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MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No. 41/1990

BUCURESTI

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June 1990

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1. INTRODUCTION

In this paper we study the exact boundary controllability for the equation modelling the small vibrations of an elastic beam in the presence of an axial force applied at one end (see [3],p.453 for a derivation of the model). More precisely we shall consider the following initial and boundary value problem:

$$\ddot{w}(x,t) + w^{(iv)}(x,t) + p(t)w''(x,t) = 0, \quad -1<x<1, \quad 0\le t\le T \tag{1.1}$$
$$w(-1,t) = w(1,t) = 0, \quad t>0 \tag{1.2}$$
$$w'(-1,t) = v_0(t); \quad w'(1,t) = v_1(t), \quad t>0 \tag{1.3}$$
$$w(x,0) = w_0(x); \quad \dot{w}(x,0) = w_1(x), \quad -1<x<1 \tag{1.4}$$

where $w(x,t)$ represents the transverse deflection of the point x of the beam at the moment t, $\dot{w}$, $\ddot{w}$, are derivatives with respect to time and $w^{(n)}$ are spatial derivatives of w. The function $p(t)$ represents the axial end thrust.

The problem of exact controllability for (1.1)-(1.4) states as follows: **given $T>0$, large enough, is it possible that for any initial data $\{w_0,w_1\}$ to find corresponding controls v_0, v_1 driving the system (1.1) to rest at time T, i.e. such that :**

$$w(x,T) = 0; \quad \dot{w}(x,T) = 0, \quad -1<x<1 \tag{1.5}$$

Such problems were recently considered in a large number of publications (see e.g. J.L.Lions [2], J.Lagnese, J.L.Lions [1], and the references therein). However, concerning the case of the time dependent coeficients very little is known (see [2] p.105). In [4] Zuazua considered the problem of exact controllability for the equation:

$$\ddot{u}(x,t) - \Delta u(x,t) + F(x,t)u(x,t) = 0$$

with $F(x,t) \in L^{\infty}(Q)$, where $Q = (-1,1) \times (0,T)$.

His approach cannot be followed for the controllability problem formulated above due to the absence of a unique continuation result for (1.1)(see [4]).

2. NOTATIONS AND PRELIMINAIRES

Let us denote with D the open interval (0,1). In this section we state three existence and uniqueness results for the problem (1.1)-(1.4). The proofs of this theorems can be easily obtained following the lines from [2], ch.4. We shall use the customary notations for Sobolev spaces.

THEOREM 2.1 If $w_0 \in H^3(D) \cap H_0^2(D)$, $w_1 \in H_0^1(D)$, $v_0=v_1=0$, $p \in C^1[0,T]$, the problem (1.1)-(1.4) has a unique solution:

$w \in C(0,T;\ H^3(D) \cap H_0^2(D)) \cap C^1(0,T;H_0^1(D))$.

We also have that there exists a constant $C>0$ such that:

$$||w||_{L^\infty(0,T;H^3(D)\cap H_0^2(D))} + ||\dot{w}||_{L^\infty(0,T;H^1(D))} \leq$$

$$\leq C\{||w_0||_{H^3(D)\cap H_0^2(D)} + ||w_1||_{H^1(D)}\} \tag{2.1}$$

THEOREM 2.2 Let us suppose that $w_0 \in H_0^2(D)$, $w_1 \in L^2(D)$, $p \in C[0,T]$ and $v_0 = v_1 = 0$. Then the problem (1.1)-(1.4) has a unique solution having the regularity

$$w \in C(0,T;H_0^2(D)) \cap C^1(0,T;L^2(D)).$$

this solution depends continously on the data, i.e. there is a constant $C>0$ such that

$$||w||_{L^\infty(0,T;H^2(D))} + ||\dot{w}||_{L^\infty(0,T;L^2(D))} \leq$$

$$\leq C(||w_0||_{H^2(D)} + ||w_1||_{L^2(D)}) \tag{2.2}$$

and it satisfies the regularity conditions

$$w''(-1,t),\ w''(1,t) \in L^2(0,T).$$

THEOREM 2.3 Let us suppose that $w_0 \in L^2(D)$, $w_1 \in H^{-2}(D)$, $p \in C[0,T]$ and $v_0, v_1 \in L^2(0,T)$. Then the problem (1.1)-(1.4) has a unique solution : $w \in C(0,T;L^2(D)) \cap C^1(0,T;H^{-2}(D))$.

