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V. NIȚICĂ & A. TÖRÖK

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V. NIŢICĂ*) and A. TÖRÖK*)

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*) *Institute of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania.*

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INTRODUCTION

Let M be a type II_1 factor, τ its unique normalized trace, and let $A \subseteq M$ be a maximal abelian von Neumann subalgebra (M.A.S.A.). Let $\mathcal{N}_M(A) = \{ u \in M \mid u$ unitary, $uAu^* = A \}$ be the normalizer of A in M and let $B = \mathcal{N}_M(A)''$ be the von Neumann subalgebra generated by $\mathcal{N}_M(A)$ in M. A is called singular if B = A and A is called regular (or Cartan) if B = M. For singular M.A.S.A.'s see [D], [P1], [P3], [Pk], [T], [Tu], and for regular M.A.S.A.'s see [C-F-W] and [P2].

The aim of this paper is to find certain singular M.A.S.A.'s in $L(F_N)$, the von Neumann algebra generated by the left regular representation of the free nonabelian group with N generators, $a_1,\dots,a_N$. Examples of singular M.A.S.A.'s in $L(F_N)$ are given in [P1] and [R]. These algebras are generated by the convolutors a_i, $1 \leq i \leq N$, in [P1], or by $a_1 + \dots + a_N + a_1^{-1} + \dots + a_N^{-1}$ in [R]. Our examples are generated by convolutors of the form

$$(1) \qquad a_1^{r_1} + \dots + a_N^{r_N} + a_1^{-r_1} + \dots + a_N^{-r_N}, \quad \text{where } r_1,\dots,r_N \in N - \{0\}.$$

In Theorem 4 of § 1, developing the technique of Pytlik ([Py]), we prove that even more general convolutors in $L(F_N)$ generate M.A.S.A.'s. Then, in Theorem 18 of § 2, we prove the singularity for convolutors given by (1), using a result of Popa ([P]) about the Pukanszky invaiant.

Let $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$ be the Hilbert norm given by τ on M, let $L^2(M,\tau)$ be the completion of M with respect to this norm, so that M acts (in the standard way)

on $L^2(M,\tau)$. Let also $J : L^2(M,\tau) \to L^2(M,\tau)$ be the canonical conjugation (given by $Jx = x^*$, for $x \in M$) and let $\mathcal{B} = (A \vee JAJ)''$ be the (abelian) von Neumann subalgebra generated in $B(L^2(M,\tau))$ by A and JAJ.

Denote by p_ξ the cyclic projection corresponding to $\xi \in L^2(M,\tau)$ in $\mathcal{B}'$ (i.e. p_ξ is the projection on $\overline{\mathcal{B}\xi}^{\|\cdot\|_2}$). Since A is a M.A.S.A., p_1 is a central projection of $\mathcal{B}'$ (see Lemma 3.1 in [P2]).

The Pukanszky invariant of A is the type of homogenity of the (type I) algebra $\mathcal{B}'$. This invariant was also considered by Ambrose-Singer.

Now, by Popa's result (Corollary 3.2 in [P2]), if $\mathcal{B}'|_{I-P_1}$ is homogeneous of type I_∞, then A is singular.

Developing the technique of Radulescu ([R]), we prove that the Pukanszky invariant of the von Neumann algebra A generated by any convolutor given by (1) is infinite.

Note that by a result of Popa ([P1]), Theorem 6.1) our result implies that the singularity holds also for von Neumann algebras generated by convolutors of the form

$$a_1^{r_1} + \dots + a_k^{r_k} + a_1^{-r_1} + \dots + a_k^{-r_k}$$

where $2 \leq k \leq N$ and $r_1,\dots,r_k \in N - \{0\}$.

One is led to conjecture that given a free family $\{v_1,\dots,v_k\}_{k\geq 2}$ in F_N, the von Neumann algebra generated by the convolutor $v_1 + \dots + v_k + v_1^{-1} + \dots + v_k^{-1}$ is singular.

Finally, we mention that similar problems can be studied in free products instead of free groups (see [B-R], where the singularity is proved for the radial algebra in some free products).

1. MAXIMAL ABELIAN SUBALGEBRAS

Let $N \geq 2$ be an integer and let F_N be free nonabelian group with N generators $a_1,\dots,a_N$. Let λ be the left regular representation of F_N on the Hilbert space $l^2(F_N)$.

Denote by $L(F_N)$ the von Neumann algebra generated by $\lambda(F_N)$ in $B(l^2(F_N))$. It is well known that $L(F_N)$ is a type II_1 factor that acts standardly on $l^2(F_N)$ and that with this identification the norm given by τ coincides with the usual norm $\|\cdot\|_2$ on $l^2(F_N)$. For a set $X \subseteq l^2(F_N)$, $\overline{X}$ will stand for the closure of X in the norm $\|\cdot\|_2$.

By $\mathbb{C}[F_N]$ we denote the group ring of F_N over $\mathbb{C}$ and by 1 we denote the unity of F_N. (However, the identity operator will usually be denoted by I.) By means of the identification of M with a subspace of $L^2(M,\tau)$, $\mathbb{C}[F_N]$ corresponds to the subspace of finitely supported functions from $l^2(F_N)$. (We shall use the same notation for the elements of F_N and for their image in $\mathbb{C}[F_N]$. If $f = \sum_{w \in F_N} \alpha_w \cdot w$ is a function in $l^2(F_N)$, then $\operatorname{supp} f = \{ w \in F_N \mid \alpha_w \neq 0 \}$.)

Each $w \in F_N \setminus \{1\}$ can be written uniquely as a finite product of generators and their inverses, $w = a_{i_1}^{\alpha_1} \dots a_{i_p}^{\alpha_p}$, such that $i_k \neq i_{k+1}$, $\alpha_k \in \mathbb{Z} \setminus \{0\}$. This is the reduced form of w. We define the beginning of w to be $l(w) = a_{i_1}^{\operatorname{sign}\alpha_1}$ and the end of w to be $r(w) = a_{i_p}^{\operatorname{sign}\alpha_p}$. The canonical length function $|\cdot|$ on F_N is defined by $|w| = |\alpha_1| + \dots + |\alpha_p|$, and $|1| = 0$. If H is a free subgroup of F_N with given generators, we shall denote by $|\cdot|_H$ the corresponding length function on H. For $v_1,\dots,v_k \in F_N$, $k \geq 2$, we say that the product $v_1 v_2 \dots v_k$ is reduced if none of the factors is 1 and for all $i = 1,\dots,k-1$, $r(v_i) \cdot l(v_{i+1}) \neq 1$. We say that the product $v_1 \dots v_k$ is reduced mod $\{v_i \mid i \in I\}$, where $I \subset \{1,\dots,k\}$, if some v_i's, $i \in I$, may equal 1 and after deleting those equal to 1, we obtain a reduced product.

