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REMARKS ON POSITIVE BLOCK-MATRICES

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1. INTRODUCTION

In the paper [5], we presented a certain structure of the positive block-matrices which proved to be useful in treating various problems (see [7] for applications to factorization theory, [9] for operator modelling theory or [3], [8] for connections with graph theory).

Motivated by some remarks concerning [7] and [3] which are described in the next section, we are faced with some new computations into the structure described in [5].

These computations are made in Section 3 and 4 and then, we return in Section 5 to some of the raised questions.

2. MOTIVATION

A. We first consider a nonstationary process, i.e. there are given a family $\{\mathcal{H}_n / n \in \mathbb{Z}\}$ of Hilbert spaces, a Hilbert space $\mathcal{K}_o$ and a family $\mathcal{V} = \{V(n)/n \in \mathbb{Z}\}$ such that $V(n)$ acts between $\mathcal{H}_n$ and the evolution space $\mathcal{K}_o$ of the process for every $n \in \mathbb{Z}$.

The <u>covariance kernel</u> $\mathcal{S} = \{S_{ij}/i,j \in \mathbb{Z}\}$ of the process is defined by $S_{ij} = V^*(i)V(j)$ for $i,j \in \mathbb{Z}$. It is obvious that $\mathcal{S}$ is a positive-definite kernel and there is no restriction in supposing $S_{ii} = I_{\mathcal{H}_i}$ for $i \in \mathbb{Z}$. One more assumption which can be made without loss of generality is that $\mathcal{K}_o = \bigvee_{n \in \mathbb{Z}} V(n)\mathcal{H}_n$.

A special interest is paid to the spaces $\mathcal{K}_p^q = \bigvee_{k=p}^{q} V(k)\mathcal{H}_k$, p,q integers or $\pm\infty$, $p < q$, and to various angles between them (for two subspaces $\mathcal{G}$ and $\mathcal{F}$ of a Hilbert space $\mathcal{H}$, the operator angle between $\mathcal{G}$ and $\mathcal{F}$ is defined by $B(\mathcal{G},\mathcal{F}) = P_{\mathcal{G}}^{\mathcal{H}} P_{\mathcal{F}}^{\mathcal{H}} P_{\mathcal{G}}^{\mathcal{H}}$, where $P_{\mathcal{G}}^{\mathcal{H}}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{G}$.)

Thus:

(a) compute the angle between the past $\mathcal{K}_1^{\infty}$ and the present $\mathcal{H}_o$.

(b) compute the angle between finite sections of past and future.

(c) compute the angle between the space $\mathcal{K}_n^{\infty} \vee \mathcal{K}_{-\infty}^{-n}$ and the space $V(p)\mathcal{H}_p$, $|p| < n$, $n > 0$.

Several methods for computing these angles (actually, for the stationary case) are developed, as those in [4], [11], [14], [15], [16], [17].

In the paper [5] a certain structure of the given process $\mathcal{V}$ is indicated. Thus, $\mathcal{V}$ is uniquely determined by a family of contractions $\mathcal{G} = \{G_{ij}/i,j \in \mathbb{Z},\ i \leq j\}$, where $G_{ii} = O_{\mathcal{H}_i}$ for $i \in \mathbb{Z}$ and for $i < j$,

$$G_{ij} \in \mathcal{L}(\mathcal{D}_{G_{i+1,j}}, \mathcal{D}_{G^*_{i,j-1}}) \tag{2.1}$$

(recall that for an operator $T \in \mathcal{L}(\mathcal{H},\mathcal{G})$ which is a contraction, i.e., $\|T\| \leq 1$, $D_T = (T - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T$ is the closure of the range of D_T and $\mathcal{L}(\mathcal{H},\mathcal{G})$ is the set of bounded operators acting between $\mathcal{H}$ and $\mathcal{G}$).

With these elements, it is obtained in([7], Theorem 3.1) the following formula:

$$I_{\mathcal{H}_0} - B(\mathcal{H}_0, \mathcal{K}_1^{\infty}) = \text{s-}\lim_{n\to\infty} (D_{G_{01}^*} \dots D^2_{G_{on}^*} \dots D_{G_{01}^*}) \tag{2.2}$$

and thus we have a solution to the problem (a). A similar result can be obtained in connection with problem (b) and it will be described in the last section of this paper. The necessary computations will be made in Section 4.

