INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

TWO PAPERS ON REFLEXIVE OPERATOR ALGEBRAS

by

Florin POP

PREPRINT SERIES IN MATHEMATICS
No. 40/1990

BUCURESTI

DERIVATIONS OF NEST-SUBALGEBRAS OF TYPE II_1 FACTORS

FLORIN POP

Institute of Mathematics

of the Romanian Academy

present address: Department of Mathematics INCREST

Bd.Pacii 220 , 79622 Bucharest, ROMANIA

DERIVATIONS OF NEST-SUBALGEBRAS OF TYPE II_1 FACTORS

Every ultraweakly continuous derivation of a nest-subalgebra of the hyperfinite type II_1 factor associated to a nest which generates a Cartan subalgebra is inner.

INTRODUCTION:

Let $B \subset B(H)$ be an arbitrary subalgebra of the algebra of all bounded linear operators on a Hilbert space H. A derivation of B is a linear mapping $\delta : B \to B$ satisfying $\delta(xy) = x\delta(y) + \delta(x)y$ for every x,y in B. δ is inner if there is an operator $b \in B$ such that

$$\delta(x) = [b,x] = bx-xb \quad (\forall)\ x \in B .$$

It is important to know about a given subalgebra $B \subset B(H)$ wether all its derivations are inner. In other words, this means that its first Hochschild cohomology group is trivial, that is $H^1(B,B) = 0$. This is known to be true for all von Neumann algebras ([5],[11]) and for all nest algebras ([1]).

In ([8]) we started to investigate this problem for nest-subalgebras of von Neumann algebras. We proved there, among other results, that derivations of nest-subalgebras of type II_1 factors which come from atomic nests are inner. The aim of this paper is to get more insight in the case when the nest generates a diffuse abelian subalgebra. We prove that if this abelian subalgebra is a Cartan subalgebra of the hyperfinite type II_1 factor, then every ultraweakly continuous derivation of the corresponding nest-subalgebra is inner.

The main ingredient is a nonselfadjoint version of the celebrated Connes-Feldman-Weiss theorem (corollary 1.2) . The result contained in 1.2(iii) is not new. Given in an even more general setting, it is due to P.Muhly and B.Solel ([7]) and it is a direct consequence of the results in ([6]). The proofs in ([6]) and ([7]) heavily used ergodic theory, as well as the original proof of the Connes-Feldman-Weiss theorem ([2]) relied on the Feldman-Moore ergodic model ([4]).

In ([10]) Sorin Popa gave a purely operator theoretical proof of the Connes-Feldman-Weiss theorem. In turn, we present an operator theoretical proof of the nonselfadjoint case.

The main result about derivations is obtained, as in ([1]) and ([8]) , by showing that nest-subalgebras of the hyperfinite type II_1 factor associated to nests which generate Cartan subalgebras have E.Christensen's 'automorphism implementation property'. This means that automorphisms close to the identity are implemented by invertible operators close to the identity (Theorem 2.3).

Let us first fix the terminology. Throughout this paper R will be the hyperfinite type II_1 factor with faithful finite normal trace τ , $\tau(I) = 1$. A nest L is a totally ordered strongly closed family of (selfadjoint) projections in B(H) containing 0 and I . If $L \subset R$, then the algebra

$$M = \{ x \in R \ ; \ (I-p)xp = 0 \quad (\forall)\ p \in L \}$$

is the nest-subalgebra of R associated to the nest L .

Recall that for an arbitrary subalgebra $B \subset B(H)$, B' denotes the commutant of B. A nest L is continuous if L'' is a diffuse abelian subalgebra. Finally, if $b \in B$ then the inner derivation of B implemented by b ,that is $\delta(x)=bx-xb$ is denoted by ad(b).

1. CONTINUOUS NESTS AND CARTAN SUBALGEBRAS :

Recall that R is the hyperfinite type II_1 factor with trace τ, $\tau(I)=1$ and $\|x\|_2 = \tau(x^*x)^{1/2}$ for every $x \in R$. If $B \subset R$ is a von Neumann subalgebra then E_B denotes the unique trace preserving normal conditional expectation of R on B. Let $A \subset R$ be a maximal abelian *-subalgebra. Then $N_R(A)$ denotes the normalizer of A in R , that is

$$N_R(A) = \{ u \in R \text{ unitary } ; u^*Au = A \}$$ and

$GN_R(A)$ denotes the normalizing grupoid of A in R

$$GN_R(A) = \{ v \in R \text{ partial isometry } ; v^*v, vv^* \in A , vAv^* = Av^*v \}$$

Note that $v \in GN_R(A)$ if and only if there is $u \in N_R(A)$ and a projection $e \in A$ such that $v=ue$ (see for example ([10])).

A is a Cartan subalgebra if $N_R(A)$ generates R as a von Neumann algebra.

The Connes-Feldman-Weiss theorem says that if A_1 and A_2 are Cartan subalgebras of R then there exists an automorphism $\varphi \in \text{Aut } R$ such $\varphi(A_1) = A_2$.

1.1 PROPOSITION:

Let $A \subset R$ be a Cartan subalgebra. If φ is a trace preserving automorphism of A then there is an automorphism $\Phi \in \text{Aut } R$ such that $\varphi(a) = \Phi(a)$ $(\forall)$ $a \in A$.

Proof: According to ([10]) , consider an increasing sequence of matrix subalgebras (type I factors) $M_n \subset R$, each of them with a set of matrix units $(e_{ij}^n)_{1 \leq i,j \leq 2^{k_n}}$ such that $e_{ii}^n \in A$, $\sum_i e_{ii}^n = I$,

$e_{ij}^n \in GN_R(A)$ $(\forall)$ i,j,n , every e_{rs}^p , $p \leq n$, is the sum of some e_{ij}^n and such that $\overline{\bigcup_n M_n}^w = R$.