Let V be a family in $F_N \setminus \{1\}$, with at least two elements, which satisfies:

(2) for any $v,w \in V^{\pm}$, if $vw \neq 1$ then vw is reduced

where $V^{\pm} := V \cup V^-$, $V^- := \{v^{-1} \mid v \in V\}$. We denote by χ_1 the convolutor $\sum_{v \in V} (v + v^{-1})$, and by H the subgroup generated by V (which is free with basis V!).

Let A be the non Neumann algebra generated by χ_1 in $L(F_N)$. We shall prove that A is a M.A.S.A. in $L(F_N)$.

In the sequel, $f \cdot g$ will stand for the canonical convolution of f and g on F_N,

for any $f,g \in l^2(F_N)$.

LEMMA 1. Let $v,v' \in V_-^+$ and $w \in F_N$ be such that vwv' is reduced mod $\{w\}$. Then, for any $\varepsilon > 0$, there exists a function $\varphi \in l^2(F_N)$ such that:

$$\| (\chi_1 \cdot \varphi - \varphi \cdot \chi_1) - (vw - wv') \|_2 < \varepsilon .$$

Proof. We use the same construction as that of Pytlik in Lemma 4.1 from [Py], but we border vwv' with words of equal H-length from H insteed of using words of equal length from F_N. ∎

LEMMA 2. Let $v,v' \in V_-^+$ and $w \in F_N \setminus \{1\}$ be such that $v'' \cdot w$ is reduced for all $v'' \in V_-^+ - \{v\}$ and wv'' is reduced for all $v'' \in V_-^+ - \{v'\}$. Then, for any $\varepsilon > 0$, there exists a function $\varphi \in l^2(F_N)$ such that:

$$\| (\chi_1 \cdot \varphi - \varphi \cdot \chi_1) - (vw - wv') \|_2 < \varepsilon .$$

Proof. $\chi_1 \cdot w - w \cdot \chi_1 = vw - wv' + \sum_{\substack{v'' \neq v \\ v'' \in V_-^+}} v''w - \sum_{\substack{v''' \neq v' \\ v''' \ V_-^+}} wv'''$.

From the two sums above, we can form $2 \cdot \operatorname{card} V - 1$ pairs $v'' \cdot w - w \cdot v'''$ with $v''wv'''$ reduced. Applying for each of them Lemma 1, we find functions $\varphi_{v'',v'''} \in l^2(F_N)$ such that:

$$\| (\chi_1 \cdot \varphi_{v'',v'''} - \varphi_{v'',v'''} \cdot \chi_1) - (v''w - wv''') \|_2 < \frac{\varepsilon}{2 \operatorname{card} V - 1}$$

Hence, for $\varphi = w - \sum_{(v'',v''')} \varphi_{v'',v'''}$ one has the desired conclusion. ∎

LEMMA 3. Let $g \in l^2(F_N)$ be such that $g \cdot \chi_1 = \chi_1 \cdot g$.

a) If $x,y \in H$ and $|x|_H = |y|_H$, then $g(x) = g(y)$.

b) If $x \notin H$, then $g(x) = 0$.

Proof. a) Use Lemma 1 above and the arguments of Lemma 4.1 and Proposition 4.2 in [Py].

b) By the uniqueness of the reduced form, there exists $h \in H$, $w \in F_N$ such that $x = hw$ is reduced mod $\{h\}$, and $|h|_H$ is maximum. If $h \neq 1$, write $h = \bar{v}_1 \dots \bar{v}_k$ in reduced form, $\bar{v}_i \in V_-^+$, and choose $\bar{w}_1, \bar{w}_2, \dots, \bar{w}_k \in V_-^+$ such that $w\bar{w}_1$, $w\bar{w}_1\bar{w}_2, \dots, w\bar{w}_1\bar{w}_2 \dots \bar{w}_k$ are reduced. (If $u \in F_N \setminus \{1\}$, there is an element $v \in V_-^+$ such that uv is reduced; indeed, condition (2) implies that the map $l : V_-^+ \to \{a_i^{+1} \mid 1 \leq i \leq N\}$ is one-to-one, and card $V_-^+ \geq 4$.) Now, from Lemma 1 and the argument of Proposition 4.2 in [Py] it follows that:

$$g(x) = g(\bar{v}_2 \dots \bar{v}_k w \bar{w}_1) = \dots = g(w\bar{w}_1 \dots \bar{w}_k) .$$

So, we can assume that $h = 1$.

The following cases appear:

I: vx is reduced for every $v \in V_-^+$;

II: there exists $v_o \in V_-^+$ such that $v_o x$ is not reduced (and then this v_o is unique, due to (2)).

Case I: There exist $\bar{v}_1, \bar{v}_2 \in V_-^+$ such that $\bar{v}_1 x$, $x\bar{v}_2$ are reduced and $\bar{v}_1 \neq \bar{v}_2^{-1}$. Then

$$g(x) = g(\bar{v}_1 x \bar{v}_2) = g(x\bar{v}_1\bar{v}_2) ,$$

where we have used Lemma 2 (with $w = x\bar{v}_2$, $v = \bar{v}_1$, $v' = \bar{v}_2^{-1}$) for the first equality, Lemma 1 (with $w = x\bar{v}_2$, $v = \bar{v}_1$, $v' = \bar{v}_1$) for the second equality and the argument of Proposition 4.2 from [Py] for both.

Since $|x| < |x\bar{v}_2\bar{v}_1|$, and $x\bar{v}_2\bar{v}_1$ is also in case I, we can iterate this trick to find an infinite set of distinct elements y from F_N such that $g(y) = g(x)$. But $g \in l^2(F_N)$, therefore $g(x) = 0$.

Case II: There exist $\bar{v}_1, \bar{v}_2 \in V_-^+$ such that $x\bar{v}_1\bar{v}_2$ is reduced and $r(v_o x\bar{v}_1) = r(\bar{v}_1)$. Then

$$g(x) = g(v_o x\bar{v}_1) = g(x\bar{v}_1\bar{v}_2) ,$$

where we have used Lemma 2 with $w = x\bar{v}_1$, $v = v_o$, $v' = \bar{v}_1^{-1}$ for the first equality and with $w = v_o x\bar{v}_1\bar{v}_2$, $v = v_o^{-1}$, $v' = \bar{v}_2^{-1}$ for the second equality (here Lemma 2 can be applied because $l(v_o x\bar{v}_1\bar{v}_2) = l(v_o)$ and $r(v_o x\bar{v}_1\bar{v}_2) = r(\bar{v}_2)$), and the argument of Proposition 4.2 from [Py].

Since $|x| < |x\bar{v}_1\bar{v}_2|$ and $x\bar{v}_1\bar{v}_2$ is also in case II, we can again iterate this trick and obtain $g(x) = 0$ too. ∎

THEOREM 4. A is a M.A.S.A. in $L(F_N)$.

