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N. BOBOC and Gh. BUCUR

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N. BOBOC*) and Gh. BUCUR**)

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*) Faculty of Mathematics, University of Bucharest, Str. Academiei no. 14, 70109 Bucharest, Romania.

**) Institute of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania.

EXCESSIVE AND SUPERMEDIAN FUNCTIONS WITH RESPECT TO SUBORDINATION RESOLVENTS OF KERNELS

N. Boboc and Gh. Bucur

In this paper $(X,\mathcal{B})$ is a measurable space $\mathcal{V}=(V_\alpha)_{\alpha>0}$ is a bounded resolvent of kernels on $(X,\mathcal{B})$ and P is a kernel on $(X,\mathcal{B})$ such that $PV_0 f \leq V_0 f$, $(\forall)\ f \geq 0$ and such that there exists a resolvent $\mathcal{U}=(U_\alpha)_{\alpha>0}$ on $(X,\mathcal{B})$ with

$$U_0 = V_0 - PV_0$$

$$U_\alpha \leq V_\alpha \quad (\forall)\ \alpha > 0.$$

We denote by $\mathcal{S}_{\mathcal{V}}$ (resp. $\mathcal{E}_{\mathcal{V}}$) the convex cone of all supermedian (resp. excessive and finite $\mathcal{V}$-a.s.) functions on X with respect to the resolvent $\mathcal{V}$.

We prove the following two results:

a) For any positive measurable function w on X we have

$$w \in \mathcal{S}_{\mathcal{U}} \Longleftrightarrow \inf(v, Pv + w) \in \mathcal{S}_{\mathcal{V}} \quad \text{for any } v \in \mathcal{E}_{\mathcal{V}}.$$

b) If

$$P(\mathcal{E}_{\mathcal{V}}) \subset \mathcal{E}_{\mathcal{V}}, \qquad \mathcal{E}_{\mathcal{V}} \subset \mathcal{E}_{\mathcal{U}}$$

then for any $w \in (\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}})_+$ we have

$$w \in \mathcal{E}_{\mathcal{U}} \Longleftrightarrow v \wedge (Pv + w) \in \mathcal{E}_{\mathcal{V}} \quad \text{for any } v \in \mathcal{E}_{\mathcal{V}}$$

whence $\wedge$ is the infimum operation in the vector lattice $(\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}}, \leq)$.

PROPOSITION 1. (G. Mokobodzki). Let w be a positive $\mathcal{B}$-measurable function on X such that $w \in \mathcal{S}_{\mathcal{U}}$. Then we have

$$\inf(v, Pv + w) \in \mathcal{S}_{\mathcal{V}}, \quad (\forall)\ v \in \mathcal{E}_{\mathcal{V}}.$$

Proof. Since $V1$ is bounded it is sufficient to prove the assertion for any $v \in \mathcal{E}_{\mathcal{V}}$ which is bounded. From hypothesis we have

$$U_\alpha \leq V_\alpha \quad , \quad (\forall)\ \alpha > 0$$

and therefore it follows ([3]) that the bounded kernel

$$G_{\alpha U_\alpha} = \sum_{n=0} (\alpha U_\alpha)^n = 1 + \alpha U$$

is subordinated to the bounded kernel

$$G_{\alpha V_\alpha} := \sum_{n=0} (\alpha V_\alpha)^n = I + \alpha V .$$

If we denote

$$P_\alpha := G_{\alpha U_\alpha} (\alpha V_\alpha - \alpha U_\alpha) = P(\alpha V_\alpha)$$

then, using the fact that

$$w \in \mathcal{S}_{\mathcal{U}} \subset \mathcal{S}_{\alpha U_\alpha} \qquad (\forall)\ \alpha > 0,$$

we get (see [3], Theorem 5)

$$\inf(v, P(\alpha V_\alpha\ v) + w) = \inf(v, P_\alpha\ v + w) \in \mathcal{S}_{\alpha V_\alpha} .$$

If α tends to $+\infty$ we obtain

$$\inf(v, Pv + w) \in \bigcap_\alpha \mathcal{S}_{\alpha V_\alpha} = \mathcal{S}_{\mathcal{V}} .$$

LEMMA 2. For any $u \in \mathcal{E}_{\mathcal{V}}$, $u < \infty$ there exists

$$u_o := Tu \in \mathcal{E}_{\mathcal{V}}$$

such that

$$u_o \leq_{\mathcal{E}_{\mathcal{V}}} u , \quad u - Pu = u_o - Pu_o$$

and such that

$$t \in \mathcal{E}_{\mathcal{V}},\ t \leq_{\mathcal{E}_{\mathcal{V}}} u_o ,\ Pt = t \Rightarrow t = 0.$$

