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by

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1. INTRODUCTION

The notion of a normed almost linear space was introduced in [1] as a generalization of the notion of a normed linear space. The almost linear spaces appeared in [5] as an abstraction of the algebraic structure of the class of all closed intervals of the real line. To compensate the weakening of the axioms of a linear space, in our definition of a normed almost linear space the norm is supposed to satisfy besides all the axioms of a norm on a linear space also an additional one which makes the framework productive. An example of a normed almost linear space is the collection of all nonempty, bounded and convex subsets of a normed linear space (see [1]).

In a series of papers we began to develop a theory for the normed almost linear spaces similar with that of the normed linear spaces. Thus, we defined the dual space of a normed almost linear space (where the functionals are no longer linear but almost linear) the bounded linear and almost linear operators between two such spaces and we obtained in this more general framework basic results from the theory of normed linear spaces. The main tool for the theory of normed almost linear spaces was given in [3] where we proved that any normed almost linear space can be embedded into a normed linear space, allowing us the use of the techniques of the normed linear spaces.

In this paper we introduce the notion of a normed almost linear algebra which generalizes the notion of a normed linear algebra. The main result of this paper is Theorem 3.6, where we prove that any normed almost linear algebra can be embedded into a normed linear algebra. As a consequence we can now use the techniques of normed linear algebras to solve certain problems in our more general framework. Examples of normed almost linear algebras are given in Section 4.

2. PRELIMINARIES

Besides notation, in this section we recall some definitions and results from previous papers, necessary for an easy understanding of this work. We assume that all spaces are over the real field R and we denote by R_+ the set $\{\lambda \in R ; \lambda \geq 0\}$.

A commutative semigroup X with zero 0 is called an almost linear space ([5]) if there is also given a mapping $(\lambda, x) \longrightarrow \lambda \circ x$ of $R \times X$ into X satisfying (i)-(v) below. Let $x,y \in X$ and $\lambda, \mu \in R$: (i) $1 \circ x = x$; (ii) $0 \circ x = 0$; (iii) $\lambda \circ (x+y) = \lambda \circ x + \lambda \circ y$; (iv) $\lambda \circ (\mu \circ x) = (\lambda\mu) \circ x$; (v) $(\lambda + \mu) \circ x = \lambda \circ x + \mu \circ x$ for $\lambda, \mu \in R_+$.

We set off the following two subsets of X ([1]):

$$V_X = \{x \in X : x + (-1 \circ x) = 0\}$$
$$W_X = \{x \in X : x = -1 \circ x\}$$

These are almost linear subspaces of X (i.e., closed under addition and multiplication by reals) and V_X is a linear space. Clearly, $V_X \cap W_X = \{0\}$. An almost linear space X is a linear space iff $X = V_X$, iff $W_X = \{0\}$.

In an almost linear space X we use the notation $\lambda \circ x$ for the multiplication of $\lambda \in R$ with $x \in X$, the notation λx being used only in a linear space.

A normed almost linear space ([1]) is an almost linear space X together with a norm $|||\cdot|||: X \longrightarrow R$ satisfying (N_1)-(N_4) below. Let $x,y \in X$, $w \in W_X$ and $\lambda \in R$: (N_1) $|||x+y||| \leq |||x||| + |||y|||$; (N_2) $|||x||| = 0$ iff $x = 0$; (N_3) $|||\lambda \circ x||| = |\lambda|\, |||x|||$; (N_4) $|||x||| \leq |||x+w|||$. Note that $|||x||| \geq 0$ for each $x \in X$. We also have([2])

$$(2.1) \qquad |||w||| \leq |||x+w||| \qquad x \in X,\ w \in W_X$$

Here we draw attention that in [1] and [2] we have worked with an equivalent definition of the norm and in [1] the last axiom of the norm is superfluous.

2.1. LEMMA ([2]). Let X be a normed almost linear space and let $x,y \in X$, $w_i \in W_X$, $v_i \in V_X$, i=1,2.

(i) If $x+y \in V_X$ then $x,y \in V_X$.

(ii) If $w_1+v_1 = w_2+v_2$ then $w_1 = w_2$ and $v_1 = v_2$.

As we observed in ([2]), Lemma 2.1 above is no longer true in an almost linear space.

Let X,Y be two normed almost linear spaces. For a mapping $T: X \longrightarrow Y$ the definitions of a linear operator and an isometry are similar with those from the linear case. We draw attention that a linear isometry is not always one-to-one. For $A \subset X$ we denote by T(A) the set $\{T(a) ; a \in A\}$.

2.2. REMARK ([3]). If T is a linear isometry of X onto Y then $T(V_X) = V_Y$ and $T(W_X) = W_Y$.

The following result is the main tool for the theory of normed almost linear spaces.

2.3. THEOREM ([3], Theorem 3.2). For any normed almost linear space $(X, |||\cdot|||)$ there exist a normed linear space $(E, \|\cdot\|)$ and a mapping $\omega : X \longrightarrow E$ with the following properties:

(i) $E = \omega(X) - \omega(X)$ and $\omega(X)$ can be organized as an almost linear space where the addition and the multiplication by non-negative reals are the same as in E.

(ii) For $z \in E$ we have

$$\|z\| = \inf \{ |||x||| + |||y|||;\ x, y \in X,\ z = \omega(x) - \omega(y) \} \tag{2.2}$$

and $(\omega(X), \|\cdot\|)$ is a normed almost linear space.

(iii) ω is a linear isometry of $(X, |||\cdot|||)$ onto $(\omega(X), \|\cdot\|)$.

2.4. COROLLARY ([3], Corollary 3.3). For any normed almost linear space $(X, |||\cdot|||)$ the function $\rho : X \times X \longrightarrow R$ defined by $\rho(x,y) = \|\omega(x) - \omega(y)\|$, $x, y \in X$ is a semi-metric on X.

The semi-metric ρ defined above generates a topology on X (which is not Hausdorff in general) and in the sequel any topological concept will be understood for this topology. Clearly, ρ is a metric on X iff ω is one-to-one. Note that for $v_1, v_2 \in V_X$ we have $\rho(v_1, v_2) = |||v_1 - v_2|||$.

The proof of the following lemma is contained in the proof of ([3], Theorem 3.2 (iv), fact I).

2.5. LEMMA. Let $(X, |||\cdot|||)$ be a normed almost linear space and let $x, y \in X$. If $\omega(x) = \omega(y)$ then for each $\varepsilon > 0$ there exist

$x_\varepsilon, y_\varepsilon, u_\varepsilon \in X$ <u>such that</u> $|||x_\varepsilon||| = |||y_\varepsilon||| < \varepsilon$ <u>and</u> $x+y_\varepsilon+u_\varepsilon = y+x_\varepsilon+u_\varepsilon$.

Let A be a subset of the normed almost linear space X and let $x \in X$. We denote the distance of x to A by dist $(x,A)(= \inf\{\rho(x,a)\,;\, a \in A\})$.

For a normed linear space E we denote by E^* the dual space of E.

3. NORMED ALMOST LINEAR ALGEBRA

An almost linear space X is called an <u>almost linear algebra</u> if there is also given a mapping $(x,y) \longrightarrow xy$ of $X \times X$ into X satisfying (A_1)-(A_3) below;

(A_1) $x(yz) = (xy)z$, $\qquad x,y,z \in X$

(A_2) $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$ $\qquad x,y,z \in X$

(A_3) $(\lambda \circ x)(\mu \circ y) = (\lambda\mu)\circ(xy)$ $\qquad \lambda,\mu \in R_+,\ x,y \in X$

By (A_3) it follows that $x0 = 0x = 0$ for each $x \in X$.

As in the linear case we call X a <u>commutative</u> almost linear algebra if $xy = yx$ for all $x,y \in X$. Also an element $e \in X$ is a <u>unit</u> of X if $ex = xe = x$ for each $x \in X$; clearly $e \in X$ is unique. When X has a unit e, an element $x \in X$ is called <u>invertible</u> if there exists an element $x^{-1} \in X$ such that $xx^{-1} = x^{-1}x = e$; clearly $x^{-1} \in X$ with the latter properties is unique.

A subset $Y \subset X$ is called an <u>almost linear subalgebra of</u> X if Y is an almost linear subspace of X and for $y_1, y_2 \in Y$ we have $y_1y_2 \in Y$.

Certain almost linear algebras satisfy one or both of the following conditions(for examples of almost linear algebras which satisfy or not one or both of these conditions see Section 4 as well as Remark 3.1 below).