But our main interest is about problem (c). The structure of the space $\mathcal{K}_n^{\infty} \vee \mathcal{K}_{-\infty}^{-n}$ suggests that it can be the evolution space of a certain subprocess of the given process. Indeed, if we define the process $\mathcal{V}^{[n]}$ given by the operators:

$$V^{[n]}(m) = \begin{cases} V(-n+m), & m \leq 0 \\ V(n+m-1), & m > 1 \end{cases} \tag{2.3}$$

we remark that $\mathcal{K}_n^{\infty} \vee \mathcal{K}_{-\infty}^{-n} = \mathcal{K}_0^{[n]}$, the evolution space of the process $\mathcal{V}^{[n]}$.

So that, it will be of interest to determine the family $\mathcal{G}^{[n]}$ of parameters associated to $\mathcal{V}^{[n]}$. This will be the question treated in the next section.

B. In [10] it was proved that a banded partial positive matrix has a positive completion. A general result in this direction was obtained in [13].

Consider a partial block-matrix $M = \{S_{ij}/1 \leq i,j \leq N\}$, where S_{ij} are bounded operators acting between the Hilbert spaces $\mathcal{H}_j$ and $\mathcal{H}_i$, in the sense that some of the elements S_{ij} are specified and some of them are not specified. Moreover, the main diagonal is specified (and we can suppose, without loss of generality, that $S_{ii} = I_{\mathcal{H}_i}$) and all the principal block-submatrices formed by specified elements are positive.

With M, an undirected graph $G = (V,E)$. is associated in the following way: $V = \{1,2,\dots,N\}$ and an edge between i and j exists ($i \neq j$) if S_{ij} is specified. G is called the associated graph of M and we say that M is subjacent to G. As a main result in [13] it is proved that a graph G has the property that all the partial block-matrices subjacent to G admit positive completions if and only if G is a chordal gaph. G is chordal if every cycle of length strictly greater then 3 has a chord, i.e. an edge joining two nonconsecutive vertices of the cycle (see [12] for all the terminology on graphs which we use here).

This result in [13] is based on the following property of chordal graphs established by Fulkerson and Gross (see [12]): there exists an ordering $\sigma = [v_1, v_2, \ldots, v_N]$ of the vertices of G such that each set $A_i = \{v_j \in Adj(v_i) / j > i\}$ is a clique. For a vertex $v \in V$, $Adj(v) = \{w \in V/(u,w) \in E\}$ and a subset of V is a clique if the induced graph is complete.

An ordering as above is called a perfect vertex elimination scheme (perfect scheme) of G and it has an useful "visual" transcription. First, a block-banded structure of a (partial) matrix is the specification (after a reordering, if necessary) of a family of index sets $B_k = \{v / m_k < v < n_k\} \subset \{1, \ldots, N\}$, $k = 1, \ldots, p$, $m_k < n_k$, $1 = m_1 < m_2 < \ldots < m_p$ and $1 < n_1 < n_2 < \ldots < n_p = N$. For an index set A, denote by M(A) the principal submatrix of a (partial) matrix corresponding to this index set. The submatrices $M(B_k)$ are the bloks of the specified block-banded structure of M.

A sequence of positive integers:

$$1 < r_1 < r_2 < \ldots < r_s = N$$

is called a completion sequence of G if satisfies the properties:

(α) there exists an ordering $\sigma(r_1)$ of r_1 vertices of V(we denote by $V(r_1)$ the set of these vertices) such that for any partial matrix M subjacent to G, $M(V(r_1))$ has a block-banded structure whose bloks consist of specified elements.

(β) for each k, $1 < k \le r_s$, there exists an ordering $\sigma(r_k)$ of r_k vertices of V ($V(r_k)$ being the set of these vertices) such that:

(i) $V(r_k) \supset V(r_{k-1})$

(ii) for any partial matrix M subjacent to G, $M(V(r_k))$ has a block-banded structure, such that $M(V(r_{k-1}))$ appears as the first block (of course, with a possible different order of the vertices) and the other blocks consist only on specified elements.