Moreover we have

$$v \in \mathcal{E}_{\mathcal{V}}\ ,\ v \leq_{\mathcal{E}_{\mathcal{V}}} u_o\ ,\ u_o - Pu_o \leq v - Pv \Longrightarrow u_o \leq v\ .$$

Proof. We consider, inductively, the sequence $(t_n)_n$ in $\mathcal{E}_{\mathcal{V}}$ defined by

$$t_o = u \bigwedge_{\mathcal{S}_{\mathcal{V}}} Pu$$

$$t_{n+1} = t_n \bigwedge_{\mathcal{S}_{\mathcal{V}}} Pt_n\ .$$

We put

$$v_o := \bigwedge_{\mathcal{S}_{\mathcal{V}}} \{t_n \mid n \in \mathbb{N}\}$$

and we have

$$v_o \in \mathcal{E}_{\mathcal{V}}\ ,\ v_o \leq_{\mathcal{E}_{\mathcal{V}}} u\ ,\ Pv_o = v_o\ .$$

Let now $u_o := u - v_o$. We have $u_o \in \mathcal{E}_{\mathcal{V}}$

$$u_o - Pu_o = u - Pu\ .$$

If $t \in \mathcal{E}_{\mathcal{V}}$ is such that $Pt = t$, $t \leq_{\mathcal{E}_{\mathcal{V}}} u_o$ then we deduce inductively

$$v_o + t \leq_{\mathcal{E}_{\mathcal{V}}} t_n \quad (\forall)\ n \in \mathbb{N}$$

and therefore

$$t = 0\ .$$

Let now $v \in \mathcal{E}_{\mathcal{V}}$ be such that

$$u_o - Pu_o \leq v - Pv\ .$$

We have

$$u_o - v \leq P(u_o - v) \leq P(R^{\mathcal{S}_{\mathcal{V}}}(u_o - v))\ ,$$

$$R^{\mathcal{S}_{\mathcal{V}}}(u_o - v) \leq P(R^{\mathcal{S}_{\mathcal{V}}}(u_o - v))\ .$$

From

$$R^{\mathcal{S}_{\mathcal{V}}}(u_o - v) \in \mathcal{E}_{\mathcal{V}}$$

we get

$$P(R^{\mathcal{S}_{\mathcal{V}}}(u_o - v)) \leq R^{\mathcal{S}_{\mathcal{V}}}(u_o - v);$$

$$P(R^{\mathcal{S}_{\mathcal{V}}}(u_o - v_2) = R^{\mathcal{S}_{\mathcal{V}}}(u_o - v) \ .$$

Since

$$R^{\mathcal{S}_{\mathcal{V}}}(u_o - v) \preceq_{\mathcal{E}_{\mathcal{V}}} u_o$$

we deduce from the above considerations

$$R^{\mathcal{S}_{\mathcal{V}}}(u_o - v) = 0, \quad u_o \leq v \ .$$

PROPOSITION 3. If $w \in \mathcal{E}_{\mathcal{U}}$ is such that there exists $u \in \mathcal{E}_{\mathcal{V}}$, $u < \infty$ with

$$w \leq u - Pu$$

then there exists $v \in \mathcal{E}_{\mathcal{V}}$, $v \leq u$ such that

$$w = v - Pv \ .$$

Proof. Let $(f_n)_n$ be a sequence of positive bounded $\mathcal{B}$-mesurable functions on X such that

$$Uf_n \uparrow w \ .$$

We have

$$Uf_n = Vf_n - PVf_n \qquad (\forall)\, n \in \mathbf{N} \ .$$

If we put

$$v_n := T(Vf_n)$$

we get

$$v_n \leq v_{n+1} \leq u$$

and therefore $w = v - Pv$ where

$$w = v - Pv \quad \text{where} \quad v = \sup_n v_n \leq u.$$

THEOREM 4. Let f be a positive $\mathcal{B}$-measurable function on X such that for any $v \in \mathcal{E}_{\mathcal{V}}$, v bounded with $v - Pv \in \mathcal{E}_{\mathcal{U}}$ we have

$$\inf(v, Pv + f) \in \mathcal{S}_{\mathcal{V}} \ .$$

Then $f \in \mathcal{S}_{\mathcal{U}}$.

