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by

V. DRAGAN and A. HALANAY

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V. DRAGAN*) and A. HALANAY**)

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*) Institute of Mathematics, Bd.Pacii nr.220, 79622 Bucharest, Romania

**) Universitatea din Bucureşti, Facultatea de Matematică, Str.Academiei nr.14, R-70109, Bucureşti, Romania.

ASYMPTOTIC EXPANSIONS FOR SINGULAR VALUES ASSOCIATED TO SINGULARLY PERTURBED CONTROL SYSTEMS

by Vasile Drăgan and Aristide Halanay

It is proved that for a singularly perturbed control system the singular values are asymptotically close to the ones associated to the reduced model and to the ones associated to the "boundary layer system " (the fast system).

1. Introduction

Consider a singularly perturbed control system

$$\begin{aligned} x_1' &= A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u(t) \\ x_2' &= A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u(t) \\ y(t) &= C_1(t)x_1(t) + C_2(t)x_2(t) \end{aligned} \tag{1}$$

Assume that $A_{22}(t)$ is invertible for all t. Then it is well known that to the given system (1) we may associate the reduced model

$$\begin{aligned} x_1' &= \widetilde{A}(t)x_1 + \widetilde{B}(t)u \\ y(t) &= \widetilde{C}(t)x_1 + \widetilde{D}(t)u \end{aligned} \tag{2}$$

$$\widetilde{A}(t) = A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t)$$
$$\widetilde{B}(t) = B_1(t) - A_{12}(t)A_{22}^{-1}(t)B_2(t)$$
$$\widetilde{C}(t) = C_1(t) - C_2(t)A_{22}^{-1}(t)A_{21}(t)$$
$$\widetilde{D}(t) = -C_2(t)A_{22}^{-1}(t)B_2(t)$$

and the"boundary layer" or "fast" system

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Universitatea din Bucureşti -Facultatea de Matematică

Str.Academiei 14 R-70109 Romania ,Bucureşti

$$x_2' = A_{22}(t)x_2 + B_2(t)u$$
$$y_2(t) = C_2(t)x_2 \qquad (3)$$

It is also known that many problems in controlling (1) are solved by considering the lower order systems (2) and (3).

In the last, years several questions related to model reduction and robustness have been considered in connection with the so called " singular values " Sh. Shokoohi, et al.(1983) E. Verriest et al. (1983). It seems to be a natural question to ask what is the relation between singular values associated to (1) and singular values associated to (2) and (3).

The main result of this paper (Theorem 3) shows that the set of singular values for (1) splits into two parts, one of which is close to the set of singular values for (2) and the other close to the set of singular values for (3).
We prove our results in the case of variable coefficients — the usual, stationary case will result as a special one.
Taking into account the role singular values play in model reduction and robustness our result may be interpreted that, say, robustness of stabilization by compensation for (1) is limited by the worst of the robustness margins for (2) or (3). (See Glover -1986).
To obtain our result we had to consider asymptotic expansions for the bounded on $\mathbb{R}$ solutions for Liapunov equations associated to a system of the form (1) ; these results may be of interest in themselves.

2. Liapunov equations and singular values

Let $A: \mathbb{R} \longrightarrow \mathcal{M}_{n \times n}$, $B: \mathbb{R} \longrightarrow \mathcal{M}_{n \times m}$, $C: \mathbb{R} \longrightarrow \mathcal{M}_{p \times n}$

($\mathcal{M}_{j \times k}$ is as usually the set of matrices with j rows and k columns). We shall assume these functions to be continuous and bounded.

Denote by X_A the evolution operator associated to A

$$\frac{d}{dt} X_A(t,s)=A(t)X_A(t,s)$$

$$X_A(s,s)=I$$

The evolution defined by A is exponentially stable if there exists $\beta>0$, $\alpha>0$ such that

$$|X_A(t,s)| \leq \beta e^{-\alpha(t-s)} \qquad -\infty<s\leq t<\infty$$

and it is called antistable if there exist $\beta>0$, $\alpha>0$ such that

$$|X_A(t,s)| \leq \beta e^{-\alpha(s-t)} \qquad -\infty<t\leq s<\infty$$

Taking into account that $X_A^*(t,s)=X_{-A^*}(s,t)$ it is seen that A defines an antistable evolution if and only if $-A^*$ defines an exponentially stable evolution.

If A defines an exponentially stable evolution we define the matrix-valued functions P,Q by

$$P(t)= \int_{-\infty}^{t} X_A(t,s)B(s)B^*(s)\, X_A^*(t,s)ds$$

$$Q(t)= \int_{t}^{\infty} X_A^*(s,t)C^*(s)C(s)X_A(s,t)ds$$

In the same way if A deffines an antistable evolution

$$P(t)= \int_{t}^{\infty} X_A(t,s)B(s)B^*(s)\, X_A^*(t,s)\, ds$$

$$Q(t)= \int_{-\infty}^{t} X_A^*(s,t)C^*(s)C(s)X_A(s,t)ds.$$

It is well known that if (A,B,C) is "uniform" that is (A,B) uniformly completely controllable and (C,A) uniformly completely observable, then $P(t)\geq k\,I$, $Q(t)\geq k\,I$.

It is also known that if A defines an antistable evolution the control $u(t)=-B^*(t)P^{-1}(t)x(t)$ is stabilizing.

It is also known that the eigenvalues of the matrix $P(t)Q(t)$ are invariant with respect to Liapunov transformations and define the singular values of the system.
The matrices P, Q defined above are solutions to Liapunov equations

$$P'=A(t)P+PA^*(t)+B(t)B^*(t) \qquad (4)$$

$$Q'=-A^*(t)Q-QA(t)-C^*(t)C(t) \qquad (5)$$

in the stable case and to

$$P'=A(t)P+PA^*(t)-B(t)B^*(t) \qquad (6)$$

$$Q'=-A^*(t)Q-QA(t)+C^*(t)C(t) \qquad (7)$$

in the antistable case.
P, Q defined above are the unique solutions of the corresponding equations bounded on all of $\mathbb{R}$ and if A,B,C are constant they are solutions to the corresponding algebraic Liapunov equations.
Our result will be based on asymptotic expansions for solutions of Liapunov equations associated to (1).

3. Asymptotic expansions for solutions of Liapunov equations associated to singularly perturbed systems

Since we want to consider mainly the antistable case we need a result of Gradštein-Klimušev-Krasovskiĭ type for antistable singularly perturbed systems.
Lemma 1. Consider the system (1) and assume $A_{ij}: \mathbb{R} \to \mathcal{M}_{n_i \times n_j}$ are bounded and uniformly continuous on $\mathbb{R}$.
Assume that $\tilde{A}$ in (2) defines an antistable evolution and assume also that there exists $\hat{\alpha} > 0$ such that for all $t \in \mathbb{R}$ the real parts of all eigenvalues of $A_{22}(t)$ exceed $2\hat{\alpha}$.

Then there exists $\hat{\varepsilon}>0$ such that for all $0<\varepsilon\leq\hat{\varepsilon}$ the evolution associated to (1) is antistable.

Moreover if $X(\cdot,\cdot,\varepsilon)$ is the evolution matrix associated to (1) and X_{ij} a corresponding partition then

$$|X_{1,1}(t,s,\varepsilon)|\leq c\, e^{-\alpha(s-t)}$$

$$|X_{1,2}(t,s,\varepsilon)|\leq \varepsilon c\, e^{-\alpha(s-t)}$$

$$|X_{2,2}(t,s,\varepsilon)|\leq c\left[e^{-\alpha(s-t)/\varepsilon}+\varepsilon e^{-\alpha(s-t)}\right]$$

for $-\infty<t\leq s<\infty$ and α, c are constants not depending upon s,t,ε.

If $A=\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22}\end{pmatrix}$ we have

$$-A^{*}=\begin{pmatrix} -A_{11}^{*} & -\frac{1}{\varepsilon}A_{21}^{*} \\ -A_{12}^{*} & -\frac{1}{\varepsilon}A_{22}^{*}\end{pmatrix}$$ and the system defined by $-A^{*}$

is

$$w_1'=-A_{11}^{*}(t)w_1-\frac{1}{\varepsilon}A_{21}^{*}(t)w_2$$

$$w_2'=-A_{12}^{*}(t)w_1-\frac{1}{\varepsilon}A_{22}^{*}(t)w_2$$

If we set $z_1=w_1$, $z_2=\frac{1}{\varepsilon}w_2$ we obtain

$$\begin{aligned} z_1' &=-A_{11}^{*}(t)z_1-A_{21}^{*}(t)z_2 \\ \varepsilon z_2' &=-A_{12}^{*}(t)z_1-A_{22}^{*}(t)z_2 \end{aligned} \qquad (8)$$

System (8) will satisfy conditions for a theorem of Gradštein-Klimušev-Krasovskiĭ Type ; for eigenvalues of $-A_{22}^{*}(t)$ the real parts will be less than $-2\hat{\alpha}<0$ for all $t\in\mathbb{R}$ and the corresponding reduced model is $z_1'=-\tilde{A}^{*}(t)z_1$ and defines an exponentially stable evolution .

We deduce the exponentially stable evolution for (8) (See for example theorem 1.2 in V.Drăgan ,A.Halanay 1983) .

Moreover if

$$\Gamma(t,s,\varepsilon)=\begin{pmatrix}\Gamma_{11}(t,s,\varepsilon) & \Gamma_{12}(t,s,\varepsilon)\\ \Gamma_{21}(t,s,\varepsilon) & \Gamma_{22}(t,s,\varepsilon)\end{pmatrix}$$

is the evolution operator associated to (8) we have the estimates (see V.Drăgan, A.Halanay 1983)

$$\begin{aligned}
&|\Gamma_{11}(t,s,\varepsilon)|\leq c\, e^{-\alpha(t-s)}\\
&|\Gamma_{12}(t,s,\varepsilon)|\leq \varepsilon c\, e^{-\alpha(t-s)}\\
&|\Gamma_{22}(t,s,\varepsilon)|\leq c(e^{-\alpha(t-s)/\varepsilon}+\varepsilon e^{-\alpha(t-s)})
\end{aligned}\tag{9}$$

$-\infty<s\leq t<\infty$.