2.1. PROPOSITION. G is chordal if and only if it has a completion sequence.

PROOF. One implication was already used in [3] (see also [2]).

Thus, take $\sigma = [v_1, v_2, \dots, v_N]$ a perfect scheme of G, define $C_k = \{v_k, \dots, r_N\}$ and R be the least integer for which C_R is a clique. Then, C_{R-1} is partitioned as:

$$C_{R-1} = \{v_{R-1}\} \cup A_{R-1} \cup D_{R-1}$$

and define $r_1 = N - R + 2$, $V(r_1) = C_{R-1}$ and

$$\sigma(r_1) = [D_{R-1}, A_{R-1}, v_{R-1}]$$

where the order in D_{R-1} and A_{R-1} ia arbitrarely choosen. It is obvious that (α) is satisfied.

Further on, we define for $k = 2, \dots, R-1$, $r_k = r_{k-1} + 1$, $V(r_k) = C_k$ and partitioning C_k as

$$C_k = \{v_k\} \cup A_k \cup D_k,$$

then

$$\sigma(r_k) = [D_k, A_k, v_k]$$

with arbitrarly choosen orders in D_k and A_k. Again, it is obvious that (β) is fulfilled.

Conversely, let $1 < r_1 < r_2 < \dots < r_s = N$ be a completion sequence of G. Let $V(r_s) - V(r_{s-1}) = [w_1, \dots, w_p]$, the ordering being that given by $\sigma(r_s)$. Then, define

$$v_1 = w_p,\ v_2 = w_{p-1}, \dots, v_p = w_1.$$

Continuing in this way for all the sets $V(r_k) - V(r_{k-1})$, $k = S-1, \dots, R$ and $V(r_1)$ alone, we obviously construct a perfect scheme of G, by the property (ii) and (α). ∎

2.2. REMARK. It is known (see [12], for instance) that the graphs for which all the subjacent partial matrices admit block-banded structures are exactly the proper interval graphs. Recall that a graph is called a proper interval graph if its vertices may be identified with a set of intervals on the real line so that an edge (i,j) occurs if and only if interval i and interval j intersect, and no interval is included in another.

The notion of completion sequence permits to introduce classes of graphs intermediating between proper interval graphs and chordal graphs. That is, take G a chordal graph and $r = \{1 < r_1 < ... < r_s = N\}$ a completion sequence of G.

The index $S = S(r)$ is called the length of this completion sequence r. Define

$$N(G) = \min\{S(r) / r \text{ a completion sequence of } G\} \tag{2.4}$$

For a connex chordal graph G, we have the following properties:

(1) $N(G) = 1$ if and only of G is proper interval graph.

(2) $\{N(G) / \text{the cardinality} |V| \text{ of } V \text{ is fixed}\} = \{1,2,...,|V| - 2\}$

(3) $N(K_{1,n}) = n - 1$, where $K_{1,n}$ is the complete bipartite graph with $n + 1$ matrices partitioned into a 1-stable set and an n-stable set.

The connections with the representations of G as an intersection graph of subtrees of a tree will be presented elsewhere. ∎

Based on the existence of a completion sequence it appears the possibility of using the structure of a positive block-matrix established in [5] in order to parametrize all the positive completions of a partial positive (block) matrix subjacent to a chordal graph.

This was already realised in [3] (see also [2]) for partial positive matrices under invertibility assumptions. Of course, in this case, the compatibility relations (2.1) are superfluous and our interest here is to see what happens in the general case. Take, for instance, the graph $G = (V,E)$, with $V = \{1,2,3,4,5,6\}$, $E = \{(1,2), (1,6), (2,3), (2,4), (2,6), (3,4), (4,5), (4,6), (5,6)\}$, then $\sigma = [1,3,5,2,4,6]$ is a perfect scheme and $r = (4,5,6)$ with $\sigma(4) = [2,4,6,5]$, $\sigma(5) = [5,6,2,4,3]$, $\sigma(6) = [3,4,5,2,6,1]$ is a completion sequence.

When passing from the parametrization of the positive completions of $M(V(4))$ to the parametrization of the positive completions of $M(V(5))$ we are faced with the problem of the modification of the parametrs of a positive block-matrix under a permutation of the indices. Further on, when passing from $M(V(5))$ to $M(V(6))$, we remark that we are faced exactly with thesame problem as in the study of the space $\mathcal{K}_n^{\infty} \vee \mathcal{K}_{-\infty}^{-n}$ of a nonstationary process.

