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AN EXCISION FREE CHERN CHARACTER FOR
P-SUMMABLE QUASIHOMOMORPHISMS

- preliminary version -

by

VICTOR NISTOR

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An excision free Chern character for p-summable quasihomomorphisms

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by Victor Nistor

Introduction

In [Co 2, Co 3] Connes has defined the cyclic cohomology $HC^*(A)$ of an algebra A over $\mathbb{C}$ and has defined a pairing

$$K_*(A) \otimes HC^*(A) \longrightarrow \mathbb{C}$$

and a Chern character

$$K^*(A) \longrightarrow HC^*(A) .$$

In [Co4] Connes has raised the problem of defining a bivariant cyclic theory and a Chern character

$$KK(A, B) \longrightarrow HC^*(A, B)$$

compatible with the product of Kasparov's bivariant K-theory.

Important steps have been made in this program. In [JK] Jones and Kassel have defined a bivariant cyclic theory enjoing all the formal properties of Kasparov's theory. In [Wa , Kas] Wang and Kassel have defined a Chern character with values in Jones-Kassel bivariant cyclic theory. However their definition is not completely satisfactory since it assumes certain excision properties (H-unitality [Wo]) which are usually not satisfied in practice. As observed by Wang one could eliminate this drawback by providing an explicit formula for the Chern character of an arbitrary p-summable quasihomomorphism à la Cuntz [Cu] ; he also provides such a formula for p = 1.

In this paper we show that Wang's formula may be adjusted to provide such formulae by a sequence of homotopies suggested by the work of

Goodwillie [G]. We also give an explicit functorial form of the operator E_D which Goodwillie has shown to exist.

Thus for a p-summable quasihomomorphism (i.e. a pair $\varphi_0, \varphi_1 : A \to B(H)\hat{\otimes}B$ such that $\varphi_0 - \varphi_1$ factors continuously as $A \to C_p\hat{\otimes}B \to B(H)\hat{\otimes}B$) we associate a sequence

$$CH_o^{2n}(\varphi_0, \varphi_1) \in HC^{2n}(A, B) \qquad n \geqslant p-1$$

such that $S\, Ch_o^{2n}(\varphi_0, \varphi_1) = Ch_o^{2n+2}(\varphi_0, \varphi_1)$ where S is the periodicity operator of the bivariant cyclic theory [JK].

Suggested by the case of the Chern character on K_0 we consider smooth C^*-algebras. For such C^*-algebras A and B we define a "smooth" variant of Kasparov's groups $K_{smooth}(A, B)$ and we show that the above Chern character Ch_o^{2n} define after tacking the limit a morphism

$$Ch_o : K_{smooth}(A, B) \to PHC_0(A, B)$$

compatible with the Chern character on K_* and the action of KK on K-theory.

The same results hold true for smooth extensions, i e extensions of the form

$$0 \to C_p\hat{\otimes}B \to E \to A \to 0$$

which have continuous linear cross sections. This may be proved along the same lines using Cuntz's description of the universal extension [Cu 2].

Our results may be viewed as generalization of some results of Quillen [Q] replacing 0-cocycles (i.e. traces) with arbitrary cocycles.

1.1 Let us recall first some basic facts about the Jones-Kassel bivariant cyclic theory [J.-K.]

A differential graded Λ-module (abreviated dg-Λ-module) is a differential module (X_n, b), $b: X_n \to X_{n-1}$, $b^2 = 0$ endowed with a degree 1 map $B: X_n \to X_{n+1}$ satisfying $B^2 = 0$, $[B,b] = Bb + bB = 0$. Here Λ is $\mathbb{C} + \mathbb{C}\varepsilon$, the algebra of dual numbers ($\varepsilon^2 = 0$) and B corresponds to the multiplication by ε.

Let $B(\Lambda) = \mathbb{C}[u]$, the polinomial algebra on a degree 2 generator u. If X is a dg-Λ-module then $B(\Lambda) \otimes X$ is endowed with the differential

$$d(u^p \otimes x) = u^p \otimes bx + u^{p-1} \otimes Bx$$

and a degree -2 map $S: B(\Lambda) \otimes X \to B(\Lambda) \otimes X$

$$S(u^p \otimes x) = u^{p-1} \otimes x, \quad \text{if } p > 0, \quad 0 \text{ otherwise} .$$

If Y is an other dg-Λ-module then $\mathrm{Hom}_S(B(\Lambda) \otimes X, B(\Lambda) \otimes Y)$ is the complex of morphisms $f: B(\Lambda) \otimes X \to B(\Lambda) \otimes Y$ commuting with S. A degree n such a morphism is realised by a sequence $(f_i)_{i \geq 0}$, $f_i: X \to Y$, f_i of degree $n + 2i$ such that

$$f(u^p \otimes x) = \sum_{i=0}^{p} u^{p-i} \otimes f_i(x)$$

formally

$$f = \sum_{i \geq 0} S^i \otimes f_i$$

The differential is $df = d\circ f - (-1)^{|f|} f\circ d$. Here $|f|$ is the degree of f. If f is as above then $df = \sum_{i\geq 0} S^i \otimes g_i$ where $g_0 = [b, f_0]$, $g_i = [B, f_{i-1}] + [b, f_i]$ $i\geq 1$ (the commutators are always graded commutators).

