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REDUCTION OF TOPOLOGICAL STABLE RANK IN INDUCTIVE LIMITS OF C^*-ALGEBRAS

Marius DĂDÂRLAT, Gabriel NAGY, András NÉMETHI and Cornel PASNICU

We consider inductive limits A of sequences $A_1 \to A_2 \to \dots$ of finite direct sums of C^*-algebras of continuous functions from compact Hausdorff spaces into full matrix algebras. We prove that A has topological stable rank (tsr) one provided that A is simple and the sequence of the dimensions of the spectra of A_i is bounded. For unital A, tsr(A) = 1 means that the set of invertible elements is dense in A. If A is infinite dimensional then, the simplicity of A implies that the sizes of the involved matrices tend to infinity, so by general arguments (see [23]) one gets that $\mathrm{tsr}(A_i) \leq 2$ for large enough i whence $\mathrm{tsr}(A) \leq 2$. The reduction of tsr from two to one requires arguments which are strongly related to this special class of C^*-algebras.

The problem of reduction of real rank (see [6]) for these algebras was recently studied in [2] in connection with some interesting features revealed in several papers ([3], [1], [15], [5], [12], [11]). The reduction of tsr and real rank for other classes of C^*-algebras was studied in [22], [21], [8], [24], [17], [25].

The paper consists of three sections :

1. Preliminaries and Notation
2. Local aspects of the connecting homomorphisms
3. The Main Result

1.

1.1. For a unital C^*-algebra A and a finitely generated projective A-module E, we denote by $End_A(E)$ the algebra of A-linear endomorphisms of E and by $GL_A(E)$ the group of units of $End_A(E)$. For $E = A^n$ we shall write GL(n,A) for $GL_A(A^n)$ and $GL^\circ(n,A)$ for the connected component of 1. Let U(A) denote the unitary group of A and $U(n) := U(\mathbf{C}^n)$. A selfadjoint idempotent element of a C^*-algebra will be simply called projection.

Recall some definitions from [23]. For a unital C^*-algebra A and a natural number n let $Lg_n(A)$ denote the set of n-tuples of elements of A which generate A as a left ideal. The topological stable rank of A is the least n (if it does not exist it will be taken by definition to be ∞) such that $Lg_n(A)$ is dense in A^n. One denotes by csr(A) the least integer n such that $GL^\circ(m,A)$ acts transitively by right multiplication on $Lg_m(A)$ for any $m \geq n$. (If no such integer exists one takes $csr(A) = \infty$). For nonunital A one takes

$tsr(A) := tsr(\tilde{A})$ and $csr(A) := csr(\tilde{A})$ where $\tilde{A}$ is the algebra obtained from A by adjoining a unit.

For a compact Hausdorff space X of finite covering dimension one has:

$$tsr(C(X)) = \left[\frac{\dim X}{2}\right] + 1$$

$$csr(C(X) \leq \left[\frac{\dim X + 1}{2}\right] + 1$$

(see [23] and [18]).

1.2. We consider C^*-inductive limits

$$A = \varinjlim(A_i, \Phi_{ij})$$

The A_i's are C^*-algebras of the form

$$A_i = \bigoplus_{t=1}^{s(i)} C(X_{it}) \otimes M_{n(i,t)}$$

where X_{it} is a Hausdorff compact space, s(i), n(i,t) are positive integers and $M_{n(i,t)}$ is the C^*-algebra of complex $n(i,t) \times n(i,t)$ matrices. The $*$-homomorphisms $\Phi_{ij}: A_i \rightarrow A_j$ are not assumed to be unital or injective. We denote by Φ_i the natural map $A_i \rightarrow A$ and by $X_i = \bigsqcup_{t=1}^{s(i)} X_{it}$ the spectrum of A_i.

We begin with a brief discussion of the $*$-homomorphisms between certain homogeneous C^*-algebras.

1.3. For given C^*-algebras C,D we denote by Hom(C,D) the space of all $*$-homomorphisms from C to D with the point-norm topology. $\mathrm{Hom}^1(C,D)$ stands for the subspace of unital $*$-homomorphisms. We shall identify

$$\mathrm{Hom}(C(X), C(Y)\otimes M_n) \text{ with } \mathrm{Map}(Y, \mathrm{Hom}(C(X), M_n))$$

where for topological spaces Y,Z, Map(Y,Z) denotes the space of continuous functions from Y to Z endowed with the comapct-open topology.