For every n consider a 2^{k_n}- partition of the unity in A $(p_n^1, \dots, p_n^{2^{k_n}})$ where $p_n^r = e_{rr}^n$ $1 \leq r \leq 2^{k_n}$.

Then $(\varphi(p_n^1), \dots, \varphi(p_n^{2^{k_n}}))$ is another 2^{k_n}-partition of the unity in A and consequently there is a unitary operator $u_n \in N_R(A)$ such that $\varphi(p_n^i) = u_n^* p_n^i u_n$ $(\forall)$ $n \geq 1$ and $1 \leq i \leq 2^{k_n}$ (see for example ([9])3.4).

Let $N_n^1 = u_n^* M_n u_n$ and $N_n^2 = u_n M_n u_n^*$

Moreover, the unitaries u_n can be chosen such that for every $n \geq 1$ and $m \geq n$

$u_m^* M_n u_m = u_n^* M_n u_n$ and $u_m M_n u_m^* = u_n M_n u_n^*$

If we define $\varphi(M_n) = N_n^1$ then, by the above remarks, φ is correctly defined and φ uniquely extends to a *-isomorphism between R and a subfactor $R_0 \subset R$ ([3] III.7.2)

More precisely, for a given countable subset $(v_i)_{i \geq 1} \subset GN_R(A)$ dense in $GN_R(A)$ in the norm $\|.\|_2$, M_n can be chosen such that $\| E_{M_n}(v_i) - v_i \|_2 < 2^{-n}$, $1 \leq i \leq n$ ([10]).

For every v_i and $\varepsilon > 0$, there is $n_\varepsilon \geq 1$ and an operator w in M_{n_ε} such that $\| v_i - w \|_2 \leq \varepsilon/2$.

Since for $n,m \geq n_\varepsilon$ $u_m u_n^* w u_n u_m^* = w$ then

$\| u_m u_n^* v_i u_n u_m^* - v_i \|_2 \leq \varepsilon$, hence $\| u_n^* v_i u_n - u_m^* v_i u_m \|_2 \leq \varepsilon$

It follows that $(u_n^* v_i u_n)_{n \geq 1}$ is Cauchy in the norm $\|.\|_2$

and since it is also bounded in the uniform norm, there is $\tilde{v}_i \in R$ such that $\lim_n \| \tilde{v}_i - u_n^* v_i u_n \|_2 = 0$.

It follows that for every $v \in GN_R(A)$ there is $\tilde{v} \in R$ such that $\lim_n \| \tilde{v} - u_n^* v u_n \|_2 = 0$, hence

$\varphi(v) = \lim_n u_n^* v u_n$ strongly for every $v \in GN_R(A)$.

On the other side, $u_m^* u_n M_n u_n^* u_m = u_m^* N_n^2 u_m = M_n$ for $m \geqslant n$ hence for every $n \geqslant 1$ and $1 \leqslant i,j \leqslant 2^{k_n}$

$u_n e_{ij}^n u_n^* \in GN_R(A)$ and $\varphi(u_n e_{ij}^n u_n^*) = e_{ij}^n$.

Consequently $R_0 \supset M_n$ $(\forall)$ $n \geqslant 1$ and this implies

$R_0 = R$, Q.E.D.

1.2. COROLLARY:

(i) Let $L_1, L_2 \subset R$ be two continuous nests such that L_1'' and L_2'' are Cartan subalgebras in R. Then there is an automorphism $\varphi \in \mathrm{Aut}\, R$ such that $\varphi(L_1) = L_2$.

(ii) If $L \subset R$ is a continuous nest such that L'' is a Cartan subalgebra, then there is an increasing sequence of matrix subalgebras $M_n \subset R$ with matrix units (e_{ij}^n) , $e_{ij}^n \in GN_R(A)$ $\overline{\bigcup_n M_n}^w = R$, satisfying in addition

$$\sum_{i=1}^{r} e_{ii}^n \in L \quad \text{for every } n \geqslant 1 \text{ and } 1 \leqslant r \leqslant 2^{k_n}$$

(iii) If $L \subset R$ is a continuous nest such that L'' is a Cartan subalgebra and if $M \subset R$ denotes the corresponding nest-subalgebra, then there is an increasing sequence of matrix nest algebras T_n with matrix units (e_{ij}^n) such that

$e_{ij}^n = 0$ for $i > j$, $\sum_{i=1}^{r} e_{ii}^n \in L$ for $n \geq 1$, $1 \leq r \leq 2^{k_n}$

$e_{ij}^n \in GN_R(A)$ and $\overline{\bigcup_n T_n}^w = M$

(iv) If $L_1, L_2 \subset R$ are continuous nests such that L_1'' and L_2'' are Cartan subalgebras and if M_1 and M_2 are their corresponding nest-subalgebras ,then there is $\varphi \in \mathrm{Aut}\, R$ such that $\varphi(M_1) = M_2$

Proof: (i) Suppose first that $L_1'' = L_2'' = A \subset R$, A Cartan subalgebra. Then for every $p \in L_1$ there is a unique $q \in L_2$ such that $\tau(p) = \tau(q)$. Define $\psi : L_1 \to L_2$ by $\psi(p) = q$. Then ψ uniquely extends to a τ-preserving automorphism of A , hence there is $\varphi \in \mathrm{Aut}\, R$ satisfying $\varphi(L_1) = L_2$. The general case reduces to the above one by the Connes-Feldman-Weiss theorem.

(ii) Let $(N_n)_{n \geq 1}$ be an increasing sequence of matrix algebras with matrix units (e_{ij}^n) , $e_{ii}^n \in A$ and $\overline{\bigcup_n N_n}^w = R$

Then the nest L_0 generated by the projections $\sum_{i=1}^{r} e_{ii}^n$, $n \geq 1$, $1 \leq r \leq 2^{k_n}$ is continuous and $L_0'' = A$.