Then $HC^n(X,Y) = H_{-n}(\mathrm{Hom}_S(B(\Lambda)\otimes X, B(\Lambda)\otimes Y))$.

There exists a degree 2 morphism $S: HC^n(X,Y) \longrightarrow HC^{n+2}(X,Y)$ comming from periodicity. If $[f] \in HC^n(X,Y)$ is represented by $f = \sum_{i\geq 0} S^i \otimes f_i$ then $S[f]$ is represented by $\sum_{i\geq 0} S^{i+1} \otimes f_i$. The composition of morphisms defines a pairing $-\circ-: HC^n(X,Y) \otimes HC^m(Y,Z) \longrightarrow HC^{n+m}(X,Z)$ the Yoneda product. If $f = \sum_{i\geq 0} S^i \otimes f_i$ and $g = \sum_{i\geq 0} S^i \otimes g_i$ then $[g]\circ[f] = [h]$ for $h = \sum_{i\geq 0} S^i \otimes h_i$, $h_k = f_0 g_k + f_1 g_{k-1} + \ldots + f_k g_0$. Also

$$S[f] = S[id_X] \circ [f] = [f] \circ S[id_Y]$$

for any $[f] \in HC^n(X,Y)$. No confusion will arrise if we shall write S instead of $S[id_X]$, $S[id_Y]$ so the previous identity becomes

$$S[f] = S\circ[f] = [f]\circ S$$

1.2. Let us also recall that the dg-Λ-module associated to a unital algebra A over $\mathbb{C}$ is $B(A) = (C_n(A), b, B)$ where $C_n(A) = A \otimes (A/\mathbb{C})^{\otimes n}$ and

$$b a_0 \otimes \ldots \otimes a_n = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$$

$$B a_0 \otimes \ldots \otimes a_n = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

Note that we have normalised from the begining .

We shall write $HC^n(A,X)$ instead of $HC^n(B(A),X)$, 1.3. Let $D:A \to A, D(1) = 0$, be an arbitrary function. Following Goodwillie [G] we define

$$e_D : C_n(A) \to C_{n-1}(A)$$

$e_D\, a_0 \otimes \ldots \otimes a_n = a_0\, D(a_1) \otimes a_2 \otimes \ldots \otimes a_n$, and

$$E_D : C_n(A) \to C_{n+1}(A)$$

$$E_D\, a_0 \otimes \ldots \otimes a_n = \sum_{i=1}^{n} 1 \otimes a_0 \otimes \ldots \otimes Da_i \otimes \ldots \otimes a_n +$$

$$+ \sum_{j=2}^{n} (-1)^{nj} \sum_{i=1}^{j-1} 1 \otimes a_j \otimes \ldots \otimes a_0 \otimes \ldots \otimes Da_i \otimes \ldots \otimes a_{j-1}$$

(D occurs only to the right of a_0 and exactly once in each term. This is an explicit form of a formula shown to exist by Goodwillie, loc. cit.)

$$L_D' : C_n(A) \to C_n(A)$$

$$L_D'\, a_0 \otimes \ldots \otimes a_n = \sum_{i=0}^{n} a_0 \otimes \ldots \otimes Da_i \otimes \ldots \otimes a_n .$$

Also let $l : A \otimes A \to A$, $l\, a \otimes b = D(ab) - aD(b) - D(a)b$

$$f_0 : C_n(A) \to C_{n-2}(A)$$

$$f_0 a_0 \otimes a_1 \otimes \ldots \otimes a_n = a_0 l(a_1, a_2) \otimes a_3 \otimes, \ldots, \otimes a_n$$

$$f_1 : C_n(A) \to C_n(A)$$

$$f_1\, a_0 \otimes a_1 \otimes, \ldots, \otimes a_n = \sum_{k=1}^{n-1} (-1)^k\, 1 \otimes a_0 \ldots \otimes l(a_k, a_{k+1}) \otimes \ldots \otimes a_n +$$

$$+ \sum_{j=2}^{n-1} \sum_{k=1}^{j-1} (-1)^{k+(n-1)j}\, 1 \otimes a_{j+1} \otimes \ldots \otimes a_0 \otimes \ldots \otimes l(a_k, a_{k+1}) \otimes \ldots \otimes a_j$$

We let $j_D = 1 \otimes e_D + S \otimes E_D$, $L_D = 1 \otimes L_D'$, $f = 1 \otimes f_0 + S \otimes f_1$

1.4. <u>Lemma</u> $dj_D = SL_D - f$.

If D is a derivation this is simply the check of [G, Theorem II.4.2]

<u>Proof</u> . We have to show that

i) $[b, e_D] = -f_0$

ii) $[B, e_D] + [b, E_D] = L_D' - f_1$

iii) $[B, E_D] = 0$

iii) is obvious since we are working with the normalised dg-Λ -module and i) is an obvious computation.