Proof. Let f be a positive function on X as in this theorem and let $v \in \mathcal{E}_{\mathcal{V}}$ be a bounded function such that $v - Pv \in \mathcal{E}_{\mathcal{U}}$. We denote by g the function on X define by

$$g = \inf(f, v - Pv) .$$

We want to show that $g \in \mathcal{S}_{\mathcal{U}}$. Let us denote

$$t := \lim_{\alpha \to \infty} \alpha U_{\alpha} (R^{\mathcal{S}_{\mathcal{U}}} g) .$$

Obviously we have

$$t \in \mathcal{E}_{\mathcal{U}} , \quad t \leq R^{\mathcal{S}_{\mathcal{U}}} g \text{ on } X, \quad t = R^{\mathcal{S}_{\mathcal{U}}} g \quad \mathcal{U}\text{- a.s. on } X .$$

We remark that

$$t = \bigwedge_{\mathcal{E}_{\mathcal{U}}} \{ w \in \mathcal{E}_{\mathcal{U}} \mid w \geq g \quad \mathcal{U}\text{- a.s. on } X \} .$$

Indeed, for any $w \in \mathcal{E}_{\mathcal{U}}$ such that $w \geq g$ $\mathcal{U}$- a.s. on X there exists a measurable subset A of X such that $U(1_A) = 0$ and such that

$$w + \infty \cdot U(1_A) \geq g \quad \text{on } X$$

Since $w + \infty \cdot U(1_A) \in \mathcal{S}_{\mathcal{U}}$ we get

$$w + \infty \cdot U(1_A) \geq R^{\mathcal{S}_{\mathcal{U}}} g, \quad w \geq t .$$

On the other hand $t \geq g$ $\mathcal{U}$- a.s. on X.

Since $t \leq v - Pv$ we deduce, using Lemma 2 and Proposition 3, that there exists $u \in \mathcal{E}_{\mathcal{V}}$, $u \leq v$ such that $t = u - Pu$ and such that for any $w \in \mathcal{E}_{\mathcal{V}}$, $t \leq w - Pw$ we have $u \leq w$.

From the preceding considerations we deduce

$$u - Pu \leq v - Pv, \quad \inf(f, u - Pu) \leq g \quad \text{on } X$$

$$g \leq \inf(f, u - Pu) \quad \mathcal{U}\text{- a.s. on } X$$

and therefore

$$g = \inf(f, u - Pu) \quad \mathcal{U}\text{- a.s. on } X$$

$$g + Pu = \inf(u, f + Pu) \quad \mathcal{U}\text{- a.s. on } X \; .$$

By hypothesis the function $\inf(u,f + Pu)$ belongs to $\mathcal{S}_{\mathcal{V}}$ and therefore the function

$$u' := \lim_{\alpha \to \infty} \alpha V_\alpha \; (\inf(u, f + Pu))$$

belongs to $\mathcal{E}_{\mathcal{V}}$ and $u' = \inf(u,f + Pu)$ $\mathcal{U}$- a.s. on X.

We have also

$$g + Pu = u' \quad \mathcal{U}\text{- a.s. on } X, \; u' \leq u,$$

$$g = u' - Pu \leq u' - Pu' \quad \mathcal{U}\text{- a.s. on } X$$

and therefore

$$u - Pu = t \leq u' - Pu' \; , \quad u \leq u', \quad u = u' \, .$$

Hence

$$u - Pu \leq f, \quad u - Pu \leq v - Pv, \quad u - Pu \leq g \quad \text{on} \quad X$$

$$g \leq t = u - Pu \quad \mathcal{U}\text{- a.s. on } X$$

and therefore $g \in \mathcal{S}_{\mathcal{U}}$.

From the above considerations we get

$$\inf(f, U\varphi) = \inf(f, V\varphi - PV\varphi)$$

for any positive, bounded, measurable function φ on X and therefore the function

$$f_o := \sup_n \inf(f, U(n))$$

belongs to $\mathcal{S}_{\mathcal{U}}$. Moreover we have

$$f_o = 0 \text{ on the set } A := [U1 = 0], \quad f_o \leq f \text{ on } X;$$

$$f_o = f \text{ on the set } X \setminus A.$$

Since $U(1_A) = 0$ we get

$$\alpha U_\alpha \; f = \alpha U_\alpha \; f_o \leq f_o \leq f \quad \text{i.e.} \quad f \in \mathcal{S}_{\mathcal{U}} \, .$$

In the sequel we suppose that

$$P(\mathcal{E}_{\mathcal{V}}) \subset \mathcal{E}_{\mathcal{V}} \quad \text{and} \quad \mathcal{E}_{\mathcal{V}} \subset \mathcal{E}_{\mathcal{U}}.$$

PROPOSITION 5. For any element $w \in \mathcal{E}_{\mathcal{U}} \cap (\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}})$ we have

$$s \wedge (Ps + w) \in \mathcal{E}_{\mathcal{V}}$$

for any $s \in \mathcal{E}_{\mathcal{V}}$ where $\wedge$ is the infimum operator in the vector lattice $(\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}}, \leq)$.