If $\hat{\Gamma}(t,s,\varepsilon)=\begin{pmatrix}\hat{\Gamma}_{11}(t,s,\varepsilon) & \hat{\Gamma}_{12}(t,s,\varepsilon)\\ \hat{\Gamma}_{21}(t,s,\varepsilon) & \hat{\Gamma}_{22}(t,s,\varepsilon)\end{pmatrix}$

is the evolution matrix associated to $-A^*$ we shall have

$\hat{\Gamma}_{11}=\Gamma_{11}$, $\hat{\Gamma}_{12}=\frac{1}{\varepsilon}\Gamma_{12}$, $\hat{\Gamma}_{21}=\varepsilon\Gamma_{21}$, $\hat{\Gamma}_{22}=\Gamma_{22}$

$$|\hat{\Gamma}_{1,1}(t,s,\varepsilon)|\leq c\, e^{-\alpha(t-s)}$$

$$|\hat{\Gamma}_{21}(t,s,\varepsilon)|\leq c\,\varepsilon\, e^{-\alpha(t-s)}$$

$$|\hat{\Gamma}_{22}(t,s,\varepsilon)|\leq c(e^{-\alpha(t-s)/\varepsilon}+\varepsilon e^{-\alpha(t-s)})$$

The estimates in the statement follow now from

$$\begin{aligned}
X_{11}(t,s,\varepsilon)&=\hat{\Gamma}^*_{11}(s,t,\varepsilon)\\
X_{12}(t,s,\varepsilon)&=\hat{\Gamma}^*_{21}(s,t,\varepsilon)\\
X_{21}(t,s,\varepsilon)&=\hat{\Gamma}^*_{12}(s,t,\varepsilon)\\
X_{22}(t,s,\varepsilon)&=\hat{\Gamma}^*_{22}(s,t,\varepsilon)
\end{aligned}$$

Consider now equation (6) corresponding to system (1), write $P=\begin{pmatrix}P_{11} & P_{12}\\ P_{12}^* & P_{22}\end{pmatrix}$, denote $\hat{P}_{22}=\frac{1}{\varepsilon}P_{22}$ and deduce the equations

$$P_{11}'=A_{11}(t)P_{11}+P_{11}A_{11}^*(t)+A_{12}(t)P_{12}^*+P_{12}A_{12}^*(t)-B_1(t)B_1^*(t)$$

$$\varepsilon P_{12}'=\varepsilon A_{11}(t)P_{11}+A_{12}(t)\hat{P}_{22}+P_{11}A_{21}^*(t)+P_{12}A_{22}^*(t)-B_1(t)B_2^*(t)$$

$$\varepsilon \hat{P}_{22}' = \varepsilon A_{21}(t)P_{12} + \varepsilon P_{12}^{*}A_{21}^{*}(t) + A_{22}(t)\hat{P}_{22} + \hat{P}_{22}A_{22}^{*}(t) - B_2(t)B_2^{*}(t)$$

Under the assumptations of Lemma 1 this system has a unique bounded on $\mathbb{R}$ solution. We are looking for the asymptotic structure of this solution. Let $\tilde{P}$ be the unique bounded on $\mathbb{R}$ solution for

$$P' = \tilde{A}(t)P + P\tilde{A}^{*}(t) - \tilde{B}(t)\tilde{B}^{*}(t) \tag{10}$$

Let $\tilde{P}_{22}$ be the unique bounded on $\mathbb{R}$ solution for

$$\varepsilon P_{22}' = A_{22}(t)P_{22} + P_{22}A_{22}^{*}(t) - B_2(t)B_2^{*}(t)$$

Under the assumptions of Lemma 1 we shall have

$$|X_{-A_{22}^{*}}(s,t,\varepsilon)| \leq \beta e^{-\alpha(s-t)/\varepsilon}$$

$-\infty < t \leq s < \infty$. (see V.Drăgan, A.Halanay 1983).

Denote $X_2(t,s,\varepsilon) = X_{\frac{1}{\varepsilon}A_{22}}(t,s,\varepsilon)$; then

$$X_2(t,s,\varepsilon) = X^{*}_{-\frac{1}{\varepsilon}A_{22}^{*}}(s,t,\varepsilon)$$

hence

$$|X_2(t,s,\varepsilon)| \leq \beta e^{-\alpha(s-t)/\varepsilon}, \quad -\infty < t \leq s < \infty.$$

We have also

$$X_2(t,s,\varepsilon) = e^{A_{22}(t)(t-s)/\varepsilon} + \hat{X}(t,s,\varepsilon)$$

$$\hat{X}(t,s,\varepsilon) = \frac{1}{\varepsilon}\int_s^t e^{A_{22}(t)(t-\tau)/\varepsilon}\left[A_{22}(\tau) - A_{22}(t)\right] X_2(\tau,s,\varepsilon)\,d\tau$$

and since

$$\tilde{P}_{22}(t,\varepsilon) = \frac{1}{\varepsilon}\int_t^{\infty} X_2(t,s,\varepsilon)B_2(s)B_2^{*}(s)X_2^{*}(t,s,\varepsilon)\,ds$$

we may write

$$\tilde{P}_{22}(t,\varepsilon) = \frac{1}{\varepsilon}\int_t^{\infty} e^{A_{22}(t)(t-s)/\varepsilon}B_2(s)B_2^{*}(s)e^{A_{22}^{*}(t)(t-s)/\varepsilon}\,ds$$

$$+ \frac{1}{\varepsilon}\int_t^{\infty}\hat{X}(t,s,\varepsilon)B_2(s)B_2^{*}(s)e^{A_{22}^{*}(t)(t-s)/\varepsilon}\,ds +$$

$$+ \frac{1}{\varepsilon}\int_t^{\infty} e^{A_{22}(t)(t-s)/\varepsilon}B_2(s)B_2^{*}(s)\hat{X}^{*}(t,s,\varepsilon)\,ds +$$

$$+ \frac{1}{\varepsilon}\int_t^{\infty}\hat{X}(t,s,\varepsilon)B_2(s)B_2^{*}(s)\hat{X}^{*}(t,s,\varepsilon)\,ds \; ;$$

we have next

$$\frac{1}{\varepsilon}\int_t^\infty e^{A_{22}(t)(t-s)/\varepsilon} B_2(s)B_2^*(s)e^{A_{22}^*(t)(t-s)/\varepsilon}\,ds =$$

$$= \int_{-\infty}^0 e^{A_{22}(t)\tau} B_2(t)B_2^*(t)e^{A_{22}^*(t)\tau}\,d\tau +$$

$$+ \frac{1}{\varepsilon}\int_t^\infty e^{A_{22}(t)(t-s)/\varepsilon} (B_2(s)B_2^*(s)-B_2(t)B_2^*(t))e^{A_{22}^*(t)(t-s)/\varepsilon}\,ds .$$

If we denote

$$R(t)= \int_{-\infty}^0 e^{A_{22}(t)\tau} B_2(t)B_2^*(t)e^{A_{22}^*(t)\tau}\,d\tau$$

we see that

$$A_{22}(t)R(t)+R(t)A_{22}^*(t)=B_2(t)B_2^*(t) .$$

If we assume that B_2 is uniformly continuous on $\mathbb{R}$ we have the estimate

$$\left|\frac{1}{\varepsilon}\int_t^\infty e^{A_{22}(t)(t-s)/\varepsilon} (B_2(s)B_2^*(s)-B_2(t)B_2^*(t))e^{A_{22}^*(t-s)/\varepsilon}\,ds\right| \leq$$

$$\left|\frac{1}{\varepsilon}\int_t^{t+\sqrt{\varepsilon}} e^{A_{22}(t)(t-s)/\varepsilon} (B_2(s)B_2^*(s)-B_2(t)B_2^*(t))\, e^{A_{22}^*(t-s)/\varepsilon}\,ds\right| +$$

$$\left|\frac{1}{\varepsilon}\int_{t+\sqrt{\varepsilon}}^\infty e^{A_{22}(t)(t-s)/\varepsilon}(B_2(s)B_2(s)-B_2(t)B_2(t))\, e^{A_{22}^*(t-s)/\varepsilon}\,ds\right| \leq$$

$$\leq c(\hat{\omega}(\sqrt{\varepsilon}) + e^{-\alpha/\sqrt{\varepsilon}})$$

By using the expression for $\hat{X}$ we deduce

$$\left|\frac{1}{\varepsilon}\int_t^\infty \hat{X}(t,s,\varepsilon)B_2(s)B_2^*(s)e^{A_{22}(t-s)/\varepsilon}\,ds\right| \leq$$

$$\leq \frac{\gamma}{\varepsilon^2}\int_t^\infty e^{\alpha(t-s)/\varepsilon} \int_t^{t+\sqrt{\varepsilon}} e^{\alpha(t-\tau)/\varepsilon} \left|A_{22}(t)-A_{22}(\tau)\right| d\tau\, ds +$$

$$+ \frac{\gamma}{\varepsilon^2}\int_t^\infty e^{\alpha(t-s)/\varepsilon} \int_{t+\sqrt{\varepsilon}}^s e^{\alpha(t-\tau)/\varepsilon} \left|A_{22}(t)-A_{22}(\tau)\right| d\tau\, ds \leq$$

$$\leq c(\omega(\sqrt{\varepsilon}) + e^{-\alpha/\sqrt{\varepsilon}})$$

if A_{22} is uniformly continuous on $\mathbb{R}$.

We estimate in the same way

$$\left|\frac{1}{\varepsilon}\int_t^\infty \hat{X}(t,s,\varepsilon)B_2(s)B_2^*(s)\hat{X}^*(t,s,\varepsilon)\,ds\right| \leq$$

$$\leq \delta(\omega(\sqrt{\varepsilon}) + e^{-\alpha/\sqrt{\varepsilon}})^2 \quad \text{with } \lim_{\varepsilon\to 0}\omega(\varepsilon) = 0 .$$

In this way we proved

Lemma 2. Under the assumpatations of Lemma 1 and assuming that A_{22}, B_2 are uniformly continuous on $\mathbb{R}$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the equation

$$\varepsilon P' = A_{22}(t)P + PA_{22}^*(t) - B_2(t)B_2^*(t)$$

has a unique bounded on $\mathbb{R}$ solution $\tilde{P}_{22}$ such that

$$\tilde{P}_{22}(t,\varepsilon) = R(t) + \check{R}(t,\varepsilon) \quad , \; |\check{R}(t,\varepsilon)| \leq \theta(\varepsilon), \; \lim_{\varepsilon\to 0}\theta(\varepsilon) = 0$$

for all $t \in \mathbb{R}$.

Let now $\tilde{P}_{12}$ be the unique bounded on $\mathbb{R}$ solution for the linear equation

$$\varepsilon P_{12}' = P_{12}A_{22}^*(t) + A_{12}(t)\tilde{P}_{22}(t,\varepsilon) + \tilde{P}_{11}(t)A_{21}^*(t) - B_1(t)B_2^*(t)$$

where $\tilde{P}_{11}$ is a given bounded on $\mathbb{R}$ function .

