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CYCLIC COHOMOLOGY OF CROSSED PRODUCTS

by

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Cyclic cohomology of crossed products

by Victor Nistor

1. One of the central problems in the K-theory of C^*-algebras is the computation of the K - theory groups of the reduced C^*-algebras associated with groups. A solution to this problem would give answers to conjectures of Novikov and Kadison [BC].

Some years ago Baum and Connes [BC] defined for a space X endowed with a continuous G action a geometric K-group $K^*(X, G)$ and an index morphism

$$\mu : K^*(X, G) \to K_*(C_0(X) \rtimes_{red} G)$$

They conjectured that μ is an isomorphism. The truth of this conjecture would give for X = point the desired computation of the K-groups of the reduced group C^*-algebras.

Despite considerable effort the problems I have mentioned are still unsolved.

Recently a new theory, cyclic cohomology [Co1], proved itself more manageable and was succesfully used by Connes and Moscovici to prove Novikov's conjecture for hyperbolic groups [CM], to mention only one of its still increasing number of applications.

Also, while for K-groups of crossed products our knoledge is quite limited the corresponding problem in cyclic cohomology has satisfactory answers provided one consideres suitable dense subalgebras [Bu, Ne, ENN, Ni1, Ni2]. The results show that, in cases we are interested, both K-theory and cyclic cohomology behave similarily. I shall mention however one important difference : there are no γ-obstruction fenomena in the cyclic cohomology of both crossed products by discrete or Lie groups.

In the following I shall present the main results I obtained in the computation of cyclic cohomology of crossed products [Ni 1, Ni 2]. In order to stress the reseamblance with the results in K-theory I shall formulate the results for cyclic homology rather than cyclic cohomology.

Our approach was strongly influenced by Connes theory of cyclic objects [Co 2]. A cyclic object $A^\natural$ is associated to any unital algebra A. Thus $A^\natural$ is the familiar symplicial object appearing in Hochschild homology endowed with a cyclic action on degree n, to be more precise an action of $\mathbb{Z}_{n+1}$ on $A^\natural_n = A^{\otimes n+1}$.

When one considered the crossed product $B = A \rtimes G$ insted of A one can easily see that the cyclic object associated with B decomposes naturaly as a direct sum of cyclic submodules indexed by the conjugacy classes of G. In order to describe this decomposition we shall denote the elements of the crossed product by $\sum_{g \in G} a_g g$ where a_g belongs to A and only finitely many do not vanish, thus I am considering the algebraic crossed product. Moreover I shall denote by α the defining morphism $G \to \mathrm{Aut}(A)$, thus $g\, a = \alpha_g(a)\, g$ for $a \in A$, $g \in G$. Also I shall denote by $\langle G \rangle$ the set of conjugacy classes of G. The direct summ decomposition looks like

$$(A \rtimes G)^\natural = \bigoplus_{x \in \langle G \rangle} L(G, x)$$

where the degree n component of $L(G,x)$ is generated linearely by tensors $a_0 g_0 \otimes \ldots \otimes a_n g_n$ such that $g_0 g_1 \ldots g_n \in x$, $a_i \in A$. It follows that

$$HC_*(A \rtimes G) = \bigoplus_{x \in \langle G \rangle} HC_*(L(G,x))$$

2. We now study $L(G,x)$. Fix h in x and let G_h denote the centralizer of h in G :

$$G_h = \{ g \in G,\ gh = hg \}$$

Inspired by a construction of Karoubi we now introduce an other simplicial object $\tilde{L}(G, h)$

$$\tilde{L}(G, h)_n = B_n(A, h) \otimes \beta_n(G)$$

Here $B(A, h) = B(A) \otimes_{A \otimes A^{opp}} A_h$ where $B(A)$ is the symplicial module giving the standard resolution of A by $A \otimes A^{opp}$- modules,

A_h is a twisted A-bimodule $a_1\ a\ a_2 = a_1\ a\alpha_h(a_2)$ $a, a_1, a_2 \in A$ (this description was suggested to me by Connes). $\beta(G)$ is the standard (Bar!) resolution of the trivial G-module $\mathbb{C}$ by free G-modules. The symplicial structure of $\tilde{L}(G, h)$ is the product one.