Proof. Since $w \in \mathcal{E}_{\mathcal{U}}$ then there exists a sequence $(g_n)_n$ of positive, bounded $\mathcal{B}$-measurable function on X such that $Ug_n \uparrow w$. It will be sufficient to show that

$$s \wedge (PS + w) \in \mathcal{E}_{\mathcal{V}} \quad (\forall)\, s \in \mathcal{E}_{\mathcal{V}}$$

where w is of the form $u - Pu$ with $u \in \mathcal{E}_{\mathcal{V}}$, u bounded.

Since $U_\alpha \leq V_\alpha \quad (\forall)\, \alpha > 0$ and since

$$U = V - PV$$

then we get, using Proposition 1,

$$t := \inf(s, Ps + u - Pu) \in \mathcal{S}_{\mathcal{V}}.$$

We have

$$t + Pu = \inf(s + Pu,\ Ps + u),$$

$$\hat{t} + Pu = (s + Pu) \wedge (Ps + u)$$

where $\wedge$ is the minimum in $\mathcal{E}_{\mathcal{V}}$ and

$$\hat{t} := \sup_{\alpha} \alpha V_\alpha t \quad ,$$

and therefore

$$\hat{t} = (s + Pu) \wedge (Ps + u) - Pu = s \wedge (Ps + u - Pu).$$

THEOREM 6. If $f \in (\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}})_+$ is such that

$$v \wedge (Pv + f) \in \mathcal{E}_{\mathcal{V}} \quad (\forall)\, v \in \mathcal{E}_{\mathcal{V}}$$

where $\wedge$ is the infimum in the vector lattice ($\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}}, \leq$), we have

$$f \in \mathcal{E}_{\mathcal{U}} .$$

Proof. Let $u_1, u_2 \in \mathcal{E}_{\mathcal{V}}$ be such that

$$f + u_2 = u_1 .$$

We denote by g the function

$$g(x) = \begin{cases} f(x) \text{ if } u_2(x) < \infty \\ \\ +\infty \text{ if } u_2(x) = +\infty . \end{cases}$$

We have for any $v \in \mathcal{E}_{\mathcal{V}}$,

$$w + u_2 = (s + u_2) \wedge (Pv + u_1)$$

where

$$w = v \wedge (Pv + f) .$$

If we put

$$t = \inf(s + u_2 , Pv + u_1)$$

we get, $g + u_2 = u_1$,

$$t = \inf(v, Pv + g) + u_2$$

and therefore

$$w + u_2 = \hat{t}$$

where

$$\hat{t} = \sup_{\alpha} \alpha V_{\alpha} \, t .$$

Hence

$$\alpha V_{\alpha} \, t = \alpha V_{\alpha} \, (\inf(v,Pv + g)) + \alpha V_{\alpha} \, u_2 = \alpha V_{\alpha} \, w + \alpha V_{\alpha} \, u_2 ,$$

and therefore

$$\alpha V_\alpha \ (\inf(v,Pv+g)) \leq \alpha V_\alpha \ w \leq w \leq \inf(v,Pv+g)$$

on the set $[u_2 < \infty]$. Since $g = +\infty$ on $[u_2 = +\infty]$ we deduce

$$\alpha V_\alpha \ (\inf(v,Pv+g)) \leq \inf(v,Pv+g) \text{ on } X$$

and therefore

$$\inf(v,Pv+g) \in \mathcal{S}_{\mathcal{V}}.$$

From Theorem 4 we get $g \in \mathcal{S}_{\mathcal{U}}$.

But from

$$g + u_2 = u_1$$

we deduce

$$\overline{g} + u_2 = u_1$$

where

$$\overline{g} = \sup_\alpha \alpha U_\alpha \ g \in \mathcal{E}_{\mathcal{U}},$$

and therefore, $f = \overline{g} \in \mathcal{E}_{\mathcal{U}}$.

REMARK. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$, $\mathcal{V} = (V_\alpha)_{\alpha>0}$, be two rezolvents on $(X,\mathcal{B})$ such that the initial kernels U and V are bounded and such that

$$U = V - PV$$

where P is a bounded kernel. Then H. Ben Saad in ([2]) proved that we have

$$U_\alpha \leq V_\alpha \qquad (\forall)\ \alpha > 0$$

iff $Pf \in \mathcal{S}_{\mathcal{U}}$ for any positive $\mathcal{B}$-measurable function f. On the other hand G. Mokobodzki proved in ([3]) that we have

$$U_\alpha \leq V_\alpha \qquad (\forall)\ \alpha > 0$$

iff

$$\inf(s, Ps + u - Pu + Pf) \in \mathcal{S}_{\mathcal{V}}$$

for any $s \in \mathcal{E}_{\mathcal{V}}$. The above Proposition 1 and Theorem 4 show how we can obtain directly the Mokobodzki result from the H. Ben Saad results and conversely.

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