Lemma 3. Under the assumptations of Lemma 1 and B_1 , B_2 uniformly continuous and bounded on $\mathbb{R}$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the solution $\tilde{P}_{12}$ has the asymptotic structure

$$\tilde{P}_{12}(t,\varepsilon) = \left[B_1(t)B_2^*(t) - A_{12}(t)R(t) - \tilde{P}_{11}(t)A_{21}^*(t)\right]\left(A_{22}^{-1}(t)\right)^* + \Pi_{12}(t,\varepsilon)$$

with $|\Pi_{12}(t,\varepsilon)| \leq \theta(\varepsilon)$, $\lim_{\varepsilon\to 0}\theta(\varepsilon) = 0$.

The proof proceeds by the usual tachniques in singular perturbations (V.Dr*gan ,A.Halanay 1983).

We look now at the equation for P_{11} and replace in it P_{12} by the principal part in $\tilde{P}_{12}$; we deduce

$$P'_{11}=A_{11}(t)P_{11}+P_{11}A^*_{11}(t)+A_{12}(t)\Big(\left[B_1(t)B^*_2(t)-A_{12}(t)R(t)-\tilde{P}_{11}(t)A^*_{21}(t)\right]\left[A^{-1}_{22}(t)\right]^*\Big)^* + \left[B_1(t)B^*_2(t)-A_{12}(t)R(t)-\tilde{P}_{11}(t)A^*_{21}(t)\right]\left[A^*_{22}(t)\right]^{-1}A^*_{12}(t)-B_1(t)B^*_1(t)$$

and if we take $\tilde{P}_{11}=P_{11}$,

$$\tilde{P}'_{11}(t)=\left\{A_{11}(t)-A_{12}(t)\left[A_{22}(t)\right]^{-1}A_{21}(t)\right\}\tilde{P}_{11}(t)+\tilde{P}_{11}(t)\left\{A_{11}(t)-A_{12}(t)\left[A_{22}(t)\right]^{-1}A_{21}(t)\right\}^*-B_1(t)\left\{B^*_1(t)-B^*_2(t)\left[A^*_{22}(t)\right]^{-1}A^*_{12}(t)\right\}+A_{12}(t)\left[A_{22}(t)\right]^{-1}B_2(t)B^*_1(t)-A_{12}(t)R(t)\left[A^*_{22}(t)\right]^{-1}A^*_{12}(t)-A_{12}(t)\left[A^*_{22}(t)\right]^{-1}R(t)A^*_{12}(t).$$

We have

$$R(t)\left[A^*_{22}(t)\right]^{-1}+\left[A_{22}(t)\right]^{-1}R(t)=\left[A_{22}(t)\right]^{-1}B_2(t)B^*_2(t)\left[A^*_{22}(t)\right]^{-1}$$

and we deduce

$$\tilde{P}'_{11}(t)=\tilde{A}(t)\tilde{P}_{11}(t)+\tilde{P}_{11}(t)\tilde{A}^*(t)-B_1(t)\tilde{B}^*(t)+\left[B_1(t)-\tilde{B}(t)\right]B^*_1(t)-A_{12}(t)\left[A_{22}(t)\right]^{-1}B_2(t)B^*_2(t)\left[A^*_{22}(t)\right]^{-1}A^*_{12}(t)=$$

$$=\tilde{A}(t)\tilde{P}_{11}(t)+\tilde{P}_{11}(t)\tilde{A}^*(t)-B_1(t)\tilde{B}^*(t)+\left[B_1(t)-\tilde{B}(t)\right]B^*_1(t)-(B_1(t)-\tilde{B}(t))(B^*_1(t)-\tilde{B}^*(t))=\tilde{A}(t)\tilde{P}_{11}(t)+\tilde{P}_{11}(t)\tilde{A}^*(t)-\tilde{B}(t)\tilde{B}^*(t)$$

and $P_{11}=\tilde{P}$ corresponding to the reduced model.

We are now in position to prove

Theorem 1. Assume all the hypotheses in Lemma 3. Then there exists $\tilde{\varepsilon}>0$ such that for $\varepsilon\ (0,\tilde{\varepsilon})$

$$P_{11}(t,\varepsilon)=\tilde{P}(t)+\check{P}_{11}(t,\varepsilon)$$
$$P_{12}(t,\varepsilon)=\tilde{P}_{12}(t,\varepsilon)+\check{P}_{12}(t,\varepsilon)$$
$$P_{22}(t,\varepsilon)=\frac{1}{\varepsilon}(\tilde{P}_{22}(t,\varepsilon)+\check{P}_{22}(t,\varepsilon))$$

where $|\check{P}_{11}(t,\varepsilon)|+|\check{P}_{12}(t,\varepsilon)|\le\theta(\varepsilon)$, $\lim_{\varepsilon\to 0}\theta(\varepsilon)=0$, $|\check{P}_{22}(t,\varepsilon)|\le\varepsilon\gamma$.

Proof. $\check{P}(t,\varepsilon)=\begin{pmatrix}\check{P}_{11}(t,\varepsilon) & \check{P}_{12}(t,\varepsilon)\\ \check{P}^*_{12}(t,\varepsilon) & \check{P}_{22}(t,\varepsilon)\end{pmatrix}$

is the bounded on $\mathbb{R}$ solution for a Liapunov equation ;we use the formula for this solution and the estimates for the solution operator and for the free term to deduce estimates for the solution.

With $A(t,\varepsilon)=\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\varepsilon}A_{21}(t) & \frac{1}{\varepsilon}A_{22}(t) \end{pmatrix}$ we have

$$\frac{d}{dt}\check{P}(t,\varepsilon)=A(t,\varepsilon)\check{P}(t,\varepsilon)+\check{P}(t,\varepsilon)A^*(t,\varepsilon)+F(t,\varepsilon)$$

$$F(t,\varepsilon)=\begin{pmatrix} F_{11}(t,\varepsilon) & F_{12}(t,\varepsilon) \\ F_{12}^*(t,\varepsilon) & \frac{1}{\varepsilon}F_{22}(t,\varepsilon) \end{pmatrix}$$

$$F_{11}(t,\varepsilon)=A_{12}(t)\Pi_{12}^*(t,\varepsilon)+\Pi_{12}(t,\varepsilon)A_{12}^*(t)$$

$$F_{12}(t,\varepsilon)=A_{11}(t)\tilde{P}_{12}(t,\varepsilon)$$

$$F_{22}(t,\varepsilon)=A_{21}(t)\tilde{P}_{12}(t,\varepsilon)+\tilde{P}_{12}^*(t,\varepsilon)A_{21}^*(t,\varepsilon)$$

$$\check{P}(t,\varepsilon)=-\int_t^\infty X(t,s,\varepsilon)F(s,\varepsilon)X^*(t,s,\varepsilon)\,ds$$

Taking into account estimates for $X_{ij}(t,s,\varepsilon)$ and $F_{ij}(s,\varepsilon)$ we obtain the estimates for $\check{P}_{ij}(t,\varepsilon)$ in the statement .

<u>Remark.</u> It is well known that if $A(\cdot)$ is antistable and $\tilde{P}(t)>0$ is the unique bounded solution of the Liapunov equation then $u(t)=-B^*(t)\tilde{P}^{-1}(t)x(t)$ defines a stabilizing feedback control.

If we apply this result to a singularly perturbed system we deduce that a stabilizing feedback will be

$$u(t)=-\left(B_1^*(t) \quad \frac{1}{\varepsilon}B_2^*(t) \right) P^{-1}(t,\varepsilon) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

If we use for $P(t,\varepsilon)$ only the principal part

$$\tilde{P}(t,\varepsilon)=\begin{pmatrix} \tilde{P}(t) & S_{12}(t) \\ S_{12}^*(t) & \frac{1}{\varepsilon}R(t) \end{pmatrix}$$

$$S_{12}(t)=\left(B_1(t)B_2^*(t)-A_{12}(t)R(t)-\tilde{P}(t)A_{21}^*(t)\right)\left[A_{22}^{-1}(t)\right]^*$$

we obtain a control of the form

$u(t)=F_1(t)x_1+F_2(t)x_2$

where $F_2(t)=-B_2^*(t)R_{22}^{-1}(t)$

$$F_1(t)=-\left[I+F_2(t)A_{22}^{-1}(t)B_2(t)\right]B_1^*(t)\tilde{P}_{11}^{-1}(t)+F_2(t)A_{22}^{-1}(t)A_{21}(t)-$$
$$-B_2^*(t)R_{22}^{-1}(t)A_{22}^{-1}(t)R_{22}(t)A_{12}^*(t)\tilde{P}_{11}^{-1}(t)$$

A direct computation shows that

$$F_1(t)=\left[I+F_2(t)A_{22}^{-1}(t)B_2(t)\right]\tilde{F}(t)+F_2(t)A_{22}^{-1}(t)A_{21}(t)$$

where $\tilde{F}(t)=-\tilde{B}^*(t)\left[\tilde{P}_{11}(t)\right]^{-1}$.

This is a two stage controller of the same form as described in a more general setting by O'Reilly (1980).

Consider now the second Liapunov equation

$$Q'=-A^*(t,\varepsilon)Q-QA(t,\varepsilon)+C^*(t)C(t)$$

Under the assumptation in Lemma 1 the equation has a unique bounded on $\mathbb{R}$ solution $Q(t,\varepsilon)\geq 0$. Denote

$$Q(t,\varepsilon)=\begin{pmatrix} Q_{11}(t,\varepsilon) & Q_{12}(t,\varepsilon) \\ Q_{12}^*(t,\varepsilon) & Q_{22}(t,\varepsilon)\end{pmatrix}.$$

and $\hat{Q}_{12}(t,\varepsilon)=\frac{1}{\varepsilon}Q_{12}(t,\varepsilon)$

$\hat{Q}_{22}(t,\varepsilon)=\frac{1}{\varepsilon}Q_{22}(t,\varepsilon)$.

Then $Q_{11},\hat{Q}_{12}$, $\hat{Q}_{22}$ satisfy

$$\frac{d}{dt}Q_{11}=-A_{11}^*(t)Q_{11}-Q_{11}A_{11}(t)-A_{21}^*(t)\hat{Q}_{12}^*-\hat{Q}_{12}A_{21}(t)+$$
$$+C_1^*(t)C_1(t)$$

$$\varepsilon\frac{d}{dt}\hat{Q}_{12}=-\varepsilon A_{11}^*(t)\hat{Q}_{12}-A_{21}^*(t)\hat{Q}_{22}-Q_{11}A_{12}(t)-\hat{Q}_{12}A_{22}(t)+$$
$$+C_1^*(t)C_2(t)$$

$$\varepsilon\frac{d}{dt}\hat{Q}_{22}=-\varepsilon A_{12}^*(t)\hat{Q}_{12}-\varepsilon\hat{Q}_{12}^*A_{12}(t)-A_{22}^*(t)\hat{Q}_{22}-\hat{Q}_{22}A_{22}(t)+$$
$$+C_2^*(t)C_2(t)$$

Let $\tilde{Q}_{11}$ be the unique bounded on $\mathbb{R}$ solution of the Liapunov equation

$$\frac{d}{dt} Q_{11}=-\tilde{A}^{*}(t)Q_{11}-Q_{11}\tilde{A}(t)+\tilde{C}^{*}(t)C(t)$$

If $\tilde{A}$ defines an antistable evolution then the above equation has a solution (bounded on $\mathbb{R}$, unique) ~~satisfies~~ $Q_{11}(t)\geq 0$ .Let $\tilde{Q}_{22}$ be the unique bounded on $\mathbb{R}$ solution of the equation

$$\varepsilon \frac{d}{dt} Q_{22}=-A_{22}^{*}(t)Q_{22}-Q_{22}A_{22}(t)+C_{2}^{*}(t)C_{2}(t) \qquad (11)$$

In the same way as for Lemma 2 we may prove

<u>Lemma 4.</u> Assume a) A_{22}, C_2 are uniformly continuous and bounded on $\mathbb{R}$

b) There exists $\hat{\alpha} > 0$ such that no eigenvalue of $A_{22}(t)$ has real part ~~is~~ less than $2\hat{\alpha} > 0$ for all $t\in\mathbb{R}$.