If $\varphi \in \mathrm{Aut}\, R$ is such that $\varphi(L_0) = L$ then $M_n = \varphi(N_n)$ satisfies the required conditions.

(iii) Let M_n be the matrix algebras from (ii) and let T_n be the upper triangular part of M_n.

If $a \in M$ then $E_{M_n}(a) \in T_n$. Indeed, for every projection $q = \sum_{i=1}^{r} e_{ii}^n$ $q \in L \cap M_n$ we have

$$(I-q)\, E_{M_n}(a)\, q = E_{M_n}((I-q)aq) = 0 \quad .$$

Since $\text{so-}\lim_n E_{M_n}(a) = a$, our assertion is proved.

(iv) Let M_n^1 and M_n^2 be two increasing sequences of matrix algebras with matrix units (e_{ij}^n) and (f_{ij}^n) satisfying

$$\sum_{i=1}^r e_{ii}^n \in L_1 \quad , \quad \sum_{i=1}^r f_{ii}^n \in L_2 \quad \text{and}$$

$\overline{\bigcup_n M_n^1} = \overline{\bigcup_n M_n^2} = R$. Let $\varphi \in \text{Aut } R$ be such that $\varphi(M_n^1) = M_n^2$, hence $\varphi(T_n^1) = T_n^2$, so that $\varphi(M_1) = M_2$.

2. AUTOMORPHISMS AND DERIVATIONS

2.1. PROPOSITION: Let N be a type II_1 factor and $L \subset N$ be a continuous nest which generates a maximal abelian *-subalgebra $A \subset N$. If M denotes the nest-subalgebra of N corresponding to L, then for every operator x in N

$$\inf\left\{ \|x-\lambda I\| , \lambda \in C \right\} \leq 7/2 \sup\left\{ \|xa-ax\| , a \in M, \|a\| \leq 1 \right\}$$

Proof: Let μ denote an invariant mean on the (abelian) unitary group U(A) . If $y = \int_{U(A)} u^*xu \, d\mu(u)$

then $y \in A' \cap N = A$. For every unitary operator $u \in U(A)$ and for every $a \in M$, $uau^* \in M$ and

$$\| [u^*xu , a] \| = \| [x , uau^*] \| .$$

Since y belongs to the ultraweakly closed convex hull of the set $\{ u^*xu \; ; \; u \in U(A) \}$ it follows that $\| ad(y)|_M \| \leq \| ad(x)|_M \|$. We prove now that for every $y \in A$

$\inf\{ \| y-\lambda I \| \} \leq 5/2 \; \| ad(y)|_M \|$. Denote by τ the trace of N , $\tau(N)=1$

If we decompose N after the unique projection $p \in L$ with $\tau(p) = 1/2$ then N is isomorphic to the algebra of 2×2 matrices over a factor N_0 with a maximal abelian $*$-subalgebra A_0 and A is isomorphic to $A_0 \oplus A_0$. Moreover, the image of M contains

$N_0 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly $y = y_1 \oplus y_2$, $y_i \in A_0$.

By hypothesis, if $s = \| ad(y)|_M \|$ then

$$\left\| \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \right\| \leq s$$

for every unitary operator $u \in N_0$. It follows that

$\| y_1 u - u y_2 \| \leq s$, hence $\| y_1 - y_2 \| \leq s$ and $\| y_1 - u y_2 u^* \| \leq s$

hence there is $\lambda_1 \in C$ such that $\| y_1 - \lambda_1 I \| \leq s$.

Similarly $\| y_2 - \lambda_2 I \| \leq s$ for some $\lambda_2 \in C$.

Consequently $| \lambda_1 - \lambda_2 | \leq \| \lambda_1 I - y_1 \| + \| y_1 - y_2 \| + \| y_2 - \lambda_2 I \| \leq 3s$

and $\| y_1 \oplus y_2 - (\lambda_1 + \lambda_2)/2 \, I \oplus (\lambda_1 + \lambda_2)/2 \, I \| \leq 5/2 \, s$

which proves our claim. It follows that

$\inf \{ \| y - \lambda I \| , \lambda \in C \} \leq 5/2 \; \| ad(x)|_M \|$

Since $\| x - y \| \leq \| ad(x)|_A \| \leq \| ad(x)|_M \|$ we conclude that

$\inf \{ \| x - \lambda I \| , \lambda \in C \} \leq 7/2 \; \| ad(x)|_M \|$ Q.E.D.

2.2. PROPOSITION: Let R, L, A and M denote the hyperfinite type II_1 factor, a continuous nest in R, a Cartan subalgebra in R and the nest-subalgebra of R

$$M = \{ x \in R \; ; \; (I-p)xp = 0 \quad (\forall) \; p \in L \}$$ respectively.

Suppose moreover that $L'' = A$. If $\varphi \in \text{Aut } M$ is a ultraweakly continuous automorphism of M such that $\|id-\varphi\| < 1$ and $\varphi(x) = x \quad (\forall) \; x \in A$ then there is an invertible operator $a \in A$ satisfying $\varphi(x) = a^{-1}xa \quad (\forall) \; x \in M$ and $\|I-a\| \leq \|id-\varphi\|$

Proof: Let $0=p_0 \leq p_1 \leq \dots \leq p_n=I$ be projections in L such that $\tau(p_{i+1}-p_i) = 1/n \quad (\forall) \; 0 \leq i \leq n-1$

and decompose R after this partition of the unity in A. Then R is isomorphic to $R \otimes B(C^n)$ (the $n \times n$ matrices over R) and A is isomorphic to $A \oplus A \oplus \dots \oplus A$ (n times).

Consider now the finite dimensional nest algebra T_n with matrix units (e_{ij}^n) where $e_{ij}^n = 0$ for $i > j$ and $e_{ij}^n = I \otimes u_{ij}^n$ for $i \leq j$ where u_{ij}^n are the matrix units in $B(C^n)$. We prove that there is an invertible operator $a \in A$ such that $\varphi(x) = a^{-1}xa \quad (\forall) \; x \in T_n$.

