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1. INTRODUCTION

This paper deals with the class of dual Banach algebras. These are Banach algebras that are duals of Banach spaces and for which the multiplication is separately weak* continuous. The study of (nonselfadjoint) dual algebras of operators on Hilbert space was initiated by S. Brown in [3] in connection with the invariant subspace problem. The Scott Brown's techniques were then applied to various classes of operators leading to several important results concerning the invariant subspaces and reflexivity of operators. See [2] for the stage of the theory until 1985 and also [1], [4], [5], [8], for recent developments.

In [6], G. Cassier proved that if the dual algebra generated by an operator T is uniform, then T has a nontrivial invariant subspace. This result is in fact obtained as a corollary of a decomposition theorem for this class of dual algebras. This theorem is proved in [6] under some extra conditions which are dropped in [7].

Our aim is to give a similar decomposition in the abstract context of dual Banach algebras.

2. PRELIMINARIES

In this section we recall some definitions and results from rational approximation theory that will be used in the sequel.

For any compact set $K \subset \mathbb{C}$, let $C(K)$ denote as usually the Banach algebra of all continuous, complex valued functions on K. Let also Rat(K) denote the algebra of rational functions with poles off K and let R(K) denote the uniform closure in C(K) of Rat(K). Let ∂K denote the (topological) boundary of K. Then R(K) is said to be Dirichlet (on ∂K) if $\mathrm{Re}R(K)|_{\partial K}$ is dense in $C_R(\partial K)$, the space of continuous real-valued functions on ∂K.

Assume R(K) is Dirichlet and let U = Int K. If U is nonempty, then for each $z \in U$ there exists a unique positive Borel measure λ_z on ∂K such that

$$f(z) = \int f d\lambda_z, \qquad f \in R(K)$$

Let $U = \bigcup_{n \geq 1} U_n$, where $\{U_n\}$ are the components of U. Then one knows (cf. [11]) that $\{U_n\}$ are simply connected. For each $n \in N$, pick $z_n \in U_n$ and define $m = \sum_{n \geq 1} 2^{-n} \lambda_{z_n}$. Then m is called the harmonic measure on ∂K and its equivalence class is independent on the choice of the points z_n. For any bounded open set $G \subset \mathbf{C}$, let $H^\infty(G)$ denote, as usual, the Banach algebra of all bounded analytic functions on G under the norm $\|f\|_\infty = \sup\{|f(z)| \, ; z \in G\}$. Since $H^\infty(G)$ is a weak* closed subspace in $L^\infty(G)$, it is the dual space of a certain factor space of $L^1(G)$. A sequence $\{f_n\} \subset H^\infty(G)$ converges weak* to zero iff it is bounded and converges pointwise to zero on G. See [14] for more about $H^\infty(G)$. The following result from [16] will be useful in the sequel.

LEMMA 2.1

Let $K \subset \mathbf{C}$ be a compact set such that R(K) is Dirichlet. Let also μ be a positive, finite Borel measure on ∂K which is singular with respect to the harmonic measure m on ∂K. Let $f \in L^\infty(\mu)$ and $g \in H^\infty(\mathrm{Int}\, K)$. Then there exists a sequence $\{f_n\} \subset R(K)$ such that:

1) $f_n \to f$ weak* in $L^\infty(\mu)$

2) $f_n \to g$ weak* in $H^\infty(\mathrm{Int}\, K)$

and

3) $\sup \|f_n\| \leq \max(\|f\|_\mu, \|g\|_\infty)$.

A subset S of a bounded open set $G \subset \mathbb{C}$ is said to be dominating in G if

$$\sup_{z \in S} |f(z)| = \sup_{z \in G} |f(z)| , \quad f \in H^{\infty}(G)$$

The next result borrowed from [16] will be also used in the proof of our main theorem. For a detailed proof, see [6].

LEMMA 2.2

Let $K \subset \mathbb{C}$ be a compact set such that $R(K) \neq C(K)$. Then there exists a compact set $L \supset K$ with $\mathrm{Int}\, L \neq \emptyset$ such that:

a) $R(L)$ is Dirichlet

and

b) $K \cap \mathrm{Int}\, L$ is dominating in $\mathrm{Int}\, L$.

3. AN EXTENSION THEOREM

In this section, we give a theorem concerning the extension of certain representations of R(K) with values in a dual algebra A. This result is related to the decomposition theorems for spectral sets, obtained in [12] and [13]. From now on, by a representation $\Phi : A \to B$, between two Banach algebras we mean a unital contractive homomorphism. The dual algebra $L^{\infty}(\mu) \oplus H^{\infty}(\mathrm{Int}\, K)$ is equipped with the norm $\|f \oplus g\| = \max \{ \|f\|_{\mu}, \|g\|_{\infty} \}$, $f \in L^{\infty}(\mu)$, $g \in H^{\infty}(\mathrm{Int}\, K)$.

