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by

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0. INTRODUCTION

In the articles [7] , [8] there have been introduced and studied some preliminary problems (including Riesz and Doob-Meyer decompositions) concerning latticial stochastic processes. In this paper we shall treat the convergence and the Itô representation problems. Let's recall the facts: consider $\leq$ an order relation and $X=(X,+,\cdot,\leq)$ a vector lattice i.e. a linear space over R which is a lattice and the operations are compatible; X is called (countable) complete if any bounded (and countable) subset of X admits $\bigvee$ and $\bigwedge$. If A is a subset of X, denote $A^{*} = \{x\in X;\ |x|\wedge|a|=0 \text{ for all } a\in A\}$ and $A^{**}=(A^{*})^{*}$. We also consider a filtration $\mathfrak{Z}=(\mathfrak{Z}_t)_{t\geq 0}$ that is an increasing family of components (see [7]) that generates a family of linear projectors $(E_t)_{t\geq 0}$ which are bounded ($Y\subset[a,b]\Rightarrow E_t(Y)\subset[-a_-,b_+]$), positive hence increasing, latticial ($E_t(\bigvee_{j\in J} x_j)=\bigvee_{j\in J} E_t(x_j)$ if the $\bigvee$ exists) and continuous $E_t((o)\text{-}\lim_{j\in J} x_j)=(o)\text{-}\lim_{j\in J} E_t(x_j)$; see [7] for order limit). Note that, if X is complete, then there exists $\bigvee_{t\geq 0} E_t$ because the operators are increasing and bounded by the identity.

1. THE CONVERGENCE PROBLEM

Denote $\mathfrak{Z}_\infty=(\bigcup_{t\geq 0}\mathfrak{Z}_t)^{**}$; of course $\mathfrak{Z}_\infty$ is a component and it is easy to prove that $\mathfrak{Z}_\infty$ is the smallest component that contains each $\mathcal{C}_t$ .Remark that every increasing family $(\mathcal{C}_t)_{t\geq 0}, \mathcal{C}_t \subset X$ generates a filtration namely $\mathcal{C}=(\mathcal{C}_t^{**})_{t\geq 0}$, so there are a lot of filtrations on a given vector lattice. For such a filtration one can think about $\mathcal{C}_\infty$ as the generator's domain for $\mathcal{C}$ in the sense of [6] . More precisely: an operator A defined on $\mathcal{D}(A)\subset X$ is called the generator of the filtration $\mathcal{C}$ if $\mathcal{D}(A)$ is the largest subset with the following property: whatever $x\in\mathcal{D}(A)$ is, then there exists $Ax=y$ such that $E_\infty(y)=x$. We have denoted E_∞ the projection on $\mathcal{C}_\infty$. It is easy to see that, in our case $\mathcal{D}(A)=\mathcal{C}_\infty$ and A is the identity on $\mathcal{C}_\infty$. The theory can't go on following [6] because on X we do not have a multiplication, so the problem of the largest algebra contained in $\mathcal{C}_\infty$ is meaningless.

Now we shall prove

LEMMA 1.1. Let be X complete vector lattice. For every $x\in X$ we have $(o)\text{-}\lim_{t\geq 0} E_t(x)=E_\infty(x)$.

Proof. Remark that $\mathcal{C}_t \subseteq \bigcup_{t\geq 0}\mathcal{C}_t \subseteq \mathcal{C}_\infty$ so, for $x\in X_+$ we have $E_t(x)\leq E_\infty(x)$ for all $t\geq 0$. In particular $E_tE_\infty=E_\infty E_t=E_t$ .Suppose that there exists a bound y smaller then $E_\infty(x)$ i.e. $E_t(x)\leq y\leq E_\infty(x)$; remark that $y\geq 0$. Apply E_t and obtain $E_t(x)\leq E_t(y)\leq E_t(x) \Rightarrow E_t(x)=E_t(y)$ $\Rightarrow y-x\in\mathcal{C}_t^*$ for all $t\geq 0 \Rightarrow y-x\in\bigcap_{t\geq 0}\mathcal{C}_t^* = (\bigcup_{t\geq 0}\mathcal{C}_t)^*$.

So $E_\infty(y-x)=0 \Rightarrow E_\infty(y)=E_\infty(x)$. Hence $y\leq E_\infty(y)$. But $y\geq E_\infty(y)$ because $y\geq 0$, so $y=E_\infty(y)=E_\infty(x)$. Thus $\bigvee_{t\geq 0} E_t(x)=E_\infty(x)$. For any $x\in X$ we have $E_t(x)=E_t(x_+)-E_t(x_-)\longrightarrow E_\infty(x_+)-E_\infty(x_-)=E_\infty(x)$ and the proof is complete.

COROLLARY 1.2. We have $(o)\text{-}\lim_{t\geq 0} E_t(x)=x \Longleftrightarrow x\in(\bigcup_{t\geq 0}\mathcal{C}_t)^{**}$

Indeed, both statements are equivalent to $E_\infty(x)=x$.

We also have a dual problem. Let be $(\mathcal{C}_t)_{t\geq 0}$ a decreasing family of components. An arbitrary intersection of components is also a component, hence $\bigcap_{t\geq 0}\mathcal{C}_t$ is; denote it by $\mathcal{C}_{-\infty}$. Remark that $(\mathcal{C}_t^*)_{t\geq 0}$ is a filtration(increasing) so $E_t^*(x)$ converges (see

lemma 1.1.) to the projection of x on $\left(\bigcup_{t\geq 0}\mathcal{B}_t^*\right)^{**}=\left(\bigcap_{t\geq 0}\mathcal{B}_t\right)^{***}=\bigcap_{t\geq 0}\mathcal{B}_t^*=\mathcal{B}_{-\infty}^*$.
Hence $(o)\text{-}\lim_{t\geq 0} E_t(x)=x-(o)\text{-}\lim_{t\geq 0} E_t^*(x)=x-E_{-\infty}^*=E_{-\infty}(x)$.