Proof. It is an easy consequence of Lemma 3 (see Theorem 4.3 in [Py]). ∎

2. SINGULAR SUBALGEBRAS

In this paragraph we take V to be:

$$V = \{a_1^{r_1}, a_2^{r_2}, \ldots, a_N^{r_N}\}, \text{ where } r_i \in N - \{0\}.$$

It is clear that V satisfies (2).

For $\mathcal{H}_1$ and A as above, let $\mathcal{A} = A \vee JAJ$. We shall prove that $\mathcal{A}'$ is of homogenous type I_∞ on $I - p_1$.

DEFINITION. We say that $w \in G$ is a core if $w = 1$ or if the reduced form of w neither begins nor ends with a word from H (i.e., if $w = h_1 v h_2$ is reduced mod$\{h_1, v, h_2\}$ with $h_1, h_2 \in H$, then $h_1 = h_2 = 1$). We put the cores in families of related cores. These families are of the following three type:

(3) $$C_o = \{1\}$$

(4) $$C(i,j,\alpha,\beta,w) = \{a_i^\alpha w a_j^\beta, a_i^{\alpha - r_i} w a_j^\beta, a_i^\alpha w a_j^{\beta - r_j}, a_i^{\alpha - r_i} w a_j^{\beta - r_j}\}$$

where $1 \leq i,j \leq N$ are such that $r_i \geq 2$, $r_j \geq 2$, $1 \leq \alpha \leq r_i - 1$, $1 \leq \beta \leq r_j - 1$ and $w \in F_N$ is such that $l(w) \neq a_i^{\pm 1}$, $r(w) \neq a_j^{\pm 1}$ and $w \neq 1$ if $i = j$.

(5) $C(i,\alpha) = \{a_i^{\alpha}, a_i^{\alpha - r_i}\}$

where $1 \leq i \leq N$ is such that $r_i \geq 2$ and $1 \leq \alpha \leq r_i - 1$.

We remark that if we denote by $\mathcal{C}$ the colection of the families of cores described above, then:

(6) $\mathcal{C}$ is a partition of the set of cores;

(7) $\mathrm{span}\{H \cdot C \cdot H\}$ is orthogonal to $\mathrm{span}\{H \cdot C' \cdot H\}$ for every $C, C' \in \mathcal{C}$, $C \neq C'$;

(8) $\bigoplus_{C \in \mathcal{C}} \overline{\mathrm{span}}\{H \cdot C \cdot H\} = l^2(F_N)$;

(9) $\overline{\mathrm{span}}\{H \cdot C \cdot H\}$ is an invariant subspace for $\lambda(\chi_1)$ and $J\lambda(\chi_1)J$ for every $C \in \mathcal{C}$.

The core of type (3) was studied by Rădulescu. One has:

THEOREM 5 (see [R]). The space $\overline{\mathrm{span}}\, H = \overline{\mathrm{span}}\, H \cdot C_o \cdot H$ is a sum of orthogonal cyclic projections $p_1 + \sum_{\xi \in I_o} p_\xi$, where $I_o \subset l^2(F_N)$ is an infinite family of norm one vectors and p_ξ is equivalent to p_η for any $\xi, \eta \in I_o$. ∎

In order to prove that the homogenity type of $\mathcal{A}'$ on $I - p_1$ is I_∞, it is sufficient to show that the central support of p_ξ, $\xi \in I_o$, is $I - p_1$. Indeed:

LEMMA 6. Let M be a von Neumann algebra of type I, let $\{e_n\}_{n \geq 1}$ be an infinite family of mutually equivalent and orthogonal projections and let $q = z(e_1)$ be their central support. Then the reduced algebra M_q is of homogeneous type I_∞.

Proof. By the contrary, let $0 \neq p \leq q$ be a central projection such that M_p is of homogeneous type I_k, with $k < \infty$. Then $\{pe_n\}_{n \geq 1} \subset M_p$ is an infinite family of nonzero (because $p \leq z(e_n)$) mutually orthogonal and equivalent projections. This is a contradiction with M_p being of homogenous type I_k, with $k < \infty$ (see [S-Zs], § 4). ∎

In fact, we shall show that for each family of cores $C \in \mathcal{C} \setminus \{C_0\}$, there exists a family of norm one vectors $I(C) \subset l^2(F_N)$ such that:

$$\overline{\text{span}}\{H \cdot C \cdot H\} = \bigvee_{\xi \in I(C)} p_\xi ,$$

and p_ξ is equivalent to p_ω for any $\xi \in I(C)$. Here ω is a fixed vector from I_0, which will be described below. By (7), (8) and the result quoted in Theorem 5, this imply that the central support of p_ω is $I - p_1$.

If $H \subset K$ are Hilbert spaces, we shall denote the orthogonal projection on H by proj_H^K.

Let us fix some $C \in \mathcal{C}$. All the notations we introduce from now on depend on C, even if it does not appear explicitly. The family C we are refering to will always be clear.

For any integers $k, k', k'' \geq 0$ define

$$V_{k',k''} := \text{span}\{\overline{w} = w_1 \cdot w \cdot w_2 \mid \overline{w} \text{ is reduce}, w \in C, w_1, w_2 \in H,$$
$$|w_1|_H = k', |w_2|_H = k''\};$$

$$V_k := \text{span}\{V_{k',k''} \mid k' + k'' = k\};$$

q_k is the orthogonal projection on $V_k \subset l^2(F_N)$

$$S_k := \text{span}\{q_k(\chi_1 \cdot w), q_k(w \cdot \chi_1) \mid w \in V_{k-1}\};$$

$$L_k := q_k \cdot \lambda(\chi_1) \cdot q_k, \quad R_k := q_k \cdot J\lambda(\chi_1)J \cdot q_k .$$

We introduce the convolutors

$$\chi_n = \begin{cases} \sum_{w \in H, |w|_H = n} w & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \\ 0 & \text{if } n < 0 . \end{cases}$$

For $\gamma \in V_k$ $(k \geq 0)$ and $n, m \in \mathbb{Z}$, define

$$\gamma_{n,m} = \begin{cases} q_{n+m+k}(\chi_n \cdot \gamma \cdot \chi_m) & \text{if } n,m \geq 0, \\ 0 & \text{if } n < 0 \text{ or } m < 0. \end{cases}$$

The following formulae are easy to check (see [C] or [R]):

$$(10) \quad \begin{cases} \chi_1 \cdot \chi_1 = \chi_2 + 2N ; \\ \chi_1 \cdot \chi_n = \chi_n \cdot \chi_1 = \chi_{n+1} + (2N-1)\chi_{n-1}, & \text{for } n \geq 2; \end{cases}$$

$$(11) \quad \begin{cases} \chi_1 \cdot \gamma_{n,m} = \gamma_{n+1,m} + (2N-1)\gamma_{n-1,m}, & \text{for } n \geq 1,\ m \geq 0; \\ \gamma_{n,m} \cdot \chi_1 = \gamma_{n,m+1} + (2N-1)\gamma_{n,m-1}, & \text{for } n \geq 0,\ m \geq 1. \end{cases}$$

