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This short note is subsequent to the work [4] and based on the results of Dimca [1]. As shown in [1] and [4], one may compute the Betti numbers of a hypersurface V in a weighted projective space $\mathbb{P}(w)$, which has 1-dimensional singularities in the affine cone. We are interested now in the following problem:

Let $f:\mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a weighted homogeneous polinomyal of degree d and weights w with 1-dimensional singularity. Describe the monodromy on the i-th complex cohomology of the Milnor fibre:

$$h^* : H^i(F) \longrightarrow H^i(F)$$

induced by the geometric monodromy $h:F \longrightarrow F$,

$$x \longmapsto t.x \text{ , with } t^d=1.$$

The monodromy matrix is diagonalisable and has eigenvalues $t_a=\exp(2\pi i a)$, with $a=0,1,\ldots,d-1$. Denote by $H^i(F)_a$ the eigenspaces corresponding to these eigenvalues.

Note that $H^{\cdot}(F)_0 \simeq H^{\cdot}(U)$, where $U:=\mathbb{P}(w) - V$ and V is the hypersurface defined by $f=0$.

We shall describe here a method to compute the dimensions of the eigenspaces.

Then we shall take one of the Zariski's examples, namely $f=(x^2+y^2)^3+(y^3+z^3)^2$, and make the computations.

Also we give examples of curves $V \subset P^2$ for which the corresponding monodromy h^* is nontrivial. Among them, a class of irreducible curves with only one singularity. They are important because the fundamental group $\pi_1(U)$ is nonabelian, which is a consequence of the fact that $h^* \neq 1$, via one observation in [2].

1. <u>The monodromy eigenspaces</u>

In the following we shall use the notations from [1] and [4]. The projection $p:F \to U$ is a d-fold ramified covering. Using the isomorphism $p_* \Omega^{\cdot}_F \simeq \bigoplus_{a=0}^{d-1} \Omega^{\cdot}_U(-a)$ between the complexes of algebraic forms (see [1, pag.3]) one can compute the eigenspaces $H^i(F)_a$ by a spectral sequence, as in [1] . Another way of finding these subspaces is the following.

Let $f''=f + f'$, where f' is a polinomyal of degree d in another set of variables $\{y_0,\dots,y_m\}$. By a Sebastiani-Thom-type property (see [1]), we have:

$$\widetilde{H}^k(F'')_0 = \bigoplus_{\substack{c=0 \\ s+t=k-1}}^{d-1} \widetilde{H}^s(F)_c \otimes \widetilde{H}^t(F')_{d-c}$$

and for the particular case $f'=y_0^d$:

$$\tilde{H}^k(F'')_o = \bigoplus_{c=1}^{d-1} \tilde{H}^{k-1}(F)_c.$$

Here F' and F'' mean the Milnor fibre of f', resp. f''. We may define a monodromy-type action on $\tilde{H}^k(F'')_o$ induced by a geometric one:

$$h' : F'' \longrightarrow F'' , \quad (x, y_o) \longrightarrow (x, t_a \cdot y_o)$$

Then $\tilde{H}^{k-1}(F)_c$ will be isomorphic to the eigenspace which corresponds to the eigenvalue t_{d-c} of the "partial monodromy"

$$(h')^* : \tilde{H}^k(F'') \longrightarrow \tilde{H}^k(F'').$$

Taking in account the isomorphism $H^k(F'')_o \simeq H^k(\tilde{U})$, where $\tilde{U} = P(w) \setminus \{f''=0\}$, we shall denote these subspaces by $H^k(\tilde{U})_a$, so

$$H^\cdot(\tilde{U}) = \bigoplus_{a=1}^{d-1} H^\cdot(\tilde{U})_a.$$

We may use the pole filtration and the spectral sequence of the algebraic de Rham complex $A^\cdot = \Gamma(\tilde{U}, \Omega^\cdot_{\tilde{U}})$ to compute $H^\cdot(\tilde{U})_a$. For this aim we have to work with forms $\omega = \dfrac{\gamma}{(f'')^s}$, where $\gamma \in \tilde{\Omega}^i_{sd}$ and γ has in each term y_o to some power congruent with a, modulo d. (We remind that Ω^i is the graded module of algebraic i-differentials and $\tilde{\Omega}^i := \ker(\Delta : \Omega^i \longrightarrow \Omega^{i-1})$, where Δ is the contraction with the Euler vector field).

Now simply write down in the spirit of [1], using the spectral sequence isomorphism from [1, Prop.1.7], the formula for the dimension of $H^n(\tilde{U})_a$:

$$b_n(\tilde{U})_a = \sum_{\substack{s=0\\ s+t=n}}^{n} \dim E^{s,t}_{n-s}(f'')_a = \sum_{\substack{s=0\\ s+t=n}}^{n} \dim({}'E^{s,t}_{n-s})_a \ ,$$

where ${}'E^{s,t}_1 := H^{s+t}(Gr^s_F\ B^{\cdot})$, (see [1] for the notations and definitions) and the eigenspace $({}'E^{s,t}_1)_a$ corresponding to a is the subspace of those forms which have y_o to some power congruent with a, modulo d, in each term.