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ equation (11) has a unique bounded on $\mathbb{R}$ solution satisfying $\tilde{Q}_{22}(t,\varepsilon)=S_{22}(t)+T_{22}(t,\varepsilon)$ where $A_{22}^{*}(t)S_{22}(t)+S_{22}(t)A_{22}(t)= =C_{2}^{*}(t)C_{2}(t)$ and $\left|T_{22}(t,\varepsilon)\right| \leq \theta(\varepsilon)$,$\lim\limits_{\varepsilon\to 0} \theta(\varepsilon)=0$.

Let now $\tilde{Q}_{12}$ be the unique bounded on $\mathbb{R}$ solution of the equation

$$\varepsilon\frac{d}{dt} Q_{12}=-Q_{12}A_{22}(t)+C_{1}^{*}(t)C_{2}(t)-\tilde{Q}_{11}(t)A_{12}(t)-A_{12}^{*}(t)\tilde{Q}_{22}(t,\varepsilon)$$

<u>Lemma 5.</u> Assume A_{ij}, C_i are uniformly continuous and bounded on $\mathbb{R}$, assume also assumptions in Lemma 1 hold. Then there exists $\varepsilon_0>0$ such that for all $\varepsilon \in (0,\varepsilon_0)$

$$\tilde{Q}_{12}(t,\varepsilon)=S_{12}(t)+T_{12}(t,\varepsilon)$$

$$S_{12}(t)=\left[C_{1}^{*}(t)C_{2}(t)-\tilde{Q}_{11}(t)A_{12}(t)-A_{21}^{*}(t)S_{22}(t)\right]\left[A_{22}(t)\right]^{-1}$$

$$\left|T_{12}(t,\varepsilon)\right| \leq \theta(\varepsilon) \qquad \lim_{\varepsilon\to 0} \theta(\varepsilon)=0 .$$

The proof proceeds by the usual singular perturbations techniques as above. Denote now

$$\check{Q}_{11}(t,\varepsilon)=Q_{11}(t,\varepsilon)-\tilde{Q}_{11}(t)$$
$$\check{Q}_{12}(t,\varepsilon)=\hat{Q}_{12}(t,\varepsilon)-\tilde{Q}_{12}(t,\varepsilon)$$
$$\check{Q}_{22}(t,\varepsilon)=\hat{Q}_{22}(t,\varepsilon)-\tilde{Q}_{22}(t,\varepsilon) .$$

We deduce that

$\frac{d}{dt}$

$$\frac{d}{dt}\begin{pmatrix} \check{Q}_{11}(t,\varepsilon) & \varepsilon\check{Q}_{12}(t,\varepsilon) \\ \varepsilon\check{Q}^*_{12}(t,\varepsilon) & \varepsilon\check{Q}_{22}(t,\varepsilon)\end{pmatrix} = -A^*(t,\varepsilon)\begin{pmatrix} \check{Q}_{11}(t,\varepsilon) & \varepsilon\check{Q}_{12}(t,\varepsilon) \\ \varepsilon\check{Q}^*_{12}(t,\varepsilon) & \varepsilon\check{Q}_{22}(t,\varepsilon)\end{pmatrix}$$

$$-\begin{pmatrix} \check{Q}_{11}(t,\varepsilon) & \varepsilon\check{Q}_{12}(t,\varepsilon) \\ \varepsilon Q^*_{12}(t,\varepsilon) & \varepsilon\check{Q}_{22}(t,\varepsilon)\end{pmatrix} A(t,\varepsilon)+$$

$$+\begin{pmatrix} G_{11}(t,\ ) & \varepsilon G_{12}(t,\varepsilon) \\ \varepsilon G^*_{12}(t,\varepsilon) & \varepsilon G_{22}(t,\varepsilon)\end{pmatrix}$$

$G_{11}(t,\varepsilon)=-A^*_{21}(t)T^*_{12}(t,\varepsilon)-T_{12}(t,\varepsilon)A_{21}(t)$

$G_{12}(t,\varepsilon)=-A^*_{11}(t)T_{12}(t,\varepsilon)$

$G_{22}(t,\varepsilon)=A^*_{12}(t)T_{12}(t,\varepsilon)-T^*_{12}(t,\varepsilon)A_{12}(t)$

$|G_{11}(t,\varepsilon)| \le \theta(\varepsilon)$, $|\dot{G}_{12}(t,\varepsilon)| + |G_{22}(t,\varepsilon)| \le c$

We use again reprezentation formulae and estimates to get

$|\check{Q}_{11}(t,\varepsilon)| + |\check{Q}_{12}(t,\varepsilon)| + |Q_{22}(t,\varepsilon)| \le \theta(\varepsilon)$

$\lim_{\varepsilon\to 0} \theta(\varepsilon)=0$, We have thus

Theorem 2. Under the assumptations in Lemma 5 there exists $\varepsilon_0>0$ such that for $\varepsilon \in (0,\varepsilon_0)$

$Q_{11}(t,\varepsilon)=\tilde{Q}_{11}(t)+\check{Q}_{11}(t,\varepsilon)$

$Q_{12}(t,\varepsilon)=\varepsilon S_{12}(t)+\varepsilon T_{12}(t,\varepsilon)+\varepsilon\check{Q}_{12}(t,\varepsilon)$

$Q_{22}(t,\varepsilon)=\varepsilon S_{22}(t)+\varepsilon T_{22}(t,\varepsilon)+\varepsilon\check{Q}_{12}(t,\varepsilon)$

where the remainders $\check{Q}_{ij}$ are estimated as above.

4. Asymptotic expansions for the singular values

We state now the main result

Theorem 3 Assume A_{ij}, B_i, C_i are uniformly continuous and bounded on $\mathbb{R}$. Assume also A_{22} and $\tilde{A}$ satisfy conditions in Lemma 1.

Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the singular values associated to (1) have the following asymptotic structure

a) n_1 singular values are of the form

$$\lambda_j(t,\varepsilon) = \tilde{\lambda}_j(t) + \theta(\varepsilon) \quad , \lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$$

and $\tilde{\lambda}_j$ are the singular values for (2)

b) n_2 singular values are of the form

$$\lambda_{n_1+j}(t,\varepsilon) = \mu_j(t) + \theta(\varepsilon), \lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$$

$\cancel{(A_{22}, B_2, C_2)}$ ~~with~~ and μ_j are the singular values for $(A_{22}(t), B_2(t), C_2(t))$ with frozen time.

<u>Proof.</u> Denote $P(t,\varepsilon)Q(t,\varepsilon) = V(t,\varepsilon) = \begin{pmatrix} V_{11}(t,\varepsilon) & V_{12}(t,\varepsilon) \\ V_{21}(t,\varepsilon) & V_{22}(t,\varepsilon) \end{pmatrix}$

Taking into account the asymptotic structure of $P(t,\varepsilon)$ given by Theorem 1 and of $Q(t,\varepsilon)$ given by Theorem 2 we have

$$V_{11}(t,\varepsilon) = \tilde{P}_{11}(t)\tilde{Q}_{11}(t) + \check{V}_{11}(t,\varepsilon)$$

$$V_{21}(t,\varepsilon) = R^*_{12}(t)\tilde{Q}_{11}(t) + R_{22}(t)S^*_{12}(t) + \check{V}_{21}(t,\varepsilon)$$

$$V_{22}(t,\varepsilon) = R_{22}(t)S_{22}(t) + \check{V}_{22}(t,\varepsilon)$$

with $|\check{V}_{11}(t,\varepsilon)| + |\check{V}_{12}(t,\varepsilon)| + |\check{V}_{21}(t,\varepsilon)| + |\check{V}_{22}(t,\varepsilon)| \leq \theta(\varepsilon)$

$\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$.

It follows that the principal part of the eigenvalues of $P(t,\varepsilon)Q(t,\varepsilon)$ is given by the singular values of (2) and the singular values of the system defined by

$$x'(\tau) = A_{22}(t)x(\tau) + B_{22}(t)u(\tau)$$

$$y(\tau) = C_{22}(t)x(\tau)$$

Let us remark that the singular values associated to A_{22}, B_2, C_2 with frozen time are in fact the principal parts of the singular values associated to (3) .

Remark finally that the results above do contain the corresponding expansions for stationary systems and also that similar results may be obtained in the case of exponentially stable evolutions.

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UNIFORM CONTROLLABILITY
FOR SYSTEMS WITH TWO TIME- SCALES

by V.Drăgan ,A.Halanay

It is a general program promoted primarily by P.V.Kokotovic and his coworkers to obtain global results for systems with two time scales from the analysis of the "slow" and "fast systems " associated (see V.A.Saksena,J.O'Reilly,P.V.Kokotovic 1984).It is the propose of the present paper to discuss from this view point the problem of uniform controllability for time-varying linear systems. While the result is quite natural, to prove it we had to use specific asymptotic expansions for the evolution operator associated to a singularly perturbed linear system (V.Drăgan and A.Halanay 1983,1985).

1.Main Results.

Consider the linear control system

$$\begin{aligned} x_1' &= A_{11}(t)x_1+A_{12}(t)x_2+B_1(t)u \\ \varepsilon x_2' &= A_{21}(t)x_1+A_{22}(t)x_2+B_2(t)u \end{aligned} \qquad (1)$$

Theorem 1 Assume that A_{ij}, B_i are defined on $\mathbb{R}$ and are uniformly Lipschitz and bounded. Assume there exist $\tau_2>0$, $\varkappa_2>0$ such that for all $t\in\mathbb{R}$

$$\int_0^{\tau_2} e^{-A_{22}(t)s} B_2(t)B_2^*(t)e^{-A_{22}^*(t)s}\,ds \geq \varkappa_2 I$$

Let $A_{22}(t)$ be invertible , with bounded inverse and let

$$\tilde{A}(t)=A_{11}(t)-A_{12}(t)A_{22}^{-1}(t)A_{21}(t)$$

$$\tilde{B}(t)=B_1(t)-A_{12}(t)A_{22}^{-1}(t)B_2(t)$$

and assume that $(\tilde{A},\tilde{B})$ is uniformly completely controllable. Then there exists $\tilde{\varepsilon}>0$ such that for all $0<\varepsilon\leq\tilde{\varepsilon}$ the given system (1) is uniformly completely controllable.