(A_4) $x(-1\circ w) = -1\circ(xw)$ (equivalently, $xw \in W_X$) , $x \in X$, $w \in W_X$

(A_5) $(-1\circ w)x = -1\circ(wx)$ (equivalently, $wx \in W_X$) , $x \in X$, $w \in W_X$

3.1. REMARK. (i) If $X = W_X$ then (A_4) and (A_5) are always satisfied. (ii) If X is commutative then (A_4) and (A_5) are simultaneously satisfied or not. (iii) Suppose X satisfies (A_4) and (A_5) is not satisfied. Then we can organize X as an almost linear algebra such that (A_5) is satisfied and (A_4) is not satisfied. Indeed, for $x,y \in X$ define the product $p(x,y)$ by $p(x,y) = yx$, $x,y \in X$. Then the almost linear space X together with the product $p(\cdot,\cdot)$ is an almost linear algebra which satisfies (A_5) but not (A_4). Note that if X has the unit e then e is the unit for X together with the product $p(\cdot,\cdot)$. Due to this remark, certain counterexamples given to show that (A_i) can not be dropped may be used to show that (A_j) can not be dropped, $i \neq j$, $i,j \in \{4,5\}$.

3.2. REMARK. Suppose X satisfies one of the conditions (A_4),(A_5). We have;

(i) W_X is an almost linear subalgebra of X.

Suppose in addition that e is the unit of X. We have;

(ii) $e \in W_X$ iff $X = W_X$.

(iii) If an element $w_o \in W_X$ is invertible in X then $X = W_X$.

The statements in the above remark are no longer true in general, when X does not satisfy (A_4) and (A_5)(see Examples 4.6 and 4.5(Case 4)).

3.3. REMARK. Let X be an almost linear algebra such that there exists a norm on X and let $x \in X$, $v \in V_X$. We have :

(i) $xv, vx \in V_X$ and $x(-1 \circ v) = -1 \circ (xv)$, $(-1 \circ v)x = -1 \circ (vx)$. Indeed, we have $0 = x0 = x(v+(-1 \circ v)) = xv+x(-1 \circ v)$. By Lemma 2.1(i) we get $xv, x(-1 \circ v) \in V_X$ and so $x(-1 \circ v) = -1 \circ (xv)$. Similarly we prove the other statements. Consequently, for $\lambda \in R_+$, $\mu \in R$ we have $(\lambda \circ x)(\mu \circ v) = (\lambda\mu) \circ (xv)$ and $(\mu \circ v)(\lambda \circ x) = (\lambda\mu) \circ (vx)$.

(ii) V_X is an almost linear subalgebra of X which is a linear algebra(use (i) above).

Suppose in addition that e is the unit of X. We have :

(iii) $e \in V_X$ iff $X = V_X$. This follows from (i) above.

(iv) If an element $v_o \in V_X$ is invertible in X then $X = V_X$. This follows from (i) and (iii) above.

The assumption that there exists a norm on X can not be dropped (see Example 4.8).

3.4. REMARK. Let X be an almost linear algebra such that there exists a norm on X. Suppose X satisfies (A_4)((A_5), resp.) and let $x \in X$, $v \in V_X$ and $w \in W_X$. We have :

(i) $vw = 0$ ($wv = 0$, resp.). For the proof use (A_4)((A_5), resp.), Remark 3.3 (i) and the fact that $W_X \cap V_X = \{0\}$. Consequently, when X satisfies both (A_4) and (A_5) and $x_i = w_i+v_i$, $w_i \in W_X$, $v_i \in V_X$, $i=1,2$, then $x_1x_2 = w_1w_2+v_1v_2$.

(ii) If $x_o \in W_X+V_X$ then $xx_o \in W_X+V_X$ ($x_ox \in W_X+V_X$, resp.). For the proof use (A_4)((A_5), resp.) and Remark 3.3 (i). Consequently, W_X+V_X is an almost linear subalgebra of X.

Suppose in addition that e is the unit of X. We have :

(iii) $e \in W_X+V_X$ iff $X = W_X+V_X$. This follows from (ii) above.

(iv) If an element $x_o \in W_X+V_X$ is invertible in X then $X = W_X+V_X$. This follows from (ii) and (iii) above.

(v) If X satisfies both (A_4) and (A_5) and $X = W_X+V_X$, $X \neq W_X$, $X \neq V_X$ then W_X and V_X are almost linear algebras with units. Indeed, if $e = w_o+v_o$, $w_o \in W_X$, $v_o \in V_X$ then by Lemma 2.1 (ii), w_o and v_o are uniquely determined. By our assumptions on X, Remark 3.2 (ii) and Remark 3.3 (iii) we get $v_o \neq 0$, $w_o \neq 0$. Simple computations show that w_o is the unit of W_X and v_o is the unit of V_X . If $x \in X$, say, $x = w+v$, $w \in W_X$, $v \in V_X$ is invertible in X and $x^{-1} = w_1+v_1$, $w_1 \in W_X$, $v_1 \in V_X$ then w is invertible in W_X and we have $ww_1 = w_1w = w_o$ and v is invertible in V_X and we have $vv_1 = v_1v = v_o$.

For some counterexamples that the assumptions on X can not be dropped see Examples 4.3, 4.6 (case 4), 4.7, 4.8.

3.5. REMARK. Let X be a normed almost linear space such that $X = W_X+V_X$, $X \neq W_X$, $X \neq V_X$. If W_X and V_X are almost linear algebras, then X can be organized as an almost linear algebra satisfying (A_4) and (A_5) and such that W_X and V_X be almost linear subalgebras of X. Indeed, if $x_1,x_2 \in X$, $x_i = w_i+v_i$, $w_i \in W_X$, $v_i \in V_X$, i=1,2, then define $x_1x_2 = w_1w_2+v_1v_2$. By Lemma 2.1 (ii) this product is well defined and X satisfies (A_1)-(A_5). If both W_X and V_X are commutative, then X is commutative. Note that if W_X has a unit w_o and V_X has a unit v_o then $e = w_o+v_o$ is a unit of X. Moreover, if w is invertible in W_X , i.e., there exists $w_1 \in W_X$ such that $ww_1 = w_1w = w_o$ and v is invertible in V_X , i.e., there exists $v_1 \in V_X$ such that $vv_1 = v_1v = v_o$, then $x = w+v$ is invertible in X and we have $x^{-1} = w_1+v_1$.

A *normed almost linear algebra* is an almost linear algebra X together with a norm $|||\cdot|||:X \longrightarrow R$ satisfying besides (N_1)-(N_4) also the following condition ;

(N_5) $\quad |||xy||| \leq |||x|||\,|||y||| \qquad (x,y \in X)$

Clearly, if $X(\neq \{0\})$ is with unit e, then $|||e||| \geq 1$.

We now state the main result of this paper ;

3.6. THEOREM. *For any normed almost linear algebra* $(X, |||\cdot|||)$ *there exist a normed linear algebra* $(E, \|\cdot\|)$ *and a mapping* $\omega : X \longrightarrow E$ *with the following properties* ;

(i) $E = \omega(X) - \omega(X)$ *and* $\omega(X)$ *can be organized as an almost linear algebra where the addition, the multiplication by non-negative reals and the product are the same as in* E. *If* X *satisfies* $(A_4)((A_5))$ *then* $\omega(X)$ *satisfies* $(A_4)((A_5))$.

(ii) *For* $z \in E$, $\|z\|$ *is given by* (2.2) *and* $(\omega(X), \|\cdot\|)$ *is a normed almost linear algebra.*

(iii) ω *is a linear isometry of* $(X, |||\cdot|||)$ *onto* $(\omega(X), \|\cdot\|)$ *and for* $x,y \in X$ *we have* $\omega(xy) = \omega(x)\omega(y)$.

(iv) *If* X *has the unit* e *then* $\omega(e)$ *is the unit of both* E *and* $\omega(X)$.

(v) *If* X *is commutative then both* E *and* $\omega(X)$ *are commutative.*

(vi) *If* $(E_1, \|\cdot\|_1)$ *and* $\omega_1 : X \longrightarrow E_1$ *satisfy* (i)-(iii) *above, then there exists a linear isometry* T *of* E *onto* E_1 *such that* $T(z_1 z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in E$.