For ii) let $D_i : C_n(A) \longrightarrow C_n(A)$ $D_i a_0 \otimes \ldots \otimes a_n = a_0 \otimes \ldots \otimes D(a_i) \otimes \ldots \otimes a_n$

$d_k a_0 \otimes \ldots \otimes a_n = (-1)^k a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_n$, $k = 0, \ldots, n-1$.

$d_n a_0 \otimes \ldots \otimes a_n = (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$

$t : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$

$t\, a_0 \otimes \ldots \otimes a_n = (-1)^n a_1 \otimes a_2 \otimes \ldots \otimes a_n$.

(Note that the above definitions differ to the usual ones mainly by signs).

$$b' = \sum_{k=0}^{n-1} d_k$$

of course $b = b' + d_n$

Finally, let $s : C_n(A) \longrightarrow C_{n+1}(A)$

$s\, a_0 \otimes \ldots \otimes a_n = 1 \otimes a_0 \otimes \ldots \otimes a_n$

Recall [Co3, LQ] that

$b's + sb' = 1$

and $B = s \sum_{j=0}^{n} t^j$. Then

$e_D = d_0 D_1$

$$Be_D = s\sum_{j=0}^{n-1} t^j d_0 D_1$$

$$e_D\, s = D_0$$

$$e_D B = \sum_{j=0}^{n} D_0 t^j$$

$$L_D = \sum_{i=0}^{n} D_i$$

$E_D = s\, E_D'$ where

$$E_D' = \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} t^j D_i .$$

The relation $bs = 1-t^{-1} - sb'$ shows that

$$bE_D = (1 - t^{-1} - sb')E_D' = L_D' - \sum_{j=0}^{n} t^j D_j - sb'E_D'$$

Let us observe that $t^j D_j = D_0 t^j$ and hence $\sum_{j=0}^{n} t^j D_j = \sum_{j=0}^{n} D_0 t^j$.

We obtain that ii) reduces to $f_1 = s\, f_1'$ where

$$f_1' = b'E_D' - E_D' b - \sum_{j=0}^{n-1} t^j d_0 D_1$$

we compute

$$b'E_D' = \sum_{k=0}^{n-1} \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} d_k\, t^j\, D_i$$

now

$$d_k\, t^j = \begin{cases} t^j\, d_{k+j} & k+j \leq n \\ t^{j-1}\, d_{k+j-n-1} & k+j \geq n+1 \end{cases}$$

Next we divide the sum in two parts , the first one contains the terms with $k+j \leq n$ and the second one contains the terms with $k+j \geq n+1$. We replace $k+j$ by k in the first part and k+j-n-1 by k and j-1 by j in the second.

We obtain

$$b'E_D' = \sum_{k=2}^{n} \sum_{j=2}^{k} \sum_{i=1}^{j-1} t^j d_k D_i + \sum_{k=0}^{n-1} \sum_{j=k+1}^{n} \sum_{i=1}^{j} t^j d_k D_i$$

$$E_D' b = \sum_{k=0}^{n} \sum_{j=2}^{n} \sum_{i=1}^{j-1} t^j D_i d_k$$

Let us observe that if we denote $(-1)^k a_0 \otimes \ldots \otimes 1(a_k, a_{k+1}) \otimes \ldots \otimes a_n$ by $1_k\, a_0 \otimes \ldots \otimes a_n$ we obtain

$$D_i d_k = \begin{cases} d_k D_i & 1 \leqslant i \leqslant k-1 \\ d_k(D_k + D_{k+1}) + 1_k & i = k \\ d_k D_{i+1} & i \geqslant k+1 \end{cases}$$

Using these relations in the formula for $E_D' b$ we get

$$E_D' b = \sum_{j=2}^{n} \sum_{i=2}^{j} t^j d_0 D_i + \sum_{k=2}^{n} \sum_{j=2}^{k} \sum_{i=1}^{j-1} t^j d_k D_i +$$

$$+ \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \sum_{i=1}^{j} t^j d_k D_i + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} t^j 1_k$$

Then

$$f_1' = b'E_D' - E_D' b - \sum_{j=0}^{n-1} t^j d_0 D_1 = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} t^j 1_k$$

The rest is obvious.

1.5. As in [Wa, Wo], given an ideal $I \subset A$ we consider the filtration of $B(A) = F_0(A, I) \supset F_{-1}(A, I) \supset F_{-2}(A, I) \supset \ldots$

where

$$F_{-n}(A,I)_k = \sum_{\nu_0+\dots+\nu_k=n} I^{\nu_0} \otimes I^{\nu_1} \otimes \dots \otimes I^{\nu_k} \qquad (I^0 = A).$$

This filtration depends functorialy on the pair (A, I) in the sense that any unital morphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ gives rise to a morphism $F_{-n}(A, I) \to F_{-n}(B,J)$ of dg-Λ-modules .