We can now describe the vector $\omega \in I_o$. It is chosen such that for $n,m,p,q \geq 0$ the following formulae hold:

$$(12) \quad \begin{cases} \chi_n \cdot \omega \cdot \chi_m = \omega_{n,m} - \omega_{n-2,m} - \omega_{n,m-2} + \omega_{n-2,m-2} ; \\ \omega_{n,m} = \sum_{s,t \geq 0} \chi_{n-2s} \cdot \omega \cdot \chi_{m-2t} ; \end{cases}$$

$$(13) \quad \begin{cases} \text{for } (n,m) \neq (p,q),\ \omega_{n,m} \text{ is orthogonal to } \omega_{p,q} ; \\ \| \omega_{n,m} \|^2 = (2N-1)^{n+m} \end{cases}$$

(See [R], Lemma 2 (a), Lemma 3 (a) and the proof of Theorem 7; almost all the vectors from I_o satisfy these formulae).

Our aim is to show that each V_k, $k \geq 0$, can be covered with cyclic projections p_ξ, which are equivalent to p_ω (i.e. $q_k \leq \bigvee_\alpha p_{\xi_\alpha}$, with p_{ξ_α} equivalent to p_ω). The case $k = 0$ is made by ad-hoc methods and will be developed later, in Lemmas 13-17. For $k \geq 1$, we shall use an induction argument, which relies on Lemmas 7-12 and is described in the Remark following Lemma 12.

LEMMA 7. Let $k \geq 1$. If $V_{k'}$, $0 \leq k' \leq k-1$, can be covered with projections, which are equivalent to p_ω, then these projections cover S_k too.

Proof. This follows from the fact that for any $w \in V_{k-1}$,

$$q_k(\chi_1 \cdot w) = \chi_1 \cdot w - \bar{w}$$

where $\bar{w} \in \operatorname{span}\{V_{k-2}, V_{k-1}\}$.

Therefore in proving the induction step for $k \geq 1$, we have to deal only with the space $V_k \ominus S_k$. ∎

LEMMA 8. Let $k \geq 1$.

(a) If $\gamma \in V_k \ominus S_k$ and $s \geq 1$, then

$$\chi_1 \cdot \gamma_{o,s} = \gamma_{1,s} + (L_k \gamma)_{o,s} ;$$

$$\gamma_{s,o} \cdot \chi_1 = \gamma_{s,1} + (R_k \gamma)_{s,o} .$$

(b) $R_k \cdot L_k = L_k \cdot R_k = 0$, hence L_k and R_k commute.

(c) $V_k \ominus S_k$ is a reducing subspace for L_k and R_k.

(d) L_k^2 is the orthogonal projection on $V_k \ominus \operatorname{span}\{V_{s,t} \mid s + t = k,\ s \geq 1\}$. In particular, the spectra of L_k, R_k, $L_k | V_k \ominus S_k$ and $R_k | V_k \ominus S_k$ are contained in $\{-1,0,1\}$ (and all these values can appear).

Proof. (a) Let $\eta \in V_{k'}$ with $k' \geq 0$ and let $s \geq 0$. Then

$$\begin{aligned} &\langle \chi_1 \cdot \gamma_{o,s} | \eta \rangle = \langle \gamma_{o,s} | \chi_1 \cdot \eta \rangle = \langle q_{k+s}(\gamma \chi_s) | \chi_1 \cdot \eta \rangle = \\ (14) \qquad &= \langle \gamma \cdot \chi_s | q_{k+s}(\chi_1 \cdot \eta) \rangle = \langle \gamma | q_{k+s}(\chi_1 \cdot \eta) \cdot \chi_s \rangle = \\ &= \langle \gamma | q_k(q_{k+s}(\chi_1 \cdot \eta) \cdot \chi_s) \rangle . \end{aligned}$$

For $k' < k + s - 1$, $q_{k+s}(\chi_1 \cdot \eta) = 0$, hence the last member of (14) is zero.

For $k' = k + s - 1$, $q_k(q_{k+s}(\chi_1 \cdot \eta)\chi_s) = q_k(\chi_1 \cdot q_{k-1}(\eta \cdot \chi_s))$, as one can see comparing the length of the words which appear. Hence the last member of (14) is zero again.

For $k' = k + s$, $q_k(q_{k+s}(\chi_1 \cdot \eta) \cdot \chi_s) = q_k(\chi_1 \cdot q_k(\eta \cdot \chi_s))$, hence the last

member of (14) becomes:

$$\langle \gamma | q_k(\chi_1 \cdot q_k(\eta \cdot \chi_s)) \rangle = \langle q_k(\chi_1 \cdot \gamma) | \eta \cdot \chi_s \rangle =$$

$$= \langle q_k(\chi_1 \cdot \gamma) \cdot \chi_s | \eta \rangle = \langle q_{k+s}(q_k(\chi_1 \cdot \gamma)\chi_s) | \eta \rangle =$$

$$= \langle [q_k(\chi_1 \cdot \gamma)]_{o,s} | \eta \rangle = \langle (L_k \gamma)_{o,s} | \eta \rangle$$

So $\chi_1 \cdot \gamma_{o,s} = \gamma_{1,s} + (L_k \gamma)_{o,s}$.

The second formula can be proved similarly.

(b), (d) It is enough to verify the formulae for $\gamma \in V_k \cap F_N$, for each type of family of cores (because $V_k \cap F_N$ is an orthogonal basis of V_k). For example, $L_k | V_{s,t} = 0$ whenever, $s \geq 1$ and $s + t = k$.

(c) The inclusion $L_k(S_k) \subset S_k$ follows from:

$$L_k q_k(\chi_1 \cdot v) = 0, \quad \text{for } v \in V_{k-1},$$

$$L_k \; q_k(v \cdot \chi_1) = q_k(q_{k-1}(\chi_1 \cdot v) \cdot \chi_1), \quad \text{for } v \in V_{k-1}.$$

Since L_k is a selfadjoint operator, $V_k \ominus S_k$ is reducing for it. R_k can be dealt with analogously. ∎

From (b) and (c) of the previous Lemma each $V_k \ominus S_k$, $k \geq 1$, has a basis made from eigenvectors for both L_k and R_k. If $\gamma \in V_k \ominus S_k$ is an eigenvector such that $L_k \gamma = \lambda_1 \cdot \gamma$, $R_k \gamma = \lambda_2 \cdot \gamma$, then from (b) and (d) of Lemma 8:

$$(\lambda_1, \lambda_2) \in \{(\varepsilon, 0), (0,0), (0, \varepsilon) \mid \varepsilon \in \{-1, 1\}\}.$$

LEMMA 9. Let $k \geq 0$, and let $\gamma \in V_k \ominus S_k$ be an eigenvector for both L_k and R_k, of norm one, with eigenvalues (λ_1, λ_2).