PROOF. For the normed almost linear space $(X, |||\cdot|||)$ let $(E, \|\cdot\|)$ and $\omega : X \longrightarrow E$ be given by Theorem 2.3. In the sequel we shall use the properties of $E, \|\cdot\|$ and ω given in Theorem 2.3. For $z_i \in E$, $z_i = \omega(x_i) - \omega(y_i)$, $x_i, y_i \in X$, $i=1,2$ define

$$z_1 z_2 = \omega(x_1x_2+y_1y_2) - \omega(x_1y_2+y_1x_2) \tag{3.1}$$

To show that (3.1) is well defined, suppose we have for $x_i, y_i \in X$, $1 \leq i \leq 4$

$$z_1 = \omega(x_1) - \omega(y_1) = \omega(x_3) - \omega(y_3) \tag{3.2}$$

$$z_2 = \omega(x_2) - \omega(y_2) = \omega(x_4) - \omega(y_4) \tag{3.3}$$

and it is enough to prove (due to the properties of ω) that we have ;

$$\omega(x_1x_2+y_1y_2+x_3y_4+y_3x_4) = \omega(x_1y_2+y_1x_2+x_3x_4+y_3y_4) \tag{3.4}$$

By (3.2) and (3.3) we get

$$\omega(x_1+y_3) = \omega(x_3+y_1)$$
$$\omega(x_2+y_4) = \omega(x_4+y_2)$$

By Lemma 2.5, for each integer $n > 4$ there exist $x_n, y_n, u_n \in X$ and $x_n', y_n', u_n' \in X$ such that

$$x_1+y_3+x_n+u_n = x_3+y_1+y_n+u_n \tag{3.5}$$

$$|||x_n||| = |||y_n||| < 1/n \tag{3.5'}$$

$$x_2+y_4+x_n'+u_n' = x_4+y_2+y_n'+u_n' \tag{3.6}$$

$$|||x_n'||| = |||y_n'||| < 1/n \tag{3.6'}$$

Using (3.5) and (3.6) we get

$$(x_1+y_3+x_n+u_n)x_2+(x_3+y_1+y_n+u_n)y_2+y_3(x_4+y_2+y_n'+u_n')+x_3(x_2+y_4+x_n'+u_n')$$
$$=(x_3+y_1+y_n+u_n)x_2+(x_1+y_3+x_n+u_n)y_2+y_3(x_2+y_4+x_n'+u_n')+x_3(x_4+y_2+y_n'+u_n')$$

Making all the products and then applying ω we obtain (using the properties of ω and the fact that we work now in the linear space E)

$$\omega(x_1x_2+y_1y_2+x_3y_4+y_3x_4)-\omega(x_1y_2+y_1x_2+x_3x_4+y_3y_4)=$$
$$=\omega(y_nx_2+x_ny_2+y_3x_n'+x_3y_n')-\omega(x_nx_2+y_ny_2+y_3y_n'+x_3x_n')$$

Using the fact that ω is a linear isometry, X a normed almost linear algebra as well as (3.5') and (3.6') we get

$$\|\omega(x_1x_2+y_1y_2+x_3y_4+y_3x_4)-\omega(x_1y_2+y_1x_2+x_3x_4+y_3y_4)\|\leq$$
$$\leq 2n^{-1}(|||x_2|||+|||x_3|||+|||y_2|||+|||y_3|||)\longrightarrow 0$$

as $n\to\infty$, which proves (3.4).

Straightforward computations show that $(A_1),(A_2)$ hold for the product on E given by (3.1). For the proof of (A_3), let $\lambda,\mu\in R_+$ and $z_i\in E$, $z_i=\omega(x_i)-\omega(y_i)$, $x_i,y_i\in X$, i=1,2. Then

$$(\lambda z_1)(\mu z_2)=(\lambda\omega(x_1)-\lambda\omega(y_1))(\mu\omega(x_2)-\mu\omega(y_2))=$$
$$=(\lambda\circ\omega(x_1)-\lambda\circ\omega(y_1))(\mu\circ\omega(x_2)-\mu\circ\omega(y_2))=$$
$$=(\omega(\lambda\circ x_1)-\omega(\lambda\circ y_1))(\omega(\mu\circ x_2)-\omega(\mu\circ y_2))=$$
$$=\omega((\lambda\circ x_1)(\mu\circ x_2)+(\lambda\circ y_1)(\mu\circ y_2))-$$
$$-\omega((\lambda\circ x_1)(\mu\circ y_2)+(\lambda\circ y_1)(\mu\circ x_2))=$$
$$=\omega((\lambda\mu)\circ(x_1x_2+y_1y_2))-\omega((\lambda\mu)\circ(x_1y_2+y_1x_2))=$$
$$=(\lambda\mu)\circ\omega(x_1x_2+y_1y_2)-(\lambda\mu)\circ\omega(x_1y_2+y_1x_2))=$$
$$=(\lambda\mu)\omega(x_1x_2+y_1y_2)-(\lambda\mu)\omega(x_1y_2+y_1x_2))=$$

$$= (\lambda\mu)(z_1 z_2)$$

As it is known (or by Remark 3.3 (i)), it follows that E is a linear algebra.

Finally we show that (N_5) holds, which will complete the proof that E is a normed linear algebra. Let $z_i \in E$, i=1,2. By (2.2), for $\varepsilon > 0$ there exist $x_i, y_i \in X$, i=1,2 such that $z_i = \omega(x_i) - \omega(y_i)$ and $|||x_i||| + |||y_i||| \leq \|z_i\| + \varepsilon$, i=1,2. We have

$$\|z_1 z_2\| = \|\omega(x_1x_2+y_1y_2) - \omega(x_1y_2+y_1x_2)\| \leq$$
$$\leq |||x_1x_2+y_1y_2||| + |||x_1y_2+y_1x_2||| \leq$$
$$\leq |||x_1|||\,|||x_2||| + |||y_1|||\,|||y_2||| + |||x_1|||\,|||y_2||| + |||y_1|||\,|||x_2|||$$
$$\leq |||x_1|||(\|z_2\| + \varepsilon) + |||y_1|||(\|z_2\| + \varepsilon)$$
$$\leq (\|z_1\| + \varepsilon)(\|z_2\| + \varepsilon) \longrightarrow \|z_1\|\|z_2\|$$

as $\varepsilon \to 0$. Consequently, $(E, \|\cdot\|)$ is a normed linear algebra.

Let $\bar{x}_i \in \omega(X)$, $\bar{x}_i = \omega(x_i) = \omega(x_i) - \omega(0)$, $x_i \in X$, i=1,2. Then by (3.1) we obtain

$$(3.7) \qquad \bar{x}_1\bar{x}_2 = \omega(x_1)\omega(x_2) = \omega(x_1x_2) \in \omega(X)$$

Clearly, $\omega(X)$ with the product defined by (3.7) is a normed almost linear algebra where the product is the same as in E. Suppose now X satisfies (A_4) and let $\bar{x} \in \omega(X)$ and $\bar{w} \in W_{\omega(X)}$. Then $\bar{x} = \omega(x)$, for some $x \in X$ and by Remark 2.2 there exists $w \in W_X$ such that $\bar{w} = \omega(w)$. By (3.7) we have $\bar{x}\bar{w} = \omega(x)\omega(w) = \omega(xw) \in \omega(W_X) = W_{\omega(X)}$, i.e., $\omega(X)$ satisfies (A_4). Similarly, if X satisfies (A_5) then $\omega(X)$ satisfies (A_5). Using Theorem 2.3 it follows that we have (i)-(iii) in Theorem 3.6.

Since (iv) and (v) are simple computations, we prove now (vi). Let $z \in E$, say, $z = \omega(x) - \omega(y)$, $x, y \in X$ and define $T; E \longrightarrow E_1$ by

$$(3.8) \qquad T(z) = \omega_1(x)-\omega_1(y)$$

To show that T is well defined suppose $z = \omega(x)-\omega(y) = \omega(x')-\omega(y')$, $x,y,x',y' \in X$. Then $\omega(x+y') = \omega(x'+y)$, whence by Lemma 2.5, for each $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon, u_\varepsilon \in X$ such that

$$(3.9) \qquad x+y'+x_\varepsilon+u_\varepsilon = x'+y+y_\varepsilon+u_\varepsilon$$

$$(3.9') \qquad |||x_\varepsilon||| = |||y_\varepsilon||| < \varepsilon$$

By (3.9) and (iii) for ω_1 we get $\omega_1(x)+\omega_1(y')+\omega_1(x_\varepsilon)+\omega_1(u_\varepsilon) = \omega_1(x')+\omega_1(y)+\omega_1(y_\varepsilon)+\omega_1(u_\varepsilon)$ whence

$$\|(\omega_1(x)-\omega_1(y))-(\omega_1(x')-\omega_1(y'))\|_1 = \|\omega_1(y_\varepsilon)-\omega_1(x_\varepsilon)\|_1 \leq \|\omega_1(y_\varepsilon)\|_1+\|\omega_1(x_\varepsilon)\|_1 = |||y_\varepsilon||| + |||x_\varepsilon||| < 2\varepsilon$$

As $\varepsilon \to 0$ we get $\omega_1(x)-\omega_1(y) = \omega_1(x')-\omega_1(y')$, i.e., T is well defined . It is easy to show that T is a linear operator.

We show now that T is one-to-one and onto E_1 . Let $z_i = \omega(x_i)-\omega(y_i) \in E$, $x_i, y_i \in X$, $i=1,2$ be such that $T(z_1) = T(z_2)$. Then $\omega_1(x_1)-\omega_1(y_1) = \omega_1(x_2)-\omega_1(y_2)$ and so $\omega_1(x_1+y_2) = \omega_1(x_2+y_1)$. By Lemma 2.5, for each $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon, u_\varepsilon \in X$ such that $x_1+y_2+x_\varepsilon+u_\varepsilon = x_2+y_1+y_\varepsilon+u_\varepsilon$ and $|||x_\varepsilon||| = |||y_\varepsilon||| < \varepsilon$. Then $\|(\omega(x_1)-\omega(y_1))-(\omega(x_2)-\omega(y_2))\| = \|\omega(y_\varepsilon)-\omega(x_\varepsilon)\| \leq \|\omega(y_\varepsilon)\|+\|\omega(x_\varepsilon)\| = |||y_\varepsilon||| + |||x_\varepsilon||| < 2\varepsilon$, whence as $\varepsilon \to 0$ we get $z_1 = z_2$ which proves that T is one-to-one. Let now $z_1 = \omega_1(x)-\omega_1(y) \in E_1$, $x,y \in X$ and let $z = \omega(x)-\omega(y) \in E$. Since $T(z) = z_1$, this proves that T is onto E_1 .