For arbitrary $1 \leq i < j \leq n$ $\quad q_i = p_i - p_{i-1}$, $q_j = p_j - p_{j-1}$

If we identify Aq_i and Aq_j with A then for every $a, b \in A$ one has $\varphi(a)=a$, $\varphi(b)=b$ and $\varphi(e_{ij}^n) \in q_iRq_j$

Moreover, for every $x \in q_iRq_j \subset M$ one has $\varphi(ax)=a\varphi(x)$ and $\varphi(xb)=\varphi(x)b$ hence $a\varphi(e_{ij}^n) = \varphi(e_{ij}^n)a \quad (\forall) \; a \in A$ and this implies $\varphi(e_{ij}^n) = a_{ij}^n \in A$.

Define $\varphi(e_{12}^{n}) = a_1 , \ldots , \varphi(e_{1n}^{n}) = a_{n-1}$.

Clearly $\|I-a_i\| \leq \|id-\varphi\| < 1$ hence the operator

$a = I \oplus a_1 \oplus \ldots \oplus a_{n-1} \in A$ is invertible and a routine computation shows that $\varphi(x) = a^{-1}xa \quad (\forall)\ x \in T_n$

(see for example ([8]) II 5 corollary)

We have therefore obtained a sequence $(a_n)_{n \geq 1}$ of invertible operators in A satisfying $\|I-a_n\| \leq \|id-\varphi\| < 1$

and $\varphi(x) = a_n^{-1}xa_n \quad (\forall)\ x \in T_n$, and, as one can easily see

$\varphi(x) = a_m^{-1}xa_m \quad (\forall)\ x \in T_n$ and $m \geq n$.

Let now $(n_k)_{k \geq 1}$ be an increasing sequence such that

$\overline{\bigcup_k T_{2^{n_k}}}^{w} = M$ (1.2 (iii)) and $\lim_k a_{2^{n_k}} = a$

ultraweakly. Then $\|I-a\| \leq \|id-\varphi\| < 1$ so that a is invertible and for every $x \in T_{2^{n_k}}$ and $l \geq k$

$a_{2^{n_l}} \varphi(x) = x\, a_{2^{n_l}}$, $a\varphi(x) = xa \quad (\forall)\ x \in T_{2^{n_k}}$

But φ is ultraweakly continuous, hence $a\varphi(x) = xa$ $(\forall)\ x \in M$, $\varphi(x) = a^{-1}xa \quad (\forall)\ x \in M$ Q.E.D.

2.3.THEOREM: With the notation in Proposition 2.2 ', for every ultraweakly continuous automorphism $\varphi \in$ Aut M such that $\|id-\varphi\| < 1/21$, there exists an invertible operator $x \in M$ such that $\|I-x\| < 15\,\|id-\varphi\|$ and $\varphi(z) = xzx^{-1} \quad (\forall)\ z \in M$.

Proof: Define $y = \int_{U(A)} \varphi(u)u^* \, d\mu(u)$

Then $\| I-y\| \leq \| id-\varphi\|$ hence y is invertible and the automorphism $\psi \in \mathrm{Aut}\, M$ $\psi(x) = y^{-1}\varphi(x)y$ satisfies $\psi(x) = x$ $(\forall)\ x \in A$. Moreover, if $\|id-\varphi\| = t$ then $\|I-y\| \leq t$, $\|y^{-1}\| \leq (1-t)^{-1}$, $\|y\| \leq 1+t$, $\|I-y^{-1}\| = \|y^{-1}(I-y)\| \leq t(1-t)^{-1}$ and for every $a \in M$

$\|\psi(a)-a\| \leq \|y^{-1}(\varphi(a)-a)y\| + \|y^{-1}ay-ay\| + \|ay-a\| \leq$

$\leq (3t+t^2)(1-t)^{-1}\|a\|$ and since $t < 1/21$ we obtain

$\|id-\psi\| < 16/5\ \|id-\varphi\|$. Since ψ leaves A elementwise fixed, there is an invertible operator $u \in A$, $\|u\| = 1$ such that $\psi(x) = uxu^{-1}$ $(\forall)\ x \in M$.

Now, for $a \in M$, $\|ua-au\| \leq \|u\| \cdot \|a-uau^{-1}\|$ hence

$\| ad(u)|_M \| \leq \|id-\psi\| < 16/5\ \|id-\varphi\|$

Choose, by Prop. 2.1 $\lambda \in C$ such that

$\|u-\lambda I\| \leq 5/2\ \|ad(u)|_M\| < 8\|id-\varphi\|$. It follows that

$|\lambda| > 1-8t > 13/21$ and $\|\lambda^{-1}u-I\| = |\lambda^{-1}| \cdot \|u-\lambda I\| < 13t < 1$

If we define $x = \lambda^{-1}yu$ then $x \in M$ and

$\|I-x\| \leq \|y(I-\lambda^{-1}u)\| + \|y-I\| \leq (1+t)13t + t = (13t+14)t < 15t$

and $\varphi(z) = xzx^{-1}$ $(\forall)\ z \in M$ Q.E.D.

REMARK: Following the terminology of ([1]), M has the (1/21, 15)- Automorphism Implementation Property.

2.4. COROLLARY: With the notation in Proposition 2.2, for every ultraweakly continuous derivation $\delta : M \to M$ there is an operator $a \in M$ such that $\delta(x) = ax-xa$ $(\forall)\ x \in M$.