(a) If $(\lambda_1, \lambda_2) = (0,0)$, then:

$$\chi_n \cdot \gamma \cdot \chi_m = \gamma_{n,m} - (\gamma_{n,m-2} + \gamma_{n-2,m}) + \gamma_{n-2,m-2},$$

$$\gamma_{n,m} = \sum_{s,t \geq 0} \chi_{n-2s} \cdot \gamma \cdot \chi_{m-2t}, \quad \text{for } n,m \geq 0.$$

(b) If $(\lambda_1, \lambda_2) = (\varepsilon, 0)$, $\varepsilon \in \{-1,1\}$, then:

$$\chi_n \cdot \gamma \cdot \chi_m = \gamma_{n,m} + \varepsilon \cdot \gamma_{n-1,m} - \gamma_{n,m-2} - \varepsilon \cdot \gamma_{n-1,m-2},$$

$$\gamma_{n,m} = \sum_{s,t \geq 0} (-\varepsilon)^s \cdot \chi_{n-s} \cdot \gamma \cdot \chi_{m-2t}, \quad \text{for } n,m \geq 0.$$

(c) If $(\lambda_1, \lambda_2) = (0, \varepsilon)$, $\varepsilon \in \{-1,1\}$, then:

$$\chi_n \cdot \gamma \cdot \chi_m = \gamma_{n,m} + \varepsilon \cdot \gamma_{n,m-1} - \gamma_{n-2,m} - \varepsilon \cdot \gamma_{n-2,m-1},$$

$$\gamma_{n,m} = \sum_{s,t \geq 0} (-\varepsilon)^t \cdot \chi_{n-2s} \cdot \gamma \cdot \chi_{m-t}, \quad n,m \geq 0.$$

Proof. In each of (a), (b), (c), the second formula is an easy consequence of the first.

For the first formula we use (a) from Lemma 8 and induction based on formulae (10) and (11). For example, in case (a):

$$\chi_1 \cdot \gamma \cdot \chi_1 = \chi_1 \cdot \gamma_{o,1} = \gamma_{1,1},$$

$$\chi_2 \cdot \gamma \cdot \chi_1 \overset{(10)}{=} \chi_1(\chi_1 \cdot \gamma \cdot \chi_1) - 2N\, \gamma \cdot \chi_1 \overset{(11)}{=}$$

$$= \gamma_{2,1} + (2N-1)\gamma_{o,1} - 2N \cdot \gamma_{o,1} = \gamma_{2,1} - \gamma_{o,1}. \qquad \blacksquare$$

LEMMA 10. Let $k \geq 1$ and let γ be like in Lemma 9. Then $\gamma_{n,m}$ is orthogonal to $\gamma_{p,q}$ for $(n,m) \neq (p,q)$.

Proof. Since $\gamma_{n,m} \in V_{k+n+m}$, the assertion is true for $n+m \neq p+q$. If $n+m = p+q$, then:

$$\langle \gamma_{n,m} | \gamma_{p,q} \rangle = \langle \gamma_{n,m} | q_{k+n+m}(\chi_p \cdot \gamma \cdot \chi_q) \rangle =$$

$$= \langle \chi_p \cdot \gamma_{n,m} \cdot \chi_q | \gamma \rangle.$$

But for every $m,n,p,q \geq 0$

$$\chi_p \cdot \gamma_{n,m} \cdot \chi_q \in \operatorname{span}\{\gamma_{s,t} \mid |n-s| \leq p,\ |m-t| \leq q\},$$

as one can see by induction using (10) and the fact that

$$\chi_1 \cdot \gamma_{n,m} \in \operatorname{span}\{\gamma_{n-1,m}; \gamma_{n,m}; \gamma_{n+1,m}\},$$

which is clear from (11) and Lemma 8 (a). Hence, if $p < n$, then:

$$\chi_p \cdot \gamma_{n,m} \cdot \chi_q \in \operatorname{span}\{\gamma_{s,t} \mid s \geq n - p > 0\} \subset V_k^{\perp}$$

and similarly for $p > n$, when $q < m$. ∎

REMARK. The last two lemmas show that for γ as in Lemma 9, the family $\{\gamma_{n,m}\}_{n,m\geq 0}$ is an orthogonal basis for $\overline{\mathcal{A}\gamma}$.

LEMMA 11. Let $k,m,n \geq 0$, $\gamma \in V_k$. Then:

$$\|\gamma_{n,m}\|_2^2 = (2N-1)^{n+m} \cdot \|\gamma\|_2^2 .$$

Proof. Let $\gamma = \sum_{g \in V_k \cap F_N} \lambda_g \cdot g$. For $g, g' \in V_k \cap F_N$, $g \neq g'$, $\operatorname{supp} g_{n,m} \cap \operatorname{supp} g'_{n,m} = \emptyset$, hence $g_{n,m}$ is orthogonal to $g'_{n,m}$. Therefore it is enough to prove the formula for $\gamma \in V_k \cap F_N$. In this case, $\gamma_{n,m}$ is the characteristic function of its support, and the cardinal of the support is $(2N-1)^{n+m}$. ∎

LEMMA 12. Let γ be like in Lemma 9 and let $T_o : \mathcal{A}\gamma \to \mathcal{A}\omega$ be the linear mapping defined by $T_o(\chi_n \cdot \gamma \cdot \chi_m) = \chi_n \cdot \omega \cdot \chi_m$. Then T_o is well defined and it extends to an invertible bounded liner operator $T: \overline{\mathcal{A}\gamma} \to \overline{\mathcal{A}\omega}$. Since $T \in \mathcal{A}'$, one obtains that p_γ and p_ω are equivalent.

Proof. The vectors $\{\chi_n \gamma \chi_m\}_{n,m\geq 0}$ are linearly independent because for any integer $k \geq 0$, $\{\chi_n \gamma \chi_m \mid n + m \leq k\}$ is a family of generators for $\operatorname{span}\{\gamma_{n,m} \mid n + m \leq k\}$ (see Lemma 9) and has the same cardinal as the dimension of the space it generates.

If γ has eigenvalues $(0,0)$, then $T_o \gamma_{n,m} = \omega_{n,m}$ (use Lemma 9 and (12)), hence T is unitary (due to (12), (13) and to Lemmas 10 and 11).