3. DELETING ROWS AND COLUMNS

We consider a family of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ and a covariance kernel $\mathcal{S} = \{S_{ij}/i,j\in\mathbb{Z}\}$. This means that the operators:

$$M_{mn}(\mathcal{S}) = M_{mn} : \bigoplus_{k=m}^{n} \mathcal{H}_k \to \bigoplus_{k=m}^{n} \mathcal{H}_k$$

$$M_{mn} = (S_{ij}/m \leq i,j \leq n) \tag{3.1}$$

are positive for $m,n\in\mathbb{Z}$, $m \leq n$. We suppose, without loss of generality, that $S_{ii} = I_{\mathcal{H}_i}$ for $i\in\mathbb{Z}$.

By Theorem 1.3 in [5], $\mathcal{S}$ is uniquely determined by a family of contractions $\mathcal{G}(\mathcal{S}) = \mathcal{G} = \{G_{ij}/i,j\in\mathbb{Z},\ i\leq j\}$, where $G_{ii} = O_{\mathcal{H}_i}$ for $i\in\mathbb{Z}$ and for $i<j$, G_{ij} satisfies (2.1).

Using the remark that T is a contraction if and only if $\begin{bmatrix} I & T \\ T^* & I \end{bmatrix}$ is positive, we obtain from the above result the structure of block-contractions. That is, suppose $T = (T_{ij}/i,j \geq 1)$ is a block-contraction in $\mathcal{L}(\bigoplus_{i=1}^{\infty}\mathcal{H}_i, \bigoplus_{j=1}^{\infty}\mathcal{H}'_j)$. Then, T is uniquely determined by a family of contractions $\mathcal{G}(\begin{bmatrix} I & T \\ T^* & I \end{bmatrix})$. Most of them are zero and we retain only the family of contractions $\mathcal{G}(T) = \{G_{ij}/0 \leq i,j\}$, with $G_{io} = 0$ as operators in $\mathcal{L}(\mathcal{H}'_{i-1},\mathcal{H}'_i)$ for $i \geq 1$ and $\mathcal{H}'_o = O$, $G_{oj} = O$ as operators in $\mathcal{L}(\mathcal{H}_j,\mathcal{H}_{j-1})$ for $j \geq 1$ and $\mathcal{H}_o = O$, and else where,

$$G_{ij} \in \mathcal{L}(\mathcal{D}_{G_{i-1,j}}, \mathcal{D}_{G^*_{i,j-1}}) \tag{3.2}$$

We will write $T = T(G_{ij}/1 \leq i,j)$ in order to explain the dependence of T on parameters.

Further on, fix a covariance kernel $\mathcal{S}$ and $\mathcal{G} = \{G_{ij}/i,j\in\mathbb{Z},\ i\leq j\}$ are the parameters of $\mathcal{S}$. For every $i\in\mathbb{Z}$, define the contractions:

$$R_i : \bigoplus_{k=i+1}^{\infty} \mathcal{D}_{G_{i+1,k}} \longrightarrow \mathcal{H}_i \tag{3.3}$$

$$R_i = T(G_{i,k}/i > k)$$

and consider the spaces

$$K_i = \bigoplus_{j=-\infty}^{i-1} \mathcal{D}_*(R_j) \oplus \mathcal{H}_i \oplus \mathcal{D}(R_i) \tag{3.4}$$

and the unitary operators:

$$W_i : \mathcal{K}_{i+1} \to \mathcal{K}_i$$

$$W_i = I \oplus \begin{bmatrix} I & O \\ O & \alpha(R_i) \end{bmatrix} J(R_i) \begin{bmatrix} O & I \\ \beta^*(R_i) & O \end{bmatrix} \tag{3.5}$$

where $\mathcal{D}_*(R_j)$ is the range of the unitary operator $\beta(R_j) : \mathcal{D}_{R_j^*} \to \mathcal{D}_*(R_j)$ defined by (2.7) in [5] and $\mathcal{D}(R_i)$ is the range of the unitary operator $\alpha(R_i) : \mathcal{D}_{R_i} \to \mathcal{D}(R_i)$ defined by (2.5) in [5]. Moreover, for a contraction $T \in \mathcal{L}(\mathcal{H}, \mathcal{G})$, we define the <u>elementary rotation</u> of T,