Of course $F_{-1}(A, I)$ is the kernel of $B(A) \to B(A/I)$.

From now on we shall no longer assume A to have a unit.

We shall use these remarks in the following specific situation. Let $QA = A * A$ be the free product and qA the kernel of $A * A \to A$ [Cu1]. Let $i_0, i_1 : A \to QA$ be the canonical embedings and $qa = i_0(a) - i_1(a)$. Recall [Cu1] that these simbols satisfy

$$q(ab) = aq(b) + q(a)b - q(a)q(b)$$

(we have identified A with $i_0(A)$) and that QA and ΩA (the universal differential graded algebra of A [A, Ka1]) are linearely isomorphic [CC] via

$$a_0 q\, a_1 \dots q\, a_n \to a_0\, da_1 \dots da_n, \quad qa_1 \dots qa_n \to da_1 \dots da_n$$

From now on we shall identify QA with $\Omega A = A \oplus \bigoplus_{n>0} A^+ \otimes A^{\otimes n}$, A^+ being the algebra with adjoint unit. We shall endow qA with the negative of the grading inherited from ΩA(i.e. $|qa| = -1$) .

QA becomes isomorphic with ΩA for the following multiplication

$$\omega_1 \circ_1 \omega_2 = \begin{cases} \omega_1 \omega_2 & |\omega_1| \text{ even} \\ \omega_1 \omega_2 + \omega_1\, d\omega_2 & |\omega_1| \text{ odd} \end{cases} \qquad [CC]$$

Note that QA (as well as qA and ΩA) is not unital, so we shall have to consider QA^+ , the algebra with a degree 0 adjoint unit . Let $D : QA^+ \to QA^+$, $D\omega = -|\omega|\omega$ for homogeneous ω. Then $l(a, b) = D(ab) - aD(b) - D(a)b = + adb$ if $|a|$ is odd, 0 otherwise , l is a degree -1 map.

1.6. i_0, i_1 extend to morphisms $A^+ \to QA^+$ denoted also i_0 and i_1 The corresponding morphisms $B(i_k) : B(A^+) \to B(QA^+)$ have the property that $(B(i_0) - B(i_1))(B(A^+)) \subset F_{-1}(QA^+, qA)$ and $B(i_0) = B(i_1)$ on $B(\mathbb{C}) \subset B(A^+)$ and hence $B(i_0) - B(i_1)$ defines an element

$$Ch_0^0(A) \in HC^0(B(A^+) / B(\mathbb{C}), F_{-1}(QA^+, qA))$$

According to lemma 1.4. we may define $S_n = S - n^{-1}dj_D \in Hom_S(F_{-n}, F_{-n-1})$ $(F_{-k} = F_{-k}(QA^+, qA)$, $k \geq 1)$ Let $i : F_{-n-1} \to F_{-n}$ be the inclusion

1.7. Lemma. a) $[S_n] \circ [i] = S$ in $HC^2(F_{-n}, F_{-n})$

b) $[i] \circ [S_n] = S$ in $HC^2(F_{-n-1}, F_{-n-1})$

c) If $x \in HC^2(F_{-n}, F_{-n-1})$ is an other element satisfying either a) of b) then $Sx = S[S_n]$.

<u>Proof</u> a) and b) are consequences of lemma 1.4.

Let x satisfy $x[i] = S$ then $Sx = x S = x[i] \circ [S_n] = S[S_n]$.

1.8. <u>Theorem</u>. (Definition and existence of the Chern character of the universal quasihomomorphism)

There exists $Ch_0^{2n}(A) = Ch_D^{2n}(A, i_0, i_1)$

a) $Ch_0^{2n}(A) \in HC^{2n}(B(A^+) / B(\mathbb{C}), F_{-n-1}(QA^+, qA))$

b) $Ch_0^0(A)$ is as defined in 1.6.

c). $Ch_0^{2n}(A)\circ[i] = S\, Ch_0^{2n-2}(A)$, $n \geqslant 1$

d) Given a morphism $\varphi : A \to B$ then

$$Ch_0^{2n}(A)\circ[\varphi''] = [\varphi']\circ Ch_0^{2n}(B)$$

where $\varphi' : B(A^+) / B(\mathbb{C}) \to B(B^+) / B(\mathbb{C})$ and $\varphi'' : F_{-n-1}(QA^+, qA) \to F_{-n-1}(QB^+, qB)$ are defined by φ.

1.9. Theorem. (Uniqueness of the Chern character of the universal quasihomomorphism up to stabilisation)

If $Ch_0'^{2n}$ satisfies a), b) and c) then $S\, Ch_0'^{2n} = S\, Ch_0^{2n}(A)$.

Proof of 1.8.

Let $Ch_0^{2n}(A) = Ch_0^0(A)\circ[S_1]\circ\ldots\circ[S_n]$ where $Ch_0^0(A)$ is as defined in 1.6.

c) follows from lemma 1.7. d) follows from the naturality of the definition of S_n.

