in the Gaussian elimination appeared in connection with an inheritance principle, i.e. no fill-in to appear when performing the elimination. So that, it is quite natural to expect some other inheritance properties when matrices subjacent to chordal graphs are considered-see for instance [12], [13], [6], [17], [5], [2].

From now on, only symmetric partial matrices are taken into account (once t_{ij} is specified so is t_{ji} also).

Consider $G = (V,E)$ a chordal graph, $|V| = N$ and $W \subset V$. We say that G_W the graph induced by W, has property (A) if for any partial matrix $T \in \mathcal{PT}$ subjacent to G, we have, with the notation in Section 4, that

(5.1) $$T(W)^{o} = T^{o}(W).$$

Define

$$\mathrm{Adj}(W) = \{ v \in V \,|\, (v,w) \in E_W \text{ for a certain } w \in W\} - W$$

and

$$K(G_W) = (V,F), \text{ where } F = E \cup \{(u,v) \,|\, u,v \in W\}.$$

We obtain the following result which extends Proposition 4.4 in [10].

5.1. THEOREM. G_W has property (A) if and only if $K(G_W)$ is a chordal graph and for each $v \in \mathrm{Adj}(W)$, $\mathrm{Adj}(v) \cap W$ is a clique.

PROOF. Suppose there exists a sequence $G = G_o, G_1, \ldots, G_t = K_N$ of chordal graphs such that each $G_j = (V,E_j)$ is obtained from G_{j-1} by adding exactly one new edge and such that there exists $r < t$ with $V_j \subset W$ for $j = 1, \ldots, r$ and $(G_r)_W$ is the complete graph with vertex set W (recall that V_j is the unique maximal clique in G^j not a clique in G_{j-1}).

In view of the parametrization of the completions in $\mathcal{T}$ of a partial matrix in $\mathcal{PT}$ subjacent to G given in the proof of Theorem 4.1 it follows that (5.1) holds.

Following the proof of Proposition 4.3 in [10] we show that, conversely, if (5.1) holds, then a sequence of chordal graphs as above exists indeed.

For this purpose, remark that there exist complex numbers $a_2, \dots, a_N$ such that the hermitian partial matrix $T = (t_{ij})_{1 \leq i,j \leq N}$, $t_{ii} = 1$, $i = 1, \dots, N$, $t_{12} = 0$, $t_{1N} = a_N$, $t_{i,i+1} = a_i$, $i = 2, N-1$ has at least two completions in $\mathcal{T}$ for which their inverses have O on the unspecified positions of T. Indeed, an obvious permutation in the partial matrices constructed in the proof of Theorem 4.1 yields the required T.

So that, if Case 1 in the proof of Proposition 4.3 in [10] holds, G does not satisfy (5.1).

For the Case 2 in the proof of Proposition 4.3 in [10], we have to remark that based on Theorem 2.4, the same arguments works as well as there.

Finally, the equivalence between the property of G_W that there exists a sequence $G = G_o, G_1, \dots, G_t = K_N$ of chordal graphs as at beginning of the proof and the properties that $K(G_W)$ is chordal graph and for each $v \in \mathrm{Adj}(W)$, $\mathrm{Adj}(v) \cap W$ is a clique is exactly the proof of Proposition 4.4 in [10]. ∎

5.2. REMARKS. (a) When a positive partial matrix x is taken into account (i.e. all principal submatrices formed with specified elements are positive), T^o is a positive completion and by Theorem 2 in [16], it is the maximum determinant completion over all the positive completions of the given partial matrix.

Thus, by Remark 4.5 in [10] several known inheritance principles appear as cases of Theorem 5.1.

(b) Proposition 3.4 obviously appears as a particular case of Theorem 5.1. ∎

6. SOME DETERMINANTAL FORMULAE

Determinantal formulae for matrices with sparse inverses are long known and recent results connect them with chordal graphs - [3], [4], [5].

In [2], determinantal formulae for all positive completions of a positive partial matrix subjacent to a chordal graph were obtained and the perfect Gaussian elimination process is pointed out by expressing the maximum determinant (i.e. $\det T^{o}$ with the notation on in previous sections) in terms of the determinants of some principal submatrices with specified elements according to a rule determined by a perfect vertex elimination scheme.