Thus if $\tau = a_0 \otimes \ldots \otimes a_n \otimes [g_0, \ldots, g_n]$ is an elementary tensor of $L(G, h)_n$ then

$$d_0\tau = a_0\alpha_h(a_1) \otimes a_2 \otimes \ldots \otimes a_n \otimes [g_1, \ldots, g_n]$$

$$d_i\tau = a_0 \otimes \ldots \otimes a_i\ a_{i+1} \otimes \ldots \otimes a_n \otimes [g_0, \ldots, \hat{g}_i, \ldots, g_n]$$

Them main feature of $\tilde{L}(G, h)$ is that G_h acts on it by simplicial automorphisms (it acts on both $B(A, h)$ and $\beta(G)$) such that

$$\tilde{L}(G, h) \otimes_{G_h} \mathbb{C} \simeq L(G, x)$$

The isomorphism is given by

$$\tau \to \alpha^{-1}_{g_n}(a_0)g_n^{-1}\ h\ g_0 \otimes \alpha^{-1}_{g_0}(a_1)g_0^{-1}\ g_1 \otimes \ldots \otimes \alpha^{-1}_{g_{n-1}}(a_n)g_{n-1}^{-1}\ g_n$$

Denote this map by p and let

$$T_{n+1}\tau = a_n \otimes \alpha_h^{-1}(a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes [h^{-1}\ g_n, g_0, \ldots, g_{n-1}]$$

Then T_{n+1} also commutes with the action of G_h and satisfies

$$p\, T_{n+1} = t_{n+1}\, p$$

where t_{n+1} is the standard cyclic permuter.

Moreover $\tilde{L}(G, h)$ with its symplicial structure and the action of T_{n+1} is very close of being a cyclic object. It satisfies all requirements except $T_{n+1}^{n+1} = 1$ which has to be replaced by

$$(2\ 1) \qquad T_{n+1}^{n+1} = h$$

(The action of h commes from the action of G_h.)

However for $h = e$, the identity of G, $\tilde{L}(G, e)$ is a cyclic object and hence standard homological algebra gives

3. Theorem There exists a spectral sequence with $E^2_{pq} = H_p(G, HC_q(A))$ convergent to $HC_{p+q}(L(G, \{e\}))$.

4. For the other conjugacy classes the general situation is not so nice. However there are definite results under some mild assumption for $A = C^\infty(X)$ where X is smooth compact manifold smoothly acted upon by G. Let X_h be the set of fixed points of h. We assume that X_h is a smooth manifold such that there is no nonzero vector in the normal bundle of X_h fixed by the isotropy representation of h. Then using Connes' description of $\tilde{L}(G, h)$ we obtain :

5. Proposition With the above assumptions.

$$HH_*(\tilde{L}(G, h)) \simeq H^*(X_h, \mathbb{C}).$$

This isomorphism is given by

$$a_0 \otimes \dots \otimes a_n \otimes [g_0, \dots, g_n] \longrightarrow i^*(a_0\, d a_1 \dots d a_n)$$

where $i : X_h \to X$ is the inclusion. This is proved using localisation.

For $L(G, x)$ the result will involve homotopy quotients and the cotangent bundle as expected from the Baum Connes conjecture.

6. <u>Proposition</u>

a) $HH_*(L(G, x)) \simeq H_*(B^* X_h // G_h, S^* X_h // G_h) \otimes \mathbb{C}$

b) For $* \geq \dim X_h$ we have $HC_*(L(G, x)) \simeq HH_*(L(G, x)) \otimes HC_*(\mathbb{C})$

for torsion h and

$HC_*(L(G, x)) \simeq H_*(B^* X_h // N_h , S^* X_h // N_h) \otimes \mathbb{C}$ for torsion free h

Here $S^* X_h$ is the cosphere bundle, $B^* X_h$ is the compactification of $T^* X_h$ such that $B^* X_h \setminus S^* X_h = T^* X_h$, // denotes the homotopy quotient and $N_h = G_h / \mathbb{Z}h$

This proposition is proved using proposition 5 and a spectral sequence comparison argument. The form of the result is due to the fact that while for $HH_*(L(G, x))$ we have a homology spectral sequence for $H^*(X//G_h)$ we have a cohomology spectral sequence so the Loday and Quillen arguments for the computation of the cyclic homology of commutative smooth algebras does not apply immediately and we need to consider a Diffeo (X_h) - equivariant dualisation of the de Rham complex of X_h. This explains the use of the cotangent bundle. Observe that this is in the spirit of [BC] where starting from K-theory elements of the pair $(B^* M, S^* M)$ there are constructed K-homology elements for M a smooth manifold. Thus this computation may be viewed as a cohomology form of the Baum Connes conjecture if we ignore torsion free elements.