Universitatea din Bucureşti ,Facultatea de Matematică ,Str. Academiei 14, R-70109 ,Bucureşti ,Romania

A corresponding result holds for uniform observability.

Theorem 2. Consider the system

$$\begin{aligned} x_1' &= A_{11}(t)x_1 + A_{12}(t)x_2 \\ \varepsilon x_2' &= A_{21}(t)x_1 + A_{22}(t)x_2 \\ y &= C_1(t)x_1 + C_2(t)x_2 \end{aligned} \qquad (2)$$

Assume A_{ij} as above, C_1, C_2 uniformly Lipschitz and bounded on $\mathbb{R}$, let $\tilde{C}(t)=C_1(t)-C_2(t)A_{22}^{-1}(t)A_{21}(t)$.

Assume there exist $\delta_2>0$, $k_2>0$ such that

$$\int_0^{\delta_2} e^{A_{22}^*(t)s} C_2^*(t)C_2(t)\, e^{A_{22}(t)s}\, ds \geq k_2 I$$

for all $t\in\mathbb{R}$ and assume also that $(\tilde{C}, \tilde{A})$ is uniformly completely observable. Then there exists $\tilde{\varepsilon}>0$ such that for all $0<\varepsilon<\tilde{\varepsilon}$ system (2) is uniformly completely observable.

Theorem 2 is dual to theorem 1 and it is proved in the same way. We shall prove in the next sections Theorem 1 in two steps, starting with the assumption that there exists $\alpha>0$ such that the real parts of the eigenvalues of $A_{22}(t)$ are larger than 2α for all $t\in\mathbb{R}$; in the second step we shall remove this special assumption.

2. Asymptotic expansions

Proposition 1. Assume A_{ij} are uniformly Lipschitz and bounded on $\mathbb{R}$; assume also that there exists $\alpha>0$ such that the real parts of the eigenvalues of $A_{22}(t)$ are larger than $2\alpha>0$ for all $t\in\mathbb{R}$. Consider the evolution matrix associated to the linear system

$$\begin{aligned} x_1' &= A_{11}(t)x_1 + A_{12}(t)x_2 \\ \varepsilon x_2' &= A_{21}(t)x_1 + A_{22}(t)x_2 \end{aligned} \qquad (3)$$

and write it in block form

$$X(t,s,\varepsilon)=\begin{pmatrix} X_{11}(t,s,\varepsilon) & X_{12}(t,s,\varepsilon) \\ X_{21}(t,s,\varepsilon) & X_{22}(t,s,\varepsilon) \end{pmatrix} \qquad (4)$$

Then for every $r>0$ there exists $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ the following asymptotic structure holds for $-\infty<t\leq s\leq t+r$

$$X_{11}(t,s,\varepsilon)=X_{\tilde{A}}(t,s)+\varepsilon\hat{X}_{11}(t,s,\varepsilon)$$

$$X_{21}(t,s,\varepsilon)=\frac{1}{\varepsilon}\int_s^t X_2(t,\sigma,\varepsilon)A_{21}(\sigma)X_{\tilde{A}}(\sigma,s)d\sigma+\varepsilon\hat{X}_{21}(t,s,\varepsilon)$$

$$X_{12}(t,s,\varepsilon)=-\varepsilon\left[X_{\tilde{A}}(t,s)A_{12}(s)A_{22}^{-1}(s)-A_{12}(s)A_{22}^{-1}(s)X_2(t,s,\varepsilon)\right]+$$
$$+\varepsilon^2\hat{X}_{12}(t,s,\varepsilon)$$

$$X_{22}(t,s,\varepsilon)=X_2(t,s,\varepsilon)+\varepsilon\hat{X}_{22}(t,s,\varepsilon)$$

where $X_{\tilde{A}}$ is the evolution operator associated to $\tilde{A}$, X_2 is the evolution operator associated to $\frac{1}{\varepsilon}A_{22}$ and

$$|\hat{X}_{ij}(t,s,\varepsilon)|\leq\hat{\beta}\,e^{-\hat{\alpha}(s-t)}$$

<u>Proof</u>. Denote $A(t,\varepsilon)=\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\varepsilon}A_{21}(t) & \frac{1}{\varepsilon}A_{22}(t) \end{pmatrix}$

The system associated to $-A^*(t,\varepsilon)$ is

$$w_1'=-A_{11}^*(t)w_1-\frac{1}{\varepsilon}A_{21}^*(t)w_2$$

$$w_2'=-A_{12}^*(t)w_1-\frac{1}{\varepsilon}A_{22}^*(t)w_2$$

and after the scalling $z_1=w_1$, $z_2=\frac{1}{\varepsilon}w_2$ becomes

$$z_1'=-A_{11}^*(t)z_1-A_{21}^*(t)z_2$$

$$\varepsilon z_2'=-A_{12}^*(t)z_1-A_{22}^*(t)z_2$$

This last system is in the conditions considered in V.Drăgan, A.Halanay 1983,1983 (for instance see V.Drăgan,A.Halanay 1985, section 2.E).

If we denote by $\Gamma(t,s,\varepsilon)$ the corresponding evolution operator and Γ_{ij} a corresponding partition of this operator we have the asymptotic expansions

$$\Gamma_{11}(t,s,\varepsilon)=\tilde{\Gamma}(t,s) + \varepsilon\, \hat{\Gamma}_{11}(t,s,\varepsilon)$$

$$\Gamma_{21}(t,s,\varepsilon) = -\frac{1}{\varepsilon}\int_s^t \Gamma_2(t,\sigma,\varepsilon)A_{12}^*(\sigma)\,\tilde{\Gamma}(\sigma,s)\,d\sigma + \varepsilon\,\hat{\Gamma}_{21}(t,s,\varepsilon)$$

$$\Gamma_{12}(t,s,\varepsilon)=-\int_s^t \tilde{\Gamma}(t,\sigma)A_{21}^*(\sigma)\Gamma_2(\sigma,s,\varepsilon)\,d\sigma + \varepsilon^2\,\hat{\Gamma}_{12}(t,s,\varepsilon)$$

$$\Gamma_{22}(t,s,\varepsilon)=\Gamma_2(t,s,\varepsilon)+\varepsilon\,\Gamma_{22}^1(t,s,\varepsilon)+\varepsilon^2\hat{\Gamma}_{22}(t,s,\varepsilon)$$

where $\tilde{\Gamma}$ is the evolution operator for

$$x_1'=-\tilde{A}^*(t)x_1$$

Γ_2 is the evolution operator for $\varepsilon x_2'=-A_2^*(t)x_2$, Γ_{22}^1 is defined by

$$\varepsilon\frac{d}{dt}\Gamma_{22}^1=\frac{1}{\varepsilon}A_{12}^*(t)\int_s^t\tilde{\Gamma}(t,\sigma)A_{21}^*(\sigma)\Gamma_2(\sigma,s,\varepsilon)\,d\sigma - A_{22}^*(t)\Gamma_{22}^1$$

$$\Gamma_{22}^1(s,s,\varepsilon)=0$$

and

$$\left|\hat{\Gamma}_{ij}(t,s,\varepsilon)\right| \quad c\; e^{\lambda(s-t)}$$

We have further

$$X_{11}(t,s,\varepsilon)=\Gamma_{11}^*(s,t,\varepsilon)$$

$$X_{21}(t,s,\varepsilon)=\frac{1}{\varepsilon}\Gamma_{12}^*(s,t,\varepsilon)$$

$$X_{12}(t,s,\varepsilon)=\varepsilon\, X_{21}^*(s,t,\varepsilon)$$

$$X_{22}(t,s,\varepsilon)=\Gamma_{22}(s,t,\varepsilon)$$ and we deduce

$$X_{11}(t,s,\varepsilon)=\tilde{\Gamma}^*(s,t,\varepsilon)+\varepsilon\,\hat{\Gamma}_{11}^*(s,t,\varepsilon)$$

$$X_{21}(t,s,\varepsilon)=\frac{1}{\varepsilon}\int_s^t\Gamma_2^*(\sigma,t,\varepsilon)A_{21}(\sigma)\,\tilde{\Gamma}^*(s,\sigma)\,d\sigma+\varepsilon\hat{\Gamma}_{12}^*(s,t,\varepsilon)$$

$$X_{12}(t,s,\varepsilon)=\int_s^t\tilde{\Gamma}^*(\sigma,t)A_{12}(\sigma)\,\Gamma_2^*(s,\sigma,\varepsilon)\,d\sigma+\varepsilon^2\hat{\Gamma}_{21}^*(s,t,\varepsilon)$$

$$X_{22}(t,s,\varepsilon)=\Gamma_2^*(s,t,\varepsilon)+\varepsilon\left[\Gamma_{22}^1(s,t,\varepsilon)\right]^*+\varepsilon\,\hat{\Gamma}_{22}^*(s,t,\varepsilon)$$

Remark that $\tilde{\Gamma}^*(\sigma,t)=X_{\tilde{A}}(t,\sigma)$, $\Gamma_2^*(s,\sigma,\varepsilon)=X_2(\sigma,s,\varepsilon)$.

We have firther

$$\int_s^t X_{\tilde{A}}(t,\sigma)A_{12}(\sigma)X_2(\sigma,s,\varepsilon)\,d\sigma = \check{X}_{12}(t,s,\varepsilon) + \int_s^t X_{\tilde{A}}(t,\sigma)A_{12}(s)A_{22}^{-1}(s)\cdot$$

$$\cdot A_{22}(\sigma)X_2(\sigma,s,\varepsilon)\,d\sigma$$

$$|\check{X}_{12}(t,s,\varepsilon)| \le \hat{\beta}\,\varepsilon^2 e^{\hat{\alpha}(s-t)} \quad , \quad -\infty < t \le s \le t+\tau$$

and

$$\int_s^t X_{\tilde{A}}(t,\sigma)A_{12}(s)A_{22}^{-1}(s)A_{22}(\sigma)X_2(\sigma,s,\varepsilon)\,d\sigma =$$

$$= \varepsilon\left[A_{12}(s)A_{22}^{-1}(s)X_2(t,s,\varepsilon) - X_{\tilde{A}}(t,s)A_{12}(s)A_{22}^{-1}(s)\right] -$$

$$- \quad \varepsilon\int_s^t \left[\frac{d}{d\sigma}X_{\tilde{A}}(t,\sigma)\right]A_{12}(s)A_{22}^{-1}(s)X_2(\sigma,s,\varepsilon)\,d\sigma \quad .$$

The above formulae lead directly to the asymptotic expansions in the statement .