In cohomological terms, $H^1_w(M,M) = 0$

Proof: It follows from Theorem 2.3 and from Theorem 3.2 in ([1])

REFERENCES

1. E.CHRISTENSEN Derivations of nest algebras
Math.Ann. 229(1977) , 155-161

2. A.CONNES, J.FELDMAN and B.WEISS An amenable equivalence relation is generated by a single transformation
Ergodic Theory Dynamical Systems 1(1981), 431-450

3. J.DIXMIER Les algèbres d'opérateurs dans l'espace hilbertien
Gauthier-Villars ,Paris 1957

4. J.FELDMAN and C.C.MOORE Ergodic equivalence relations, cohomology and von Neumann algebras I,II
Trans.Amer.Math.Soc. 234(1977) , 289-359

5. R.V.KADISON Derivations of operator algebras
Ann.Math. 83(1986) , 280-293

6. P.MUHLY, K.S.SAITO and B.SOLEL Coordinates for triangular operator algebras Ann.Math.127(1988),245-278

7. P.MUHLY and B.SOLEL Dilations for representations of triangular operator algebras Bull.London Math.Soc. 21(1989)

8. F.POP Derivations of certain nest-subalgebras of von Neumann algebras in Operator Theory: Advances and Applications vol.43(1990) Birkhäuser Verlag Basel, 279-288

9. S.POPA On a problem of R.V.Kadison on maximal abelian *-subalgebras in factors Invent.Math.65(1981), 269-281

10. S.POPA Notes on Cartan subalgebras in type II_1 factors
Math.Scand. 57(1985) , 171-188

11. S.SAKAI Derivations of W*-algebras
Ann.Math. 83(1966) , 273-279

DISTANCE ESTIMATES FOR TENSOR PRODUCTS OF NEST ALGEBRAS

FLORIN POP [1)]

Institute of Mathematics
of the Romanian Academy

1) Present address: Department of Mathematics INCREST
Bd.Pacii 220 , 79622 Bucharest , ROMANIA

DISTANCE ESTIMATES FOR TENSOR PRODUCTS OF NEST ALGEBRAS

Let H be a Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$ and let p_n denote the orthogonal projection on the Hilbert subspace generated by $\{e_1,\ldots,e_n\}$. Then $A=\{x\in B(H);\ (I-p_n)xp_n=0\ .\ (\forall)n\geq 1\}$ is the nest algebra associated to the family $(p_n)_{n\geq 1}$.

We prove that the algebra $A\otimes A\subset B(H\otimes H)$ is hyperreflexive and its distance constant satisfies $K(A\otimes A)\leq 3$.

INTRODUCTION

The purpose of this paper is to provide a solution to a problem of W.B.Arveson concerning hyperreflexivity of tensor products of nest algebras ([3]). Let us first recall some basic facts.

A nest is a totally ordered strongly closed family L of selfadjoint projections on a Hilbert space H, containing 0 and I. The nest algebra associated to L is

$$\mathrm{Alg}\ L=\{x\in B(H)\ ;\ (I-p)xp=0\ \ (\forall)\ p\in L\}$$

In [2] W.B.Arveson proved the remarkable result that for every operator $a\in B(H)$ the distance to $\mathrm{Alg}\ L$ is given by the formula

$$\mathrm{dist}\ (a,\mathrm{Alg}\ L)=\sup_{p\in L}\|(I-p)ap\|$$

(See also [15] and [21] for different proofs)

The operator algebra $A \subset B(H)$ is said to be hyperreflexive if there is a positive number $K \geq 1$ such that

$$\mathrm{dist}\,(a, A) \leq K \sup \{ \|(I-p)ap\|;\ p \in \mathrm{Lat}\, A \}$$

where $\mathrm{Lat}\, A = \{ p=p^2=p^* \in B(H);\ (I-p)ap=0 \quad (\forall)\ a \in A \}$.

The smallest constant K satisfying the above condition, denoted by K(A) ,is called the distance constant of A. Hyperreflexivity is a powerful tool of investigation for perturbation problems, automorphisms, similarities and derivations. (See [1],[4],[5],[7], [10],[13],[14],[16],[17],[19],[20],[22],[23]) for more details about these topics .)

Algebras of type Alg L for some commutative strongly closed lattice of projections L containing 0 and I are called CSL-algebras. It is worth mentionning that Lat Alg L = L if L is commutative ([3],[6]). Every nest algebra is a CSL-algebra but the latter class is considerably larger. It also contains nonhyperreflexive algebras ([9]).

A natural example of CSL-algebra is the tensor product of two nest algebras. Are these algebras hyperreflexive ? This question is originally due to Arveson ([3]) and it is also presented in K.Davidson´s book ([8] ,chapter 25, problem 4).

In what follows we give a positive answer in the simplest case of the problem. Namely, if H is a Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$ and if p_n denotes the projection on the subspace generated by $\{ e_1, \ldots, e_n \}$ then, for $L=(p_n)_{n\geq 1} \cup \{0, I\}$ consider the nest algebra $A = \mathrm{Alg}\, L$.

Define $A \otimes A \subset B(H \otimes H)$ to be the ultraweak closure of the algebra generated by the operators $a \otimes b$, $a \in A$, $b \in A$. If $L \otimes L$ denotes the (commutative) strongly closed lattice generated by the projections $p_m \otimes p_n$ ($m,n \geq 0$, $p_0 = 0$), then $A \otimes A = \mathrm{Alg}\, L \otimes \mathrm{Alg}\, L = \mathrm{Alg}(L \otimes L)$ as one can easily verify.

Our main result is that $K(A \otimes A) \leq 3$ (Th.2.1) The proof relies on the idea of relative hyperreflexivity. The algebra $A \otimes A$ is found to be hyperreflexive with respect to the algebra of upper triangular operators on $H \otimes H$, and since this one has the distance constant equal to 1, $A \otimes A$ will be itself hyperreflexive. A standard argument reduces the problem to the matrix algebra case (Lemma 1.4), where things are more tractable.

In particular, Proposition 1.2 implies that every nest-subalgebra of a hyperreflexive von Neumann algebra is hyperreflexive. (See [11] and [12] for more information on these algebras). This extends a result of F.Gilfeather and D.R.Larson . ([12])

It is therefore our feeling that relative hyperreflexivity which we introduce below might become a successful instrument for further investigation.