To show that T is an isometry of E onto E_1 , let $z \in E$ and let $\varepsilon > 0$. By (2.2) there exist $x, y, x', y' \in X$ such that

$$z = \omega(x) - \omega(y)$$
$$|||x||| + |||y||| \leq \|z\| + \varepsilon$$
$$T(z) = \omega_1(x') - \omega_1(y')$$
$$|||x'||| + |||y'||| \leq \|T(z)\|_1 + \varepsilon$$

Let $z' = \omega(x') - \omega(y') \in E$. Then $z = z'$ since $T(z) = T(z')$ and T is one-to-one. We have $\|T(z)\|_1 = \|\omega_1(x) - \omega_1(y)\|_1 \leq \|\omega_1(x)\|_1 + \|\omega_1(y)\|_1 = |||x||| + |||y||| \leq \|z\| + \varepsilon = \|z'\| + \varepsilon = \|\omega(x') - \omega(y')\| + \varepsilon \leq \|\omega(x')\| + \|\omega(y')\| + \varepsilon = |||x'||| + |||y'||| + \varepsilon \leq \|T(z)\|_1 + 2\varepsilon$. As $\varepsilon \to 0$ we get $\|T(z)\|_1 = \|z\|$ and so T is a linear isometry of E onto E_1.

Finally, we show that $T(z_1 z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in E$. Suppose $z_i = \omega(x_i) - \omega(y_i)$, $x_i, y_i \in X$, $i=1,2$. By (3.1), the definition of T and (iii) we get $T(z_1 z_2) = \omega_1(x_1x_2 + y_1y_2) - \omega_1(x_1y_2 + y_1x_2) = \omega_1(x_1)\omega_1(x_2) + \omega_1(y_1)\omega_1(y_2) - \omega_1(x_1)\omega_1(y_2) - \omega_1(y_1)\omega_1(x_2) = (\omega_1(x_1) - \omega_1(y_1))(\omega_1(x_2) - \omega_1(y_2)) = T(z_1)T(z_2)$, which completes the proof.

By the above proof of (vi) it follows the following completion of Theorem 3.2 of [3].

3.7. THEOREM. <u>In Theorem</u> 2.3 <u>the normed linear space</u> $(E, \|\cdot\|)$ <u>is unique up to a linear isometry</u>.

Example 4.10 shows that if we do not require for the norm of $z \in E$ to satisfy (2.2), then Theorem 3.7 and (vi) in Theorem 3.6 are no longer true.

A consequence of Theorem 3.6 is the following:

3.8. COROLLARY. *Let* X *be a normed almost linear algebra with unit* e *and such that* E *is a Banach algebra*.

(i) *If* $X \neq V_X$ (*equivalently*, $\omega(X) \neq V_{\omega(X)}$) *then* $\text{dist}\,(e,V_X) \geq 1$.

(ii) *If* $\omega(X) \neq W_{\omega(X)}$ (*in particular, if* $X \neq W_X$ *and* ω *is one-to-one*) *and* X *satisfies one of the conditions* $(A_4),(A_5)$, *then* $\text{dist}\,(e,W_X) \geq 1$.

PROOF. We first show that for any normed almost linear space X we have $X \neq V_X$ iff $\omega(X) \neq V_{\omega(X)}$. Suppose $X = V_X$. Then $\omega(X) = \omega(V_X)$, whence by Remark 2.2 we have $\omega(V_X) = V_{\omega(X)}$. Conversely, suppose $\omega(X) = V_{\omega(X)}$ and let $x \in X$. Then $\omega(x) \in V_{\omega(X)}$ and so $\omega(x)+(-1\circ\omega(x)) = 0$. Then $\omega(x+(-1\circ x)) = 0 = \omega(0)$, hence $x+(-1\circ x) = 0$, i.e., $x \in V_X$.

(i) Let $v \in V_X$ such that $\rho(e,v) < 1$. Then $\|\omega(e)-\omega(v)\| < 1$ and since by Theorem 3.6 the Banach algebra E has the unit $\omega(e)$, the element $\omega(v)$ is invertible in E, i.e., there exists $z \in E$ such that $z\omega(v) = \omega(v)z = \omega(e)$. Suppose $z = \omega(x)-\omega(y)$, $x,y \in X$. Then $z\omega(v) = \omega(x)\omega(v)-\omega(y)\omega(v) = \omega(e)$. By Theorem 3.6, Remark 2.2 and Remark 3.3 (i) we get $\omega(x)\omega(v), \omega(y)\omega(v) \in V_{\omega(X)}$ and so $\omega(e) \in V_{\omega(X)}$. By Remark 3.3 (iii) it follows $\omega(X) = V_{\omega(X)}$ contradicting the hypothesis. Consequently, $\text{dist}\,(e,V_X) \geq 1$.

(ii) Suppose X satisfies (A_4), $\omega(X) \neq W_{\omega(X)}$ and let $w \in W_X$ such that $\rho(e,w) < 1$. As in (i) above, for $\omega(w)$ there exists $z = \omega(x)-\omega(y) \in E$, $x,y \in X$ such that $z\omega(w) = \omega(w)z = \omega(e)$. Then $\omega(x)\omega(w)-\omega(y)\omega(w) = \omega(e)$ and by Theorem 3.6, Remark 2.2 and (A_4) we get $\omega(x)\omega(w) = \overline{w}_1 \in W_{\omega(X)}$, $\omega(y)\omega(w) = \overline{w}_2 \in W_{\omega(X)}$. Since $\overline{w}_1 = \overline{w}_2+\omega(e)$, multiplying by -1 in the almost linear space $\omega(X)$ we get $\overline{w}_1 = \overline{w}_2+(-1\circ\omega(e))$. Consequently $\overline{w}_2+\omega(e)=\overline{w}_2+(-1\circ\omega(e))$

and since this relation holds also in E, we get $\omega(e) = -1\circ\omega(e)$, i.e., $\omega(e)\in W_{\omega(X)}$. By Remark 3.2 (ii) it follows $\omega(X) = W_{\omega(X)}$ a contradiction. Consequently dist $(e,W_X) \geqslant 1$. The proof for the case when X satisfies (A_5) is similar.

The assumption in (ii) that X satisfies one of the conditions $(A_4),(A_5)$ is essential (see Example 4.3).

For another application of Theorem 3.6 we need the following lemma which hold in a normed almost linear space.

3.9. LEMMA. Let X be a normed almost linear space and let $f\in E^{*}\setminus\{0\}$. If we have

$$(3.10)\qquad \{z\in E:f(z)>0\}\subset\omega(X)\setminus V_{\omega(X)}$$

then the equality sign holds in (3.10).

PROOF. We first show that if we have (3.10), then

$$(3.11)\qquad \omega(X)\setminus V_{\omega(X)}\subset\{z\in E:f(z)\geqslant 0\}$$

Indeed, let $\bar{x}\in\omega(X)\setminus V_{\omega(X)}$ and suppose that $f(\bar{x})<0$. Then $f(-\bar{x})>0$ whence by (3.10) we get $-\bar{x}\in\omega(X)\setminus V_{\omega(X)}$. Since $\bar{x}+(-\bar{x}) = 0$, by Lemma 2.1 (i) we get $\bar{x}\in V_{\omega(X)}$, a contradiction. Consequently we have (3.11).