A stochastic process $(m_t)_{t\geq 0}$ is a sequence of elements from X indexed by $[0,+\infty)$. The stochastic process $(m_t)_{t\geq 0}$ is called $\mathcal{B}$-adapted if $m_t\in\mathcal{B}_t$ for $t\geq 0$. Remark that, if an adapted process is order convergent, then $(o)\text{-}\lim_{t\geq 0} m_t\in\mathcal{B}_\infty$ because
$E_\infty((o)\text{-}\lim_{t\geq 0} m_t)=(o)\text{-}\lim_{t\geq 0} E_\infty(m_t)=(o)\text{-}\lim_{t\geq 0} m_t$.

The stochastic process $(m_t)_{t\geq 0}$ is called $\mathcal{B}$-martingale if $m_s=E_s(m_t)$ for $0\leq s\leq t$. It follows that any martingale is adapted. For example, the process $(E_t(x))_{t\geq 0}$ with $x\in X$ is a martingale (because $\mathcal{B}_t$ increases with t) which we shall call basic martingale; they play the role of the uniformly integrable martingales in classical theory (see [1]). We shall prove

PROPOSITION 1.3. Let be X complete vector lattice and $(m_t)_{t\geq 0}$ a martingale. The following statements are equivalent

i) $(m_t)_{t\geq 0}$ is order convergent

ii) $(m_t)_{t\geq 0}$ is bounded

iii) there exists $x\in X$ such that $m_t=E_t(x)$

In this case $(o)\text{-}\lim_{t\geq 0} m_t=E_\infty(x)$.

Proof. i) $\Rightarrow$ ii) is obvious and iii) $\Rightarrow$ i) follows from lemma 1.1. Now let's see ii) $\Rightarrow$ iii). We know that there exists $a,b\in X$ such that $a\leq m_t\leq b$ for $t\geq 0$. In particular $E_t(a)\leq m_t$; denote $c_t:=m_t-E_t(a)$ and apply E_s $(s\leq t)$, so $E_s(c_t)=m_s-E_s(a)=c_s$ $\Rightarrow$ $(c_t)_{t\geq 0}$ is a positive martingale, hence increasing(see [7]).
In the same time $c_t\leq b-E_t(a)\leq b+E_t(a_-)\leq b+a_-$ so $(c_t)_{t\geq 0}$ is bounded and, as X is complete, it follows that there exists $c:=\bigvee_{t\geq 0} c_t=(o)\text{-}\lim_{t\geq 0} c_t$; hence $m_t=E_t(a)+c_t$ is order convergent. Let be s fixed and $s<t$; we obtain $m_s=E_s(m_t)\Rightarrow m_s=(o)\text{-}\lim_{t\geq s} E_s(m_t)=$
$=(o)\text{-}\lim_{t\geq 0} E_s(m_t)=E_s((o)\text{-}\lim_{t\geq 0} m_t)=E_s((o)\text{-}\lim_{t\geq 0} E_t(a)+(o)\text{-}\lim_{t\geq 0} c_t)=E_s(E_\infty(a)+c)=E_s(a+c$
Take $x=a+c$ and finish the proof.

If the family $(\mathcal{B}_t)_{t\geq o}$ decreases, one can define the "reversed martingales" (see [2]). A reversed martingale is a process $(m_t)_{t\geq o}$ such that, for $o\leq s\leq t$ we have $E_t(m_s)=m_t$. Remark that $(m_t)_{t\geq o}$ is adapted and $m_t=E_t(m_o)$ for $t\geq o$, hence it is order convergent ($\mathcal{B}_t$ decreases with t). As we have seen above, $(o)\text{-}\lim\limits_{t\geq o} m_t=(o)\text{-}\lim\limits_{t\geq o} E_t(m_o)=E_{-\infty}(m_o)$.

For $x\in X$, the set $\{x\}^{**}$ is a component of X; denote by x^{**} the projection on this subset. If $(x_t)_{t\geq o}$ is increasing, then $(\{x_t\}^{**})_{t\geq o}$ is a filtration; denote the associated projectors by x_t^{**}.

Example. Consider the vector lattice $X=\{f:[o,+\infty)\to R\}$ and fix $h\in X$. Denote $a_t(u)=\max\{t-h(u),o\}$ and consider the filtration $(\{a_t\}^{**})_{t\geq o}$. We obtain $a_t^*(f)=f\chi_{\{h\geq t\}}$ and $a_t^{**}(f)=f\chi_{\{h<t\}}$; so the martingales $(f_t)_{t\geq o}\subset X$ have the form $f_t=f\chi_{\{h<t\}}$ with $f\in X$. Also remark that $-f_-\leq f\leq f_+$ so, in this case all martingales are basic and $(o)\text{-}\lim\limits_{t\geq o} f_t=f$ i.e. $(\bigcup\limits_t a_t^{**})^{**}=X$.

Let's give an example of a martingale that is not basic. Consider X complete vector lattice, $(\mathcal{B}_t)_{t\geq o}$ a filtration; let be $t_o\geq o$ the smallest index such that $\mathcal{B}_{t_o}\neq\{o\}$ and $o\neq x\in\mathcal{B}_{t_o}$. Define the martingale $m_t=t\cdot\chi_{[t_o,+\infty)}(t)x$. Suppose that m_t is bounded: $a\leq m_t\leq b\Rightarrow tx\leq b$ for $t\geq t_o\Rightarrow x\leq\frac{1}{t}b$; as X is archimedian, we obtain $x\leq\bigwedge\limits_{t\geq t_o}\frac{1}{t}b=o$. From $a\leq m_t\Rightarrow a\leq tx$ for $t\geq t_o$ hence $o\leq x$. That is x=o-contradiction-

A stochastic process $(x_t)_{t\geq o}$ is called sub(super)martingale if it is adapted and $x_s\leq E_s(x_t)$ (resp. $x_s\geq E_s(x_t)$) for $o\leq s\leq t$.