1. RELATIVE HYPERREFLEXIVITY

1.1 DEFINITION: Let $A \subset B \subset B(H)$ be two operator algebras. Then A is said to be hyperreflexive with respect to B if there is $K \geq 1$ such that

$$\mathrm{dist}(b, A) \leq K \sup \left\{ \|(I-p)bp\| \;,\; p \in \mathrm{Lat}\, A \right\}$$

for every operator $b \in B$.

The smallest K satisfying the above inequality, denoted by $K_B(A)$, is called the relative distance constant of A with respect to B.

1.2 PROPOSITION: Let $A \subset B \subset B(H)$ be two operator algebras. If B is hyperreflexive and if A is hyperreflexive with respect to B, then A is hyperreflexive and

$$K(A) \leq K(B) + K_B(A) + K(B)K_B(A)$$

Proof: For given x in B(H) and $\varepsilon > 0$ choose b_0 in B such that $\|x-b_0\| \leq \operatorname{dist}(x,B) + \varepsilon$. Then for every $a \in A$

$\|x - a\| \leq \|x - b_0\| + \|b_0 - a\|$ hence

$\operatorname{dist}(x, A) \leq \|x - b_0\| + \operatorname{dist}(b_0,A)$

$\operatorname{dist}(x, A) \leq \operatorname{dist}(x,B) + \operatorname{dist}(b_0,A) + \varepsilon \leq$

$\leq K(B) \sup\{\|(I-p)xp\| \; p \in \operatorname{Lat}B\} + K_B(A) \sup\{\|(I-p)b_0p\|; p \in \operatorname{Lat} A\} + \varepsilon$

Since $\operatorname{Lat} B \subset \operatorname{Lat} A$ it follows that

$$\begin{aligned}
\operatorname{dist}(x,A) &\leq K(B) \sup\{\|(I-p)xp\| ; p \in \operatorname{Lat}A\} + \\
&+ K_B(A) \sup\{\|(I-p)(b_0-x)p\| ; p \in \operatorname{Lat} A\} + \\
&+ K_B(A) \sup\{\|(I-p)xp\|; p \in \operatorname{Lat} A\} + \varepsilon \leq \\
&\leq (K(B) + K_B(A)) \sup\{\|(I-p)xp\| ; p \in \operatorname{Lat} A\} + K_B(A) \|b_0-x\| + \varepsilon \leq \\
&\leq (K(B) + K_B(A)) \sup\{\|(I-p)xp\| ; p \in \operatorname{Lat} A\} + K_B(A)\operatorname{dist}(x,B) + \\
&+ \varepsilon(1 + K_B(A)) \leq (K(B) + K_B(A)) \sup\{\|(I-p)xp\| ; p \in \operatorname{Lat} A\} + \\
&+ K_B(A)K(B) \sup\{\|(I-p)xp\| ; p \in \operatorname{Lat} B\} + \varepsilon(1 + K_B(A)) \leq \\
&\leq (K(B) + K_B(A) + K(B)K_B(A)) \sup\{\|(I-p)xp\| ; p \in \operatorname{Lat} A\} \\
&+ \varepsilon(1 + K_B(A))
\end{aligned}$$

. Since $\varepsilon > 0$ was arbitrary, it follows that

$$K(A) \leq K(B) + K_B(A) + K(B)K_B(A)$$

which concludes the proof.

1.3 COROLLARY: Let $M \subset B(H)$ be a hyperreflexive von Neumann algebra and let $A \subset M$ be a nest-subalgebra(i.e. $A = M \cap \text{Alg } L$ for some nest $L \subset M$). Then A is hyperreflexive and

$$K(A) \leq 2K(M) + 1$$

Proof: It follows from [19] that $K_M(A) = 1$ hence

$$K(A) \leq 2K(M) + 1$$

REMARK: This result was obtained by F.Gilfeather and D.R.Larson for M injective ([12]).

We conclude this paragraph with a technical lemma.

1.4 LEMMA: Let $A \subset B(H)$ be a CSL-algebra and let $(p_n)_{n \geq 1}$ be an increasing sequence of projections in Lat A converging strongly to the identity. If $A_n = p_n A p_n \subset B(p_n H)$ and if

$$K(A_n) \leq K \quad \text{then} \quad K(A) \leq K :$$

REMARK: For every CSL-algebra A , Lat $A \subset A$. Indeed, for any $p, q \in \text{Lat } A$ $(I-q)pq = 0$. It follows that $p \in \text{Alg Lat } A = A$.

Proof of 1.4 : For given $x \in B(H)$ and $\varepsilon > 0$ choose $a_n \in A_n$ such that $\| x_n - a_n \| \leq \text{dist}(x_n, A_n) + \varepsilon$, where $x_n = p_n x p_n$.

Since $(a_n)_{n \geq 1}$ is bounded , let $(n_k)_{k \geq 1}$ be an increasing sequence such that the ultraweak limit $\lim_k a_{n_k} = a$ exists.

Clearly $a \in A$ since $p_n \in A$ and $\lim_n x_n = x$ ultraweakly.

Since the norm is ultraweakly lower semicontinuous, there is k_0 such that $\| x - a \| \leq \| x_{n_k} - a_{n_k} \| + \varepsilon \leq \text{dist}(x_{n_k}, A_{n_k}) + 2\varepsilon$

for $k \geq k_0$. Consequently $\text{dist}(x, A) \leq 2\varepsilon +$

$$+ K(A_{n_k}) \sup \{ \|(I-p) x_{n_k} p\| ; \ p \in \text{Lat } A_{n_k} \} \leq$$

$$\leq 2\varepsilon + K \sup \left\{ \|(I-p)p_{n_k} x p_{n_k} p\| \; ; \; p \in \text{Lat } A_{n_k} \right\} \leq$$

$$\leq 2\varepsilon + K \sup \left\{ \|(I-p)xp\| \; ; \; p \in \text{Lat } A \right\}$$

Since $\varepsilon > 0$ was arbitrary, the lemma is proved.