If γ has eigenvalues $(\varepsilon, 0)$, $\varepsilon \in \{-1, 1\}$, then:

$$T_o \cdot \gamma_{n,m} = \omega_{n,m} - \varepsilon \cdot \omega_{n-1,m}$$

(use Lemma 9 and (12)). We define the operators $U_o : \mathcal{A}\gamma \to \mathcal{A}\omega$ and $S_1^o : \mathcal{A}\gamma \to \mathcal{A}\gamma$ given by $U_o \gamma_{n,m} = \omega_{n,m}$ and $S_1^o \gamma_{n,m} = \gamma_{n-1,m}$. U_o clearly extends to a unitary $U : \overline{\mathcal{A}\gamma} \to \overline{\mathcal{A}\omega}$. If we consider the unitary $W : \overline{\mathcal{A}\gamma} \to l^2(N) \otimes l^2(N)$ given by

$$W(\gamma_{n,m}) = (2N-1)^{-\frac{n+m}{2}} \delta_n \otimes \delta_m ,$$

then

$$WS_1^o W^* = (2N-1)^{-\frac{1}{2}} S^* \otimes I ,$$

where both sides are restricted to the space $W(\mathcal{A}\gamma)$ and S is the unilateral shift. Therefore S_1^o has an extension $S_1 : \overline{\mathcal{A}\gamma} \to \overline{\mathcal{A}\omega}$ too, and $\| S_1 \| = (2N-1)^{-\frac{1}{2}}$.

Since $T_o = U(I - \varepsilon S_1)|_{\mathcal{A}\gamma}$, the conclusion follows.

The case of eigenvalues $(0, \varepsilon)$, $\varepsilon \in \{-1,1\}$ is similar. ■

REMARK. The induction step is now completely proved. For $k \geq 1$ we can choose a basis for $V_k \ominus S_k$ made from eigenvectors for both L_k and R_k (by Lemma 8), and Lemma 12 shows that the cyclic projections associated to the vectors of this basis are equivalent to p_ω. Hence, if we admit that span $\{V_{k'} \mid k' \leq k-1\}$ is covered by projections equivalent to p_ω, then Lemma 7 and the above arguments show that V_k can be covered by projections equivalent to p_ω too.

We shall study now the spaces V_o for the families of cores $C \in \mathcal{C} \setminus \{C_o\}$.

Let $V_o = \text{span}\, C(i,j,\alpha,\beta,w)$ (see (4)). Consider the vectors of norm one:

$$\xi^{\varepsilon,\delta} = \tfrac{1}{2}(a_i^{\alpha} + \varepsilon \cdot a_i^{\alpha - r_i})\, w\, (a_j^{\beta} + \delta \cdot a_j^{\beta - r_j}),$$

where $\varepsilon, \delta \in \{-1,1\}$. It is obvious that:

$$\text{span}\{\xi^{\varepsilon,\delta} \mid \varepsilon, \delta \in \{-1,1\}\} = V_o .$$

Fix now a pair $(\varepsilon, \delta) \in \{-1,1\}^2$ and denote $\xi^{\varepsilon,\delta}$ by ξ. We shall prove that p_ξ is equivalent to p_ω.

LEMMA 13. (a) $\chi_n \cdot \xi \cdot \chi_m = \xi_{n,m} + \varepsilon \cdot \xi_{n-1,m} + \delta \cdot \xi_{n,m-1} +$

$$+\varepsilon \cdot \delta \cdot \xi_{n-1,m-1},$$

$$\xi_{n,m} = \sum_{s,t\geq 0}' (-\varepsilon)^s \cdot (-\delta)^t \chi_{n-s} \cdot \xi \cdot \chi_{m-t}, \quad \text{for } n,m \geq 0.$$

(b) If $(m,n) \neq (p,q)$, where $m,n,p,q \in \mathbf{N}$, then $\xi_{n,m}$ is orthogonal to $\xi_{p,q}$, hence the family $\{\xi_{n,m}\}_{n,m\geq 0}$ is an orthogonal basis for $\overline{\mathcal{A}\xi}$.

(c) $\|\xi_{n,m}\|^2 = (2N-1)^{n+m}$, for $n,m \geq 0$.

Proof. (a) Denote $\eta = a_i^{\alpha} + \varepsilon \cdot a_i^{\alpha - r_i}$, $\zeta = a_j^{\beta} + \delta\, a_j^{\beta - r_j}$. Since all the words which appear in the support of $\chi_p \cdot \eta$ end with a nonzero power of a_i (and similarly for $\zeta \cdot \chi_q$), one has:

(15) $$(\eta \cdot w \cdot \zeta)_{p,q} = \eta_{p,o} \cdot w \cdot \zeta_{o,q}, \quad \text{for } p,q \text{ integers.}$$

For $1 \leq k \leq r_i - 1$ one has:

$$\chi_n \cdot a_i^k = (a_i^k)_{n,o} + (a_i^{k-r_i})_{n-1,0},$$

hence:

(16) $$\chi_n \cdot \eta = \eta_{n,o} + \varepsilon \cdot \eta_{n-1,o},$$

and similar formulae hold for $\zeta \cdot \chi_m$.

Now, (15) and (16) give the first formula. The second follows from the first.

(b) See the proof of Lemma 10.

(c) See Lemma 11. ∎

LEMMA 14. Consider the linear mapping $T_o : \mathcal{A}\xi \to \mathcal{A}\omega$ defined by $T_o(\chi_n \cdot \gamma \cdot \chi_m) = \chi_n \cdot \omega \cdot \chi_m$. Then T_o is well defined and it extends to an invertible bounded linear operator $T: \overline{\mathcal{A}\xi} \to \overline{\mathcal{A}\omega}$. Since $T \in \mathcal{A}'$, p_γ and p_ω are

equivalent.

Proof. That T_o is well defined follows like in Lemma 12.

The formula of T_o with respect to the basis $\{\xi_{n,m}\}_{n,m\geq 0}$ is

$$T_o \cdot \xi_{n,m} = \omega_{n,m} - \varepsilon\cdot\omega_{n-1,m} - \delta\cdot\omega_{n,m-1} + \varepsilon\cdot\delta\cdot\omega_{n-1,m-1}$$

Consider the operators $S_1, S_2 : \overline{\mathcal{A}\xi} \to \overline{\mathcal{A}\xi}$, $U: \overline{\mathcal{A}\xi} \to \overline{\mathcal{A}\omega}$ given by

$$S_1 \xi_{n,m} = \xi_{n-1,m}, \quad S_2 \xi_{n,m} = \xi_{n,m-1}, \quad U\xi_{n,m} = \omega_{n,m},$$

for $n,m \geq 0$.

As in the proof of Lemma 12, U is unitary and $\|S_1\| = \|S_2\| = (2N-1)^{-\frac{1}{2}}$. Since:

$$T_o = U(I - \varepsilon S_1)(I - \delta S_2)\,|_{\mathcal{A}\xi},$$

the lemma holds. ∎

Take now $V_o = \text{span}\, C(i, \alpha)$ (see (5)). We introduce the vectors of norm one:

$$\eta^{\varepsilon} = \frac{1}{\sqrt{2}}(a_i^{\alpha} + \varepsilon\cdot a_i^{\alpha - r_i}), \quad \text{where } \varepsilon \in \{-1,1\}.$$

It is obvious that:

$$V_o = \text{span}\,\{\eta^{\varepsilon} \mid \varepsilon \in \{-1,1\}\}.$$

Fix $\varepsilon \in \{-1,1\}$ and denote η^{ε} by η. We shall prove that p_{η} is equivalent to p_{ω}, but in this case, instead of an invertible operator $T : \overline{\mathcal{A}\eta} \to \overline{\mathcal{A}\omega}$ from $\mathcal{A}'$, we have only a closed, densely defined injective operator with dense range, which is affiliated to $\mathcal{A}'$.