Let us remark that we obtained the expansions for the nonstandard in singular perturbations case of an antistable A_{22}.

3. A first controlability result.

We shall prove Theorem 1 under the assumptation on A_{22} stated in Proposition 1. Uniform controllability depends upon the behaviour of the matrix $\begin{pmatrix} H_{11}(t,\delta,\varepsilon) & H_{12}(t,\delta,\varepsilon) \\ H_{21}(t,\delta,\varepsilon) & H_{22}(t,\delta,\varepsilon) \end{pmatrix}$

.

$$H_{ij}(t,\delta,\varepsilon) = \int_t^{t+\delta}\left[X_{ij}(t,s,\varepsilon)B_1(s) + \frac{1}{\varepsilon}X_{i2}(t,s,\varepsilon)B_2(s)\right]\left[B_1^*(s)X_{1j}^*(t,s,\varepsilon)\right.$$

$$\left. + \frac{1}{\varepsilon}B_2^*(s)X_{21}^*(t,s,\varepsilon)\right]ds$$

Using the asymptotic structure described in Proposition 1 we have

$$H_{11}(t,\delta,\varepsilon) = \int_t^{t+\delta}\tilde{X}(t,s)\tilde{B}(s)\tilde{B}^*(s)\tilde{X}^*(t,s,\varepsilon)\,ds + \varepsilon\,\hat{H}_{11}(t,\delta,\varepsilon)$$

$$|\hat{H}_{11}(t,\delta,\varepsilon)| \le c$$

$$|H_{12}(t,\delta,\varepsilon)| \le c$$

$$H_{22}(t,\delta,\varepsilon) = \frac{1}{\varepsilon^2}\int_t^{t+\delta}X_2(t,s,\varepsilon)B_2(s)B_2^*(s)X_2^*(t,s,\varepsilon)\,ds + \hat{H}_{22}(t,s,\varepsilon)$$

$$|\hat{H}_{22}(t,s,\varepsilon)| \le c$$

Since $X_2(t,s,\varepsilon)=e^{A_{22}(t)(t-s)/\varepsilon}+\hat{X}(t,s,\varepsilon)$

we deduce that

$$\frac{1}{\varepsilon^2}\int_t^{t+\delta} X_2(t,s,\varepsilon)B_2(s)B_2^*(s)X_2^*(t,s,\varepsilon)\,ds =$$

$$=\frac{1}{\varepsilon^2}\int_t^{t+\delta} e^{A_{22}(t)(t-s)/\varepsilon} B_2(t)B_2^*(t)\, e^{A_{22}^*(t)(t-s)/\varepsilon}\,ds \quad +\check{H}_{22}(t,\delta,\varepsilon)$$

Usual in singular perturbations procedures lead to the estimate

$$|\check{H}(t,\delta,\varepsilon)| \leq c .$$

We deduce next that

$$\frac{1}{\varepsilon^2}\int_t^{t+\delta} e^{A_{22}(t)(t-s)/\varepsilon} B_2(t)B_2^*(t)\, e^{-A_{22}^*(t)(t-s)/\varepsilon}\,ds \geq$$

$$\geq \frac{1}{\varepsilon}\int_0^{\delta_2} e^{-A_{22}(t)s} B_2(t)B_2^*(t)\, e^{-A_{22}^*(t)s}\,ds \geq \frac{1}{\varepsilon}\,\kappa_2\, I$$

and finally that $|H_{22}(t,s,\varepsilon)| \geq \frac{1}{2\varepsilon}\kappa_2 I$.

We have further

$$H_{11}(t,\delta,\varepsilon)-H_{12}(t,\delta,\varepsilon)H_{22}^{-1}(t,\delta,\varepsilon)H_{12}^*(t,\delta,\varepsilon) =$$

$$= \int_t^{t+\delta} X_{\tilde{A}}(t,s)\tilde{B}(s)\tilde{B}^*(s)X_{\tilde{A}}^*(t,s)\,ds + \varepsilon\,\tilde{H}_{11}(t,\delta,\varepsilon)$$

$|H_{11}(t,\delta,\varepsilon)| \leq c$, and we deduce that

$$H_{11}(t,\delta,\varepsilon)-H_{12}(t,\delta,\varepsilon)H_{22}^{-1}(t,\delta,\varepsilon)H_{12}^*(t,\delta,\varepsilon) \geq \hat{k}\, I$$

This property proves the uniform controllability.

4, <u>The general case</u>.

Since $|A_{22}(t)| \leq \gamma$ we have $\left|e^{A_{22}(t)s}\right| \geq e^{-\gamma s}$ for $s > 0$ and the controllabilty property in the statement of Theorem 1 implies

$$\int_0^{\delta_2} e^{A_{22}(t)s} B_2(t)B_2^*(t)\, e^{A_{22}^*(t)s}\,ds \geq \hat{k}_2\, e^{-2\gamma\delta_2}\, I$$

Denoting $W(t,\alpha)= \int_0^{\delta_2} e^{-4\alpha s}\, e^{A_{22}(t)s} B_2(t)B_2^*(t)e^{A_{22}^*(t)s}\,ds$,

$\alpha > 0$, we have $W(t,\alpha) \geq e^{-2(\gamma+2\alpha)\delta_2}\hat{k}_2\, I$.

Take $F_2(t)=-B_2^*(t)W^{-1}(t,\alpha)$.

<u>Lemma 1.</u> The real parts of the eigenvalues of the matrix $A_{22}(t)+B_2(t)F_2(t)$ are not less than 2α.

<u>Proof.</u> Consider a solution of the system

$$x'(\tau)=[A(t)+B(t)F(t)]x(\tau)$$

and let $V(\tau)=x^*(\tau)W^{-1}(t,\alpha)x(\tau)$.

A direct calculation shows that

$$\tilde{V}'(\tau)=x^*(\tau)\left\{[A(t)-2\alpha I]^*W^{-1}(t,\alpha)+W^{-1}(t,\alpha)[A(t)-2\alpha I\right\}x(\tau)+4\alpha\tilde{V}(\tau)+2x^*(\tau)W^{-1}(t,\alpha)B(t)B^*(t)W^{-1}(t,\alpha)x(\tau)$$

$$\geqslant 4\alpha\tilde{V}(\tau) \qquad \text{hence} \quad \tilde{V}(\tau)\geqslant e^{4\alpha\tau}\tilde{V}(0)$$

Since $k_m I\leqslant W^{-1}(t,\alpha)\leqslant k_M I$ we deduce that

$$|x(\tau)|\geqslant\sqrt{\frac{k_m}{k_M}}\,e^{2\alpha\tau}|x(0)|,\quad \tau\geqslant 0$$

and such estimate proves the lemma.

Consider now the new system

$$x_1'=A_{11}(t)x_1+[A_{12}(t)+B_1(t)F_2(t)]x_2+B_1(t)u \qquad (5)$$

$$\varepsilon x_2'=A_{21}(t)x_1+[A_{22}(t)+B_2(t)F_2(t)]x_2+B_2(t)u$$

Uniform controllability of this system is equivalent to uniform controllability for (1).

We have now to check that (5) satisfies all assumptions in section 3 to get the final result.

The smoothness properties are obviously satisfied.

By using a reasoning in (L.M.Silverman, B.D.O.Anderson 1968), we show that there exist $\delta_2>0$, $\hat{k}_2>0$ such that

$$\int_0^{\delta_2} e^{-(A_{22}(t)+B_2(t)F_2(t))s}B_2(t)B_2^*(t)e^{-(A_{22}(t)+B_2(t)F_2(t))^*s}\,ds\geqslant\hat{k}_2 I.$$

Assume this is not true; then for every $\mu>0$ there exist $x_2^\mu(t)$, $|x_2^\mu(t)|=1$ such that

$$\int_0^{\delta_2} \left| (x_2^\mu(t))^* e^{-(A_{22}(t)+B_2(t)F_2(t))s} B_2(t) \right|^2 ds < \mu$$

Denote $z_2^\mu(t)=e^{[A_{22}(t)+B_2(t)F_2(t)]\delta_2} x_2^\mu(t)$.

Condider now the control

$$u^\mu(s)=B_2^*(t)e^{A_{22}^*(t)(\delta_2 -s)} \left(\int_0^{\delta_2} e^{A_{22}(t)(\delta_2-\sigma)} B_2(t)B_2^*(t)e^{A_{22}^*(t)(\delta_2-\sigma)} d\sigma \right)^{-1} z_2^\mu$$

and use it in the system

$$x_2'(s)=A_{22}(t)x_2(s)+B_2(t)u^\mu(s)$$

If we take $x_2(0)=0$ we shall have

$$x_2(\delta_2)= \int_0^{\delta_2} e^{A_{22}(\delta_2-s)} B_2(t)B_2^*(t)e^{A_{22}^*(\delta_2-s)} ds \cdot$$

$$\cdot \left(\int_0^{\delta_2} e^{A_{22}(t)(\delta_2-\sigma)} B_2(t)B_2^*(t)e^{A_{22}^*(\delta_2-\sigma)} d\sigma \right)^{-1} z_2^\mu(t) = z_2^\mu(t)$$

and for the same $x_2(s)$ as above the control

$v^\mu(s)=u^\mu(s)-F_2(t)x_2(s)$ used in the system

$$x_2'(s)= [A_{22}(t)+B_2(t)F_2(t)] x_2(s)+B_2(t) v^\mu(s)$$

will lead again to $x_2(\delta_2)= z_2^\mu(t)$.

We have also $|v^\mu(s)| \leq \rho(\delta_2)$ for all $s \in [0, \delta_2]$ where the function ρ does not depend upon t, μ.