Each $\Psi \in \mathrm{Hom}(C(X), M_n)$ has the form

$$\Psi(f) = \sum f(x_r)p_r, \quad f \in C(X)$$

for suitable points $x_r \in X$ and mutually orthogonal projections p_r in M_n. Let L_Ψ be the set of all x_r's that appear in the above formula. More generaly, each $\Phi \in \mathrm{Hom}(C(X), C(Y)\otimes M_n)$ is identified with a map $\bar{\Phi}: Y \rightarrow \mathrm{Hom}(C(X), M_m)$ and we define for each $y \in Y$, $L_\Phi(y) := L_{\bar{\Phi}(y)}$. In the same way for given

$$\Phi \in \mathrm{Hom}(\oplus C(X_\alpha)\otimes M_{n(\alpha)}, \oplus C(Y_\beta) \times M_{m(\beta)})$$

and $y \in Y$ we define

$$L_\Phi(y) = \bigsqcup_\alpha L_{\Phi_{\alpha,\beta}}(y)$$

where $\Phi_{\alpha,\beta}$ denotes the component of Φ from $C(X_\alpha) \subset C(X_\alpha)\otimes$ $\otimes M_{n(\alpha)}$ to $C(Y_\beta) \times M_{m(\beta)}$.

The map $y \mapsto L_{\Phi}(y)$ has useful semicontinuity properties:

a) if $L_{\Phi}(y)$ is contained in some open set U then $L_{\Phi}(z) \subset U$ for any z in some neighborhood of y

b) the set $\{y : L_{\Phi}(y) \cap U \neq \emptyset\}$ is open for each open set U (see [9] and [19]).

1.4. The space $\mathrm{Hom}(C, M_m)$, $C = \bigoplus C(X_\alpha) \times M_{n(\alpha)}$, decomposes as a union of disjoint open sets labeled by the systems of positive integers $K = (k_\alpha)$ such that the quantity $k_o := m - \sum_\alpha k_\alpha n(\alpha)$ is positive. Let $\mathrm{Hom}_K(C, M_m)$ denote the open subset corresponding to K. Then there is (see [10]) a principal bundle

$$E(K) \xrightarrow{p} \mathrm{Hom}_K(C, M_m)$$

with

$$G(K) = \prod_\alpha U(k_\alpha) \times U(k_o) \quad \text{as structural group}$$

$$E(K) = \prod_\alpha \mathrm{Hom}^1(C(X_\alpha), M_{k_\alpha}) \times U(m)$$

$$p((\Phi_\alpha), u) = u(\bigoplus \Phi_\alpha \otimes \mathrm{id}(M_{n(\alpha)}) \oplus O_{k_o})u^*$$

This description of p corresponds to the canonical embeding

$$\bigoplus M_{k_\alpha} \otimes M_{n(\alpha)} \oplus M_{k_o} \subset M_m$$

2.

We begin by giving two criteria of simplicity for C^*-algebras A as above, which extend the corresponding results for AF-algebras [4] and Bunce-Deddens algebras [7].

2.1. Proposition. Let $A = \varinjlim(A_i, \Phi_{ij})$ be as in 1.1 and assume that the connecting homorphisms Φ_{ij} are injective,. Then the following conditions are equivalent:

(i) A is simple

(ii) For any positive integer i and any open nonempty subset U of X_i there is a j_o such that $L_{\Phi_{ij}}(x) \cap U \neq \emptyset$ for any $j \geq j_o$ and $x \in X_j$.

(iii) For any nonzero $a \in A_i$ there is a j_o such that

$$\Phi_{ij}(a)(x) \neq 0 \text{ for each } j \geq j_o \text{ and } x \in X_j.$$

Proof. (ii)$\Leftrightarrow$(iii). This is clear since for given $a \in A_i$ one has

$$\Phi_{ij}(a)(x) = 0 \text{ if and only if } a = 0 \text{ on } L_{\Phi_{ij}}(x)$$

(i)$\Rightarrow$(ii). Assume that (ii) does not hold for some i and some open nonempty $U \subsetneq X_i$. Passing to a subsequence, if necessary, we may assume that for any $j \geq i$ the set $F_j = \{x \in X_j ;\ L_{\Phi_{ij}}(x) \cap U = \emptyset\}$ is nonempty and $F_j \neq X_j$. By the last part of 1.3 F_j is closed. Therefore the family $(J_j)_{j \geq i}$ where

$$J_j = \{a \in A_j : a = 0 \text{ on } F_j\}$$

defines a closed two sided ideal J in A. (Note that $\Phi_{jk}(J_j) \subset J_k$ since $L_{\Phi_{ij}}(y) \subset L_{\Phi_{ik}}(x)$ for any $y \in L_{\Phi_{jk}}(x)$). Also $J \neq A$ since if e_i is the unit of A_i then $\mathrm{dist}(\Phi_{ij}(e_i), J_j) = 1$ for any $j \geq i$ and so $e_i \notin J$. The existence of J contradicts (i).