We say that an adapted process $(x_t)_{t\geq o}$ admits a basic Doob-Meyer decomposition if $x_t=E_t(a)+c_t$ for $t\geq o$, where $a\in X$ and $(c_t)_{t\geq o}$ is an increasing process. As we have shown in [8], $(c_t)_{t\geq o}$ is adapted and the decomposition is not unique. We also have the following characterisation result: in a complete vector lattice, the adapted process $(x_t)_{t\geq o}$ admits a basic Doob-Meyer decomposition if and only if $(x_t)_{t\geq o}$ is a bounded below submartingale. From this we deduce

PROPOSITION 1.4. Any bounded submartingale is convergent.

Proof. $(x_t)_{t\geq o}$ is bounded below, so it admits a basic Doob-Meyer decomposition $x_t=E_t(a)+c_t$ for $t\geq o$; but $(x_t)_{t\geq o}$ is also bounded above $x_t\leq b$ so $c_t=x_t-E_t(a)\leq b+a_-$, hence there exists $c:=\bigvee_t c_t=(o)\text{-}\lim_{t\geq o} c_t$ so $\exists\,(o)\text{-}\lim_{t\geq o} x_t=(o)\text{-}\lim_{t\geq o} E_t(a)+(o)\text{-}\lim_{t\geq o} c_t=E_\infty(a)+c.$

The converse of this proposition is obviously true.

Example. For the same concrete lattice and filtration as above, we have that the submartingales $(f_t)_{t\geq o}$ have the form $f_t(u)=o$ if $h(u)\geq t$ and $f_t(u)$ increases in respect to t if $h(u)<t$. Consider a bounded below submartingale $f_t\geq a$ with $a\in X$. Its basic Doob-Meyer decomposition is $f_t=a\chi_{\{h<t\}}+c_t$. The process $(c_t)_{t\geq o}$ is increasing because, for $o\leq s<t$, $c_t(u)-c_s(u)$ equals o if $s<t\leq h(u)$; $f_t(u)-a(u)$ if $s\leq h(u)<t$; $f_t(u)-f_s(u)$ if $h(u)<s<t$. If, in addition, the submartingale $(f_t)_{t\geq o}$ is bounded above, then there exists $\bigvee_{t\geq o} c_t$ and $\exists\,(o)\text{-}\lim_{t\geq o} f_t=a+\bigvee_{t\geq o} c_t$.

A submartingale $(z_t)_{t\geq o}$ is called a potential if, for $t\geq o$, there exists $(o)\text{-}\lim_{p\geq o} E_t(z_p)=o$. In particular $z_t\leq o$. Remark that, if the potential is order convergent, then $(o)\text{-}\lim_{t\geq o} z_t=o$. Indeed, let be $z=(o)\text{-}\lim_{t\geq o} z_t \Rightarrow 0=(o)\text{-}\lim_t \lim_p E_t(z_p)=\lim_t E_t(\lim_p z_p)=\lim_t E_t(z)=E_\infty(z)\Rightarrow z\in\mathcal{B}_\infty^*$. But $(z_t)_{t\geq o}$ is adapted so $z\in\mathcal{B}_\infty$; hence $z=o$. We have

PROPOSITION 1.5. Let be X complete vector lattice. The potential $(z_t)_{t\geq o}$ admits a basic Doob-Meyer decomposition if and only if there exists $(a_t)_{t\geq o}$ adapted, increasing and bounded above process such that $z_t=a_t-E_t(a)$. where $a=\bigvee_{t\geq o} a_t$.

Proof. If $z_t=E_t(b)+a_t$ then $a_t=z_t-E_t(b)\leq -E_t(b)\leq b_-$ so there exists $a:=\bigvee_{t\geq o} a_t$; for fixed t we have $E_t(z_p)=E_t(E_p(b))+E_t(a_p)\Rightarrow$ $(o)\text{-}\lim_{p\geq o} E_t(z_p)=(o)\text{-}\lim_{p\geq t} E_t(E_p(b))+(o)\text{-}\lim_{p\geq o} E_t(a_p)\Rightarrow 0=(o)\text{-}\lim_{p\geq o} E_t(b)+$ $E_t((o)\text{-}\lim_{p\geq o} a_p)\Rightarrow 0=E_t(b)+E_t(a)$ hence $z_t=E_t(-a)+a_t$.

Conversely $z_t=E_t(-a)+a_t$ is a submartingale and $(o)\text{-}\lim_p E_t(z_p) =$
$(o)\text{-}\lim_{p\geqslant t} E_t(E_p(-a)) + (o)\text{-}\lim_{p\geqslant o} E_t(a_p) = (o)\text{-}\lim_p E_t(-a) + E_t((o)\text{-}\lim_p a_p) = E_t(-a) + E_t(a) = o$

For example, in our concrete situation, the processes that are increasing, simple convergent to o, are potentials and admit the Doob-Meyer property.

COROLLARY 1.6. Let be $(z_t)_{t\geqslant o}$ a bounded (above) potential. The following statements are equivalent

i) $(z_t)_{t\geqslant o}$ admits a Doob-Meyer decomposition

ii) $(z_t)_{t\geqslant o}$ is bounded below

iii) $(z_t)_{t\geqslant o}$ is order convergent

In this case $(o)\text{-}\lim_{t\geqslant o} z_t=o$.

Proof. i) $\Rightarrow$ ii) $\Leftrightarrow$ iii) are clear. See ii) $\Rightarrow$ i). $z_t\geqslant a \Rightarrow$
$c_t := z_t - E_t(a) \geqslant o$ and $c_t \geqslant E_s(c_t) = E_s(z_t) - E_s(a) \geqslant z_s - E_s(a) = c_s$,
so $(c_t)_{t\geqslant o}$ is a positive submartingale hence ([8]) increasing.

The potentials that admit a Doob-Meyer decomposition (in wiew of prop.5) play the role of "potentials generated by increasing processes" from the classical theory (see [4]) and, in our case, (by corollary 7) these potentials are the order convergent ones.