LEMMA 15. Let $\tau = \frac{1}{\sqrt{2}}(a_i^{\alpha + r_i} + \varepsilon\cdot a_i^{\alpha - r_i}) \in V_1$ and introduce the notations:

$$\overline{\eta}_{n,m} = \eta_{n,m} - \varepsilon\cdot\tau_{n-1,m-1};$$

$$\overline{\overline{\eta}}_{n,m} = \sum_{s\geq 0} \overline{\eta}_{n-s,m-s}.$$

Then

(a) $\chi_n \cdot \eta \cdot \chi_m = \bar{\bar{\eta}}_{n,m} + \varepsilon(\bar{\bar{\eta}}_{n-1,m} + \bar{\bar{\eta}}_{n,m-1}) + \bar{\bar{\eta}}_{n-1,m-1}$

$$\bar{\eta}_{n,m} = \chi_n \cdot \eta \cdot \chi_m + \sum_{s\geq 1} (-\varepsilon)^s \cdot (\chi_{n-s} \cdot \eta \cdot \chi_m + \chi_n \cdot \eta \cdot \chi_{m-s})$$

(b) $$\langle \eta_{n,m} | \eta_{p,q} \rangle = \begin{cases} 0 & \text{if } n+m \neq p+q, \\ 2(2N-1)^{n+m-|n-p|}, & \text{if } n+m = p+q. \end{cases}$$

(c) $\{\bar{\eta}_{n,m}\}_{n,m\geq 0}$ is a basis for $\mathcal{A}\eta$;

(d) If $|(n+m)-(p+q)| \geq 2$, then $\bar{\eta}_{n,m}$ is orthogonal to $\bar{\eta}_{p,q}$.

Proof. (a) We prove succesively the following formulae (each of them has one similar for the right action of χ_1, which we do not write):

(i) $\chi_1 \cdot \eta_{o,o} = \eta_{1,o} + \varepsilon \cdot \eta_{o,o}$;

(ii) $\chi_1 \cdot \eta_{o,1} = \eta_{1,1} + \varepsilon \cdot \eta_{o,1} + \eta_{o,o} - \varepsilon \cdot \tau_{o,o}$;

(iii) $\chi_1 \cdot \tau_{o,o} = \tau_{1,o} + \eta_{o,o}$;

(iv) $\chi_1 \cdot \tau_{o,1} = \tau_{1,1} + \eta_{o,1}$;

(v) $\chi_1 \cdot \tau_{o,n} = \tau_{1,n} + \eta_{o,n}$, for $n \geq 0$;

(by induction, using (iv) and (11))

(vi) $\chi_1 \cdot \eta_{0,n} = \eta_{1,n} + \varepsilon \cdot \eta_{o,n} + \eta_{o,n-1} - \varepsilon \cdot \tau_{o,n-1}$, for $n \geq 0$;

(by induction, using (i), (ii), (iii) and (11)).

Using (v) and (vi), we can proceed like in Lemma 9 and we obtain the first formula. The second formula is a consequence of the first.

(b) The case $n+m \neq p+q$ is obvious. For $n+m = p+q$, since $\eta_{s,t}$ is the characteristic function of its support, we have to compute the cardinal of the intersection $\operatorname{supp} \eta_{n,m} \cap \operatorname{supp} \eta_{p,q}$. But:

$$\operatorname{supp} \eta_{s,t} = \operatorname{supp}(a_i^{\alpha})_{s,t} \cup \operatorname{supp}(a_i^{\alpha - r_i})_{s,t},$$

the union being disjoint,

$$\mathrm{supp}(a_i^{\alpha})_{n,m} \cap \mathrm{supp}(a_i^{\alpha-r_i})_{p,q} = \emptyset ,$$

$$\mathrm{card}[\mathrm{supp}(a_i^{\alpha})_{n,m} \cap \mathrm{supp}(a_i^{\alpha})_{p,q}] = (2N-1)^{n+m-|n-p|} ,$$

$$\mathrm{card}[\mathrm{supp}(a_i^{\alpha-r_i})_{n,m} \cap \mathrm{supp}(a_i^{\alpha-r_i})_{p,q}] = (2N-1)^{n+m-|n-p|}$$

(c) The family $\{\overline{\eta}_{n,m}\}_{n,m\geq 0}$ is a system of generators for $\mathcal{H}_\eta$ due to the relations from (a).

Assume that:

$$\sum_{k=0}^{K} \sum_{s=0}^{k} \lambda_{s,k-s} \cdot \overline{\eta}_{s,k-s} = 0 , \text{ for } K \geq 0 \text{ and } \lambda_{s,t} \in \mathbf{C} .$$

We apply q_K, the projection on V_K, and we obtain:

$$\sum_{s=0}^{K} \lambda_{s,K-s} \cdot \eta_{s,K-s} = 0$$

But the family $\{\eta_{s,k-s}\}_{0\leq s\leq k}$ is linearly independent for any $k \geq 0$ (because $\det(\langle \eta_{s,k-s} | \eta_{t,k-t} \rangle)_{0\leq s,t\leq k}$ is nonzero by [R], Lemma 5), hence $\lambda_{s,K-s} = 0$ for any $0 \leq s \leq K$. Therefore, an induction argument shows that the given family is linearly independent. ∎

LEMMA 16. Consider the linear mapping $W_o : \mathcal{H}_\eta \to \mathcal{H}_\omega$ defined by $W_o \cdot \overline{\eta}_{n,m} = \omega_{n,m}$, for $n,m \geq 0$. Then W_o is a well defined closable operator on $\overline{\mathcal{H}_\eta}$ into $\overline{\mathcal{H}_\omega}$, and its closure W is densely defined, injective, with dense range.

Proof. By Lemma 15, (c), and (12), (13), W_o is bijective. We shall prove that W_o and W_o^{-1} are closable and then the lemma follows. (If W_o is closable and bijective, then W is injective if and only if W_o^{-1} is closable).

W_o and W_o^{-1} would clearly be closable if $\{\overline{\eta}_{n,m}\}_{n,m\geq 0}$ was an orthogonal basis of $\mathcal{H}_\eta$. Fortunately, our situation is not too far from this.