(iii)$\Rightarrow$(i). Let J be a two-sided closed nonzero ideal of A. One has $J = \overline{\bigcup (J \cap A_i)}$ (see [4]). We shall prove that $J \cap A_j = A_j$ for large enough j. Take $a \in J \cap A_i$, $a \neq 0$. By (iii) there is a j_o such that $\Phi_{ij}(a)(x) \neq 0$ for all $j \geq j_o$ and $x \in X_j$. Since $\Phi_{ij}(J \cap A_i) \subset J \cap A_j$ we find that $\Phi_{ij}(a) \in J \cap A_j$ for $j \geq j_o$. Since $\Phi_{ij}(a)$ does not vanish at any point of X_j this forces $J \cap A_j = A_j$. □

Let $A = \varinjlim(A_i, \Phi_{ij})$ be as above. For a noninvertible element $a \in A_i$ there are $x_o \in X_i$, $u \in U(A_i)$ and a projection $p \in A_i$ (both u and p "scalars") such that $ua(x_o)p = pua(x_o) = 0$.

For simple A the following sequence of Lemmas enables us to obtain something similar for $\Phi_{ij}(a)$ (for some $j \geq i$) locally around any point of X_j, after a small perturbation of a.

2.2. LEMMA. Let $\Phi \in \mathrm{Hom}(C(X), C(Y)\otimes M_m)$, k a positive integer , U a nonempty open subset of X. Suppose that $L_\Phi(y) \cap U$ has at least k points for some $y \in Y$. Then there is a neighbourhood W of y and a projection valued continuous map $q_W : W \to M_m$ with rank $q_W(z) \geq k$ for any $z \in W$ such that for any $f \in C(X)$ with $f = 0$ on U one has

$$\Phi(f)q_W = q_W \Phi(f) = 0 \text{ on } W.$$

Proof. Take open sets U_1, U_2 having disjoint closures sch that

$$L_\Phi(y) \cap U \subset U_1 \subset U$$

$$L_\Phi(y) \cap (Y \setminus U) \subset U_2$$

Using the semicontinuity of L_Φ (see 1.3) we find a neighbourhood W of y such that $L_\Phi(z) \subset U_1 \cup U_2$ for any $z \in W$. Take a continuous map $g : X \to [0,1]$ such that $g = 1$ on U_1 and $g = 0$ on U_2 and define q_W as the restriction of $\Phi(g)$ to W. It is clear that q_W is projection valued since $g^2 = g$ on $L_\Phi(W)$. The continuity of the map $z \mapsto \mathrm{tr}\, \Phi(g)(z)$ shows that for z close to y one has rank $q_W(z) \geq k$. The proof is complete since $fg = 0$ on $L_\Phi(W)$. □

2.3. LEMMA. Let $C = \bigoplus_{i=1}^{s} C(Z_i)\otimes M_{n(i)}$, $\Phi \in \mathrm{Hom}(C, C(Y)\otimes M_m)$, k a positive integer, U is a nonempty open subset of Z_1 and $y \in Y$ such that $L_\Phi(y) \cap U$ has at least k points.

Then there is a neighbourhood W of y and a projection valued map $p_W : W \to M_m$ with rank $p_W(x) \geq k$ for x W such that for any $a \in C$ satisfying

$$ae_{11} = e_{11}a = 0 \text{ on } U$$

(here (e_{ij}) is the canonical system of matrix units of $M_{n(1)}$)
one has

$$\Phi(a)p_W = p_W\Phi(a) = 0 \text{ on } W.$$

Proof. Viewing $\bar{\Phi}$ as a map $\bar{\Phi} : Y \to \mathrm{Hom}(C, M_m)$ let Y_K be the preimage of $\mathrm{Hom}_K(C, M_m)$ under this map (see 1.4). The family (Y_K) is a partition of Y by open sets. Choose $K = (k_i)$ such that $y \in Y_K$. Let $W \subset Y_K$ be a neighbourhood of y such that the restriction of Φ to W has a lifting $\Phi' : W \to E(K)$. This is possible since $E(K) \to \mathrm{Hom}_K(C, M_m)$ is a locally trivial fibration (see 1.4). Therefore we find continuous maps