2. THE REPRESENTATION PROBLEM

For this purpose we need a stochastic integral. Let's try with a classical one (as it is done is [5]).

Let be X vector lattice, $(\mathcal{G}_t)_{t\geqslant o}$ filtration, $(m_t)_{t\geqslant o}$ adapted process and f: $[o,+\infty) \longrightarrow R$ simple i.e. there exists a division $o=t_o<t_1<\ldots<t_n<t_{n+1}=+\infty$ such that $f(t)=f(t_k)$ for $t_k\leq t<t_{k+1}$.
We define the integral of f in respect to $(m_t)_{t\geqslant o}$ by $\int_0^t f(s)\,dm_s =$
$\sum_{i=0}^{k-1} f(t_i)(m_{t_{i+1}}-m_{t_i}) + f(t_k)(m_t - m_{t_k})$ if $k\geqslant 1$; for k=o, the sum in the right term vanishes. From this definition it follows that the process $(\int_0^t f(s)\,dm_s)_{t\geqslant o}$ is adapted and, if f(t)=c=constant, then

$\int_0^t f(s)\,dm_s = c(m_t - m_0)$.

Remark that, if $(m_t)_{t\geq 0}$ is a martingale and f is simple, then $(\int_0^t f(s)\,dm_s)_{t\geq 0}$ is a martingale, too. Indeed, let be $0\leq s<t$ and $0\leq j\leq k$ such that $t_j\leq s<t_{j+1}$. We have $E_s(\int_0^t f(u)\,dm_u) =$
$\sum_{i=0}^{k-1} E_s(f(t_i)(m_{t_{i+1}}-m_{t_i})) + E_s(f(t_k)(m_t-m_{t_k})) = \sum_{i=0}^{j-1}(\dots) + E_s(f(t_j)(m_{t_{j+1}}-m_{t_j})) +$
$\sum_{i=j+1}^{k-1}(\dots) + E_s(f(t_k)(m_t-m_{t_k})) = \sum_{i=0}^{j-1}(\dots) + f(t_j)(m_s-m_{t_j}) = \int_0^s f(u)\,dm_u$.

Now extend the above definition in the next situation: X= countable complete, $f\in \mathcal{J}$ where $\mathcal{J}=\{f:[0,+\infty)\to R;\ \forall t\geq 0\ \exists (f_n)_n$ simple s.t. $\lim_n f_n(s)=f(s)$ on $[0,t]\}$. Obviously, if f is Borel measurable, then $f\in \mathcal{J}$
Let be $t\geq 0$ fixed; for $l\in N$, as $(f_n)_n$ is Cauchy, $\exists l_0\in N$ such that $|f_m(t)-f_n(t)|\leq \frac{1}{l}$ for $m,n\geq l_0$. Let be $m,n\geq l_0$. Put the division points for f_m, f_n in a single division $(t_k)_k$. If $t_k\leq t<t_{k+1}$ we obtain
$|\int_0^t f_m(u)\,dm_u - \int_0^t f_n(u)\,dm_u| = |\sum_{i=0}^{k-1}(f_m(t_i)-f_n(t_i))(m_{t_{i+1}}-m_{t_i}) + (f_m(t_k)-f_n(t_k))\cdot$
$(m_t-m_{t_k}) \leq \frac{1}{l}(\sum_{i=0}^{k-1}|m_{t_{i+1}}-m_{t_i}| + |m_t-m_{t_k}|)$.
The right term goes to 0 because X is archimedian, so the sequence $(\int_0^t f_n(u)\,dm_u)_n$ is order Cauchy, hence $\exists\ (0)\text{-}\lim_{n\in N}\int_0^t f_n(u)\,dm_u$.
Put by definition $\int_0^t f(u)\,dm_u = (0)\text{-}\lim_n \int_0^t f_n(u)\,dm_u$. The definition does not depend on $(f_n)_n$ (put any two such sequences in a single one and apply the above considerations).

Remark that if $(m_t)_{t\geq 0}$ is a martingale and $f\in\mathcal{J}$ then $(\int_0^t f(u)\,dm_u)_{t\geq 0}$ is a martingale, too. Indeed, let be $f_n \to f$. Denote $b_{t,n}=\int_0^t f_n(u)\,dm_u$; E_s is order continuous operator so $E_s(\int_0^t f(u)\,dm_u)=$
$E_s((0)\text{-}\lim_n \int_0^t f_n(u)\,dm_u) = E_s((0)\text{-}\lim_n b_{t,n}) = (0)\text{-}\lim_n E_s(b_{t,n}) = (\text{see above}) =$
$(0)\text{-}\lim_n b_{s,n} = (0)\text{-}\lim_n \int_0^s f_n(u)\,dm_u = \int_0^s f(u)\,dm_u$.

Put $\int_s^t = \int_0^t - \int_0^s$ and see that $\int_t^t=0$; $\int_t^s = -\int_s^t$ and, if $(m_t)_{t\geq 0}$ is a martingale, then $E_r(\int_s^t f(u)\,dm_u) = \int_s^{\mathrm{mid}(r,s,t)} f(u)\,dm_u$ for $s\leq t$ where mid(r,s,t) is the middle term of the triple (r,s,t). Indeed, if $r\leq s\leq t \Rightarrow E_r(\int_s^t) = E_r(\int_0^t) - E_r(\int_0^s) = \int_0^r - \int_0^r = 0$; if $s\leq r\leq t \Rightarrow$

$E_r(\int_s^t)=\int_0^r-\int_0^s=\int_s^r$ and if $s\le t\le r \Rightarrow E_r(\int_s^t)=\int_0^t-\int_0^s=\int_s^t$. In particular $E_s(\int_s^t)=0 \Rightarrow \int_s^t\in \mathcal{F}_s^*$ and $E_t(\int_s^t)=\int_s^t \Rightarrow \int_s^t\in\mathcal{F}_t$.