A perfect vertex elimination scheme of a chordal graph $G = (V,E)$ (which exists by a theorem of Fulkerson and Gross see [15]) is an ordering $\sigma = [v_1, \dots, v_N]$ of the vertices such that each set $A_i = \{ v_j \in Adj(v_i) \mid j > i \}$ is a clique.

The following construction is quite useful in connection with a fixed perfect vertex elimination scheme $\sigma = [v_1, \dots, v_N]$ of G.

Define $B_k = \{ v_k, \dots, v_N \}$ and R be the least integer for which B_R is a clique. Then B_{R-1} is a partitioned as

$$B_{R-1} = \{v_{R-1}\} \cup A_{R-1} \cup (B_{R-1} - (\{v_{R-1}\} \cup A_{R-1})).$$

If $B_{R-1} - (\{v_{R-1}\} \cup A_{R-1}) = [w_1, \dots, w_S]$, where the order is that in σ, we define $E_1 = E \cup \{(v_{R-1}, w_S)\}$, $E_2 = E_1 \cup \{(v_{R-1}, w_{S-1})\}, \dots, E_S = E_{S-1} \cup \{(v_{R-1}, w_1)\}$. Further on, for $k = 2, \dots, \ - 1$,

$$B_{R-k} = \{v_{R-k}\} \cup A_{R-k} \cup (B_{R-k} - (\{v_{R-k}\} \cup A_{R-k}))$$

and keeping in B_{R-k} the same order as in σ, we continue to define a sequence of chordal graphs by succesively connecting each v_{R-k} with the vertices in $(B_{R-k} - (\{v_{R-k}\} \cup A_{R-k}))$, but taken in the reversed order. As σ is a perfect vertex elimination scheme of G, a sequence $G = G_o, G_1, \dots, G_t = K_N$ of chordal graphs is obtained, such that G_j is obtained from G_{j-1} by adding exactly one new edge.

Let T be a partial matrix in $\mathcal{PT}$ subjacent to G and $\{g(u_j,v_j), g(v_j,u_j)/j = 1,\ldots,t\}$ are the parameters of $\tilde{T}$, a completion of T in $\mathcal{T}$, associated to the above sequence of chordal graphs as in the proof of Theorem 4.1.

As an extension of a result in [2], we obtain

6.1. THEOREM. _With the above notations,_

$$\mathbf{det}\,\tilde{T} = K\prod_{j=1}^{t}(\varepsilon_1^{(j)} - g(u_j,v_j)\,\varepsilon_2^{(j)} g(v_j,u_j))\,\mathbf{det}\,T^o$$

and $\mathbf{det}\,T^o = \prod_{m=1}^{n} (\mathbf{det}\,\tilde{T}(\{v_m\} \cup A_m)/\mathbf{det}\,T(A_m))$

where $|\varepsilon_1^{(j)}| = |\varepsilon_2^{(j)}| = 1$ _and_ $K\prod_{j=1}^{t}\varepsilon_1^{(j)} = 1.$

PROOF. We follow the some line as in the proof of Proposition 4.1 in [2], but using formula in Corollary 2.5 when necessary. For identifying

$$\mathbf{det}\,\tilde{T}/K\prod_{j=1}^{t}(\varepsilon_1^{(j)} - g(u_j,v_j)\,\varepsilon_2^{(j)} g(v_j,u_j))$$

with $\mathbf{det}\,T^o$, we have to use the result obtained in Theorem 4.1 that T^o is given by the parameters $g^o(u_j,v_j) = g^o(v_j,u_j) = 0$. ∎

6.2. REMARK. (a) A formula for $\mathbf{det}\,T^o$ was previously obtained in [4] expressing this determinant in terms of a spanning tree of G - see also [5].

One more formula can be obtained as in Proposition 4.1 in [10].

(b) It is easy to see that the formula for det T in Theorem 6.1 holds with respect to any other sequence of chordal graphs $G = G_o, G_1, \ldots, G_t = K_N$ where each G_j is obtained from G_{j-1} by adding exactly one new sdge.

For this, we can follow the line of the proof of Theorem 4.6 in [2]. ∎

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