$$J(T) : \mathcal{H} \oplus \mathcal{D}_{T^*} \to \mathcal{G} \oplus \mathcal{D}_T$$

$$J(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \tag{3.6}$$

By Theorem 2.4 in [5], if we define

$$V(n) : \mathcal{H}_n \to \mathcal{K}_o$$

$$V(n) = \begin{cases} W^*_{-1} W^*_{-2} \ldots W^*_n / \mathcal{H}_n, & n < 0 \\ P^{\mathcal{K}_o}_{\mathcal{H}_o} / \mathcal{H}_o, & n = 0 \\ W_o W_1 \ldots W_{n-1} / \mathcal{H}_n, & n > 0 \end{cases} \tag{3.7}$$

then

$$S_{ij} = V^*(i)V(j) \qquad i,j \in \mathbf{Z} \tag{3.8}$$

Defining:

$$V_{ij} : \bigoplus_{k=i+1}^{j} \mathcal{D}_{G_{i+1,k}} \oplus \mathcal{D}_{G^*_{ij}} \to \bigoplus_{k=i}^{j-1} \mathcal{D}_{G_{ik}} \oplus \mathcal{D}_{G_{ij}}$$

$$V_{ij} = J_j(G_{i,i+1}) J_j(G_{i,i+2}) \ldots J_j(G_{ij}) \tag{3.9}$$

where the subscript j at $J(G_{i,i+k})$ means that the elementary rotation of $G_{i,i+k}$ was augmented with the identity operator on the corresponding spaces and

$$U_{ij} : \bigoplus_{k=j}^{-i} \mathcal{D}_{G^*_{-k,j}} \longrightarrow \bigoplus_{k=i}^{j} \mathcal{D}_{G_{ik}} \tag{3.10}$$

$$U_{ij} = V_{ij}(U_{i+1,j} \oplus I_{\mathcal{D}_{G^*_{ij}}})$$

we have by Theorem 1.3 in [5] that

$$S_{ij} = T(G_{ik}/k = i+1,\dots,j)U_{i+1,j-1}T(G_{kj}/k = i,\dots,j-1) + \tag{3.11}$$

$$+ D_{G^*_{i,i+1}} \dots D_{G^*_{i,j-1}} G_{ij} D_{G_{i+1,j}} \dots D_{G_{j-1,j}}$$

Two special features of the structure of S_{ij} are implied by the formulas (3.8) and (3.11). That is, by (3.11) we obtain exactly the parameters on which S_{ij} depends.

On the other hand, Lemma 2.1 in [1] shows a certain multiplicative structure of W_i in which the main role is played by the elementary rotations of the parameters.

Motivated by these remarks we have the following results.

3.1. LEMMA. For $i < j$, there exist operators E_{ij}, $\hat{E}_{ij}$ and $A = A(S_{ij})$ such that:

$$E_{ij}S^*_{ij} = A(S_{ij})\hat{E}^*_{ij}$$

$$S^*_{ij}\hat{E}_{ij} = E^*_{ij}A(S_{ij})$$

$$A^*(S_{ij})A(S_{ij}) + \hat{E}^*_{ij}\hat{E}_{ij} = I.$$

PROOF. Define E_{ij} such that

$$\begin{bmatrix} S_{ij} \\ E_{ij} \end{bmatrix} = \begin{bmatrix} U_{i,j-1}T(G_{-k,j}/-j < k < -i) \\ D_{G_{ij}} \dots D_{G_{j-1,j}} \end{bmatrix}$$

(an explicite formula for E_{ij} can be easily obtained using [5]). Then, define $\hat{E}_{ij}$ according to the following general rule: for a formula $(G_{mn}/i \leq m,n \leq j)$,

$$\widehat{\text{formula}} = \text{formula}\,(G^*_{j-n,j-m}/i \le m,n \le j)^* \qquad (3.12)$$