The purpose of this integral is to solve equations like this: for $f\in \mathcal{J}$, find $(m_t)_{t\geq 0}$ such that

(1) $m_t=m_0+\int_0^t f(u)\,dE_u(x)$ with $x\in X$ fixed

i.e. which are the processes that can be represented as stochastic integrals in respect to a basic martingale?

As we have seen above, the process $(m_t)_{t\geq 0}$ must be a martingale. Conversely, suppose first that f is simple. If f=1 on $[t_k,t_{k+1})$ then any $m_t=E_t(x)$ is a solution for (1). If $f\neq 1$ on $[t_k,t_{k+1})$ then $m_t=\frac{1}{1-f(t_k)}\{m_0-f(t_k)E_{t_k}(x)+\sum_{i=0}^{k-1}f(t_i)(E_{t_{i+1}}(x)-E_{t_i}(x))\}$ i.e. $m_t=$ constant on each $[t_k,t_{k+1})$. Now suppose that f is Borel measurable; we can choose a sequence $(f_n)_n$ of simple functions $f_n\longrightarrow f$ such that $f_n\neq 1$, $\forall\, n\in N$. For example take $f_n=-n$ on $\{f<-n\}$; $\frac{k+\sqrt{2}}{2^n}$ on $\{\frac{k}{2^n}\le f<\frac{k+1}{2^n}\}$ for $-n\le\frac{k}{2^n}\le n$; $n+2$ on $\{f\geq n\}$. On each of these partitions m_t=constant, so the solution for (1) is a locally constant basic martingale: $m_t=\sum_k \chi_{[t_k,t_{k+1})}(t)\cdot c_{t_k}\cdot E_{t_k}(x)$. From the martingale property it follows that $(c_t-c_s)E_s(x)=0$ $s\le t$ but this is too restrictive. The unicity also fails because $\int_0^t f(u)\,dE_u(x)=\int_0^t f(u)\,dE_u(x+b)$ for $b\in X$. Finally remark that, if m_t=constant, then x=0 hence $m_t=0$.

So, the problem (1) is not "well posed" and this is because of the classical type integral. In the sequel let's construct an extension of the stochastic integral concept, useful to us.

Consider X=countable complete vector lattice with unity 1 i.e. 1 is total ($1^*=\{0\}$) and $1\geq 0$, a filtration $(\mathcal{F}_t)_{t\geq 0}$, $m_{t,u}\in X$, $t\geq 0$, $u\in R$ such that $t\longmapsto m_{t,u}$ is adapted for all $u\in R$ and $f:R\longrightarrow R$ a function. Consider $\dots<u_{-1}^n<u_0^n=0<u_1^n<\dots$ a sequence of divisions for R and $r_i^n\in[u_i^n,u_{i+1}^n)$ for $i\in Z$. If there exists

$\lim\limits_{\|\Delta^n\|\to 0} \sum\limits_{i\in Z} f(r_i^n)(m_{t,u_{i+1}^n} - m_{t,u_i^n})$ for every r_i^n, then this limit will be the integral of f in respect to $(m_{t,u})$; denote it by $\int f(u)\,dm_{t,u}$

As E_t are order continuous, remark that, if $t \longrightarrow m_{t,u}$ is a martingale for every $u \in R$, then the process $t \longrightarrow \int f(u)\,dm_{t,u}$ is also a martingale for every u because $E_s\left(\lim\limits_{\|\Delta^n\|\to 0} \sum\limits_{i\in Z} f(r_i^n)(m_{t,u_{i+1}^n} - m_{t,u_i^n})\right) = \lim\limits_{\|\Delta^n\|\to 0} \sum\limits_{i\in Z} f(r_i^n)(m_{s,u_{i+1}^n} - m_{s,u_i^n})$.

This kind of integral generalises the first one because if $f: [o,+\infty) \to R$ is continuous and $(m_t)_{t\geq o}$ is adapted (suppose $m_o=o$) then, if we denote $m_{t,u} = \chi_{[o,t]}(u)m_u$ then we obtain $\int f(u)\,dm_{t,u} = \lim\limits_{\|\Delta^n\|\to 0} \sum\limits_{i=1}^{t_n} f(r_i^n)(m_{u_{i+1}^n} - m_{u_i^n})$ where $o = u_o^n < u_1^n < \dots < u_{t_n}^n = t$ are divisions for $[o,t]$. But taking $f_n(u) = f(r_i^n)$ for $u \in [u_i^n, u_{i+1}^n)$ with fixed $r_i^n \in [u_i^n, u_{i+1}^n)$ $i=o,1,\dots,t_n-1$ and, as f is uniformly continuous on $[o,t]$, we have $f_n \longrightarrow f$. So, the last integral equals $\lim\limits_n \int_o^t f_n(u)\,dm_u = \int_o^t f(u)\,dm_u$. Also remark that $\int f(u)\,dm_u = \int f(u)\,d(m_u + x)$ for $x \in X$.

Now, the problem is: given f and $(m_t)_{t\geq o}$ find conditions on f such that the process $(m_t)_{t\geq o}$ admits the latticial representation

(2) $m_t = m_o + \int f(u)\,dm_{t,u}$.

In other words, solve the equation (2) knowing f and $(m_t)_{t\geq o}$, in the case that f is integrable in respect to $(m_{t,u})$.

THEOREM 2.1. Let be X countable complete vector lattice with unity 1, $(m_t)_{t\geq o} \subset X$ stochastic process and $f: R \to R$ monotone, continuous and $\lim\limits_{t\to\pm\infty} f(t) = \pm\infty$. Then there exists $(m_{t,u}) \subset X$, $t \geq o$, $u \in R$ such that f is $m_{t,u}$-integrable and $m_{t,u}$ is a solution for (2).