7. Actually we do not lose very much if we ignore these torsion free elements. Indeed we have the following theorem.

<u>Theorem</u> There exists a natural $H^*(N_h, \mathbb{C})$-module structure on $HC^*(L(G, x))$ such that the action of Connes periodicity operator corresponds to the multiplication with the element $\xi_h \in H^2(N_h) \otimes \mathbb{C}$ corresponding to the class of the central extension

$$0 \to \mathbb{Z}h \to G_h \to N_h \to 0$$

for torsion free h.

To prove this one carefully looks at the structure of $\tilde{L}(G, h)$ First it is a symplicial object, second it carries an action of G_h compatible with the first structure and third there exists in each degree an action of $\mathbb{Z}$ that relates both structures ; the relation with the symplicial structure is the same as in the cyclic category except for the cyclic identity which has to be replaced by (2.1) as explained before.

We may express this by saying that $\tilde{L}(G, h)$ is a $\Lambda \rtimes G_h$ object, where $\Lambda \rtimes G_h$ is a new category, in analogy with the notions of symplicial or cyclic objects. $\Lambda \rtimes G_h$ may be viewed as an extension of Λ by G_h ; it contains Δ, the symplicial category and has the same objects as Δ. If $\varphi : m \to n$ a morphism in $\Lambda \rtimes G_h$ then φ may be uniquely expressed as a product $\varphi_0 \varphi_1$ where φ_0 is in Δ and $\varphi_1 \in \mathrm{Aut}(m)$. $\mathrm{Aut}(m)$ is an extension of G_h

$$0 \to G_h \to \mathrm{Aut}(m) \to \mathbb{Z}_{m+1} \to 0$$

Our interest in this structure lies in the following isomorphisms

$$HC^*(L(G,x)) \simeq \mathrm{Ext}^*_{\Lambda}(L(G,x), \mathbb{C}^{\natural}) \simeq \mathrm{Ext}^*_{\Lambda \rtimes G_h}(\tilde{L}(G,h), \mathbb{C}^{\natural})$$

The first isomorphism is due to Connes. The second isomorphism is compatible with the natural morphism

$$(7.1) \qquad \mathrm{Ext}^*_{\Lambda}(\mathbb{C}^\natural, \mathbb{C}^\natural) \longrightarrow \mathrm{Ext}^*_{\Lambda \rtimes G_h}(\mathbb{C}^\natural, \mathbb{C}^\natural)$$

defined by the functor $\Lambda \rtimes G_h \to \Lambda$. These isomorphisms define an $\mathrm{Ext}^*_{\Lambda \rtimes G_h}(\mathbb{C}^\natural, \mathbb{C}^\natural)$ module structure on $HC^*(L(G, x))$ and determine the action of S. The theorem will follow from the computation of the morphism (7.1). Now this morphism is the same as the morphism

$$H^*(B\Lambda) \to H^*(B(\Lambda \rtimes G_h)) .$$

defined by the map of classifying spaces $B(\Lambda \rtimes G_h) \to B\Lambda$ determined by the above functor.

There exists a functor $\Lambda \rtimes G_h \to N_h$ such that the homotopy fiber of $B(\Lambda \rtimes G_h) \to B\,N_h$ is homotopy equivalent to $B(\Lambda \rtimes \mathbb{Z})$. Using results of Quillen [Q] it is easy to show that $B(\Lambda \rtimes Z)$ is contractible and hence $B(\Lambda \rtimes G_h) \to B\,N_h$ is a homotopy equivalence. Looking at the commutative diagram of categories and functors

$$\begin{tikzcd}
& \Delta \times G_h \arrow[r] \arrow[dl] \arrow[d] & \Delta \arrow[d] \\
N_h & \Lambda \rtimes G_h \arrow[l] \arrow[r] & \Lambda
\end{tikzcd}$$

we obtain a commutative diagram of S^1 - fibrations

$$\begin{tikzcd}
B\mathbb{Z} = S^1 \arrow[r] \arrow[d] & S^1 \arrow[d] \\
BG_h \arrow[r] \arrow[d] & B\Delta = ES^1 \arrow[d] \\
BN_h \arrow[r] & B\Lambda = BS^1
\end{tikzcd}$$

where equalities are homotopy equivalences.

Since S viewed as an element of $\mathrm{Ext}^2_\Lambda(\mathbb{C}^\natural, \mathbb{C}^\natural) = H^2(B\Lambda)$ is the characteristic class of the right fibration it follows that the immage of S in $\mathrm{Ext}^2_{\Lambda \rtimes G_h}(\mathbb{C}^\natural, \mathbb{C}^\natural) \simeq H^*(N_h) \otimes \mathbb{C}$ is the characteristic class of the left fibration which is well known to be the class of the central extension

$$0 \to \mathbb{Z} \to G_h \to N_h \to 0$$

It is not difficult to see that this theorem entails quite often the vanishing of the periodic cyclic cohomology group $PHC^*(L(G, x))$ and this is an abstract Selberg principle as Connes pointed out.