We deduce that we may write

$$z_2^\mu(t) = \int_0^{\delta_2} e^{[A_{22}(t)+B_2(t)F_2(t)](\delta_2-s)} B_2(t)v^\mu(s)ds$$

hence

$$x_2^\mu(t) = \int_0^{\delta_2} e^{-[A_{22}(t)+B_2(t)F_2(t)]s} B_2(t)v^\mu(s)ds$$

$$1= [x_2^\mu(t)]^* x_2^\mu(t) = \int_0^{\delta_2} [x_2^\mu(t)]^* e^{-[A_{22}(t)+B_2(t)F_2(t)]s} B_2(t)v^\mu(s)ds \leq$$

$$\leq \left(\int_0^{\delta_2} \left| (x_2^\mu(t))^* e^{-(A_{22}(t)+B_2(t)F_2(t))s} B_2(t)\right|^2 ds \right)^{1/2} \left(\int_0^{\delta_2} |v^\mu(s)|^2 ds \right)^{1/2} \leq$$

$$\leq (\mu\delta_2)^{1/2} \rho(\delta_2) \ ;$$

since μ is arbitrary we have obtained a contradiction.
We have now to check the uniform controllability for the couple

$$\Big(A_{11}(t)-\left[A_{12}(t)+B_1(t)F_2(t)\right]\left[A_{22}(t)+B_2(t)F_2(t)\right]^{-1}A_{21}(t)\ ,$$

$$B_1(t)-\left[A_{12}(t)+B_1(t)F_2(t)\right]\left[A_{22}(t)+B_2(t)F_2(t)\right]^{-1}B_2(t)\Big)$$

A direct calculation shows that this couple may be written as $(\tilde{A}(t)+\tilde{B}(t)\tilde{F}(t)\ ,\tilde{B}(t)(I+F_2(t)A_{22}^{-1}(t)B_2(t))^{-1}\ )$ (we have $(I+F_2A_{22}^{-1}B_2)^{-1}=I-F_2(A_{22}+B_2F_2)^{-1}B_2)$.
This last couple id obtained from $(\tilde{A},\ \tilde{B})$ by a feedback transformation and a change of coordinates in the control space , not influencing the uniform controllability .
This ends the proof.

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ASYMPTOTIC EXPANSIONS OF THE BOUNDED SOLUTIONS OF RICCATI EQUATIONS ASSOCIATED TO SYSTEMS WITH TWO TIME SCALES

by Vasile Drăgan and Aristide Halanay

Consider a control system with two time scales. In previous papers we have studied asymptotic expansions for associated Liapunov equations as well as conditions for uniform controllability and uniform observability in terms of the corresponding "slow " and "fast" systems. In this paper we discuss the asymptotic structure of the bounded on $\mathbb{R}$ stabilizing solutions for the associated Riccati eqations.

As an application we deduce the asymptotic structure of the corresponding invariants introduced by Jonckhere and Silverman 1983 .

1. The problem . Main result.

Consider the system

$$
\begin{aligned}
x_1' &= A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u \\
\varepsilon x_2' &= A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u \qquad (1) \\
y &= C_1(t)x_1 + C_2(t)x_2
\end{aligned}
$$

Denote $A(t,\varepsilon) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\varepsilon}A_{21}(t) & \frac{1}{\varepsilon}A_{22}(t) \end{pmatrix}$ $B(t,\varepsilon) = \begin{pmatrix} B_1(t) \\ \frac{1}{\varepsilon}B_2(t) \end{pmatrix}$

$C(t) = (\ C_1(t) \quad C_2(t)\ )$

Associate the Riccati equation

$$W' + A^*(t,\varepsilon)W + WA(t,\varepsilon) - WB(t,\varepsilon)B^*(t,\varepsilon) + C^*(t)C(t) = 0 \qquad (2)$$

Facultatea de Matematică, Universitatea din Bucureşti -str. Academiei 14, R-70109 , Romania

Under suitable conditions for the "slow" system defined by $(\tilde{A},\tilde{B},\tilde{C})$

$$\tilde{A}(t)=A_{11}(t)-A_{12}(t)A_{22}^{-1}(t)A_{21}(t)$$
$$\tilde{B}(t)=B_1(t)-A_{12}(t)A_{22}^{-1}(t)B_2(t)$$
$$\tilde{C}(t)=C_1(t)-C_2(t)A_{22}^{-1}(t)A_{21}(t)$$

and for the "fast" system defined by (A_{22},B_2,C_2), the triple $(A(t,\varepsilon),B(t,\varepsilon),C(t))$ is uniformly controllable and uniformly observable and the Riccati equation has a unique ,bounded on $\mathbb{R}$ pozitive definite stabilizing solution $W^+(t,\varepsilon)$.If we write

$$W^+(t,\varepsilon)=\begin{pmatrix} W_{11}^+(t,\varepsilon) & W_{12}^+(t,\varepsilon) \\ (W_{12}^+(t,\varepsilon))^* & W_{22}^+(t,\varepsilon) \end{pmatrix},$$

We shall prove that

$$W_{11}^+(t,\varepsilon)=\tilde{P}_{11}^+(t)+\varepsilon\check{P}_{11}^+(t,\varepsilon)$$
$$W_{12}^+(t,\varepsilon)=\varepsilon\tilde{P}_{12}^+(t,\varepsilon)+\varepsilon^2\check{P}_{12}^+(t,\varepsilon)$$
$$W_{22}^+(t,\varepsilon)=\varepsilon\tilde{P}_{22}^+(t,\varepsilon)+\varepsilon^2\check{P}_{22}^+(t,\varepsilon)$$

where

$$|\check{P}_{ij}^+(t,\varepsilon)|\leq c \quad \text{for all } t\in R \text{ and } \varepsilon>0 \text{ sufficiently small.}$$

Similar expansions hold for the negative definite,antistabilizing solution $W^-(t,\varepsilon)$ and from here we deduce the asymptotic structure for the eigenvalues of the matrix $-[W^-(t,\varepsilon)]^{-1}W^+(t,\varepsilon)$; these eigenvalues are invariant with respect to Liapunov transformations in the state space and define the invariants introduced by Jonckheere and Silverman 1983 .

The conclusion is that these invariants split into two sets, one of them close to the invariants associated to the "slow" system and the second one close to the invariants associated to the "fast" system.

2. Parametrized algebraic Riccati equation

Theorem 1. Let A,B,C be continuous and bounded on $\mathbb{R}$.Assume there exist $\delta>0$, $K>0$ such that for all $t\in\mathbb{R}$

$$\int_0^\delta e^{A(t)s}B(t)B^*(t)e^{A^*(t)s}\,ds \geq KI$$

$$\int_0^\delta e^{A^*(t)s}C_2^*(t)C_2(t)e^{A(t)s}\,ds \geq KI$$

Then the equation

$$A^*(t)P+PA(t)-PB(t)B^*(t)P+C^*(t)C(t)=0$$

has a unique solution $\hat{P}(t)$ such that $\rho_m I \leq \hat{P}(t) \leq \rho_M I$, $\rho_m > 0$ and the real parts of the eigenvalues of the matrix $A(t)-B(t)B^*(t)\hat{P}(t)$ are less than -2α, $\alpha>0$.

Proof. Denote $\hat{A}(t)=A(t)-B(t)B^*(t)\hat{P}(t)$; existence and uniqueness of $\hat{P}(t)$ are standard ; we have to prove only the uniform with respect to t positivity and boundedness, and the properties of the eigenvalues of $\hat{A}(t)$.

a) We have the standard representation formula

$$\hat{P}(t)=\int_0^\infty e^{\hat{A}^*(t)s}[C^*(t)C(t)+\hat{P}(t)B(t)B^*(t)\hat{P}(t)]e^{\hat{A}(t)s}ds.$$

Assume there is no $\rho>0$ such that for all $t\in\mathbb{R}$ $\hat{P}(t)\geq\rho I$; there for every $\rho>0$ there exists $t_\rho\in\mathbb{R}$ and x_ρ with $|x_\rho|=1$ such that $x_\rho^*\hat{P}(t_\rho)x_\rho<\rho$. From the representation formula it would follow that

$$\int_0^\infty |C(t_\rho)e^{\hat{A}(t_\rho)s}x_\rho|^2ds+\int_0^\infty|B^*(t_\rho)\hat{P}(t_\rho)e^{\hat{A}(t_\rho)s}x_\rho|^2ds<\rho$$

Since

$$e^{\hat{A}(t)s}=e^{A(t)s}-\int_0^s e^{A(t)(s-\sigma)}B(t)B^*(t)\hat{P}(t)e^{\hat{A}(t)\sigma}d\sigma$$

We deduce that

$$\int_0^\infty|C(t_\rho)e^{\hat{A}(t_\rho)s}x_\rho|^2ds\geq\int_0^\delta|C(t_\rho)e^{A(t_\rho)s}x_\rho|^2ds-$$

$$-x_\rho^*\int_0^\delta e^{A(t_\rho)s}C^*(t_\rho)C(t_\rho)\int_0^s e^{A(t_\rho)(s-\sigma)}B(t_\rho)B^*(t_\rho)\hat{P}(t_\rho)e^{\hat{A}(t_\rho)\sigma}d\sigma x_\rho\,ds-$$

$$-x_\rho^*\int_0^\delta\left(\int_0^s e^{A(t_\rho)(s-\sigma)}B(t_\rho)B^*(t_\rho)\hat{P}(t_\rho)e^{\hat{A}(t)\sigma}d\sigma\right)^*C^*(t_\rho)C(t_\rho)e^{A(t_\rho)s}ds\,x_\rho$$

By using the Cauchy-Schwarz inequality we have the estimate

$$\left|x_\rho^*\int_0^\delta e^{A(t_\rho)s}C^*(t_\rho)C(t_\rho)\left(\int_0^s e^{A(t_\rho)(s-\sigma)}B(t_\rho)B^*(t_\rho)\hat{P}(t_\rho)e^{\hat{A}(t)\sigma}d\sigma\right)ds\,x_\rho\right|\leq$$

$$\leq\left(\int_0^\delta\left|x_\rho^*\int_\sigma^\delta e^{A(t_\rho)s}C^*(t_\rho)C(t_\rho)e^{A(t_\rho)(s-\sigma)}ds\,B(t_\rho)\right|^2d\sigma\right)^{1/2}\cdot$$

$$\cdot\left(\int_0^\delta|B^*(t_\rho)\hat{P}(t_\rho)e^{\hat{A}(t_\rho)\sigma}x_\rho|^2d\sigma\right)^{1/2}\leq\beta\sqrt{\rho}$$

and then

$$\rho > \int_0^\infty |C(t_\rho)e^{\hat{A}(t_\rho)s} x_\rho|^2 ds \geq \int_0^\delta |C(t_\rho)e^{A(t)s} x_\rho|^2 ds - 2\beta\sqrt{\rho} \geq$$

$$\geq K - 2\beta\sqrt{\rho} \quad , \text{ a contradiction.}$$

The upper estimate for $\hat{P}$ follows from the reprezentation formula

$$\hat{P}^{-1}(t) = \int_0^\infty e^{\hat{A}(t)s} \hat{P}^{-1}(t) C^*(t) C(t) \hat{P}^{-1}(t) e^{\hat{A}^*(t)s} ds +$$

$$+ \int_0^\infty e^{\hat{A}(t)s} B(t)B^*(t) e^{\hat{A}^*(t)s} ds$$

and from $\int_0^\delta e^{\hat{A}(t)s} B(t)B^*(t) e^{\hat{A}^*(t)s} ds \geq \hat{k} I$.

b) Let $\lambda(t)$ an eigenvalue of $\hat{A}(t)$, $x(t)$ with $|x(t)| = 1$, $x^*(t)A^*(t) = \lambda(t)x^*(t)$. We deduce from $\hat{A}^*(t)\hat{P}^{-1}(t) + \hat{P}^{-1}(t)\hat{A}(t) =$ $= -\hat{P}^{-1}(t)C^*(t)C(t)\hat{P}^{-1}(t) - B(t)B^*(t)$ that

$|2\mathrm{Re}\,\lambda(t)|\, x^*(t)\hat{P}^{-1}(t)x(t) \geq x^*(t)B(t)B^*(t)x(t)$

We prove that $\gamma > 0$ exists such that for all $t \in \mathbb{R}$ $x^*(t)B(t)B^*(t)x(t) \geq \gamma$. If for every γ there were a t_j such that $x^*(t_j)B(t_j)B^*(t_j)x(t_j) < \gamma$, then

$$\hat{k} \leq \int_0^\delta x^*(t_j) e^{\hat{A}(t_j)s} B(t_j) B^*(t_j) e^{\hat{A}(t_j)s} x(t_j) ds = x^*(t_j)B(t_j)B^*(t_j)x(t_j) \cdot$$

$$\cdot \int_0^\delta e^{2\mathrm{Re}\,\lambda(t_j)s} ds \leq \gamma\delta$$

a contradiction.