Proof. We can suppose that $m_o=o$. Let be f increasing and denote $m_{t,u} = (f(u)1 - m_t)_+^{**}(1)$. Remark that $u \to (f(u)\cdot 1 - m_t)_+^{**}$ is increasing and $(f(u)\cdot 1 - m_t)_+^{**}(f(u)\cdot 1 - m_t) = (f(u)\cdot 1 - m_t)_+$ so
$f(u)\cdot(f(u)\cdot 1 - m_t)_+^{**}(1) = (f(u)\cdot 1 - m_t)_+^{**}(m_t) + (f(u)\cdot 1 - m_t)_+ \geq (f(u)\cdot 1 - m_t)_+^{**}(-x_-) + f(u)\cdot 1 - m_t = -(f(u)\cdot 1 - m_t)_+^{**}(x_-) + f(u)\cdot 1 - m_t = -\lim\limits_n (m_t)_- \wedge n(f(u)\cdot 1 - m_t)_+ + f(u)\cdot 1 - m_t \geq -(m_t)_- + f(u)\cdot 1 - m_t = f(u)\cdot 1 - (m_t)_+$. So, for $u > o \Rightarrow f(u) > o$

(by hypothesis) $\Rightarrow (f(u)\cdot 1-m_t)_+^{**}(1) \geq 1-\frac{1}{f(u)}(m_t)_+$. As X is archimedian $\Rightarrow \bigvee_u (f(u)\cdot 1-m_t)_+^{**}(1)\geq 1$; but $(f(u)\cdot 1-m_t)_+^{**}(1)\leq 1$ so the sup is 1. We also have $f(u)\cdot(f(u)1-m_t)_+^{**}(1)=(f(u)\cdot 1-m_t)_+^{**}(m_t)+(f(u)\cdot 1-m_t)_+\geq$ $(f(u)\cdot 1-m_t)_+^{**}(m_t)\geq -(m_t)_-$. So, for $u<0 \Rightarrow f(u)<0$ and $(f(u)\cdot 1-m_t)_+^{**}(1)\leq$ $-\frac{1}{f(u)}(m_t)_- \Rightarrow \bigwedge_u (f(u)\cdot 1-m_t)_+^{**}(1)\leq 0$. But $(f(u)\cdot 1-m_t)_+^{**}(1)\geq 0$ thus the inf equals o[1]. Consider $\ldots<u_{-1}<u_0<u_1<\ldots$ a division for R. We have $\sum_{i=0}^{K}(m_{t,u_{i+1}}-m_{t,u_i})=m_{t,u_{K+1}}-m_{t,0}\uparrow_K 1-m_{t,0}$ and $\sum_{i=-1}^{-K}(m_{t,u_{i+1}}-m_{t,u_i})=$ $m_{t,0}-m_{t,u_{-K}}\searrow_K m_{t,0}$ so $\sum_{i\in Z}(m_{t,u_{i+1}}-m_{t,u_i})=1-m_{t,0}+m_{t,0}=1$. As 1 is total $\Rightarrow 1^{**}(m_t)=m_t$ so, as $\mathcal{T}=E(1)^{**}$ for $\mathcal{T}$ =component, $\sum_{i\in Z}(m^{**}_{t,u_{i+1}}-m^{**}_{t,u_i})(m_t)=m_t \Rightarrow$ (3) $\sum_{i\in Z}(m^{**}_{t,u_{i+1}}(m_t)-m^{**}_{t,u_i}(m_t))=m_t$.

From $f(u_i)\cdot 1-m_t-(f(u_i)\cdot 1-m_t)_+\leq 0 \Rightarrow m^{**}_{t,u_{i+1}}(f(u_i)\cdot 1-m_t)-$ $(f(u_i)\cdot 1-m_t)_+\leq 0 \Rightarrow m^{**}_{t,u_{i+1}}(f(u_i)\cdot 1-m_t)\leq m^{**}_{t,u_{i+1}}(f(u_i)\cdot 1-m_t)\Rightarrow$

(4) $f(u_i)m_{t,u_{i+1}}-f(u_i)m_{t,u_i}\leq m^{**}_{t,u_{i+1}}(m_t)-m^{**}_{t,u_i}(m_t)$

From $m_{t,u_i}\leq m_{t,u_{i+1}} \Rightarrow m^{**}_{t,u_i}(f(u_{i+1})\cdot 1-m_t)\leq m^{**}_{t,u_{i+1}}(f(u_{i+1})\cdot 1-m_t)\Rightarrow$

(5) $f(u_{i+1})\cdot m_{t,u_i}-m^{**}_{t,u_i}(m_t)\leq f(u_{i+1})\cdot m_{t,u_{i+1}}-m^{**}_{t,u_{i+1}}(m_t)$

From (4) and (5) we obtain

$f(u_i)(m_{t,u_{i+1}}-m_{t,u_i})\leq m^{**}_{t,u_{i+1}}(m_t)-m^{**}_{t,u_i}(m_t)\leq f(u_{i+1})(m_{t,u_{i+1}}-m_{t,u_i})$.

In this relation add $-f(r_i)(m_{t,u_{i+1}}-m_{t,u_i})$ with $u_i\leq r_i<u_{i+1}$

We obtain $(f(u_i)-f(r_i))(m_{t,u_{i+1}}-m_{t,u_i})\leq m^{**}_{t,u_{i+1}}(m_t)-m^{**}_{t,u_i}(m_t)-$ $f(r_i)(m_{t,u_{i+1}}-m_{t,u_i})\leq(f(u_{i+1})-f(r_i))(m_{t,u_{i+1}}-m_{t,u_i})$.

Now take $(u_i^n)_{i\in Z}$ a sequence of divisions for R with $\|\Delta^n\|\to 0$. As f is continuous on each $[u_i^n,u_{i+1}^n)$ we have $\forall\varepsilon>0\ \exists\delta(\varepsilon)>0: |u_{i+1}^n-u_i^n|<\delta(\varepsilon)\Rightarrow|f(u_{i+1}^n)-f(u_i^n)|<\varepsilon$; in the same time, as $\|\Delta^n\|\to 0$ we obtain that for $\delta(\varepsilon)>0\ \exists n_0: \max_i|u_{i+1}^n-u_i^n|<\delta(\varepsilon)$ for $n\geq n_0$. Take $\varepsilon=\|\Delta^n\|$ and $n\geq n_0$; as f is increasing, we have $-\|\Delta^n\|\leq f(u_i^n)-f(u_{i+1}^n)\leq f(u_i^n)-f(r_i^n)$ and

1) In this proof, m_t was ocasionally denoted by x and the component generated by an element of X, $\{x\}^{**}$, was denoted as the corresponding projector on this component.