Proof of 1.9.

We proceed by induction on n. For n = 0 it is part of the hypothesis. Suppose now that $S\, Ch_0'^{2n-2}(A)\circ[i] = S\, Ch_0^{2n-2}(A)$ then $Ch_0'^{2n}\circ[i] = Ch_0^{2n}(A)\circ[i]$

Multiplying by S_n the right we get the conclusion.

1.10. The next step is the definition of the Chern character of a p-summable quasihomomorphism.

Let A, B be complete locally convex algebras.

Recall [Co 2, Cu1, Wa] that a p-summable quasihomomorphism (φ_0, φ_1) is a pair of continuous morphisms $\varphi_0, \varphi_1 : A \to L(H)\hat{\otimes} B$ such that $\varphi_0 - \varphi_1 : A \to C_p \hat{\otimes} B$ is well defined and continuous. We shall write $(\varphi_0, \varphi_1) : A \to L(H)\hat{\otimes} B \rhd C_p \hat{\otimes} B$. Given such a quasihomomorphism (φ_0, φ_1) there are defined morphisms $QA \to L(H)\hat{\otimes} B$, $qA \to C_p \hat{\otimes} B$ [Cu1]. Following [Wa] there exists for any $n \geq p - 1$ a morphism of dg-Λ-modules

$$tr : F_{-n-1}(L(H)\hat{\otimes} B^+, C_p \hat{\otimes} B) \to B(B^+)/B(\mathbb{C})$$

defined by $tr(T_0 \otimes b_0, \ldots, T_n \otimes b_n) = tr(T_0 \ldots T_n) b_0 \otimes \ldots \otimes b_n$.

(Here L(H) is the C^*-algebra of bounded operators in the Hilbert space H, $C_p \subset L(H)$ is the subspace of those $T \in B(H)$ such that $tr(T^*T)^{p/2} < \infty$ [Si] and $\hat{\otimes}$ is the projective tensor product [Gr].)

Let $g_n = S_n \circ S_{n-1} \circ \ldots \circ S_1 \circ (B(i_0)) - B(i_1)) : B(A^+)/B(\mathbb{C}) \to F_{-n-1}(QA^+, qA)$

$$\chi_n : F_{-n-1}(QA^+, qA) \to F_{-n-1}(L(H)\hat{\otimes} B, C_p \hat{\otimes} B)$$

1.11. Definition

$$Ch_0^{2n}(\varphi_0, \varphi_1) = Ch_0^{2n}(A) \circ [\chi_n] \circ [tr] \in HC^{2n}(A,B), \qquad n \geq p - 1$$

If one whishes one may consider the topological bivariant cyclic theory. It is defined as before but considering continuous morphisms [JK]. Then $Ch_0^{2n}(\varphi_0, \varphi_1)$ is the class of $tr \circ \chi_n \circ g_n$.

Here $HC^*(A,B) = HC^*(B(A^+)/B(\mathbb{C}), B(B^+)/B(\mathbb{C}))$.

1.12. Proposition. Let φ_0, φ_1 be a p-summable quasihomomorphism as above then

a) $S\, Ch_0^{2n}(\varphi_0,\varphi_1) = Ch_0^{2n+2}(\varphi_0,\varphi_1)$, $n \geq p-1$

b) Ch_0^{2n} depends functorialy on A and B i.e.

$$Ch_0^{2n}(\varphi_0 \circ \psi, \varphi_1 \circ \psi) = [B(\psi)] \circ Ch_0^{2n}(\varphi_0,\varphi_1) \qquad \psi : A' \longrightarrow A$$

$$Ch_0^{2n}((1\otimes\psi')\circ\varphi_0, (1\otimes\psi')\circ\varphi_1) = Ch_0^{2n}(\varphi_0,\varphi_1)\circ B(\psi') \quad , \psi' : B \to B'$$

ψ and ψ' being continuous.

Proof.

Let $i : F_{-n-2}(QA^+, qA) \to F_{-n-1}(QA^+, qA)$ be the inclusion, then

$$[\chi_{n+1}]\circ[tr] = [i]\circ[\chi_n]\circ[tr]$$

and hence

$$Ch_0^{2n+2}(\varphi_0,\varphi_1) = Ch_0^{2n+2}(A)\circ[\chi_{n+1}]\circ[tr] = Ch_0^{2n+2}(A)[i]\circ[\chi_n]\circ[tr] =$$

$= S\, Ch_0^{2n}(\varphi_0,\varphi_1)$ by Theorem 1.8. c).

The functoriality is clear from definition.

1.13. Corollary. $(Ch_0^{2n}(\varphi_0,\varphi_1))_{n\geq p-1}$ gives a well defined element

$Ch_0(\varphi_0,\varphi_1) \in PHC^0(A,B) = \varinjlim (PHC^{2n}(A,B), S)$.

2. The relation with the Chern character in K-theory

2.1. We want now to define something like KK(A,B) if A, B are C^*-algebras as a natural domain of our Chern character. We stay very close to the definitions in [Ka, Cu1].