2. THE MAIN RESULT

2.1 THEOREM: Let H be a Hilbert space with orthonormal basis $(e_n)_{n \geq 1}$ and let p_n denote the projection on the Hilbert subspace generated by $\{ e_1, \ldots, e_n \}$. If $L = (p_n)_{n \geq 1} \cup \{0, I\}$ and $A = \text{Alg } L$ then $A \otimes A = \text{Alg } (L \otimes L)$ is hyperreflexive and $K(A \otimes A) \leq 3$.

Proof: Let A_n be the algebra of all upper triangular $n \times n$ matrices, that is, A_n is the nest algebra in $B(C^n)$ associated to the nest $L_n = \{ 0 = p_0 \leq p_1 \leq \cdots \leq p_n = I \}$ where p_i is the projection on the subspace generated by the first i vectors of the canonical basis in C^n.

By taking into account Lemma 1.4 , it is enough to prove that $K(A_n \otimes A_n) \leq 3$ $(\forall)$ $n \geq 2$. Since $K(A_n) = 1$ $(\forall)$ $n \geq 2$ this will be a consequence of Proposition 1.2 provided

$$\text{that} \quad K_{A_{n^2}}(A_n \otimes A_n) = 1 \quad \text{for every } n \geq 2.$$

We prove a stronger result, namely that $K_{A_{n_1 n_2}}(A_{n_1} \otimes A_{n_2}) = 1$ $(\forall)$ $n_1, n_2 \geq 2$. We proceed by induction. The case n=2 is trivially satisfied, as one can easily check.

Suppose that $K_{A_{n_1 n_2}}(A_{n_1} \otimes A_{n_2}) = 1$ for every $n_1, n_2 \leq n$.

We prove that $K_{A_{n_1 n_2}}(A_{n_1} \otimes A_{n_2}) = 1$ for every $n_1, n_2 \leq n+1$.

We illustrate the ideas only for $n_1=n_2=n+1$, the other cases having identical proofs.

So, we are concerned with the infimum of the norm of the operator matrix $[B_{ij}]_{1\leq i,j\leq n+1}$ where $B_{ij}=0$ for $i>j$, $B_{ii}=(x^i_{lk})_{1\leq l\leq k\leq n+1}$ and

$$B_{ij} = \begin{bmatrix} x^{ij}_{11} & x^{ij}_{12} & \cdots & x^{ij}_{1,n} & x^{ij}_{1,n+1} \\ a^{ij}_{21} & x^{ij}_{22} & \cdots & x^{ij}_{2,n} & x^{ij}_{2,n+1} \\ & & \vdots & & \\ a^{ij}_{n+1,1} & \cdots & & a^{ij}_{n+1,n} & x^{ij}_{n+1,n+1} \end{bmatrix} \quad \text{for } i<j .$$

The problem clearly reduces to the operator matrix $[A_{ij}]$ where $A_{ij}=0$ for $i>j$, $A_{ii}=(x^i_{lk})_{2\leq l\leq k\leq n}$ and

$$A_{ij} = \begin{bmatrix} a^{ij}_{21} & x^{ij}_{22} & \cdots & x^{ij}_{2,n} \\ & & \vdots & \\ a^{ij}_{n+1,1} & & \cdots & a^{ij}_{n+1,n} \end{bmatrix} \quad \text{for } i>j$$

Denote by T the operator in $B(C^{(n+1)^2})$ with entries a^{ij}_{lk} and 0 instead of x^{ij}_{lk}. All projections in $L_{n+1}\otimes L_{n+1}$ have the form $p=p_{k_1}\oplus p_{k_2}\oplus\cdots\oplus p_{k_{n+1}}$, $p_{k_i}\in L_{n+1}$, briefly denoted by $p=(p_{k_1},p_{k_2},\ldots,p_{k_{n+1}})$ where $k_1\geq k_2\geq\cdots\geq k_{n+1}\geq 0$ and $k_i=\dim p_{k_i}$ $(0\leq k_i\leq n+1)$.

In order to prove that the infimum of the norm of $[A_{ij}]$, that is $\mathrm{dist}(T, A_{n+1}\otimes A_{n+1})$, is less or equal to $\max\{\|(I-p)Tp\| ; p\in L_{n+1}\otimes L_{n+1}\}$, we will recursively elliminate one row after the other from every row block-matrix $[A_{ij}]_{1\leq j\leq n+1}$.

We first elliminate the first row from the first row block-matrix. Parrott's theorem ([18],[21]) implies that dist(T, $A_{n+1} \otimes A_{n+1}$) is less or equal to the maximum between $\|(I-q)Tq\|$ where $q=(p_1,\dots,p_1) \in L_{n+1} \otimes L_{n+1}$ and the norm of the operator obtained after elliminating the first row of the row block-matrix $[A_{11}, A_{12}, \dots, A_{1,n+1}]$, and so on. Suppose that we have elliminated the first n-k-1 rows from every row block-matrix and the (n-k)-th row from the first l row block-matrices $[A_{ri}]_{r \leq i \leq n+1}$ $(1 \leq r \leq l)$.