$f(u^n_{i+1}) - f(r^n_i) \le f(u^n_{i+1}) - f(u^n_i) \le \|\Delta^n\|$. So $|m^{**}_{t,u^n_{i+1}}(m_t) - m^{**}_{t,u^n_i}(m_t) - f(r^n_i)(m_{t,u^n_{i+1}} - m_{t,u^n_i})| \le \|\Delta^n\|(m_{t,u^n_{i+1}} - m_{t,u^n_i})$. Sum over $i \in Z$ and obtain (using (3)): $|m_t - \sum_{i\in Z} f(r^n_i)(m_{t,u^n_{i+1}} - m_{t,u^n_i})| \le \|\Delta^n\| \cdot 1$.

Pass to $\lim_n$ and obtain the conclusion.

If f decreases $\Longrightarrow$ $-f$ increases, so the equation $m_t = m_o + \int -f(u)\, dm_{t,u}$ admits the solution $(-f(u)\cdot 1 - m_t)^{**}_+(1)$ hence the equation (2) admits the solution $-(-f(u)\cdot 1 - m_t)^{**}_+(1)$.

For example, if $m_t = f(t)1$ and f is strictly increasing, then the solution will be $m_{t,u} = \chi_{(t,+\infty)}(u)\cdot 1$. The equation (2) becomes in this case $f(t)\cdot 1 = f(o)\cdot 1 + \int f(u)\, d\chi_{(t,+\infty)}(u)\cdot 1 \Longrightarrow f(t)\cdot 1 = f(o)\cdot 1 + \int_{t+}^{t\infty} f(u)\, d1$. Indeed, $m_{t,u} = (f(u)\cdot 1 - f(t)\cdot 1)^{**}_+(1) = ((f(u) - f(t))\cdot 1)^{**}_+(1)$. For $u \le t$ $\Longrightarrow f(u) \le f(t) \Rightarrow (f(u) - f(t))\cdot 1 \le o$ and if $u > t$ take into account that $(cl)^{**} = 1^{**} = X$ for any $c > o$.

In the sequel we shall consider $f(t) = t$ for all $t \geqslant o$ and X countable complete. Even if $(m_t)_{t \geqslant o}$ is a martingale, the family $t \longmapsto (u\cdot 1 - m_t)^{**}_+$ is not a filtration (i.e. for $u \le o$ it increases in respect to t, if $u > o$ it does not). Indeed, for $u \le o$ amd $o \le s \le t$ we have $(u\cdot 1 - m_t)_+ \geqslant o \Rightarrow E_s((u\cdot 1 - m_t)_+) \le (u\cdot 1 - m_t)_+$ so $(u\cdot E_s(1) - E_s(m_t))_+ \le (u\cdot 1 - m_t)_+ \Rightarrow (u\cdot E_s(1) - m_t)_+ \le (u\cdot 1 - m_t)_+$.

But $u \le o \Rightarrow u\cdot 1 \le u\cdot E_s(1) \Rightarrow (u\cdot 1 - m_s)_+ \le (u\cdot E_s(1) - m_s)_+$ For $u > o$ we have only $(u\cdot 1 - m_s)_+ \le (u\cdot 1 - m_t)_+ + u\cdot 1$ Indeed, denote $m = (u\cdot 1 - m_t)_+ \geqslant o$; $m \geqslant u\cdot 1 - m_t \geqslant -m_t$ so $m \geqslant (-m_t) \vee o$ i.e. $m \geqslant (-m_t)_+$. But any positive martingale increases so $(-m_t)_+ \geqslant (-m_s)_+ \Rightarrow m \geqslant (-m_s)_+ \Rightarrow m \geqslant -m_s \Rightarrow m + u\cdot 1 \geqslant u\cdot 1 - m_s \Rightarrow m + u\cdot 1 \geqslant (u\cdot 1 - m_s)_+$ because $m + u\cdot 1 \geqslant o$

So $m_{t,u}$ has the shape of a $(u\cdot 1 - m_t)^{**}_+$-basic martingale (in respect to t) but does not verify the martingale condition $(u\cdot 1 - m_s)^{**}_+(m_{t,u}) \ne m_{s,u} \Longleftrightarrow (u\cdot 1 - m_s)^{**}_+((u\cdot 1 - m_t)^{**}_+(1)) \ne (u\cdot 1 - m_s)^{**}_+(1)$ because $(u\cdot 1 - m_s)^{**}_+ \not\subset (u\cdot 1 - m_t)^{**}_+$

In respect to u, $m_{t,u}$ is a $(u\cdot 1 - m_t)^{**}_+$ -basic martingale because for $u < v$ we have $(u\cdot 1 - m_t)^{**}_+((v\cdot 1 - m_t)^{**}_+(1)) = (u\cdot 1 - m_t)^{**}_+(1)$

and $u \longrightarrow (u\cdot 1 - m_t)_+^{**}$ is a filtration.

Our purpose is to obtain a latticial Itô representation result (see the classical result in [3]) for martingales. This means to represent any $\mathcal{G}$ -martingale as stochastic integral in respect to basic martingales (possible in respect to other filtrations). One can write any process $(m_t)_{t \geq o}$ as the difference of two negative processes $m_t = -(m_t)_- - (-(m_t)_+) = (m_t \wedge o) - (-(m_t)_+) :=$ $a_t - b_t$ for $t \geq o$. Apply theorem 2.1. for $(a_t)_{t \geq o}$ and $(b_t)_{t \geq o}$; we obtain the existence of $\mathcal{A}_{t,u} = (u\cdot 1 - a_t)_+^{**}$, $\mathcal{B}_{t,u} = (u\cdot 1 - a_t)_+^{**}$ such that

(6) $m_t = m_o + \int u\, da_{t,u} - \int u\, db_{t,u}$

where $a_{t,u} = A_{t,u}(1)$ and $b_{t,u} = B_{t,u}(1)$.