For the hypersurfaces with 1-dimensional singular locus in the affine cone we get the other nonzero Betti number $b_{n+1}(\tilde{U})$ by computing the Euler characteristic $\chi(\tilde{U})$. In the notations of [4] :

$b_n - b_{n+1} = T + S_o$, and moreover we may take the eigenvalue a corresponding part:

$$(b_n)_a - (b_{n+1})_a = T_a + (S_o)_a$$

About the second term of this formula, there is the following:

Lemma $(S_o)_a$ is the same for every $a \in \{1,2,\dots,d-1\}$.

Proof $(S_o)_a$ is computable as a shown in [4, pag.9] from the Poincaré series associated to the Koszul complex (see [3]). In fact, the Koszul complex decomposes into a direct sum of (d-1)-complexes, each of them associated to a value $a \in \{1,2,\dots,d-1\}$ (i.e. regarding to the power of y_o, as explained above).

But these complexes are obviously isomorphic in the following explicit way:

$$\phi^{i}_{a_1,a_2} : (\Omega^i)_{a_1} \xrightarrow{\sim} (\Omega^i)_{a_2} ,$$

$$\omega^1 = \sum_{j=0}^{\infty} y_o^{d.j+a_1} . \omega_j' + d(y_o^{d.j+a_1}) \wedge \omega_j''$$

$$\downarrow$$

$$\omega^2 = \sum_{j=0}^{\infty} y_o^{d.j+a_2} . \omega_j' + d(y_o^{d.j+a_1}) \wedge \omega_j''$$

We conclude that $(S_o)_a = \frac{1}{d-1} S_o$. ∎

For simplicity, in the following we shall work only in the homogeneous case i.e. all the weights are equal to 1. From [4] we have the formula:

$$T = \sum_{\alpha \in \phi} \underbrace{\sum_{i=1}^{r(\alpha)} \dim E_i^{-i,n+i} (g_\alpha, 0)}_{T_\alpha} ,$$

where $(g_\alpha, 0)$ denote germs of the transversal singularities to the singular locus $\Sigma \subset \mathbb{C}^{n+2}$.

As we are dealing with forms in this local spectral sequence also, for each value $a \in \{1,2,\ldots,d-1\}$ we get the following:

$(T)_a = \sum_{\alpha \in \phi} \sum_{i=1}^{r(\alpha)} \dim E_i^{-i,n+i} (g_\alpha, 0)_a$, where ϕ is the set of the singular points of $\widetilde{V} = \{f'' = 0\}$.

The limits $r(\alpha)$ could be revealed by the condition:

$T_\alpha = \mu(g_\alpha, 0)$ = the Milnor number, or better:

$$\sum_{a=1}^{d-1} (T_\alpha)_a = \mu(g_\alpha, 0).$$

<u>Proposition</u>. For any transversal singularity $(g_\alpha, 0)$, the terms $(T_\alpha)_a$ are equal to $\frac{1}{d-1} T_\alpha$, for every $a \in \{1,2,\ldots,d-1\}$.

Proof. As in the preceding lemma, the local complex $(\Omega^{\cdot}_{(g_{\alpha},0)}, d)$ decomposes, regarding the action of $t=\exp 2\pi i/d$, into a direct sum of (d-1) subcomplexes which are isomorphic. Then this isomorphism induce one between spectral sequences, which at the E_1-level looks

$$E_1^{s,t}(g_{\alpha},0)_{a_1} \cong E_1^{s,t}(g_{\alpha},0)_{a_2}. \qquad \blacksquare$$

Corollary. The Euler-Poincaré characteristic for each eigenvalue a, i.e. $\sum(-1)^i b_i(F)_a$, is the same for every $a\in\{1,2,\dots,d-1\}$ and equal to $^1/_{d-1}\,\chi(F'')_0$. $\blacksquare$

2. The Zariski's example

If every transversal singularity is quasihomogeneous, then the local spectral sequence degenerates at E_2 and this fact means that we have to know only the maps $.g_{\alpha}: M(g_{\alpha}) \to M(g_{\alpha})$, the multiplication by g_{α} inside the Milnor algebras ([1, Cor.3.10]).