Let us denote by λ_0 the greatest value of for which :

$$\int_{-1}^{1} [u''(x)]^2 dx \geq \lambda_0 \int_{-1}^{1} [u'(x)]^2 dx, \qquad (2.3)$$

for any $u \in H_0^2(D)$; (λ_0 is a Friedrics' constant).

We shall suppose that :

$$p(t) \leq p_0 < \lambda_0, \quad t \in [0,T] \qquad (2.4)$$

i.e. that p is an axial compression force less than the buckling load or that p is a traction force.

Concerning the solutions of (1.1)-(1.4) existing in the conditions of the Theorem 2.2 we have the following estimate on the energy:

$$E(t) = \int_{-1}^{1} [\dot{w}(x,t)]^2 dx + \int_{-1}^{1} [w''(x,t)\}^2 dx - p_0 \int_{-1}^{1} [w'(x,t)]^2 dx \qquad (2.5)$$

LEMMA 2.1 Suppose that $w_0 \in H_0^2(D)$, $w_1 \in L^2(D)$ and (2.4) holds for any $t \in [0,T]$. Then there exist constants $K>0$, $C>0$ such that:

$$E(t) \geq K E_0 \qquad (2.6)$$

$$E(t) \leq C E_0 \qquad (2.7)$$

where $E_0 = \int_{-1}^{1} w_1^2(x)dx + \int_{-1}^{1} [w_0''(x)]^2 dx - p_0 \int_{-1}^{1} [w'(x)]^2 dx.$

Proof We shall prove this result for $w_0 \in H^3(D)$ $H_0^2(D)$, $w_1 \in H_0^2(D)$ and $p \in C^1$ (0,T), i.e. in the conditions of the Theorem 2.1. By an approximation argument and using (2.2) we obtain the conclusion of Lemma 2.4.

In the conditions of the Theorem 2.1 we derivate (2.4) with respect to time to obtain:

$$|\dot{E}(t)| = 2|p(t)-p_0|\cdot\left|\int_{-1}^{1}\dot{w}(x,t)\, w''(x,t)dx\right| \leq CE(t).$$

As from (2.3) and (2.5) we get that $E(t)>0$, we obtain that:

$$\dot{E}(t) \geq -CE(t)$$

and by integrating with respect to time we obtain (2.6). The etimate(2.7) can be proved in a similar way.

3. THE EXACT CONTROLLABILITY RESULT

The main result of the paper states as follows:

THEOREM 3.1 Let T be given large enough, $T>T_0$ say. Then for any $\{w_0,w_1\} \in L^2(D)\times H^{-2}(D)$ one can find $v_0,v_1 \in L^2(0,T)$ such that v_0,v_1 drive the system (1.1)-(1.4) starting from w_0, w_1 at time 0 to rest at time T, i.e. the solution w satisfies the relation (1.5).

In order to prove Theorem 3.1 we shall use the Hilbert Uniqueness Method (H.U.M) developed in [2], consisting the following steps:

a) Consider $\{\theta_0, \theta_1\} \in H_0^2(D)\times L^2(D)$ and let θ be the solution of the initial and boundary value problem:

$$\ddot{\theta}(x,t) + \theta^{(iv)}(x,t) + p(t)\theta''(x,t)=0,\quad 0<x<1,\quad 0\leq t\leq T \tag{3.1}$$

$$\theta(-1,t) = \theta(1,t) = \theta'(-1,t) = \theta'(1,t) = 0,\quad 0\leq t\leq T \tag{3.2}$$

$$\theta(x,0) = \theta_0(x)\ ;\quad \dot{\theta}(x,0) = \theta_1(x). \tag{3.3}$$

According to Theorem 2.2 we have that:

$$\theta \in C(0,T;H_0^2(D)) \cap C^1(0,T;L^2(D)), \tag{3.4}$$

$$\theta''(-1,.),\ \theta''(1,.) \in L^2(0,T). \tag{3.5}$$

b) We construct now the linear operator:

$$\Lambda\{\theta_0, \theta_1\} = \{\dot{y}(.,0),-y(.,0)\} \tag{3.6}$$

where $y(x,t)$ is the unique solution of the problem:

$$\ddot{y}(x,t) + y^{(iv)}(x,t) + p(t)y''(x,t) = 0, \quad -1<x<1, \quad 0\le t\le T, \tag{3.7}$$

$$y(-1,t)=y(1,t)=0; y'(-1,t)=\theta''(-1,t); y'(1,t)=\theta''(1,t), 0\le t\le T \tag{3.8}$$

$$y(x,T) = \dot{y}(x,T) = 0, \quad 0<x<1. \tag{3.9}$$

According to (3.5) and to Theorem 2.3 we get that :

$$y \in C(0,T;L^2(D)) \cap C^1(0,T;H^{-2}(D))$$

so that Λ takes values in the dual space of $H_0^2(D)\times L^2(D)$.