THEOREM 3.1

Let $K \subset \mathbb{C}$ be a compact set such that R(K) is Dirichlet and let A be a dual Banach algebra with a separable predual X. Let $\Phi: R(K) \to A$ be a representation. Then there exists a positive, finite Borel measure μ on ∂K, singular with respect to the harmonic measure m on ∂K and a weak* continuous representation

$$\tilde{\Phi} : L^{\infty}(\mu) \oplus H^{\infty}(\mathrm{Int}\, K) \to A$$

which extends Φ. Moreover, $\tilde{\Phi}$ restricted to $L^{\infty}(\mu)$ is one-to-one.

Proof

Let $\{x_n\} \subset X$ be a dense sequence in the unit ball of X. For each $n \in N$, the map $L_n : R(K) \to \mathbb{C}$ defined by $L_n(f) = \langle \Phi(f), x_n \rangle$ is a continuous, contractive linear functional. Therefore, by the Hahn-Banach theorem and the Riesz representation theorem, there exists a complex, Borel measure μ_n on ∂K such that

$$L_n(f) = \int f d\mu_n , \qquad f \in R(K)$$

Moreover, since $\| x_n \| \leq 1$, we also have $\| \mu_n \| \leq 1$. Let $\mu_n = \mu_n^s + \mu_n^a$ be the Lebesgue decomposition of μ_n with respect to the harmonic measure m on ∂K, where $\mu_n^s \perp m$ and $\mu_n^a < m$. Let $\mu = \sum 2^{-n} |\mu_n^s|$, where $|\mu_n^s|$ denotes the total variation of μ_n^s. The extension $\tilde{\Phi}$ of Φ will be constructed as follows. For any pair of functions (f,g), where $f \in L^\infty(\mu)$ and $g \in H^\infty(\text{Int } K)$, there exists, by Lemma 2.1 a sequence $\{f_j\} \subset R(K)$ such that $f_j \to f$ w^* in $L^\infty(\mu)$, $f_j \to g$ w^* in $H^\infty(\text{Int } K)$ and $\| f_j \| \leq \max \{ \| f \|_\mu, \| g \|_\infty \}$ for all n N. We claim that the sequence $\{\Phi(f_j)\} \subset A$ is weak* convergent to an element $\tilde{\Phi}(f \oplus g) \in A$. Since $\{x_n\}$ is dense in $\{x \in X, \|x\| \leq 1\}$, and the unit ball of A is weak* sequentially compact, it suffices to show that $\{\langle \Phi(f_j), x_n \rangle\}_j$ is a convergent sequence, for each $n \in N$.

Let $h_n \in L^1(\mu)$ and $g_n \in L^1(m)$ such that $d\mu_n^s = h_n d_\mu$ and $d\mu_n^a = g_n dm$. Then:

$$\langle \Phi(f_j), x_n \rangle = \int f_j d\mu_n = \int f_j h_n d\mu + \int f_j g_n dm .$$

Since $f_j \to f$ weak* in $L^\infty(\mu)$,

$$\int f_j h_n d\mu \to \int f h_n d\mu$$

On the other hand, one knows (see [11]) that the restriction map from R(K) into $H^\infty(\text{Int } K)$ extends to an isometric isomorphism and a weak* homeomorphism between $H^\infty(\partial K)$, the weak* closure of R(K) in $L^\infty(m)$ and $H^\infty(\text{Int } K)$. We identify a function $g \in H^\infty(\text{Int } K)$ with its corresponding element in $H^\infty(\partial K)$.

Since $f_j \to g$ w^* in $H^\infty(\text{Int } K)$, it follows that $f_j \to g$ w^* in $H^\infty(\partial K)$, therefore

$$\int f_j g_n dm \to \int g g_n dm$$

Finally, one obtains that:

$$\langle \Phi(f_j), x_n \rangle \to \int f d\mu_n^a + \int g d\mu_n^s, \quad n \in N$$

One easily proves that the map

$$L^\infty(\mu) \oplus H^\infty(\text{Int } K) \ni f \oplus g \to \tilde{\Phi}(f \oplus g) \in A$$

is well defined and weak* continuous.

Moreover, $\|\tilde{\Phi}(f \oplus g)\| \leq \overline{\lim} \|\Phi(f_j)\| \leq \sup \|f_j\| \leq \max\{\|f\|_\mu, \|g\|_\infty\}$, hence $\tilde{\Phi}$ is contractive. In order to show that $\tilde{\Phi}$ is multiplicative, one uses the separate weak* continuity of the multiplication map on A.