Suppose now that the equality sign does not hold in (3.10). Then there exists $\bar{x}\in\omega(X)\setminus V_{\omega(X)}$ such that (using (3.11)) $f(\bar{x})=0$. Since $-1\circ\bar{x}\in\omega(X)\setminus V_{\omega(X)}$, by (3.11) we get $f(-1\circ\bar{x})\geqslant 0$. If $f(-1\circ\bar{x})>0$, then $f(-\bar{x}+(-1\circ\bar{x}))>0$ and by (3.10) we get

$-\bar{x}+(-1\circ\bar{x}) = \bar{y} \in \omega(X) \setminus V_{\omega(X)}$. Hence $-1\circ\bar{x} = \bar{x}+\bar{y}$ and so $\bar{x} = -1\circ\bar{x}+(-1\circ\bar{y})$. Then $0 = f(\bar{x}) = f(-1\circ\bar{x})+f(-1\circ\bar{y})$, which is not possible since $f(-1\circ\bar{x}) > 0$ and $-1\circ\bar{y} \in \omega(X)\setminus V_{\omega(X)}$ whence by (3.11), $f(-1\circ\bar{y}) \geq 0$. Consequently, we have

$$f(\bar{x}) = f(-1\circ\bar{x}) = 0$$

Since $\bar{x} \notin V_{\omega(X)}$ we have $\|\bar{x}+(-1\circ\bar{x})\| \neq 0$. Choose $\lambda \in R$

$$0 < \lambda < \|\bar{x}+(-1\circ\bar{x})\|$$

and let $\bar{y} \in E$, $\|\bar{y}\| = 1$ such that $f(\bar{y}) > 0$. Since $f(\lambda\bar{y}-\bar{x}-(-1\circ\bar{x})) = \lambda f(\bar{y}) > 0$, by (3.10) we get $\lambda\bar{y}-\bar{x}-(-1\circ\bar{x}) = \bar{x}_1 \in \omega(X)\setminus V_{\omega(X)}$. Then $\lambda\bar{y} = \bar{x}_1+\bar{x}+(-1\circ\bar{x})$ and by 2.1) we have

$$\|\bar{x}+(-1\circ\bar{x})\| \leq \|\bar{x}_1+\bar{x}+(-1\circ\bar{x})\| = \|\lambda\bar{y}\| = \lambda$$

contradicting the choice of λ . Consequently $f(\bar{x}) > 0$ which completes the proof.

An inspection of the proof of Lemma 3.9 shows that we have used only the linearity of $f \neq 0$ and not its continuity. It is not difficult to show that (3.10) does not hold when f is a non-zero linear functional which does not belong to E^* .

In contrast to the linear case, in a complete normed almost linear algebra X with unit e, it is possible that some elements $x \in X$ with $\rho(x,e) < 1$ to be not invertible and also that the set of invertible elements of X to be not open (see Example 4.9). Another consequence of Theorem 3.6 is the next corollary, where we give sufficient conditions in order that those two results from the linear case to hold.

3.10. COROLLARY. *Let X be a normed almost linear algebra with unit e such that E is a Banach algebra and ω is one-to-one. Suppose there exists $f \in E^* \setminus \{0\}$ satisfying (3.10) and such that $f(z_1 z_2) = f(z_1)f(z_2)$ for $z_1, z_2 \in E$. We have:*

(i) Each $x \in X$ with $\rho(x,e) < 1$ is invertible.

(ii) The set of invertible elements is open.

PROOF. Since $f \neq 0$, by (3.10) we get

$$(3.12) \qquad \omega(X) \neq V_{\omega(X)}$$

Since $\omega(e)$ is the unit of the almost linear algebra $\omega(X)$, by Remark 3.3 (iii) and (3.12) it follows that $\omega(e) \in \omega(X) \setminus V_{\omega(X)}$. Thus, by (3.10) and Lemma 3.9 we get

$$(3.13) \qquad f(\omega(e)) > 0$$

We claim that if $x \in X$ is such that $\omega(x) \in \omega(X) \setminus V_{\omega(X)}$ and $\omega(x)$ is invertible in E, then x is invertible. Indeed, since $\omega(x) \in \omega(X) \setminus V_{\omega(X)}$, by (3.10) and Lemma 3.9 we get

$$(3.14) \qquad f(\omega(x)) > 0$$

Since $\omega(e)$ is the unit of E and $\omega(x)$ is invertible in E, there exists $z \in E$ such that $\omega(x)z = z\omega(x) = \omega(e)$. By the assumption on f it follows that $f(\omega(x))f(z) = f(e)$, whence by (3.13) and (3.14) we have $f(z) > 0$. By (3.10) we get $z \in \omega(X) \setminus V_{\omega(X)}$, hence $z = \omega(y)$, $y \in X \setminus V_X$. Consequently, by (3.7) we have $\omega(x)\omega(y) =$

$= \omega(xy) = \omega(e) = \omega(y)\omega(x) = \omega(yx)$ and since ω is one-to-one we get $xy = yx = e$, i.e., x is invertible.

(i) Let now $x \in X$ such that $\rho(x,e) < 1$. By 3.12 and Corollary 3.8 (i) we have $x \notin V_X$ and so $\omega(x) \in \omega(X) \setminus V_{\omega(X)}$. Since $\|\omega(x) - \omega(e)\| < 1$ and E is a Banach algebra with unit $\omega(e)$, it follows that $\omega(x)$ is invertible in E, whence the conclusion follows by our claim above.

(ii) Let $x \in X$ such that x is invertible. By Remark 3.3 (iv) and (3.12) we have $x \notin V_X$ and so $\omega(x) \in \omega(X) \setminus V_{\omega(X)}$. As in (i) above we get that $\omega(x)$ satisfies (3.14). Since $\omega(x)$ is invertible in $\omega(X)$, it is invertible in E and so there exists $\varepsilon_1 > 0$ such that $z \in E$ is invertible if $\|z - \omega(x)\| < \varepsilon_1$. Since $f \in E^*$, by (3.14) there exists $\varepsilon_2 > 0$ such that $f(z) > 0$ if $z \in E$ and $\|z - \omega(x)\| < \varepsilon_2$. By (3.10) it follows that $z \in \omega(X) \setminus V_{\omega(X)}$ if $z \in E$, $\|z - \omega(x)\| < \varepsilon_2$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then each $y \in X$ with $\rho(y,x) < \varepsilon$ is invertible. Indeed, if $\rho(y,x) < \varepsilon$ then $\omega(y) \in \omega(X) \setminus V_{\omega(X)}$ and $\omega(y)$ is invertible in E, whence by our claim y is invertible, which completes the proof.

The assumption on ω to be one-to-one can not be dropped in Corollary 3.10 (see Example 4.3)

In ([4], Lemma 6.1) we proved that a normed almost linear space $(X, |||\cdot|||)$ is complete iff $(E, \|\cdot\|)$ is a Banach space and $\omega(X)$ is norm-closed in E. Consequently, in Corollaries 3.8 and 3.10 the assumption on E to be a Banach algebra is weaker than the assumption on X to be a complete normed almost linear algebra.

4. EXAMPLES

In this section we give examples of normed almost linear algebras as well as the counterexamples at which we reffered in Section 3.

4.1. EXAMPLE. For $\alpha, \beta \in R$, $\alpha \leq \beta$, let $I_{\alpha,\beta} = \{\lambda \in R;\ \alpha \leq \lambda \leq \beta\}$, i.e., $I_{\alpha,\beta}$ is a closed interval (possible a singleton) of R with end points α and β. Let

$$X = \{I_{\alpha,\beta} : \alpha, \beta \in R,\ \alpha \leq \beta\}$$

For $J_1, J_2 \in X$ and $\lambda \in R$ define:

$$(4.1)\qquad J_1 + J_2 = \{\lambda_1 + \lambda_2 : \lambda_1 \in J_1,\ \lambda_2 \in J_2\}$$

$$(4.2)\qquad \lambda \circ J_1 = \{\lambda \lambda_1 : \lambda_1 \in J_1\}$$

The element $0 \in X$ is $I_{0,0} = \{0\}$. Then X is an almost linear space. We have $V_X = \{I_{\lambda,\lambda} : \lambda \in R\} = \{\{\lambda\} : \lambda \in R\}$, $W_X = \{I_{-\lambda,\lambda} : \lambda \in R_+\}$ and $X = W_X + V_X$.

For $\alpha \leq \beta$, $\gamma \leq \delta$ let us put:

$$(4.3)\qquad \lambda = \frac{\alpha\delta + \beta\gamma}{2}$$

$$(4.4)\qquad \mu = \frac{\alpha\gamma + \beta\delta}{2}.$$

For $I_{\alpha,\beta}, I_{\gamma,\delta} \in X$ we define the product

$$I_{\alpha,\beta} I_{\gamma,\delta} = I_{\lambda,\mu}$$

where λ, μ are given by (4.3),(4.4). It is easy to show that (A_1)-(A_3) are satisfied and that X is a commutative almost linear algebra with unit $e = I_{0,2}$. Note that X satisfies (A_3) for $\lambda, \mu \in R$ instead of R_+. Consequently X satisfies (A_4) and (A_5).

For $J \in X$ define

$$(4.5) \qquad |||J||| = \sup \{ |\lambda| : \lambda \in J \}$$

Then $(X, |||\cdot|||)$ is a normed almost linear algebra. The normed linear algebra $(E, \|\cdot\|)$ and the mapping $\omega : X \longrightarrow E$ given by Theorem 3.6 are the following; $E = R^2$, the product on E being the usual one, i.e., $(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \beta\delta)$, $(\alpha, \beta), (\gamma, \delta) \in E$ $\|(\alpha, \beta)\| = |\alpha| + |\beta|$, $(\alpha, \beta) \in E$ and $\omega(I_{\alpha,\beta}) = (\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2})$. Clearly, ω is one-to-one and we have $\omega(V_X) = \{(\alpha, 0) : \alpha \in R\}$, $\omega(W_X) = \{(0, \beta) : \beta \in R_+\}$ and $\omega(X) = \omega(W_X) + \omega(V_X)$.