c) We prove the surjectivity of Λ and choose:

$$\{\theta_0, \theta_1\} \in \Lambda^{-1}\{w_1, -w_0\}$$

so that the controls v_i, $i=0,1$ can be taken in the following way:

$$v_0(t) = \theta''(-1,t) \; ; \; v_1(t) = \theta''(1,t).$$

In order to accomplish the last step of H.U.M. we notice that by multiplying (3.7) with θ and integrating on Q we obtain:

$$< \Lambda\{\theta_0, \theta_1\},\{\theta_0, \theta_1\}> = \int_{-1}^{1}[\theta_0 y(x,0)-\theta_1(x)y(x,0)]dx =$$

$$= \int_0^T \{[\theta''(-1,t)]^2 + [\theta''(1,t)]^2\}dt, \tag{3.10}$$

where by $<.,.>$ we denoted the duality pairing between $H_0^2(D)\times L^2(D)$ and $H^{-2}(D)\times L^2(D)$.

From (3.10) it follows that the proof of Theorem 3.1 is complete once we have the following result:

<u>LEMMA 3.1</u> There is a constant $T_0>0$ such that for $T>T_0$ we have the estimate:

$$\int_0^T \{[\theta''(-1,t)]^2+[\theta''(1,t)]^2\}dt \ge C(T-T_0)\|\{\theta_0,\theta_1\}\|^2_{H^2(D)\times L^2(D)},$$

where C is a positive constant.

<u>Proof:</u> We use a multiplier technique. More precisely we multiply (3.1) with $x\theta'(x,t)$ and integrate on $(0,T)\times D$ to obtain:

$$(1/2)\int_0^T \{[\theta''(-1,t)]^2+[\theta''(1,t)]^2\}dt = \int_{-1}^{1} \dot\theta(x,t) \times \theta'(x,t)dx\Big|_0^T -$$

$$- (1/2)\int_0^T \int_{-1}^{1} \{[\dot\theta(x,t)]^2-[\theta''(x,t)]^2+p(t)[\theta'(x,t)]^2\}dxdt +$$

$$+\int_0^T \int_{-1}^{1} \{[\dot\theta(x,t)]^2+[\theta''(x,t)]^2\}dx\,dt \qquad (3.11)$$

As by multiplying (3.1) by θ and integrating on Q we obtain that :

$$\int_0^T \int_{-1}^{1} \{[\dot\theta(x,t)]^2-[\theta''(x,t)]^2+p(t)[\theta'(x,t)]^2\}dxdt =$$

$$= \int_{-1}^{1} \dot\theta(x,t)\ \theta(x,t)dx\Big|_0^T,$$

from (3.10) and Lemma 2.1 we obtain that :

$$(1/2)\int_0^T [\theta''(-1,t)]^2 + [\theta''(1,t)]^2 \geq$$

$$\geq C_1TE_0 + \int_{-1}^{1} x\dot\theta(x,t)\theta'(x,t)dx\Big|_0^T - (1/2)\int_{-1}^{1} \dot\theta(x,t)\theta(x,t)dx\Big|_0^T \qquad (3.12)$$

As by (2.7):

$$\left|\int_{-1}^{1} x\dot\theta(x,t)\theta'(x,t)dx\Big|_0^T - (1/2)\int_{-1}^{1} \dot\theta(x,t)\theta(x,t)\Big|_0^T\right| \leq C_2E_0,$$

the conclusion of the Lemma follows.

<u>REMARK 3.1</u> By the use of a multiplier technique one can also obtain the estimate:

$$<\Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}> \leq C\|\{\theta_0,\theta_1\}\|^2_{H^2(D)\times L^2(D)},$$

which, mixed with Lemma 3.1, gives that Λ is an isomorphism.

REMARK 3.2 Theorem 3.1 can be easily extended in order to obtain exact controllability for the case $D \subset R^n$, i.e. for the problem:

$$\ddot{w}(x,t) + \Delta^2 w(x,t) + p(t)\Delta w(x,t) = 0,$$
$$w(x,t) = 0; \quad \frac{\partial w}{\partial \nu}(x,t) = v(x,t), \ x \in \partial D, 0 \leq t \leq T$$
$$w(x,0) = w_0(x); \quad \dot{w}(x,0) = w_1(x), \ x \in .$$

However we choose the one dimensional case due to its physical relevance and in order to simplify the calculations.

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