Finally, using computations made in[5], we obtain that

$$U_{ij} = \begin{bmatrix} S_{ij} & \hat{E}_{ij} \\ E_{ij} & A(S_{ij}) \end{bmatrix} \qquad (3.13)$$

and as U_{ij} is a unitary operator, we get the required relations between E_{ij}, $\hat{E}_{ij}$ and $A(S_{ij})$. ■

3.2. LEMMA. For $i < j$, there exist operators B_{ij}, $\hat{B}_{ij}$ and $A = A(T(G_{ik}/i+1 \le k \le j))$ such that

$$B_{ij}T^*(G_{ik}/i + 1 \le k \le j) = A\hat{B}^*_{ij}$$

$$T^*(G_{ik}/i + 1 \le k \le j)\hat{B}_{ij} = B^*_{ij}A$$

$$A^*A + \hat{B}^*_{ij}\hat{B}_{ij} = I.$$

PROOF. Define B_{ij} by formula (1.7) in [5] and $\hat{B}_{ij} = D_{G^*_{i,i+1}} \ldots D_{G^*_{ij}}$. Then, by formula (1.6) in [5],

$$V_{ij} = \begin{bmatrix} T(G_{ik}/i + 1 \le k \le j) & \hat{B}_{ij} \\ B_{ij} & A \end{bmatrix} \qquad (3.14)$$

and as V_{ij} is a unitary operator, we get the required relations. ■

3.3. LEMMA. There exist operators C_{nm}, $\hat{C}_{nm}$ and $A = A(T(G_{ij}/1 \le i \le n, 1 \le j \le m))$ such that

$$C_{nm}T^*(G_{ij}/1 \le i \le m, 1 \le j \le n) = A\hat{C}^*_{nm}$$

$$T^*(G_{ij}/1 \le i \le m, 1 \le j \le n)\hat{C}_{nm} = C^*_{nm}A$$

$$A^*A + \hat{C}^*_{nm}\hat{C}_{nm} = I$$

PROOF. The proof is based on Lemma 3.2 and on the remark that, with a slight abuse of notation,

$$T(G_{ij}/1 \le i \le n,\ 1 \le j \le m) = T(T(G_{ij}/1 \le i \le n)/1 \le j \le m) =$$

$$= T(T(G_{ij}/\ 1 \le j \le m)/1 \le i \le n). \tag{3.15}$$

The abuse of notation consists in the fact that the "parameters" $T(G_{ij}/1 \le i \le n)$ and respectyively, $T(G_{ij}/1 \le j \le m)$ must be "corrected" with some obvious unitary operators. ∎

Fix now a strictly increasing sequence of integers, $\pi = (\pi_n/n \in \mathbb{Z})$ and let $\mathcal{S}(\pi)$ be the covariance kernel obtained from $\mathcal{S}$ by deleting the rows and columns indexed by π. Ordering $\mathbb{Z}-\pi$ with the natural order, $\mathbb{Z}-\pi = [\ldots,p_{-1},p_o,p_1,\ldots]$, consider the following family of contractions: for $i \in \mathbb{Z}$,

$$G_{p_i,p_{i+1}}(\ ) = S_{p_i,p_{i+1}} \tag{3.16}$$

and for $j > i + 1$,

$$G_{p_i,p_j}(\ ) = T(G_{mn}/p_i \le m < p_{i+1},\ p_{j-1} < n \le p_j). \tag{3.17}$$

Moreover, we define the following Dictionary: for $i \in \mathbb{Z}$,

$$D_{G_{p_i,p_{i+1}}(\pi)} \longrightarrow E_{p_i,p_{i+1}}$$

$$D_{G^*_{p_i,p_{i+1}}(\pi)} \longrightarrow \hat{E}_{p_i,p_{i+1}}$$

$$G^*_{p_i,p_{i+1}}(\pi) \longrightarrow A$$

where the elements in the right are defined by Lemma 3.1, and for $j > 1 + 1$,

$$D_{G_{p_i,p_j}(\pi)} \longrightarrow C_{p_i,p_j}$$

$$D_{G^*_{p_i p_j}(\pi)} \longrightarrow \hat{C}_{p_i,p_j}$$

$$G^*_{p_i,p_j}(\pi) \longrightarrow A$$

where the elements in the right are now defined by Lemma 3.3.

We can obtain the main result of this section.