The complete normed almost linear algebra X satisfies all the conditions in Corollary 3.10. Indeed, one can choose $f \in E^*$ defined by $f((\alpha, \beta)) = \alpha$, $(\alpha, \beta) \in E$. Let us note that each $I_{\alpha,\beta} \in X \setminus (W_X \cup V_X)$ is invertible and we have $(I_{\alpha,\beta})^{-1} = I_{\gamma,\delta}$ where $\gamma = (-4\alpha\beta)/(\beta^2 - \alpha^2)$ and $\delta = 4\beta^2/(\beta^2 - \alpha^2)$. Here $\beta^2 - \alpha^2 \neq 0$ since $I_{\alpha,\beta} \notin W_X \cup V_X$. By Remarks 3.2 (iii) and 3.3 (iv) the elements $I_{\alpha,\beta} \in W_X \cup V_X$ are not invertible.

4.2. EXAMPLE. For $\alpha \leq \beta$ let $I_{\alpha,\beta}$ be defined as in Example 4.1 and for $\alpha < \beta$ let $\overset{\circ}{I}_{\alpha,\beta} = \{\lambda \in R : \alpha < \lambda < \beta\}$, $I'_{\alpha,\beta} = \{\lambda \in R : \alpha \leq \lambda < \beta\}$ and $I''_{\alpha,\beta} = \{\lambda \in R : \alpha < \lambda \leq \beta\}$. Let

$$X = \{\overset{\circ}{I}_{\alpha,\beta}, I'_{\alpha,\beta}, I''_{\alpha,\beta} : \alpha, \beta \in R, \alpha < \beta\} \cup \{I_{\alpha,\beta} : \alpha, \beta \in R, \alpha \leq \beta\}$$

We organize X as an almost linear space defining for $J_1, J_2 \in X$

and $\lambda \in R$, J_1+J_2 as in (4.1), $\lambda \circ J_1$ as in (4.2) and $0 \in X$ by $I_{0,0}$. We have $V_X = \{ I_{\lambda,\lambda} : \lambda \in R \}$ and $W_X = \{ I_{-\lambda,\lambda} : \lambda \in R_+ \} \cup \{ \mathring{I}_{-\lambda,\lambda} : \lambda > 0 \}$. Clearly $X \neq W_X + V_X$.

We organize X as a commutative almost linear algebra in the following way: for $\alpha \leq \beta$, $\gamma \leq \delta$ let λ, μ be defined by (4.3), (4.4).

$$
\begin{aligned}
I_{\alpha,\beta} I_{\gamma,\delta} &= I_{\alpha,\beta} \mathring{I}_{\gamma,\delta} = I_{\alpha,\beta} I'_{\gamma,\delta} = I_{\alpha,\beta} I''_{\gamma,\delta} = I_{\lambda,\mu} \\
\mathring{I}_{\alpha,\beta} \mathring{I}_{\gamma,\delta} &= \mathring{I}_{\alpha,\beta} I'_{\gamma,\delta} = \mathring{I}_{\alpha,\beta} I''_{\gamma,\delta} = \mathring{I}_{\lambda,\mu} \\
I'_{\alpha,\beta} I'_{\gamma,\delta} &= I'_{\lambda,\mu} \\
I''_{\alpha,\beta} I''_{\gamma,\delta} &= I''_{\lambda,\mu} \\
I'_{\alpha,\beta} I''_{\gamma,\delta} &= \mathring{I}_{\lambda,\mu}
\end{aligned}
$$

We draw attention that in the first line the equalities hold when they make sense. Clearly, X satisfies (A_4), (A_5) but X has not unit.

For $J \in X$ define $|||J|||$ as in (4.5). Then $(X, |||\cdot|||)$ is a commutative (complete) normed almost linear algebra. The normed linear algebra $(E, \|\cdot\|)$ and $\omega : X \longrightarrow E$ given by Theorem 3.6 are the following: $(E, \|\cdot\|)$ is the same as in Example 4.1 and $\omega(I_{\alpha,\beta}) = \omega(\mathring{I}_{\alpha,\beta}) = \omega(I'_{\alpha,\beta}) = \omega(I''_{\alpha,\beta}) = (\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2})$ for $\alpha < \beta$ and $\omega(I_{\alpha,\alpha}) = (\alpha, 0)$, $\alpha \in R$. Clearly, ω is not one-to-one. Here $\omega(V_X)$, $\omega(W_X)$ and $\omega(X)$ are the same as in Example 4.1.

4.3. EXAMPLE. Let X be the almost linear space described in Example 4.2. We organize X as a commutative almost linear algebra in the following way: for $\alpha \leq \beta$, $\gamma \leq \delta$ let λ, μ be given by (4.3), (4.4).

$$I_{\alpha,\beta}I_{\gamma,\delta} = I_{\lambda,\mu} \qquad \alpha \le \beta , \gamma \le \delta$$
$$I_{\alpha,\beta}\mathring{I}_{\gamma,\delta} = \mathring{I}_{\lambda,\mu} \qquad \alpha < \beta , \gamma < \delta$$
$$I_{\alpha,\beta}I'_{\gamma,\delta} = I'_{\lambda,\mu} \qquad \alpha < \beta , \gamma < \delta$$
$$I_{\alpha,\beta}I''_{\gamma,\delta} = I''_{\lambda,\mu} \qquad \alpha < \beta , \gamma < \delta$$
$$\mathring{I}_{\alpha,\beta}\mathring{I}_{\gamma,\delta} = \mathring{I}_{\alpha,\beta}I'_{\gamma,\delta} = \mathring{I}_{\alpha,\beta}I''_{\gamma,\delta} = I'_{\alpha,\beta}I'_{\gamma,\delta} = I'_{\alpha,\beta}I''_{\gamma,\delta} = I''_{\alpha,\beta}I''_{\gamma,\delta} = \mathring{I}_{\lambda,\mu} \qquad \alpha < \beta , \gamma < \delta$$
$$\mathring{I}_{\alpha,\beta}I_{\gamma,\gamma} = I'_{\alpha,\beta}I_{\gamma,\gamma} = I''_{\alpha,\beta}I_{\gamma,\gamma} = I_{\lambda,\mu} \qquad \alpha < \beta , \gamma \in R$$

Note that X does not satisfy (A_4) and (A_5). Together with the norm defined by (4.5), X is a complete normed almost linear algebra. The element $e = I_{0,2}$ is the unit of X. We have $e = I_{0,2} = I_{-1,1}+I_{1,1} \in W_X+V_X$ (and e is invertible) but $X \neq W_X+V_X$.

The normed linear algebra $(E,\|\cdot\|)$ and $\omega : X \longrightarrow E$ given by Theorem 3.6 are the same as in Example 4.2. All conditions in Corollary 3.10 are satisfied, except for ω one-to-one. Indeed, $f \in E^*$ defined in Example 4.1 can be chosen. The conclusions of Corollary 3.10 are no longer true since e.g., for $x = \mathring{I}_{0,2}$ we have $\rho(x,e) = 0$ and $\mathring{I}_{0,2}$ is not invertible. Hence the set of invertible elements is not open.

4.4. EXAMPLE. Let X be a normed almost linear space and call $T: X \longrightarrow X$ an almost linear operator ([4]) if T is additive, positively homogeneous and $T(W_X) \subset W_X$. Let $\mathcal{L}(X)$ be the set of all almost linear operators $T: X \longrightarrow X$. We organize $\mathcal{L}(X)$ as an almost linear space ([4]) defining the addition and zero as in the linear case and for $T \in \mathcal{L}(X)$ and $\lambda \in R$ define $(\lambda \circ T)(x) = T(\lambda \circ x)$, $x \in X$. For $T \in \mathcal{L}(X)$ let $|||T||| = \sup\{|||T(x)||| : |||x||| \le 1\}$ and let $L(X) = \{T \in \mathcal{L}(X) : |||T||| < \infty\}$. Then $(L(X), |||\cdot|||)$ is a normed almost linear space ([4]). Similar with the linear case we organize $(L(X), |||\cdot|||)$ as a normed almost linear algebra, defining for $T_1, T_2 \in L(X)$ the product T_1T_2 by $(T_1T_2)(x) = T_1(T_2(x))$, $x \in X$. Clearly, $T_1T_2 \in L(X)$ and $|||T_1T_2||| \leq |||T_1|||\,|||T_2|||$. It is easy to show that (A_1)-(A_4) hold. Let us note that $L(X)$ satisfies the following stronger condition than (A_4), namely we have

$$T_1(-1\circ T_2) = -1\circ(T_1T_2) \qquad (T_1, T_2 \in L(X))$$

At the end of this example we show that $L(X)$ does not always satisfy (A_5).

The unit of $L(X)$ is the almost linear operator $I: X \longrightarrow X$ defined by $I(x) = x$, $x \in X$. Clearly, when $X = V_X$ (i.e., when X is a normed linear space), then $(L(X), |||\cdot|||)$ defined above is the usual normed linear algebra of all bounded linear operators $T: X \longrightarrow X$.