The most interesting things are in the case $(m_t)_{t \geq o} = \mathcal{G}$ - martingale: a_t and b_t are negative martingales, hence([7]) they are decreasing $a_s \geq a_t \Rightarrow u\cdot 1 - a_s \leq u\cdot 1 - a_t \Rightarrow \mathcal{A}_{s,u} \subseteq \mathcal{A}_{t,u}$ and also $\mathcal{B}_{s,u} \subseteq \mathcal{B}_{t,u}$ for $o \leq s \leq t$ and any $u \in R$. So $t \longmapsto a_{t,u}$ and $t \longmapsto b_{t,u}$ are basic $\mathcal{A}$ (resp. $\mathcal{B}$)-martingales. So we have proved

PROPOSITION 2.2. Let be X countable complete vector lattice with unity and $\mathcal{G}$ a filtration. Then any $\mathcal{G}$ -martingale admits a latticial representation (6) in respect to $\mathcal{A}$ (resp. $\mathcal{B}$) basic martingales.

Suppose that X is complete. Let us notice that $a_{t,u}$ is bounded by $[o,1]$ and $t \rightarrow a_{t,u}$ increases so, by the martingale convergence theorem 1.3. we have $(o)\text{-}\lim_{t \geq o} a_{t,u} = \bigvee_{t \geq o} a_{t,u} = A_{\infty,u}(1)$ where $A_{\infty,u}$ is the projection on $(\bigcup_t \mathcal{A}_{t,u})^{**} = \bigcup_{t \geq o} (u\cdot 1 - a_t)_+^{**}$. In the same time $E_s(a_{t,u}) = (u E_s(1) - a_s)_+^{**}(1)$ because a_t is $\mathcal{G}$ -martingale. The last term is $\leq (u\cdot 1 - a_s)_+^{**}$ for $u > o$ (resp. it is $\geq, = o$ for $u < o$, $u = o$) hence $t \mapsto a_{t,u}$ is $\mathcal{G}$ -supermartingale (resp. sub martingale; martingale). We saw (prop. 1.4.) that any bounded sub(super)martingale is order convergent. In our case we can indicate its limit, namely $A_{\infty,u}(1)$. Another remark in the case $(m_t)_{t\geq 0}$ =martingale is that $A_{\infty,0}=E_\infty$. Indeed, for u=o we have $E_s(a_{t,u})=a_{s,u}$. By $\lim\limits_t$ we obtain $E_s(A_{\infty,0}(1))=a_{s,0}$. By $\lim\limits_s$ we obtain $A_{\infty,0}(1)=E_\infty(A_{\infty,0}(1))\Rightarrow A_{\infty,0}(1)\in\mathcal{B}_\infty$. But $\mathcal{A}_{\infty,0}=(A_{\infty,0}(1))^{**}\Rightarrow \mathcal{A}_{\infty,0}\subseteq\mathcal{B}_\infty$. Conversely, if $x\in\mathcal{B}_\infty\Rightarrow|x|\wedge|y|=0$ for y such that $|y|\wedge|a|=0\ \forall a\in\bigcup\limits_{t\geq 0}\mathcal{B}_t$. But $\bigcup\limits_t(-a_t)_+^{**}=\bigcup\limits_t(m_t)_-^{**}\subset\bigcup\limits_t\mathcal{B}_t\Rightarrow|x|\wedge|z|=0$. for z such that $|z|\wedge|y|=0\ \forall b\in\mathcal{A}_{\infty,0}$, so $x\in\mathcal{A}_{\infty,0}$.

The things are going wrong with the representation if $(m_t)_{t\geq 0}$ is a sub(super)martingale only: one of the processes a_t, b_t is submartingale (and negative) hence the corresponding filtration is increasing; the other is a submartingale (negative) only.

But, if you look carefully to the proof of prop. 2.2., see that the point was that the process $(m_t)_{t\geq 0}$ could be decomposed into the difference of two decreasing processes. Any such process admits the latticial decomposition (6). For example, the monotone processes and the processes that admits a basic Doob-Meyer decomposition. Indeed, $m_t=E_t(a)+c_t$ with $(c_t)_{t\geq 0}$ increasing can be written as $m_t=-E_t(a_-)-(-E_t(a_+)-c_t)$ i.e. as the difference of two decreasing processes ($E_t(a_\pm)$ are positive martingales hence increasing). So, the bounded below submartingales admit this representation (also see [8]).

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REFERENCES

1. I.Cuculescu, Martingales on von Neumann algebras, J.Multivariate Anal.,vol.1,nr.1,1971
2. S.Goldstein, Norm convergence of martingales in L^p-spaces over von Neumann algebras, Rev.Roumaine Math.Pures Appl.,vol.32,nr.6,1987
3. N.Ikeda,S.Watanabe, Stochastic differential equations and diffusion processes, North Holland/Kodansha,1989
4. P.A.Meyer, Probability and potentials, Blaisdel/Waltham,1966
5. P.A.Meyer, Un cours sur les intégrales stochastiques, Sém. Prob.X,Lect.Notes Math.511,Springer-Verlag,1976
6. Gh.Stoica, On kernels which are projectors, Stud.Cerc.Mat., vol.41,nr.3,1989
7. Gh.Stoica, Martingales in vector lattices, Bull.Math.Soc.Sci. Math.Roumanie,vol.34,nr.4,1990
8. Gh.Stoica, Decompositions for stochastic processes in vector lattices, Bull.Math.Soc.Sci.Math.Roumanie (to appear)