3.4. THEOREM. The elements of the covariance kernel $\mathcal{S}(\pi)$ are computed using formula (3.11) for the family $\mathcal{G}(\pi)$ according to the rules defined by the above Dictionary.

PROOF. For the proof we use formula (3.8) and computations similar to those in the proof of Theorem 3.2 in [1]. ∎

4. COINCIDENCE OPERATORS

We extend here a result in [6]. Thus, use the rule (3.12) in order to describe a certain dual process of a given process $\mathcal{V}$. Then, take $\mathcal{S}$ the covariance kernel of $\mathcal{V}$ and $\mathcal{G}$ its parameters. We introduce now the following elements:

$$\hat{\mathcal{K}}_i = \bigoplus_{j=-\infty}^{i-1} \widehat{\mathcal{D}_*(R_j)} \oplus \mathcal{H}_{-i+1} \oplus \widehat{\mathcal{D}(R_i)} \tag{4.1}$$

$$\hat{W}_i : \hat{\mathcal{K}}_i \longrightarrow \hat{\mathcal{K}}_{i+1} \tag{4.2}$$

$$\hat{V}(n) : \mathcal{H}_n \longrightarrow \hat{\mathcal{K}}_1$$

$$\hat{V}(n) = \begin{cases} \hat{W}_1^* \hat{W}_2^* \dots \hat{W}_{-n}^* / \mathcal{H}_n, & n < 0 \\ P^{\hat{\mathcal{K}}_1}_{\mathcal{H}_0} / \mathcal{H}_0 & n = 0 \\ \hat{W}_0 \hat{W}_1, \dots \hat{W}_{-n+1} / \mathcal{H}_n, & n > 0 \end{cases} \tag{4.3}$$

and the covariance kernel of this process $\hat{\mathcal{V}}$ is also $\mathcal{S}$, in view of the properties of the rule $\hat{\ }$.

Consequently, by general dilation theory (because the minimality conditions $\mathcal{K}_0 = \bigvee_{n\in\mathbf{Z}} V(n)\mathcal{H}_n$ and $\hat{\mathcal{K}}_1 = \bigvee_{n\in\mathbf{Z}} \hat{V}(n)\mathcal{H}_n$ hold), there exists a unitary operator $\psi : \mathcal{K}_0 \longrightarrow \hat{\mathcal{K}}_1$ (called a <u>coincidance operator</u>) such that $\hat{V}(n) = \psi V(n)$ for $n \in \mathbf{Z}$.

Our purpose is to obtain a certain ψ with a wellunderstood structure.

4.1. THEOREM. <u>The operator</u> $I_{\mathcal{H}_0} \oplus J(T(G_{jk}/j < 0, k \geq 1))$ <u>induces a coincidance operator for</u> $\mathcal{V}$ <u>and</u> $\hat{\mathcal{V}}$.

PROOF. A repeated use of Lemma 2.3 in [1] together with the identities (3.15) get, with a slight abuse of notation, the following relations:

$$J(T(G_{jk}/j \leq i, k > i)) = (I_{\mathcal{H}_i} \oplus J(T(G_{jk}/j < i, k > i))W_i \tag{4.4}$$

and

$$J(T(G_{jk}/j \leq i, k > i)) = \hat{W}_{-i}(I_{\mathcal{H}_i} \oplus J(T(G_{jk}/j \leq i, k > i + 1)). \tag{4.5}$$

Here, the abuse of notation consists in the fact that the two elementary rotations involved in (4.4) (as well as those in (4.5)) must be "corrected" with some unitary operators, of the type of the operators A_n in formula (2.8) in [6].

With these relations and taking the structure of V(n) and $\hat{V}(n)$ into account, we obtain the required property of $I_{\mathcal{H}_0} \oplus J(T(G_{jk}/j < 0, k \geq 1))$. ∎

5. APPLICATIONS

A. We return to question (b) in Section 2A regarding the computation of the angles between sections of past and future. We will obtain formulas in terms of the parameter $\mathcal{G}$ of the given process. Thus, we have the following result:

5.1. THEOREM. For $q \in \{k \in \mathbb{Z}/k \leq 0\} \cup \{-\infty\}$ and $p \in \{k \in \mathbb{Z}/k \geq 1\} \cup \{\infty\}$,

$$B(\mathcal{K}_q^0, \mathcal{K}_1^p) = \psi^* \tilde{T}\tilde{T}^* \psi,$$

where

$$\tilde{T} = P^- T(G_{jk}/q \leq j \leq 0,\ p \geq k \geq 1)$$

$$P^- = P^{\hat{\mathcal{K}}_1}_{\mathcal{H}_0 \oplus \bigoplus_{k=-1}^{q} \mathcal{D}_{G^*_{ko}}}$$

and ψ is the coincidence operator described in Theorem 4.1.