For the ellimination of the (n-k)-th row from the row block-matrix $[A_{l+1,i}]_{l+1 \leq i \leq n+1}$, Parrott's theorem asserts that the distance from T to $A_{n+1} \otimes A_{n+1}$ is this time less or equal to the maximum between the norm of the operator obtained after elliminating the above (n-k)-th row and the infimum of the norm of the operator

$$
\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} =
\left[\begin{array}{cccc|ccc}
\tilde{A}_{11} & \tilde{A}_{12} & \dots & \tilde{A}_{1l} & \tilde{A}_{1,l+1} & \dots & \tilde{A}_{1,n+1} \\
0 & \tilde{A}_{22} & \dots & \tilde{A}_{2,l} & & \vdots & \\
 & & \vdots & & & & \\
0 & 0 & \dots 0 & \tilde{A}_{l,l} & \tilde{A}_{l,l+1} & \dots & \tilde{A}_{l,n+1} \\
\hline
 & & & & \tilde{A}_{l+1,l+1} & \dots & \tilde{A}_{l+1,n+1} \\
 & & 0 & & & \vdots & \\
 & & & & 0 & \dots\ 0 & \tilde{A}_{n+1,n+1}
\end{array}\right]
$$

where $\tilde{A}_{ij}$ are obtained from A_{ij} in the following way :

For $1 \leq i \leq j \leq l$ $\tilde{A}_{ij}$ is A_{ij} without its first n-k rows

For $1 \leq i \leq l$ and $j \geq l+1$ $\tilde{A}_{ij}$ is A_{ij} without its first n-k rows and without its last k columns.

For $l+1 \leq i \leq j \leq n+1$ $\tilde{A}_{ij}$ is A_{ij} without its first n-k-1 rows and without its last k columns.

Since only the block A is variable, the infimum of the norm of A is seen to be related to the distance formula for the algebra $A_{k-1} \otimes A_l$. Since the distance from an operator S to the algebra $A_{k-1} \otimes A_l$ is, by the induction assumption, less or equal to the maximum of the numbers $\|(I-q)Sq\|$, $q \in L_{k-1} \otimes L_l$, $q = (q_{n_1}, q_{n_2}, \ldots, q_{n_l})$, $q_{n_i} \in L_{k-1}$ and $k-1 \geq n_1 \geq n_2 \geq \cdots \geq n_l$, then the infimum of $\|A\|$ is less or equal to the maximum of the numbers $\|(I-q)Tq\|$, where $q = (p_{n-k+1+n_1}, p_{n-k+1+n_2}, \ldots, p_{n-k+1+n_l})$.

It follows that the infimum of the norm of the block-matrix $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is less or equal to the maximum of the numbers $\|(I-q)Tq\|$, where $q = (p_{n-k+1+n_1}, \ldots, p_{n-k+1+n_l}, \underbrace{p_{n-k}, \ldots, p_{n-k}}_{n-l+1})$

and clearly q belongs to $L_{n+1} \otimes L_{n+1}$, which concludes the proof. We have therefore shown by induction that for every $n \geq 2$ $\text{dist}(T, A_n \otimes A_n) \leq \sup\{\|(I-p)Tp\| \; ; \; p \in L_n \otimes L_n\}$

and this implies that $K_{A_{n^2}}(A_n \otimes A_n) = 1$ Q.E.D.

R E F E R E N C E S

1. N.T.ANDERSEN Compact perturbations of reflexive algebras
   J.Funct.Anal. 38(1980) 366-400
2. W.B.ARVESON Interpolation problems in nest algebras
   J.Funct.Anal. 20(1975) ,208-233
3. W.B.ARVESON Ten lectures on operator algebras
   CBMS Regional Conference Series No.55, AMS Providence , 1984
4. M.D.CHOI AND K.R.DAVIDSON Perturbations of matrix algebras
   Michigan Math.J. 33(1986) , 273-287
5. E.CHRISTENSEN Extensions of derivations II
   Math.Scand. 50(1982) , 111-122
6. K.R.DAVIDSON Commutative subspace lattices
   Indiana U.Math.J. 27(1978) , 479-490
7. K.R.DAVIDSON Perturbations of reflexive operator algebras
   J.Operator Theory 15(1986) , 289-305
8. K.R.DAVIDSON Nest Algebras , Pitman Research Notes
   in Mathematics Series No.191 , 1988
9. K.R.DAVIDSON AND S.C.POWER Failure of the distance formula
   J.London Math.Soc.(2) 32(1985) , 157-165
10. T.FALL, W.B.ARVESON AND P.MUHLY Perturbations of
    nest algebras , J.Operator Theory 1(1979) , 137-150
11. F.GILFEATHER AND D.R.LARSON Nest-subalgebras of
    von Neumann algebras , Adv.in Math. 46(1982), 171-199
12. F.GILFEATHER AND D.R.LARSON Nest-subalgebras of
    von Neumann algebras: commutants modulo compacts and
    distance estimates , J.Operator Theory 7(1982) , 279-302
13. J.KRAUS AND D.R.LARSON Some applications of a technique
    for constructing reflexive operator algebras
    J.Operator Theory 13(1985) , 227-236

14. J.KRAUS AND D.R.LARSON Reflexivity and distance formulae Proc.London Math. Soc. 53(1986) , 340-356

15. E.C.LANCE Cohomology and perturbations of nest algebras Proc.London Math.Soc. 43(1981) , 334-356

16. D.R.LARSON A solution to a problem of Ringrose Bull.Amer.Math.Soc. 7(1982) , 243-246

17. D.R.LARSON Nest algebras and similarity transformations Ann.Math. 121(1985) , 409-427

18. S.PARROTT On a quotient norm and the Sz.Nagy-Foias lifting theorem , J.Funct.Anal. 30(1978) ,311-328

19. F.POP Perturbations of nest-subalgebras of von Neumann algebras , J.Operator Theory 21(1989) , 139-144

20. F.POP AND D.T.VUZA Continuity properties of the distance constant function , J.Operator Theory 22(1989) ,73-84

21. S.C.POWER The distance to upper triangular operators Math.Proc.Camb.Phil.Soc. 88(1980) , 327-329

22. S.ROSENOER Distance estimates for von Neumann algebras Proc.Amer.Math.Soc. 86(1982) , 248-252

23. B.SOLEL Distance formula for analytic operator algebras Bull.London Math.Soc. 20(1988) , 345-349