$$u : W \to U(m)$$

$$\Phi_i : W \to \mathrm{Hom}^1(C(Z_i), M_{k_i}) \quad i = 1, \dots, s.$$

such that

$$\bar{\Phi} = u\left(\bigoplus_{i=1}^{s} \Phi_i \otimes \mathrm{id}(M_{n(i)}) \oplus 0_{k_o}\right)u^* \text{ on } W.$$

Since $L_{\Phi_1}(y) = L_{\bar{\Phi}}(y) \cap Z_1$, shrinking W if necessary we may assume that Φ_1 satisfies the conditions of Lemma 2.2 and let q_W be the corresponding map. Put $p_W = u(q_W \otimes e_{11})u^*$ viewed as a continuous map $W \to M_{k_1} \otimes M_{n(1)} \subset M_m$. A simple computation based on the conclusions of Lemma 2.2 concludes the proof. □

2.4. LEMMA. Let $C = C(X) \otimes M_n$ and let $a \in C$ such that $\det a(x) = 0$ for some $x \in X$. Then for any $\varepsilon > 0$ the are $u, v \in GL(C)$ and $b \in C$ such that

$$\| uav - b \| < \varepsilon \text{ and } be_{11} = e_{11}b = 0 \text{ on a neighbourhood of } x.$$

Proof. Take $u,v \in Gl(n,\mathbf{C})$ such that the matrix $ua(x)v$ has only zero entries on the first row and onthe first column. Now b is easily found since continuous functions vanishing at x can be uniformly approximated by continuous functions vanishing on a neighbourhood of x. □

3.

The next step toward the main result is based on the following theorem which follows from Michael's paper [16].

3.1. THEOREM. Let X be a Hausdorff compact space of dimension d, let T be a complete metric space and let Y be a map from X to the family of the nonempty closed subsets of T.

Suppose that

a) Y is lower semicontinuous i.e. for each open subset U of T the set $\{x \in X : Y(x) \cap U \neq \phi\}$ is open

b) Each Y(x) is (d + 1) - connected

c) There is $\varepsilon > 0$ such that for any $0 < r \leq \varepsilon$ and $x \in X$ the intersection of Y(x) with any closed ball of radius r in T is a contractible space.

Then there is a continuous map $\sigma : X \to T$ such that $\sigma(x) \in Y(x)$ for all $x \in X$.

Proof. The Theorem follows from Theorem 1.2 in [16] using the comments from the second part of the same paper.

3.2. PROPOSITION. Let X be a Hausdorff compact space, let $k' \geq k \geq 1$ integers, let $\mathcal{W}$ be an open cover of X and assume that for each $W \in \mathcal{W}$ it is given a continuous projection valued map $p_W : W \to M_n$ such that $\operatorname{rank} p_W(x) \geq k'$ for $x \in W$. If $\dim(X) \leq 2(k' - k) - 1$ then there is a continuous projection valued map $p : X \to M_n$ such that for $x \in X$:

$$\operatorname{rank} p(x) \geq k$$

$$p(x) \leq \bigvee \{p_W(x) : W \in \mathcal{W},\ x \in W\}$$

Proof. For $x \in X$ define $\mathcal{W}(x) = \{W \in \mathcal{W} : x \in W\}$ and $H(x) = \mathrm{span}\{p_W(x)\mathbf{C}^n : W \in \mathcal{W}(x)\}$.