2.2. Definition. Let A be a C^*-algebra. We shall say following Connes that A is a smooth C^*-algebra if there is given a dense complete self adjoint locally convex algebra $A^\infty \subset A$ such that whenever $a \in M_n(A^{\infty +})$ and f is an analytic function in a neighborhood of $\sigma(a)$ then $f(a) \in M_n(A^{\infty +})$.

A^∞ will be called a smooth subalgebra of A.

If moreover $C_p \hat{\otimes} A^\infty$ is smooth in $K \otimes A$ then A^∞ is called absolutely smooth.

2.3. Example. If X is a smooth manifold then $C_c^\infty(X)$ is an absolutely smooth subalgebra of $C_0(X)$.

2.4. Let A, B be two non commutative manifolds with $A^\infty \subset A$, $B^\infty \subset B$ the associated smooth algebras.

Let H_B be the Hilbert space on B [Ka], i.e. the completion of $B^{(N)}$ $N = 1, \ldots, \aleph_0$ for $\|(b_0, \ldots, b_n, 0, \ldots)\| = \|\sum_{k \geq 0} b_k^* b_k\|^{1/2}$.

We define $\mathcal{E}_p(A,B) = \{ (\varphi_0, \varphi_1), \varphi_0, \varphi_1$ are $*$-homomorphisms $A \to L(H_B)$ such that $\varphi_{i|A^\infty}$, $i = 0,1$ factor continuously $A^\infty \to L(H) \hat{\otimes} B^\infty \to L(H_B)$ and $\varphi_0 - \varphi_{1|A^\infty}$ factors continuously as $A^\infty \to C_p \otimes B^\infty \to L(H_B) \}$

$$\mathcal{D}_p(A, B) = \{ (\varphi_0, \varphi_0) \in \mathcal{E}_p(A, B) \} .$$

These are "smooth"-forms of the cycles of KK-theory in Cuntz's picture.

The pair (φ_0, φ_1) defines by restriction a p-summable quasihomomorphism $\varphi_0, \varphi_1 : A^\infty \to B(H) \hat{\otimes} B^\infty \triangleright C_p \hat{\otimes} B^\infty$.

We define addition as in case of KK-theory.

Homotopy is replaced by smooth homotopy : $x_0, x_1 \in \mathcal{E}_p(A, B)$ are called smoothly homotopic if there exists $x \in \mathcal{E}_p(A, C([0,1], B))$ which restricted at the end points gives x_0, respectively x_1. Here $C([0,1], B)$ is endowed with the smooth structure defined by $C^\infty([0,1], B^\infty)$.

2.5. We let on $\mathcal{E}_{smooth}(A, B) = \bigcup_{p \geq 1} \mathcal{E}_p(A, B)$ the equivalence relation $\equiv$ generated by

1) addition of degenerate elements :

$x_0 \equiv x_1$ if $x_0 + y = x_1$ for some $y \in \mathcal{D}_{smooth}(A, B) = \bigcup_{p \geq 1} \mathcal{D}_p(A, B)$.

2) smooth homotopy : $x_0 \equiv x_1$ if x_0 is smoothly homotopic to x_1.

Let $K_{smooth}(A,B) = \mathcal{E}_{smooth}(A,B) / \equiv$.

2.6. Observation a) Let $x = (\varphi_0, \varphi_1)$ and $u \in L(H) \hat{\otimes} B^\infty$ be a unitary then $(ad_u \circ \varphi_0, ad_u \circ \varphi_1)$ represents the same element as x in $K_{smooth}(A, B)$.

b) If $U \in \mathbb{C} + C_p \hat{\otimes} B^\infty$ then also $(ad_u \circ \varphi_0, \varphi_1)$ represents the same element as x.

To see this one uses a smooth homotopy from $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The following Proposition is proved as in the case of KK-group.

2.7. Proposition. $K_{smooth}(A, B)$ is a group. The unit is the class of any degenerate element, the inverse of (φ_0, φ_1) is (φ_1, φ_0).

K_{smooth} is covariant in the second variable and contravariant in the first variable for morphisms of smooth C^*-algebras.

2.8. Proposition. Ch_0 factors to a morphism

$Ch_0 : K_{smooth}(A, B) \to PHC^0(A^\infty, B^\infty)$.

Proof. Ch_0 is clearly additive and vanishes on degenerate elements.

Suppose $x_0 = (\varphi_0, \varphi_1)$ and $x_1 = (\varphi_0, \varphi_1)$ are smoothly homotopic via an element $x \in \mathcal{E}_p(A, C([0,1], B))$. Let $e_t : C([0,1], B) \to B$ be the evaluation at t, $x_i = e_{i*}(x)$. Then

$$Ch_0^{2n}(x_i) = Ch_0^{2n}(x) \circ [e_i] \qquad n \geq p - 1$$

Since $S[e_0] = S[e_1]$ [Kas, pag 54] Theorem 8.1 we obtain from proposition 1.12 a) that $Ch_0^{2n}(x_0) = Ch_0^{2n}(x_1)$, $n \geq p$.