In case of the Zariski's example $f=(x^2+y^2)^3+(y^3+z^3)^2$ we have six cusps as transversal singularities. By adjunction of y_o we get $f''=f+y_o^6$ with transversal singularities of type $g\sim x^2+y^3+y_o^6$. Hence $\mu(g)=10$ and $M(g)\simeq \mathbb{C}\langle 1,z,y_o,y_o^2,y_o^3,y_o^4,zy_o,zy_o^2,zy_o^3,zy_o^4\rangle$. So we get:

$$(T_{\alpha})_1 = \dim M(g)_0 = \dim \mathbb{C}\langle 1,z\rangle = 2,$$
$$(T_{\alpha})_2 = \dim M(g)_1 = \dim \mathbb{C}\langle y_o, zy_o\rangle = 2,$$
$$(T_{\alpha})_3 = \dim M(g)_2 = \dim \mathbb{C}\langle y_o^2, zy_o^2\rangle = 2,$$
$$(T_{\alpha})_4 = \dim M(g)_3 = \dim \mathbb{C}\langle y_o^3, zy_o^3\rangle = 2,$$
$$(T_{\alpha})_5 = \dim M(g)_4 = \dim \mathbb{C}\langle y_o^4, zy_o^4\rangle = 2,$$

and $(T)_a = \sum_{\alpha\in\phi} (T_{\alpha})_a = 6 \times 2 = 12$, for any $a\in\{1,2,\dots,5\}$.

To get S_o, we must find some coefficients c_i of the Poincaré series $P_{reg}(t)=\prod_{i=0}^{3} \frac{(1-t^5)}{1-t} =(1+t+t^2+t^3+t^4)^4$, because $S_o= -\sum_{k=1}^{3} c_{6k-4}$.

We get easily $(S_o)_a= -\frac{105}{5} = -21$, for every $a\in\{1,2,3,4,5\}$. Hence $\chi(F)_a=-\chi(\tilde{U})_a=-((T)_a+(S_o)_a)=-12+21=9$ and $\chi(F)_o=8$.

Let's find now the Betti numbers $b_1(F)_a$. First:

$$b_1(F)_o=b_1(U)= \dim E_2^{0,1}(f) + \dim E_1^{1,0}(f).$$

We have $E_1^{0,1}(f) = \bigoplus_{i=1}^{3} \mathbb{C}\langle[\omega]\rangle$, where

$$\omega = -xy^2dx\wedge dy - z^2ydy\wedge dz + xz^2dz\wedge dx,$$

but $E_2^{0,1}(f)=0$ by some computation. As $E_1^{1,0}(f)=0$ obviously, it follows that $b_1(F)_o=0$.

Further, for the other values of $a\geq 1$, $b_1(F)_a= \sum_{s=-1}^{1} \dim E_{2-s}^{s,2-1-s}(f)_a$. It is clear that $E_1^{1,0}(f)_a=0$ and also:

$$E_2^{01}(f)_a = \begin{cases} \mathbb{C}\langle[\omega]\rangle & , \text{ for } a=1 \\ 0 & , \text{ for } a\neq 1. \end{cases}$$

For $E_3^{-1,2}(f)_a$, by studying the Koszul complex of f, we get

$$E_3^{-1,2}(f)_a = \begin{cases} \mathbb{C}\langle[f\omega]\rangle, & \text{for } a=1 \\ 0 & , \text{ for } a\neq 1 \end{cases}.$$

The final result is $b_1(F)_a= \begin{cases} 2 & , \text{ for } a=5 \\ 0 & , \text{ for } a=0,2,3,4,1, \end{cases}$

$$\text{and } b_2(F)_a = \begin{cases} 8 & \text{, for } a=0 \\ 11 & \text{, for } a=5 \\ 9 & \text{, for } a=2,3,4,1 \end{cases} .$$

3. Some other examples of plane curves with $\pi_1(U)$ nonabelian

Let's look to the plane curve $f=x^3y^3+z^6$. We can see rapidly that $b_1(F)_3 \geq 1$ because there is the form $\omega = xdy\wedge dz-$ $-ydx\wedge dy$ for which $\omega\wedge df=0$ and $d\omega = 0$.
(Our V has now two singularities). By some remark in [2], this is enough to imply that $\pi_1(U)$ is nonabelian, and the same is true for the Zariski's example.

Let's give now two examples of curves V for which $\pi_1(U)$ be nonabelian .

1) $f=y^k+z^k$, which is a union of planes.

Here the forms of type $y^i dy\wedge dz$ give a nonzero class $[\omega]\in(E_2^{0,1})_a$, for $a\neq 0$.

2) $f=xy^k+z^{k+1}$, an irreducible curve, for $k\geq 2$.

The forms of type $\omega = z^k(kxdy\wedge dz + ydx\wedge dz)$ produce a non-zero class $[\omega]\in(E_3^{-1,2})_{k-1}$.

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REFERENCES

1. Dimca, A.: 'On the Milnor fibrations of weighted homogeneous polinomyals', preprint INCREST nr.16(1988), Bucharest.
2. Némethi, A.: 'On the fundamental group of the complement of certain singular plane curves', Math.Proc.Camb.Phil. Soc. 102(1987), 453-457.
3. Siersma, D.: 'Quasihomogeneous singularities with transversal type A_1', Contemporary Mathematics 90(1989), 261-294.
4. Tibăr, M.: 'Betti numbers for weighted-projective hypersurfaces with one-dimensional singularities in the affine cone', preprint INCREST nr.7(1989), Bucharest.