For any linear subspace H of $\mathbf{C}^n$ let $V(H,k)$, $k \leq \dim(H)$, denote the Stiefel manifold of k-orthogonal frames in H (see [14]). For any $x \in X$ define $\Upsilon(x) = V(H(x), k) \subset V(\mathbf{C}^n;k)$. We check that Υ satisfies the conditions of Theorem 3.1.

a) The lower semicontinuity of Υ follows from the lower semicontinuity of the map $x \mapsto H(x) \subset \mathbf{C}^n$ which is almost obvious having in mind the definition of $H(x)$.

b) $V(H,k)$ is $2(\dim(H) - k)$ - connected (see [14]). Therefore $V(H(x), k)$ is $2(k' - k)$ - connected since $\dim H(x) \geq k'$.

c) For any m, $n \geq m \geq k$, there is $\varepsilon_m > 0$ such that any closed ball of radius at most ε_m in $V(\mathbf{C}^m,k)$ is contractible. (We consider $V(\mathbf{C}^m,k)$ with the metric induced by the restriction of a $U(n)$ - invariant Riemann structure on $V(\mathbf{C}^n,k)$). In this situation $V(\mathbf{C}^m,k)$ is a totally geodesic manifold of $V(\mathbf{C}^n,k)$ and the same is true for any $V(H,k)$ with $H \subset \mathbf{C}^n$. Therefore the induced metric form from $V(\mathbf{C}^n,k)$ coincides with the metric given by the induced Riemann structure of $V(H,k)$ (see [13]). Having also the $U(n)$ - invariance of this metric one can take

$$\varepsilon = \min\{\varepsilon_m : k \leq m \leq n\}. \qquad \square$$

We also need the following approximation results:

3.3. LEMMA. Let B be a unital C^*-algebra and let $k \geq \max(\mathrm{tsr}(B), \mathrm{csr}(B))$. Then for any positive integer m and any $a \in M_m(B)$, the matrix $\begin{pmatrix} a & 0 \\ 0 & 0_k \end{pmatrix}$ belongs to the closure of $GL(m + k, B)$.

Proof. If $m \leq k$ one can take

$$b_\varepsilon = \begin{pmatrix} a & \varepsilon 1_m & 0 \\ \varepsilon 1_m & 0_m & 0 \\ 0 & 0 & \varepsilon 1_{k-m} \end{pmatrix} \in GL(m+k, B)$$

and $b_\varepsilon \longrightarrow a$ as $\varepsilon \longrightarrow 0$.

For $m \geq k$ we proceed by induction. Assume the statement holds for a fixed $m \geq k$ and let $a \in M_{m+1}(B)$. Since $m \geq \max(tsr(B), csr(B))$ it follows from [23] that for each $\varepsilon > 0$ there are $t \in GL(m+1, B)$, $a_1 \in M_m(B)$ and $b \in B^m$ such that

$$\left\| a - \begin{pmatrix} 1 & 0 \\ b & a_1 \end{pmatrix} \cdot t \right\| < \varepsilon$$

By the induction hypothesis are can approximate

$$\begin{pmatrix} 1 & 0 & 0 \\ b & a_1 & 0 \\ 0 & 0 & 0_k \end{pmatrix}$$

with an invertible matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ b & \multicolumn{2}{c}{c} \\ 0 & & \end{pmatrix}$$

Hence $\begin{pmatrix} a & 0 \\ 0 & 0_k \end{pmatrix}$ will be approximated by

$$\begin{pmatrix} 1 & 0 & 0 \\ b & \multicolumn{2}{c}{c} \\ 0 & & \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & 1_k \end{pmatrix} \qquad \square$$

3.4. REMARK. Suppose B,k are as above. Let F,G,H be finitely generated projective B-modules and put $E = F \oplus G \oplus H$. If F,G are free and $G \simeq B^k$, then a slight modification of the above arguments shows that $\mathrm{End}_B(F) \subset \overline{GL_B(E)}$.

In the proof of the main result we shall invoke the following straightforward approximation device:

3.5. LEMMA. Let $B = \overline{\bigcup B_i}$ where the B_i's form an increeasing sequence of unital C^*-algebras. Let e_i be the unit of B_i. If for any $a \in B_i$ and $\varepsilon > 0$ there is $j \geq i$ and $b \in GL(e_i B_j e_i)$ such that $\| a - b \| < \varepsilon$ then $\mathrm{tsr}(B) = 1$.

Proof. Let $\tilde{B} = B + \mathbf{C}.1$ be the algebra obtained by adjoining a unit to B. Let $x + \lambda 1 \in \tilde{B}$ with $x \in B_i$. By hypothesis there is $j \geq i$ and $y \in GL(e_i B_j e_i) \subset GL(e_i B e_i)$ such that $\| x + \lambda e_i - y \|$ is small. Choosing a non zero scalar λ' close to λ, the element $y + \lambda'(1 - e_i)$ is invertible and approximate $x + \lambda.1$. Therefore $GL(\tilde{B})$ is dense in $\tilde{B}$ which means $\mathrm{tsr}(B) = 1$. $\square$

3.6. THEOREM. Let $A = \varinjlim(A_i, \Phi_{ij})$ where

$A_i = \bigoplus_{t=1}^{s(i)} C(X_{it}) \otimes M_{n(i,t)}$, each X_{it} being a Hausdorff compact space such that $d = \sup \dim(X_{it}) < \infty$.