8. Let us describe a case in which the previous theorem applies. Suppose G acts without inversion on a tree $X = (X^0, X^1)$. Let $h \in G$ be an element acting without fixed point on X. Define $m = \min\{d(P, hP), P \in X^0\}$ then the set of points Q satisfying $d(Q, hQ) = m$ is a straight line X' on which h acts by translation [S]. It is obvious that if $\gamma \in G_h$ then γ also acts by translation on X' so there exists a morphism $\varphi : G_h \to \mathbb{Z}$, $\varphi(\gamma)$ = the magnitude of the translation defined by γ on X'. Since $\varphi(h) = \pm m$ (the sign depends upon the orientation we have chosen on X') we obtain that $m\,\xi_h = 0$ in $H^2(N_h)$. Then Theorem 7 shows that S = 0 for the cyclic object L(G, x) if x is the conjugacy class of h (note that h may not be torsion since torsion elements acting on a tree without inversion have fixed points [S]). This argument is a part of the proof of the following theorem.

<u>Theorem</u> Let G and X be as above and let Y be a fundamental domain of X/G. Then there exists a six term exact sequence

$$\begin{tikzcd}
\bigoplus_{y\in Y^{1+}} PHC_0(A\rtimes G_y) \arrow[r] & \bigoplus_{P\in Y^0} PHC_0(A\rtimes G_P) \arrow[r] & PHC_0(A\rtimes G) \arrow[d] \\
PHC_1(A\rtimes G) \arrow[u] & \bigoplus_{P\in Y^0} PHC_1(A\rtimes G_P) \arrow[l] & \bigoplus_{y\in Y^{1+}} PHC_1(A\rtimes G_y) \arrow[l]
\end{tikzcd}$$

Here G_y and G_P denote the stabilizer of the edge y and respectively of the vertex P . This exact sequence is similar to Pimsner's exact sequence for K-groups [P] and this gives more evidence for the similarity of the behaviour of K-theory and cyclic homology. It is also similar to the behaviour of cohomology groups of groups acting on trees [S].

In fact the rest of the proof is an application for G_h acting on X_h = the subtree of X fixed by h of the techiques used in [S] for the computation of cohomology groups of groups acting on trees.

9. I shall say now a few words about the computations of cyclic cohomology groups of crossed products by Lie groups which seems to be the correct setting for cohomological index formulae for transversaly elliptic operators.

If G is a Lie group the crossed product we are considering is $C_c^{\infty}(G, A)$ with the usual multiplication.

In case G is compact we have the following Weyl theorem.

10. Theorem Suppose G is connected and T is a maximal torus then

$$PHC^*(A\rtimes G) \simeq PHC^*(A\rtimes T)^W$$

For general G with a finite number of connected components we have a theorem for coverings.

11. Theorem $G_1 \to G$ is a finite covering then

$$PHC_*(A \rtimes G) \simeq PHC_*(A \rtimes G_1)^H$$

where H is the covering group.

The most interesting result is the local vanishing of the γ-obstruction. The result is the following.

12. Theorem Let G be an algebraic or a connected semisimple Lie group, $k = \dim G/K$ where K is a maximal compact subgroup of G. Then both $PHC_*(A \rtimes G)$ and $PHC_*(A \rtimes K)$ are modules over $C^\infty_{inv}(G)$ - the ring of smooth class functions of G and

$$PHC_*(A \rtimes G)_m \simeq PHC_{*+k}(A \rtimes K)_m$$

for any maximal ideal m of $C^\infty_{inv}(G)$.

Moreover the only nonvanishing contribution commes from maximal ideals corresponding to conjugacy classes that intersect K.

Considering localisation is in this case a continuous substitute for looking at a given conjugacy class.

Let me mention that the proof actually gives something more :

$$PHC_*(A \rtimes G) \simeq PHC_{*+k}(A \rtimes K)$$

however this isomorphism is not canonical since it simply is obtained by counting dimensions.

A canonical isomorphism might be obtained by considering the Chern character of the Dirac element in the setting of bivariant cyclic cohomology [JK] This can indeed be done for finitely summable quasihomomorphisms [Ni3] but in this case, as Connes pointed out to me, one needs a Chern character for θ-summable quasihomomorphisms. The definition of the Chern character in this case as well as its applications seems to be an exciting subject for further study.

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