For the injectivity of its restriction to $L^\infty(\mu)$, observe that if $\tilde{\Phi}(f \oplus 0) = 0$, then

$$\int f g \mu_n^s = 0 \text{ for all } g \in C(\partial K) \text{ and } n \in N$$

Therefore $f = 0$ $|\mu_n^s|$-almost everywhere, hence $f = 0$ in $L^\infty(\mu)$.

4. SINGLY GENERATED DUAL ALGEBRAS

If A is a dual algebra and $a \in A$, then the dual algebra generated by a is the smallest weak* closed subalgebra of A containing a and the identity. This coincides with the weak* closure in A of the set $\{p(a);\ p \text{ polynomial}\}$. By the separate weak* continuity of the multiplication, a singly generated dual algebra is always commutative. Recall now that a commutative Banach algebra is said to be uniform if the Gelfand transform is an isometry.

Let H be a separable, complex Hilbert space and let L(H) denote the algebra of all bounded linear operators on H. One knows that L(H) is the dual space of the space $C_1(H)$ of trace-class operators on H, via the pairing

$$\langle T, L \rangle = \text{tr}(TL), \quad T \in L(H), \quad L \in C_1(H)$$

If $S \in L(H)$ is a subnormal operator (i.e. restriction of a normal operator to an invariant subspace) then one knows (cf. [9]) that the dual algebra A_S generated by S in L(H) is a uniform algebra. For this special case, J. Conway and R. Olin proved, ([10]) a structure theorem with important consequences concerning the theory of subnormal operators.

Other important examples of singly generated dual uniform algebras are furnished by operators in the class A (see [2] for terminology).

The main result of the paper is the following:

THEOREM 4.1

Let A be a singly generated dual uniform algebra with separable predual. Let $a \in A$ be one of its generators and let K denote the spectrum of a. Then

α) R(K) is Dirichlet

and

β) there exists a positive, finite Borel measure μ on ∂K, singular with respect to the harmonic measure on ∂K such that the natural map

$$\mathrm{Rat}(K) \ni f \to f(a) \in A$$

extends to an isometric isomorphism and a weak* homeomorphism

$$\tilde{\Phi} : L^{\infty}(\mu) \oplus H^{\infty}(\mathrm{Int}\, K) \to A.$$

Proof

Let M_A denote the maximal ideal space of A and let $\psi \in M_A$. Then for each $f \in \mathrm{Rat}(K)$, $\psi(f(a)) = f(\psi(a))$, therefore $\|f(a)\| = \sup_{\psi \in M_A} |\psi(f(a))| \leq \sup_{z \in K} |f(z)|$. Thus the map $\mathrm{Rat}(K) \ni f \to f(a) \in A$ extends to a contractive representation

$$\Phi : R(K) \to A$$

Assume first that $R(K) \neq C(K)$. Then by Lemma 2.2, there exists a compact set $L \supset K$ such that R(L) is Dirichlet and $K \cap \mathrm{Int}\, L$ is dominating in Int L. Let $U = \mathrm{Int}\, L$ and let

$\Phi_1 : R(L) \to A$ defined by $\Phi_1(f) = \Phi(f|_K)$, $f \in R(L)$. By Theorem 2.2, Φ_1 extends to a weak* continuous contractive representation

$$\tilde{\Phi} : L^\infty(\mu) \oplus H^\infty(U) \to A$$

for some positive, finite Borel measure μ on ∂L, singular with respect to the harmonic measure. We will show that $\tilde{\Phi}$ is isometric. Let us denote $a_o = \tilde{\Phi}(z \oplus 0)$ and $a_1 = \tilde{\Phi}(0 \oplus z)$. Then, obviously, $a = a_o + a_1$ and $a_o a_1 = 0$. For any $\psi \in M_A$, $\psi(a) = \psi(a_o) + \psi(a_1)$ and $\psi(a_o)\psi(a_1) = 0$, hence $\psi(a) = 0$ if and only if $\psi(a_o) = 0$ or $\psi(a_1) = 0$.. It follows that $\sigma(a) = \sigma(a_o) \cup \sigma(a_1)$. Moreover, since the measure μ is supported on ∂L and $\tilde{\Phi}(z \oplus 0) = a_o$, it follows that $\sigma(a_o) \subset \partial L$.

Let $f \in H^\infty(U)$ and let $\lambda \in K \cap U$. Then

$$f(z) - f(\lambda) = g(z)(z - \lambda), \quad z \in U$$

for some $g \in H^\infty(U)$.