2.9. There exists an obvious morphism $K_{smooth}(A,B) \to KK(A,B)$ and hence, given $x = K_{smooth}(A, B)$ the Kasparov product defines a morphism

$$K_0(A) \xrightarrow{-\otimes x} K_0(B)$$

Recall [Co 1] that $K_0(A) \simeq K_0(A^\infty)$, $K_0(B) \simeq K_0(B^\infty)$.

Theorem Suppose A and B are smooth C^∞-algebras and B^∞ is absolutely smooth, then the diagram

$$\begin{tikzcd}
K_0(A^\infty) \simeq K_0(A) \arrow[r, "-\otimes x"] \arrow[d, "Ch_0"'] & K_0(B) \simeq K_0(B^\infty) \arrow[d, "Ch_0"] \\
PHC_0(A^\infty) \arrow[r, "-\circ Ch_0(x)"] & PHC_0(B^\infty)
\end{tikzcd}$$

is commutative.

Proof. Let $e \in A^{\infty}$ be a projection (a selfadjoint idempotent). Recall that Ch(e) is defined as $PHC_0(\varphi)(1) \in PHC_0(A^{\infty})$ where 1 stands for the generator of $PHC_0(\mathbb{C}) \simeq \mathbb{C}$ and $\varphi : \mathbb{C} \to A^{\infty}$ is given by $1 \to e$ [Co 3, Kar2]. The partial Chern character $Ch_0^{2n} : K_0(A^{\infty}) \to HC_{2n}(A^{\infty})$ is defined similarily and $Ch_0^{2l'} = S\, Ch_0^{2l+2}$.

Suppose x is represented by $(\varphi_0, \varphi_1) \in \mathcal{E}_p(A, B)$.

Then our diagram is the inverse limit of

$$\begin{array}{ccc} K_0(A^{\infty}) \simeq K_0(A) & \xrightarrow{-\otimes x} & K_0(B) \simeq K_0(B^{\infty}) \\ \downarrow {\scriptstyle Ch_0^{2l}} & & \downarrow {\scriptstyle Ch_0^{2l-2n}} \\ HC_{2l}(A^{\infty}) & \xrightarrow{-\circ Ch_0^{2n}(x)} & HC_{2l-2n}(B^{\infty}) \end{array}$$

$l \geq n \geq p-1$. Assume we have fixed $Ch_0^{2l}([1]) \in HC_{2l}(\mathbb{C})$ to be a distinguished set of generators. Then from the definition of the Chern character on K_0 and the naturality of the Chern character for quasihomomorphisms we see that we may assume $A = \mathbb{C}$.

Claim. $K_{smooth}(\mathbb{C}, B) \simeq K_0(B)$. Let us first conclude the proof using the claim. Let $x \in K_{smooth}(\mathbb{C}, B)$, the claim shows that we may suppose that x is represented by (φ_0, φ_1) such that $\varphi_0(1), \varphi_1(1) \in M_N(B)$ for some large N. Then $[1] \otimes x = [\varphi_0(1)] - [\varphi_1(1)] \in K_0(B)$ and

$$Ch_0^{2l}([1])\ Ch_0^{2n}(x) = S^n Ch_0^{2l}([1]) \circ Ch_0^{0}(x) = S^n(Ch_0^{2l}([\varphi_0(1)]) - Ch_0^{2l}([\varphi_1(1)])) =$$

$$= Ch_0^{2l-2n}([1] \otimes x) .$$

Proof of the claim . It is obvious that $K_{smooth}(\mathbb{C}, B) \to K_0(B)$ is onto. In order to prove that it is into it is enough to prove that every element $x \in K_{smooth}(\mathbb{C}, B)$ may be represented by a quasihomomorphism (φ_0, φ_1) such that $\varphi_0(1), \varphi_1(1) \in M_N(B^{\infty})$ for some large N.

Let $e_0, e_1 \in L(H) \hat{\otimes} B^{\infty}$ be such that $e_0 - e_1 \in C_p \otimes B^{\infty}$.

We shall use a trick of Atiyah and Singer [AS] to fill in some gap in $e_1e_0 \cdot e_1e_0$ viewed as an element of $L(e_0H_B, e_1H_B)$ is an essential isometry (an isometry modulo compact operators). This shows that there exists a finite n_0 and a linear operator $R_0 \in L(B^{n_0}, e_1H_B)$ such that

$e_1e_0 \oplus R : e_0H_B \oplus B^{n_0} \longrightarrow e_1B$ is onto . Similarily choose R_1 such that

$(1-e_1)(1-e_0) \oplus R_1 : (1-e_0)\, H_B \oplus B^{n_1} \longrightarrow (1-e_1)H_B$ is onto.