<u>Remark.</u> The solution $\hat{P}(t)$ may be considered locally as being obtained from the implicit function theorem ; uniqueness shows that if A, B, C are C^1 then $\hat{P}$ is C^1. By differentiating the equation with respect to t, it is seen that $P'(t)$ solves a Liapunov equation and we deduce that if A, B, C are C^1 with bounded derivative , the same is true for $\hat{P}$.

3. Singularly perturbed Riccati equations

<u>Theorem 2.</u> Assume A, B, C as in Theorem 1 and moreover of class C^1 with bounded on $\mathbb{R}$ derivatives . Consider the equation

$$\varepsilon P' + A^*(t)P + PA(t) - PB(t)B^*(t)P + C^*(t)C(t) = 0$$

Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ this equation has a unique stabilizing positive definite bounded on $\mathbb{R}$ solution $\tilde{P}$ such that $|\tilde{P}(t,\varepsilon) - \hat{P}(t)| \leq c\varepsilon$ for all $t \in \mathbb{R}$.

Proof. It is known (see for example Coppel 1975 that $\tilde{P}(t,\varepsilon)=\lim_{T\to\infty} P(t,T,\varepsilon)$ where $P(t,T,\varepsilon)$ is the solution of the same equation with $P(T,T,\varepsilon)=0$.

By repeating the same calculations as in Drăgan and Halanay 1980, we deduce that

$$|P(t,T,\varepsilon)-\hat{P}(t)| \le \varepsilon k_0 + \frac{k_1}{\varepsilon}\int_t^T e^{-2\alpha(s-t)/\varepsilon} |P(s,T,\varepsilon)-\hat{P}(s)|^2 ds$$

Denote $v_T(t,\varepsilon)=\varepsilon k_0 + \frac{k_1}{\varepsilon}\int_t^T e^{-2\alpha(s-t)/\varepsilon} |P(s,T,\varepsilon)-\hat{P}(s)|^2 ds$

A direct calculation shows that

$$\varepsilon \frac{d}{dt} v_T(t,\varepsilon) > \frac{2\alpha}{\varepsilon}(v_T(t,\varepsilon)-\varepsilon k_0) - \frac{k_1}{\varepsilon} v_T^2(t,\varepsilon) \quad , \quad t\le T$$

and that $v_T(t,\varepsilon) \le w_T(t,\varepsilon)$

where

$$\varepsilon \frac{d}{dt} w_T(t,\varepsilon) = 2\alpha w_T(t,\varepsilon) - k_1 w_T^2(t,\varepsilon) - 2\alpha k_0 \varepsilon \quad ; \quad w_T(T,\varepsilon)=\varepsilon k_0 .$$

It is seen that $\lim_{T\to\infty} w_T(t,\varepsilon) = w_0 < 2\varepsilon k_0$

and then $|\tilde{P}(t,\varepsilon)-\hat{P}(t)| \le 2\varepsilon k_0$ for all t.

4, Bounded, positive definite, global stabilizing solution for a Riccati equation

Proposition 1. Let $\tilde{A},\tilde{B},\tilde{C},\tilde{L},\tilde{K}$ be continuous and bounded on $\mathbb{R}$, $\tilde{K}(t) \ge kI$, $(\tilde{A},\tilde{B})$ uniformly controllable. Then the Riccati equation

$$P' + \tilde{A}^*(t)P + P\tilde{A}(t) - [P\tilde{B}(t)+\tilde{L}(t)]\tilde{K}^{-1}(t)[\tilde{B}^*(t)P + \tilde{L}^*(t)] + \tilde{C}^*(t)\tilde{C}(t) = 0$$

has a unique solution, defined on $\mathbb{R}$, such that for all $t\in\mathbb{R}$

$\tilde{\rho}_m I \le \tilde{P}(t) \le \tilde{\rho}_M I$, $\tilde{\rho}_M > 0$ and with

$$\tilde{F}(t) = \tilde{K}^{-1}(t)[\tilde{B}^*(t)\tilde{P}(t) + \tilde{L}^*(t)]$$

the matrix $\tilde{A}-\tilde{B}\tilde{F}$ defines an exponentially stable evolution.

Proof. Remark that uniform controllability of $(\tilde{A},\tilde{B})$ implies uniform controllability for $(\tilde{A}-\tilde{B}\tilde{K}^{-1}\tilde{L}^*, \tilde{B})$. The Riccati equation may be written as

$$P' + [\tilde{A}(t)-\tilde{B}(t)\tilde{K}^{-1}(t)\tilde{L}^*(t)]^*P + P[\tilde{A}(t)-\tilde{B}(t)\tilde{K}^{-1}(t)\tilde{L}^*(t)] -$$

$$-P\tilde{B}(t)\tilde{K}^{-1}(t)\tilde{B}^*(t)P + \tilde{C}^*(t)\tilde{C}(t) - \tilde{L}(t)\tilde{K}^{-1}(t)\tilde{L}^*(t) = 0$$

and the conclusion follows from well-known facts.

5. Proof of the main result

We start by proving a useful result

Lemma 1. Assume A_{ij}, B_i, C_i are uniformly Lipschitz and bounded on R and moreover

a) There exist $\delta_2>0$, $k_2>0$ such that

$$\int_0^{\delta_2} e^{A_{22}(t)s} B_2(t)B_2^*(t) e^{A_{22}^*(t)s} ds \geq k_2 I$$

$$\int_0^{\delta_2} e^{A_{22}^*(t)s} B_2^*(t)C_2(t) e^{A_{22}(t)s} ds \geq k_2 I$$

b) $(\tilde{A},\tilde{B},\tilde{C})$ as defined in Section 1 is uniformlly controllable and uniformlly observable.

Then there exist $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ the Riccati equation (2) has a unique bounded solution $W^+(\cdot,\varepsilon)$ which is positive definite and stabilizing. Moreover $W^+(t,\varepsilon)\leq c$ for all $t\in\mathbb{R}$ with c not depending upon ε.

Proof. We use results in Drăgan & Halanay 1990 to see that a) and b) imply uniform controllability and uniform observability for $(A(\cdot,\varepsilon),B(\cdot,\varepsilon),C(\cdot,\varepsilon))$ hence existence of the solution $W^+(\cdot,\varepsilon)$. To see that W^+ is bounded by a constant not depending on ε we have to remark that there exist F_1, F_2 not depending on ε such that ~~there exist~~ with $F=(F_1,F_2)$ $A(\cdot,\varepsilon)+B(\cdot,\varepsilon)F(\cdot)$ defines an exponentially stable evolution. (see O'Reilly 1980) Since W^+ is the optimal value of the cost for the linear-quadratic problem it is majorated by the cost corresponding to the feedback control

$F(t)=F_1(t)x_1(t)+F_2(t)x_2(t)$ and this cost is easily estimated by

$$x_0^* V^+(t,\varepsilon)x_0=\int_t^\infty \Gamma^*(s,t,\varepsilon)[C^*(s)C(s)+F^*(s)F(s)]\Gamma(s,t,\varepsilon)ds$$

where $\Gamma(s,t,\varepsilon)$ is the evolution operator associated to $A(\cdot,\varepsilon)+B(\cdot,\varepsilon)F(\cdot)$.

The proof of the main result proceeds by writing the equations for W_{11}^+, W_{12}^+, W_{22}^+ then by denoting $P_{11}=W_{11}^+$, $P_{12}=\frac{1}{\varepsilon}W_{12}^+$, $P_{22}=\frac{1}{\varepsilon}W_{22}^+$ and obtaining the corresponding equations for P_{ij}.

Under the assumption in Lemma 1 the functions (A_{22},B_2,C_2) are as in Theorem 1 and there exists $\widehat{P}_{22}$ with the corresponding properties ; we use next Theorem 2 to get $\widetilde{P}_{22}$ with corresponding properties . We use proposition 1 with

$$\widetilde{K}(t) = I + B_2^*(t)\,[A_{22}^*(t)]^{-1}C_2^*(t)\,C_2(t)\,A_{22}^{-1}(t)\,B_2(t)$$

$$\widetilde{L}(t) = -\widetilde{C}^*(t)\,C_2(t)\,A_{22}^{-1}(t)\,B_2(t)$$

to obtain $\widetilde{P}_{11}(t)$ and define $\widetilde{P}_{12}$ as the unique bounded on $\mathbb{R}$ solution for the singulatly perturbed linear system

$$\varepsilon P_{12}' = -P_{12}\,[A_{22}(t) - B_2(t)B_2^*(t)\widetilde{P}_{22}(t,\varepsilon)] -$$

$$- A_{21}^*(t)\,\widetilde{P}_{22}(t,\varepsilon) - \widetilde{P}_{11}(t)\,[A_{12}(t) - B_1(t)B_2^*(t)\widetilde{P}_{22}(t,\varepsilon)]$$

It is in this way that we have described the principal parts in the asymptotic expansions.
With the notation in Section 1 we write down complicated equations for $\check{P}_{ij}$; after long computations we obtain for the corresponding bounded on $\mathbb{R}$ solutions representation formulae describing $\check{P}_{ij}$ as solutions for certain integral equations, more precisely as fixed points for certain integral operators.
By using the Theorem of Schauder-Tichonov such fixed points are shown to exist in a given ball and uniqueness arguments show that $\check{P}_{ij}$ are bounded by constants not depending upon ε .
The arguments are completely similar to the ones in Drăgan 1983.

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