PROOF. Define $P^+ = P^{\mathcal{K}_1}_{\mathcal{H}_1 \oplus \bigoplus_{k=2}^{p} \mathcal{D}_{G_{1k}}}$ and remark that

$$\mathcal{K}_q^0 = \psi^* \bigvee_{k=q}^{0} \hat{V}(k)\mathcal{H}_k = \psi^*(\mathcal{H}_0 \oplus \bigoplus_{k=-1}^{q} \mathcal{D}_{G^*_{ko}}).$$

Then,

$$\mathcal{K}_1^p = W_0(\mathcal{H}_1 \oplus \bigoplus_{k=2}^{p} \mathcal{D}_{G_{1k}})$$

and we obtain:

$$B(\mathcal{K}_q^0, \mathcal{K}_1^p) = \psi^* P^- \psi W_0 P^+ W_0^* \psi^* P^- \psi.$$

The last remark is that we can use formula (4.4) and the proof is finished. ∎

5.2. REMARK. For $q = 0$ and $p = \infty$, we obtain formula (2.2).

Moreover, the formula obtained in above theorem explains the results in Lemma 3.1, Theorem 3.2, Corollary 3.3 and Theorem 3.4 in [1]. Formulas in Corollary 5.5 and Corollary 5.7 of [1] are also consequences of Theorem 5.1. ∎

B. Now, consider question (c) in Section 2.A regarding the computation of the space $\mathcal{K}_n^\infty \vee \mathcal{K}_{-\infty}^{-n}$. We already remarked that $\mathcal{K}_n^\infty \vee \mathcal{K}_{-\infty}^{-n}$ is the evolution space of the process $\mathcal{V}^{[n]}$ defined by (2.3). Let $\mathcal{S}$ be the covariance kernel of the process $\mathcal{V}$ and $\mathcal{G}$

its parameter, and let $\mathcal{S}^{[n]}$ be the covariance kernel of the process $\mathcal{V}^{[n]}$. Remark that, with the notation in Section 3, $\mathcal{S}^{[n]} = \mathcal{S}[\pi]$, where $\pi = [-n+1, -n+2, \dots, n-1]$. Using Theorem 3.4, we get

5.3. COROLLARY.

$$W^*_{-1} \dots W^*_{-n}(\mathcal{K}^{\infty}_{n} \vee \mathcal{K}^{-n}_{-\infty}) =$$

$$= \bigoplus_{j=-\infty}^{-n-1} \mathcal{D}_*(R_j) \oplus \mathcal{D}^{[n]}_* \oplus \mathcal{H}_{-n} \oplus \overline{E_{-n,n}\mathcal{H}_n} \oplus \bigoplus_{k=n+1}^{\infty} \mathcal{G}_{-n,k},$$

where $\mathcal{D}^{[n]}_*$ is the identification of the defect space of the column-contraction of parameters $(S_{-n,n}, T(G_{-n-1,k}/-n+1 \le k \le n), \dots)$ and $E_{-n,n}$ is that defined in Lemma 3.1. ∎

5.4. REMARK. Using Theorem 3.4 and Theorem 5.1, we can obtain a computation of the operator angles $B(\mathcal{K}^{o}_{q}, \mathcal{K}^{p}_{n})$ for $q \in \{k \in \mathbf{Z}/k \le 0\} \cup \{-\infty\}$, $p \in \{k \in \mathbf{Z}/k \ge 1\} \cup \{\infty\}$ and $n \ge 1$, $n < p$. The formulas are similar to those in Theorem 5.1 with $\widetilde{T}$ replaced by a $\widetilde{T}^{[n]}$ based on the parameters of the process $\mathcal{V}^{[n]}$. ∎

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