If A is simple then $\mathrm{tsr}(A) = 1$.

Proof. Replacing each A_i by its image in A one may suppose that all the Φ_{ij}'s are injective. We shall verify the conditions from lemma 3.5. Let $a \in A_i$ be a noninvertible element and put $Z = \{ x \in X_i : \det a(x) = 0 \}$. If Z consists only of isolated points of X_i then it is obvious that $a \in \overline{GL(A_i)}$. Thus we may assume that there is $x \in Z$ such that each neighbourhood of x is an infinite set.

Moreover by Lemma 2.4 we may suppose that $ae_{11}^t = e_{11}^t a = 0$ on some neighbourhood U of x for some t. Fix integers k',k such that

$$k \geq 2d + 4$$

$$2(k' - k) + 1 \geq d$$

Since U is an infinite open set and the C^*-algebras A is simple it follows by Proposition 2.1 that there is $j \geq i$ such that $L_{\bar{\Phi}_{ij}}(y) \cap U$ has at least k' elements for any $y \in X_j$. This enables us by using the proof of Lemma 2.3 to find an open covering $\mathcal{W}$ of X_j suh that for each $W \in \mathcal{W}$ there is $p_W \in A_j$ satisfying

1) p_W is projection valued on W

2) rank $p_W(y) \geq k'$ for any $y \in W$

3) $p_W \Phi_{ij}(a) = \Phi_{ij}(a) p_W = 0$ on W

4) $p_W \leq \Phi_{ij}(e_i)$ where e_i is the unit of A_i.

Proposition 3.2 provides us a projection $p \in A_j$ such that

a) $p(x) \leq \bigvee \{p_W(x) : W \in \mathcal{W}, x \in W\}$ for all $x \in X_j$.

b) rank $p(x) \geq k$ for all $x \in X_j$.

Of course 4) and a) imply that $p \leq \Phi_{ij}(e_i)$.

We have also

c) $\Phi_{ij}(a)p = p\Phi_{ij}(a) = 0$

as a consequence of 3) and a).

Let $b := \Phi_{ij}(a)$ have the components (b_t) with $b_t \in C(X_{jt}) \otimes M_{n(j,t)}$. We shall use Remark 3.4 in order to approximate each b_t by invertible elements in $\mathrm{End}_{C(X_{jt})}(E_t)$ where $E_t := \Phi_{ij}(e_i)C(X_{jt})^{n(j,t)}$. Consider also the finitely generated projective $C(X_{jt})$-modules

$$P_t = pC(X_{jt})^{n(j,t)}$$

$$Q_t = (\Phi_{ij}(e_i) - p)C(X_{jt})^{n(j,t)}.$$

It is clear that $E_t \simeq P_t \oplus Q_t$. Using the stability properties of vector bundles (see [14, Chapter 8]) one can find a finitely generated projective $C(X_{jt})$-module R_t of rank at most $d/2 + 1$ such that $Q_t \oplus R_t$ is free. Since $k \leq \mathrm{rank}\, P_t$ is large enough using again the results of [14, Chapter 8] we find finitely generated projective $C(X_{jt})$-modules G_t and H_t such that G_t is free of rank greater than $[(d+1)/2] + 1 \geq \max\left(\text{tsr}\, C(X_{jt}), \text{csr}\, C(X_{jt})\right)$ and P_t is isomorphic to $R_t \oplus G_t \oplus H_t$. Puting $F_t = Q_t \oplus R_t$ we have $b_t \in \overline{\text{End}_{C(X_{jt})}(Q_t)} \subset \text{End}_{C(X_{jt})}(F_t)$ by c). Using Remark 3.4 we find that $b_t \in \overline{GL_{C(X_{jt})}(E_t)}$.. It follows

$$\Phi_{ij}(a) = b \in \overline{GL_{C(X_j)}(\oplus E_t)} = \overline{GL(\Phi_{ij}(e_i)A_j\Phi_{ij}(e_i))}$$

and we are done by Lemma 3.5

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