Let $(E, \|\cdot\|)$ and ω be given by Theorem 2.3 for the normed almost linear space X. For simplicity, in the sequel we suppose ω one-to-one (for the general case use Theorem 3.6 of [4] and Theorem 3.6 above). Let K be the cone of the normed linear algebra $L(E)$ defined by

$$K = \{\tilde{T} \in L(E) : \tilde{T}(\omega(X)) \subset \omega(X),\ \tilde{T}(\omega(W_X)) \subset \omega(W_X)\}$$

Then the normed linear algebra $(E_{L(X)}, \|\cdot\|_{L(X)})$ and $\omega_{L(X)}: L(X) \longrightarrow E_{L(X)}$ given by Theorem 3.6 are the following; $E_{L(X)} = K-K$ equipped with the norm

$$\|\tilde{T}\|_{E_{L(X)}} = \inf\{\|\tilde{T}_1\|_{L(E)} + \|\tilde{T}_2\|_{L(E)} : \tilde{T} = \tilde{T}_1 - \tilde{T}_2,\ \tilde{T}_1, \tilde{T}_2 \in K\},\ \tilde{T} \in E_{L(X)}$$

and for $T \in L(X)$ we define $\omega_{L(X)}(T) = \tilde{T} \in K$, where for $z = \omega(x) - \omega(y) \in E$, $x, y \in X$, $\tilde{T}(z) = \omega(T(x)) - \omega(T(y))$ (which does not depend on the representation of z). Here $\omega_{L(X)}$ is one-to-one and $\omega_{L(X)}(L(X)) = K$. Let us also note that when $\omega(x) = x$ for each $x \in X$ then $L(X) = \{\tilde{T}|X : \tilde{T} \in K\}$.

Finally we show that $L(X)$ does not always satisfy (A_5). Let $X = \{(\alpha, \beta) \in R^2 ; \beta \geq |\alpha|\}$. Define the addition and the multiplication by non-negative reals as in R^2 and for $(\alpha, \beta) \in X$ and $\lambda < 0$ define $\lambda \circ (\alpha, \beta) = (\lambda\alpha, |\lambda|\beta)$. Then X is an almost linear space and we have $V_X = \{(0,0)\}$, $W_X = \{(0, \beta) \in R^2 : \beta \in R_+\}$ and $X \neq W_X + V_X$. For $(\alpha, \beta) \in X$ define $|||(\alpha, \beta)||| = |\alpha| + \beta$. Then X is a normed almost linear space. Let $T_1, T_2 \in L(X)$ be defined by

$$T_1((\alpha, \beta)) = (0, 2\beta) \qquad (\alpha, \beta) \in X$$

$$T_2((\alpha, \beta)) = (0, \beta - \alpha) \qquad (\alpha, \beta) \in X$$

Then $T_1 \in W_{L(X)}$ and for $(\alpha, \beta) \in X \setminus W_X$ we have $(T_1 T_2)((\alpha, \beta)) =$ $= (0, 2(\beta - \alpha))$ and $(-1 \circ (T_1 T_2))(\alpha, \beta) = (T_1 T_2)((-\alpha, \beta)) =$ $= (0, 2(\beta + \alpha))$, i.e., $L(X)$ does not satisfy (A_5).

4.5 EXAMPLE. Let $M_{n \times n}$ be the Banach algebra of all $n \times n$ matrices (α_{ij}) equipped with the norm

$$\|(\alpha_{ij})\| = \sum_{i,j} |\alpha_{ij}| \tag{4.6}$$

Let $J = \{(i,j) ; 1 \leq i, j \leq n\}$ and let J_+, J_- be two (possible empty) disjoint subsets of J. Let

$$Y=Y(J_+,J_-)=\{(\alpha_{ij})\in M_{n\times n}: \alpha_{ij}\geq 0,(i,j)\in J_+ ,\alpha_{ij}\leq 0,(i,j)\in J_- \}$$

We organize Y as an almost linear space where the addition and the multiplication by non-negative reals are the same as in $M_{n\times n}$ and for $\lambda < 0$ and $(\alpha_{ij})\in Y$ we define $\lambda\circ(\alpha_{ij}) = (\beta_{ij})\in Y$, where

$$\beta_{ij} = \lambda\alpha_{ij} \qquad (i,j)\notin J_+\cup J_-$$
$$\beta_{ij} = |\lambda|\alpha_{ij} \qquad (i,j)\in J_+\cup J_-$$

We have

$$V_Y = \{(\alpha_{ij})\in Y; \alpha_{ij} = 0 \text{ for } (i,j)\in J_+\cup J_- \}$$
$$W_Y = \{(\alpha_{ij})\in Y; \alpha_{ij} = 0 \text{ for } (i,j)\notin J_+\cup J_-$$

Clearly, $Y = W_Y+V_Y$ and we have $Y = V_Y$ iff $J_+ = J_- = \emptyset$ and in this case $Y = M_{n\times n}$. For $(\alpha_{ij})\in Y$ we define $|||(\alpha_{ij})|||$ as in (4.6). Then $(Y,|||\cdot|||)$ is a normed almost linear space. The normed linear space $(E_Y,\|\cdot\|_{E_Y})$ and the mapping $\omega_Y: Y\longrightarrow E_Y$ given by Theorem 2.3 are the following: $E_Y = M_{n\times n}$ equipped with the norm (4.6) and $\omega_Y(y) = y$ for each $y\in Y$.

If we define the product of two matrices of Y as in the linear case, then clearly this product does not always belong to Y. That is why we are now interested in those almost linear subspaces X of Y which are almost linear algebras for the product of two matrices defined as in $M_{n\times n}$. Being hard in general, we shall consider here the case n = 2. We denote by e the unit matrix of $M_{2\times 2}$. Due to the large number of cases (there are 81 cases to be considered for obtaining maximal almost linear algebras X of Y for all possible choices of J_+ and J_-) we shall list below nine normed almost linear algebras $X_1,X_2,\dots,X_9$, all with unit e, obtained for certain pairs J_+, J_- . All the other cases will be almost linear subalgebras of X_i , $1 \leq i \leq 9$. We recall that Y stands for $Y(J_+, J_-)$.

Case 1. $J_+ = J_- (= \emptyset)$.

$$X_1 = V_{X_1} = Y = M_{2 \times 2}$$

Case 2. $J_+ = \{(1,1)\}$, $J_- = \emptyset$

$$X_2 = \{(\alpha_{ij}) \in Y : \alpha_{12} = 0\}$$
$$X_3 = \{(\alpha_{ij}) \in Y : \alpha_{21} = 0\}$$

We have

$$V_{X_i} = \{(\alpha_{ij}) \in X_i : \alpha_{11} = 0\} \qquad i=2,3$$
$$W_{X_2} = \{(\alpha_{ij}) \in X_2 : \alpha_{21} = \alpha_{22} = 0\}$$
$$W_{X_3} = \{(\alpha_{ij}) \in X_3 : \alpha_{12} = \alpha_{22} = 0\}$$

and $X_i = W_{X_i} + V_{X_i}$, i=2,3. Here we draw attention that if a normed almost linear space Y is of the form $Y = W_Y + V_Y$ and Y_1 is an almost linear subspace of Y , then we do not have in general $Y_1 = W_{Y_1} + V_{Y_1}$.

Case 3. $J_+ = \{(2,2)\}$, $J_- = \emptyset$.

$$X_4 = \{(\alpha_{ij}) \in Y : \alpha_{12} = 0\}$$
$$X_5 = \{(\alpha_{ij}) \in Y : \alpha_{21} = 0\}$$

We have

$$V_{X_i} = \{(\alpha_{ij}) \in X_i : \alpha_{22} = 0\} \qquad i=4,5$$
$$W_{X_4} = \{(\alpha_{ij}) \in X_4 : \alpha_{11} = \alpha_{21} = 0\}$$

$$W_{X_5} = \{(\alpha_{ij}) \in X_5 : \alpha_{11} = \alpha_{12} = 0\}$$

and we have $X_i = W_{X_i} + V_{X_i}$, i=4,5.

<u>Case</u> 4. $J_+ = \{(1,1),(2,2)\}$, $J_- = \emptyset$.

$$X_6 = \{(\alpha_{ij}) \in Y ; \alpha_{12} = 0\}$$
$$X_7 = \{(\alpha_{ij}) \in Y ; \alpha_{21} = 0\}$$

We have

$$V_{X_i} = \{(\alpha_{ij}) \in X_i : \alpha_{11} = \alpha_{22} = 0\} \qquad i=6,7$$
$$W_{X_6} = \{(\alpha_{ij}) \in X_6 : \alpha_{21} = 0\}$$
$$W_{X_7} = \{(\alpha_{ij}) \in X_7 : \alpha_{12} = 0\}$$

and $X_i = W_{X_i} + V_{X_i}$, i=6,7. Note that $e \in W_{X_i}$, i=6,7 and $X_i \neq W_{X_i}$. Moreover for $v \in V_{X_i} \setminus \{0\}$, i=6,7 we have $ev = ve = v \neq 0$. Also $e \in W_{X_i}$ is invertible but $X_i \neq W_{X_i}$, i=6,7. Here X_6 and X_7 do not satisfy (A_4) and (A_5).