By applying $\tilde{\Phi}$, we obtain:

$$\tilde{\Phi}(0 \oplus f) - f(\lambda)p_1 = \tilde{\Phi}(0 \oplus g)(a_1 - \lambda)$$

where $p_1 = \tilde{\Phi}(0 \oplus 1)$.

Let $\psi \in M_A$ such that $\psi(a) = \lambda$. Since $\sigma(a_o) \subset \partial L$ and $\lambda \in U$, we have that $\psi(a_1) = \lambda$, hence

$$\psi(\tilde{\Phi}(0 \oplus f)) = f(\lambda)\psi(p_1)$$

Suppose now that $\psi(p_1) = 0$. Then $\psi(\tilde{\Phi}(0 \oplus h)) = 0$ for every $h \in H^\infty(U)$, in particular for $h(z) = z$.

This means that $\psi(a_1) = 0$ and $\psi(a_o) = \lambda$, which contradicts the fact that $\sigma(a_o) \subset \partial L$. Therefore, $\psi(p_1) = 1$ and $\psi(\tilde{\Phi}(0 \oplus f)) = f(\lambda)$.

Since $K \cap U$ is dominating in U, one obtains

$$\|f\|_\infty = \sup_{\lambda \in K \cap U} |f(\lambda)| \leq \|\tilde{\Phi}(0 \oplus f)\| \leq \|f\|_\infty$$

Therefore $\tilde{\Phi}$ restricted to $H^\infty(U)$ is isometric. On the other hand, since $\tilde{\Phi}$

restricted to $L^\infty(\mu)$ is one-to-one, by Theorem 2.2, and A is a uniform algebra, it follows easily that $\tilde{\Phi}$ restricted to $L^\infty(\mu)$ is isometric, hence $\tilde{\Phi}$ itself is isometric on $L^\infty(\mu) \oplus H^\infty(U)$. By a standard application of the Krein-Smilian theorem, one obtains that $\tilde{\Phi}$ is an isometric isomomorphism and a weak* homeomorphism from $L^\infty(\mu) \oplus H^\infty(U)$ onto A. Moreover, we get $\sigma(a) = \bar{U}$, therefore, since by construction, L is K union with some holes, we have K = L. The case R(K) = C(K) is easier to prove following ideas from Theorem 3.1. The proof is finished.

Since the components of Int K are simply connected provided R(K) is Dirichlet, the following consequence of Theorem 4.1 holds:

COROLLARY 4.2.

Let A be a singly generated uniform algebra and let $a \in A$ be one of its generators. Then there exist:

1) a finite, positive Borel measure μ, supported on the boundary of the spectrum of a

2) simply connected regions $G_1, G_2, \ldots$ in $\mathbb{C}$ and

3) an isometric isomorphi and weak* homeomorphism

$$\Phi : L^\infty(\mu) \oplus \left(\bigoplus_{n \geq 1} H^\infty(G_n)\right) \to A$$

sucht that $\Phi(z) = a$.

The first such decomposition appear in [16], in which it is given a characterization for $P^\infty(\mu)$, the weak* closure of polynomials in $L^\infty(\mu)$. For the case of dual algebras generated by subnormal operators, this decomposition is proved in [10], as we have mentioned in the first part of the paper.

We close with some few remarks concerning the above corollary. The first is that, since a is a weak* generator of A, the measure μ satisfies $P^\infty(\mu) = L^\infty(\mu)$. Similarly one gets that polynomials are weak* dense in $H^\infty(G_n)$ for each $n \geq 1$. Therefore, if

$h_n : D \to G_n$ is a conformal mapping, then h_n is a weak* generator of $H^\infty(D)$. For a topological characterization of such domains, see [15].

If A_T is a singly generated dual uniform subalgebra of L(H), for some separable Hilbert space, then the idempotents P_o and P_1 corresponding to the functions $1 \oplus 0$ and $0 \oplus 1$ in $L^\infty(\mu) \oplus H^\infty(\text{Int } K)$ are orthogonal projections in L(H) with $P_o P_1 = 0$. Therefore $T = T_o \oplus T_1$ with T_o a reductive normal operator and T_1 having an isometric functional calculus with functions in $H^\infty(\text{Int } K)$.

As a final remark, in both Theorems 3.1 and 4.1 one of the two summands of the dual algebra $L^\infty(\mu) \oplus H^\infty(\text{Int } K)$ may be absent. For example, if A has no nontrivial projections, then either $A = \mathbb{C}$ or Int K is connected and therefore A is isometric with $H^\infty(\text{Int } K)$. On the other hand, if $R(\sigma(a)) = C(\sigma(a))$ for some generator $a \in A$, then $A = L^\infty(\mu)$, for some positive measure on $\sigma(a)$, hence A is a W^*-algebra.

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