Let

$$V_0 = \begin{bmatrix} e_1e_0 + (1-e_1)(1-e_0) & R_0 & R_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in L(H_B \oplus B^{n_0} \oplus B^{n_1})$$

It satisfies $V_0V_0^* \geq c\, 1_{H_B}$ for some $c > 0$. Choosing some convenient small perturbation of R_0 and R_1 we may suppose that $V_0 \in 1 + C_p(H \oplus C^{n_0+n_1}) \otimes B^{\infty}$.

Let $\chi(0) = 0$, $\chi(z) = z^{-1/2}$, $z \neq 0$ be an analytic function defined in a small neighborhood of $\sigma(V_0V_0^*)$. Then $V = \chi(V_0V_0^*)V_0$ is a partial isometry.

Let

$$e_k' = \begin{bmatrix} e_k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

. Then (e_0', e_1') represents the same element as (e_0, e_1) in $K_{smooth}(\mathbb{C}, B)$. (They differ by a degenerate element).

$$\text{Let } U = \begin{bmatrix} V_0 & 1-V_0V_0^* \\ 1-V_0^*V_0 & V_0^* \end{bmatrix}$$

then $e_0'' = Ue_0' \oplus 0 U^*$ commutes with e_1. Since also $(e_0'', e_1' \oplus 0)$ represents x by Observation 2.7 we have reduced the problem to the case when e_0 and e_1 commute. But then $[(e_0, e_1)] = [e_0(1 - e_1), e_1(1 - e_0)] + [e_0e_1, e_0e_1]$ and $e_0(1 - e_1)$, $e_1(1 - e_0)$ are homotopic to projections in some matrix algebras.

2.10. We now treat the case of K_1.

Let $[u] \in K_1(A^\infty)$, then, choosing a representative u we obtain a morphism $\varphi : \mathbb{C}[\mathbb{Z}] \to A^\infty$. Recall that $HC_p(\mathbb{C}[\mathbb{Z}]) \simeq \mathbb{C}$ for $p \geq 1$ [Bu, Co, LQ] and that $S : HC_p(\mathbb{C}[\mathbb{Z}]) \to HC_{p-2}(\mathbb{C}[\mathbb{Z}])$ is an isomorphism for $p \geq 3$. Let v be the generator of $\mathbb{Z}$; v is invertible in $\mathbb{C}[\mathbb{Z}]$. Then [Kar 2, Co] $Ch^{2p-1}([v]) =$
$= S\, Ch^{2p+1}([v]) \neq 0$ and $Ch^{2p+1}([u]) = Ch^{2p+1}(\varphi_*[v]) = HC_{2p+1}(\varphi)Ch^{2p+1}([v])$.

Theorem. Suppose A and B are smooth C^*-algebras and B^∞ is absolutely smooth. Let $x \in K_{smooth}(A, B)$ then the following diagram is commutative

$$\begin{tikzcd}
K_1(A^\infty) \simeq K_1(A) \arrow[r, "-\otimes x"] \arrow[d, "Ch_0"'] & K_1(B) \simeq K_1(B^\infty) \arrow[d, "Ch_0"] \\
PHC_1(A^\infty) \arrow[r, "-\circ Ch(x)"] & PHC_1(B^\infty)
\end{tikzcd}$$

Proof. We may by functoriality suppose that $A = C_0(\mathbb{R})$, $A^\infty = \mathcal{S}(\mathbb{R})$. Let $x = [(\varphi_0, \varphi_1)]$ for a p-summable quasihomomorphism (φ_0, φ_1). Also let $u \in \mathcal{S}(\mathbb{R})^+$ generate $K_1(A)$ be such that $u : \mathbb{R} \to \mathbb{T} \setminus \{1\}$ is one-to-one. Then $[u] \otimes x$ is represented by $\varphi_0(u)\varphi_1(u)^{-1}$ in $K_1(C_p \otimes B^\infty) \simeq K_1(B)$ since B^∞ is absolutely smooth. Observe that φ_0 and φ_1 are determined by $\varphi_0(u)$ and $\varphi_1(u)$ and we shall identify φ_i with $\varphi_i(u)$.

Then $(\varphi_0(u) \oplus \varphi_1(u)^{-1}, \varphi_1(u) \oplus \varphi_1(u)^{-1})$ also represents x and is homotopic to $(\varphi_0(u)\varphi_1(u)^{-1} \oplus 1, 1 \oplus 1)$ .Using again homotopy and cutting out a degenerate element we may suppose that $\varphi_0(u) \in M_N(B^{\infty+})$ for some large N and $\varphi_1(u) = 1$. Then the result follows from the functoriality of the Chern character on K_1.

2.11. Remarck. The previous theorems where based on the algebraic properties of the Chern character (functoriality) and on two analytic facts :

$$K_{smooth}(\mathbb{C}, B) \simeq K_0(B) \quad \text{and}$$

$$K_{smooth}(C_0(\mathbb{R}), B) \simeq K_1(B)$$

which in turn depended on the fact that the pairs (φ_0, φ_1) of smooth morphisms $\varphi_0, \varphi_1 : A \to M_N(B)$

form a complete set of representives for the above K_{smooth} -groups.

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