<u>Case</u> 5. $J_+ = \{(1,1),(1,2),(2,1),(2,2)\}$, $J_- = \emptyset$.

$$X_8 = W_{X_8} = Y$$

<u>Case</u> 6. $J_+ = \{(1,1),(2,2)\}$, $J_- = \{(1,2),(2,1)\}$.

$$X_9 = W_{X_9} = Y$$

As exemplification we describe now the maximal almost linear algebras of Y for two pairs of J_+, J_- .

<u>Case</u> (a). $J_+ = \emptyset$, $J_- = \{(1,1)\}$.

$$X_{10} = V_{X_{10}} = \{(\alpha_{ij}) \in X_1 : \alpha_{11} = \alpha_{12} = 0\}$$
$$X_{11} = V_{X_{11}} = \{(\alpha_{ij}) \in X_1 : \alpha_{11} = \alpha_{21} = 0\}$$

Case (b). $J_+ = \{(1,1),(2,1)\}$, $J_- = \{(1,2)\}$.

$$X_{12} = W_{X_{12}} = \{(\alpha_{ij}) \in X_8 : \alpha_{12} = \alpha_{22} = 0\}$$
$$X_{13} = W_{X_{13}} = \{(\alpha_{ij}) \in X_9 : \alpha_{21} = \alpha_{22} = 0\}$$
$$X_{14} = W_{X_{14}} + V_{X_{14}} = \{(\alpha_{ij}) \in X_2 : \alpha_{21} = 0\} =$$
$$= \{(\alpha_{ij}) \in X_3 : \alpha_{12} = 0\}$$

Finally, the normed linear algebra $(E_{X_i}, \|\cdot\|_{E_{X_i}})$ and $\omega_{X_i} : X_i \longrightarrow E_{X_i}$ given by Theorem 3.6 are the following: $E_{X_i} = X_i - X_i$, $\|\cdot\|_{E_{X_i}}$ is given by (4.6) and $\omega_{X_i}(x) = x$, $x \in X_i$, $1 \leq i \leq 14$.

4.6. EXAMPLE. Let

$$X = \{(\alpha, \beta) \in R^2 : \beta \geq 0\}$$

Define the addition and the multiplication by non-negative reals as in R^2 and for $(\alpha, \beta) \in X$ and $\lambda < 0$ define $\lambda \circ (\alpha, \beta) = (\lambda\alpha, |\lambda|\beta)$. Then X is an almost linear space and we have $V_X = \{(\alpha, 0) : \alpha \in R\}$, $W_X = \{(0, \beta) : \beta \in R_+\}$ and $X = W_X + V_X$. For $(\alpha_i, \beta_i) \in X$, $i=1,2$ define the product

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\beta_1\beta_2, \beta_1\beta_2)$$

Then X is an almost linear algebra which does not satisfy (A_4) and (A_5). Clearly, W_X is not an almost linear subalgebra of X.

4.7. EXAMPLE. Let X be the almost linear space described in Example 4.6. Equipped with the norm $|||(\alpha,\beta)||| = |\alpha| + \beta$, $(\alpha,\beta) \in X$, it is a normed almost linear space. For $(\alpha_i, \beta_i) \in X$, $i=1,2$ define the product

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\beta_1 \alpha_2, \beta_1 \beta_2)$$

Then X satisfies (A_1)-(A_4) but (A_5) is not satisfied. For $w \in W_X \setminus \{0\}$, $v \in V_X \setminus \{0\}$ we have $wv \neq 0$.

4.8. EXAMPLE. Let $X = R^3$. We organize X as an almost linear space where the addition and the multiplication by non-negative reals are the same as in R^3 and for $(\alpha_1, \alpha_2, \alpha_3) \in X$ and $\lambda < 0$ we define $\lambda \circ (\alpha_1, \alpha_2, \alpha_3) = (\lambda\alpha_1, \lambda\alpha_2, |\lambda|\alpha_3)$. We have

$$V_X = \{(\alpha_1, \alpha_2, 0) ; \alpha_1, \alpha_2 \in R\}$$
$$W_X = \{(0,0,\alpha_3) : \alpha_3 \in R\}$$

and $X = W_X + V_X$. There exists no norm on X, since otherwise, for $w_1 = (0,0,1) \in W_X$, $w_2 = (0,0,-1) \in W_X$ we have by (N_4), $0 \leq |||w_1||| \leq$ $\leq |||w_1 + w_2||| = 0$, contradicting (N_2).

For $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in X$ define the product

$$(\alpha_1,\alpha_2,\alpha_3)(\beta_1,\beta_2,\beta_3) = (\alpha_1\beta_1, \alpha_1\beta_2 + \alpha_2(\beta_1+\beta_2), \alpha_1\beta_3 + \alpha_3(\beta_1+\beta_2))$$

It is easy to show that (A_1)-(A_5) hold and X is an almost linear algebra with unit $e = (1,0,0)$.

We have $e \in V_X$ and $X \neq V_X$. Moreover, for $x \in X \setminus V_X$ we have $xe = x \notin V_X$ and $x(-1 \circ e) \neq -1 \circ (xe) = -1 \circ x$. Clearly $e \in V_X$ is invertible and $X \neq V_X$. For $w \in W_X \setminus \{0\}$ we have $ew = we = w \neq 0$. Note that V_X is an almost linear subalgebra of X.

If we change the product defined above with the following

$$(\alpha_1,\alpha_2,\alpha_3)(\beta_1,\beta_2,\beta_3) = (\alpha_1\beta_1,\alpha_1\beta_2+\alpha_2(\beta_1+\beta_2),\alpha_1\beta_3+\alpha_2(\beta_1+\beta_2))$$

then (A_1)-(A_5) hold, but X has no unit. Here V_X is not an almost linear subalgebra of X since e.g., for $v_1 = v_2 = (1,1,0) \in V_X$ we have $v_1v_2 = (1,3,2) \notin V_X$.

4.9. EXAMPLE. Let $X = \{(\alpha,\beta)\in R^2 ;\ \beta \geq |\alpha|\}$. We organize X as an almost linear space similar with Example 4.6. We have $V_X = \{(0,0)\}$, $W_X = \{(0,\beta) : \beta \in R_+\}$ and $X \neq W_X+V_X$. For $(\alpha_i,\beta_i)\in X$, i=1,2 define the product $(\alpha_1,\beta_1)(\alpha_2,\beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2)$. Then X is a commutative almost linear algebra, satisfying (A_4),(A_5) and e = (1,1) is the unit of X. For the norm $|||(\alpha,\beta)||| = |\alpha|+\beta$, $(\alpha,\beta)\in X$, X is a normed almost linear algebra. The normed linear algebra $(E,\|\cdot\|)$ and $\omega : X \longrightarrow E$ given by Theorem 3.6 are the following : $E = R^2$ equipped with the norm given by (2.2) and $\omega(x) = x$, $x \in X$.

The set of invertible elements of X is $\{\lambda\circ e ;\ \lambda \in R\setminus\{0\}\}$ which is not an open subset of X. For $x = (\frac{1}{2},1)\in X$ we have $\rho(x,e) = \frac{1}{2} < 1$ and x is not invertible.

4.10. EXAMPLE. Let X be the normed almost linear algebra described in Example 4.9. Let $(E,\|\cdot\|)$ and ω be given by Theorem 3.6. Let E_1 = E equipped with the norm $\|(\alpha,\beta)\|_1 = |\alpha|+|\beta|$, $(\alpha,\beta)\in E_1$ and let $\omega_1 = \omega$. Then $(E_1,\|\cdot\|_1)$ and ω_1 satisfy (i)-(iii) in Theorem 3.6 except for (2.2) and $(E,\|\cdot\|)$ is not isometric with $(E_1,\|\cdot\|_1)$.

REFERENCES

1. GODINI, G.: A framework for best simultaneous approximation: Normed almost linear spaces. J. Approximation Theory 43, 338-358 (1985)

2. GODINI, G. An approach to generalizing Banach spaces: Normed almost linear spaces. Proceedings of the 12th Winter School on Abstract analysis (Srni 1984). Suppl. Rend. Circ. Mat. Palermo II. Ser. 5, 33-50 (1984)

3. GODINI, G. On normed almost linear spaces. Math. Ann., 449-455 (1988)

4. GODINI, G. Operators in normed almost linear spaces. Proceedings of the 14th Winter School on Abstract Analysis (Srni 1986). Suppl. Rend. Circ. Mat. Palermo II. Ser 14, 309-328 (1987)

5. MAYER, O. Algebraische Strukturen in der Intervallrechnung und einige Anwendungen. Computing 5, 144-162 (1970)

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