For $n \geq 1$, denote:

$$P_n = \mathrm{proj}_{\overline{\mathrm{span}}\{\overline{\eta}_{s,t} \mid s+t \leq n\}}^{\overline{\mathcal{H}\eta}} ,$$

$$P'_n = \mathrm{proj}_{\overline{\mathrm{span}}\{\overline{\eta}_{s,t} \mid s+t \geq n\}}^{\overline{\mathcal{H}\eta}} ,$$

$$Q_n = \mathrm{proj}_{\mathrm{span}\{\omega_{s,t} \mid s+t \leq n\}}^{\overline{\mathcal{H}\omega}} ,$$

and put I for the identity operator both on $\overline{\mathcal{H}\eta}$ and on $\overline{\mathcal{H}\omega}$.

It is not hard to see that P_n and P'_{n+2} are orthogonal projection and $I - P_n - P'_{n+2}$ is a finite dimensional projection.

Note that $P'_n(\mathcal{H}\eta) \subset \mathcal{H}\eta$ because for all $s,t \geq 0$,

$$P'_n(\overline{\eta}_{s,t}) = \begin{cases} 0 & \text{if } s+t \leq n-2, \\ \mathrm{proj}_{\overline{\mathrm{span}}\{\overline{\eta}_{p,q} \mid 0 \leq p,q,\ p+q=n\}}^{\overline{\mathcal{H}\eta}} (\overline{\eta}_{s,t}), & \text{if } s+t = n-1, \\ \overline{\eta}_{s,t} & \text{if } s+t \geq n. \end{cases}$$

In the following we shall consider all the operators restricted to $\mathcal{H}\eta$, respectively $\mathcal{H}\omega$.

It is easy to verify that:

$$W_o \cdot P_n = Q_n \cdot W_o \cdot P_n ,$$

$$Q_n \cdot W_o \cdot P'_{n+1} = 0, \qquad \text{hence } Q_n \cdot W_o \cdot P'_{n+2} = 0 .$$

Then:

$$(17) \quad Q_n \cdot W_o = Q_n \cdot W_o \cdot P_n + Q_n \cdot W_o \cdot P'_{n+2} + Q_n \cdot W_o(I - P_n - P'_{n+2}) =$$

$$= W_o \cdot P_n + X_n ,$$

where $X_n = Q_n \cdot W_o(I - P_n - P'_{n+2})$ and $W_o P_n$ are operators of finite rank, hence bounded.

Since $W_o^{-1} \cdot \omega_{p,q} = \eta_{p,q}$, one has:

$$W_o^{-1}Q_n = P_n W_o^{-1} Q_n$$

$$P_n \cdot W_o^{-1}(I - Q_{n+1}) = 0,$$

hence

$$P_n \cdot W_o^{-1} = P_n \cdot W_o^{-1} Q_n + P_n \cdot W_o^{-1}(Q_{n+1} - Q_n) + P_n \cdot W_o^{-1}(I - Q_{n+1}) =$$

$$= W_o^{-1} \cdot Q_n + Y_n$$

where $Y_n = P_n \cdot W_o^{-1}(Q_{n+1} - Q_n)$ and $W_o^{-1} \cdot Q_n$ are again operators of finite rank, hence bounded.

We can show now that W_o and W_o^{-1} are closable operators. The proof is made only for W_o, for W_o^{-1} the arguments being similar.

Take $x_i \in \mathcal{A}\xi$ $(i \geq 0)$ such that $\lim_{i\to\infty} x_i = 0$ and $y_i = W_o \; x_i \xrightarrow[i]{} y$. We have to show that $y = 0$. Fix $n \geq 1$ arbitrarly. Then, by (17)

$$Q_n \; y_i = Q_n \cdot W_o \cdot x_i = (W_o P_n + X_n) x_i \xrightarrow[i]{} 0$$

because $W_o \cdot P_n$ and X_n are bounded operators. But $y_i \xrightarrow[i]{} y$, hence

$$Q_n \; y = \lim_{i\to\infty} Q_n y_i = 0 .$$

Since $Q_n \xrightarrow[n]{s.o.} I$, one has $y = 0$. ∎

LEMMA 17. Consider the linear mapping $T_o : \mathcal{A}\eta \to \mathcal{A}\omega$ given by $T_o(\chi_n \cdot \eta \cdot \chi_m) = \chi_n \cdot \omega \cdot \chi_m$. Then T_o is a well defined closable operator. Its closure $T : \mathcal{D}_T \subset \overline{\mathcal{A}\eta} \to \overline{\mathcal{A}\omega}$ is densely defined, injective, with dense range. Since T is affiliated to $\mathcal{A}'$, p_η and p_ω are equivalent.

Proof. That T_o is well defined follows like in Lemma 12, using Lemma 15 (a) and (c). The formula of T_o with respect to the basis $\{\bar{\eta}_{n,m}\}_{n,m \geq 0}$ of $\mathcal{A}\eta$ is:

$$T_o \bar{\eta}_{n,m} = \omega_{n,m} - \varepsilon(\omega_{n-1,m} + \omega_{n,m-1}) +$$

$$+ \varepsilon(\omega_{n-1,m-2} + \omega_{n-2,m-1}) - \omega_{n-2,m-2} .$$

Consider the operators: $S_1, S_2 : \mathcal{H}\omega \to \mathcal{H}\omega$, $S_1\omega_{n,m} = \omega_{n-1,m}$, $S_2\omega_{n,m} = \omega_{n,m-1}$, and the operator $W_o : \mathcal{H}\eta \to \mathcal{H}\omega$, $W_o\eta_{n,m} = \omega_{n,m}$ considered in Lemma 16. Then $\| S_1 \| = \| S_2 \| = (2N-1)^{-\frac{1}{2}}$ as in Lemma 12, and:

$$T_o = [I - \varepsilon (S_1 + S_2) + \varepsilon S_1 S_2 (S_1 + S_2) - S_1^2 S_2^2] \cdot W_o \,|_{\mathcal{H}\eta} =$$

$$= (I - S_1 S_2)(I - \varepsilon S_1)(I - \varepsilon S_2) W_o \,|_{\mathcal{H}\eta} .$$

Since $\| S_1 \| = \| S_2 \| < 1$, $(I - S_1 S_2)(I - \varepsilon S_1)(I - \varepsilon S_2)$ is invertible and then Lemma 16 gives the desired conclusion. ■

From the hole § 2, one obtains:

THEOREM 18. The von Neumann algebra $\mathcal{H}' = \{\lambda(\chi_1), J\lambda(\chi_1)J\}' \subset \subset B(l^2(F_N))$ is of homogenous type I_∞ on $I - p_1$. Consequently, the algebra $A = \{\lambda(\chi_1)\}'' \subset L(F_N)$ is singular. ■

MOREL MATICA and ANDREI TOBOC

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VIOREL NIȚICĂ and ANDREI TÖRÖK

The Mathematical Institute of the
Romanian Academy
str. Academiei 14
70109 Bucharest, Romania

current address:
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802
U.S.A.