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Abstract. The small-amplitude cellular convection in a porous layer is studied in the critical regime, when the buoyancy force balance the viscosity of the darcean fluid. We give rigorous proofs on the fact that all the nodal properties of the appearance of convection are associated with the planform eigensolutions. Detailed results which were predicted by former studies (the point spectrum of the linearized problem, the value of the critical Rayleigh number, the non-existence of subcritical convective solutions) are recast.

1. Introduction

The occurence of convective motions in a horizontal porous layer bounded by two isothermal planes, maintaining a temperature gradient opposite to the direction of gravity, is usually studied by the mathematical model furnished by the DOB (Darcy-Oberbeck-Boussinesq) equations. It seems that this is the mathematically simplest system of the nonlinear fluid mechanics which agrees with the observed flows (see [2] §70). The conduction solution is easily obtained and exists for all values of the temperature gradient, the darcean fluid remaining at rest. Also, as in this model the density of the gravitational force varies affinely with the temperature (the buoyancy force), it yields the convection phenomenon: that on exceeding some critical temperature gradient the fluid starts to move and thus the heat transfer is increased.

From the mathematical point of view, below the critical Rayleigh number the boundary conditions are sufficient to determine the conduction solution uniquely, while more stationary solutions are possible above that value. It is generally accepted that the convective motion occurs in a periodic cell pattern, where a cell is defined by an impermeable and insulated closed boundary with vertical walls, while the periodicity means that all the cells are identical.

The early linear theories (see for instance [1]), determined the criterion for the stability of the conduction state and predicted some features of the convective flow. Elementary separation of variables leaded to the use of the planform eigensolutions and vice-versa.

In order to study the small-amplitude steady convective solutions of the full nonlinear problem, the assumption that the smallest eigenvalue of the linearized problem is associated with a planform eigensolution is crucial even when using the energy method of stability (see [2] § 73) or the formal expansion method of Gorkov-Malkus-Veronis (see [2] § 78 or [4] Ch.3).

The main result of the present paper is that all the nodal properties of the small-amplitude convection are indeed determined by the planform eigensolutions. It is contained in Sec.3 and it is proved using properties of the elliptic systems.

In Sec.2 we present the conduction solution and formulate the problem of the convective solutions. Sec.2 also contains the non-dimensionalizing transformations of the DOB system which set the linearized equations into the favorable form of Euler's equations of a variational problem.

In Sec.4 we establish the uniqueness of the conduction solution for subcritical Rayleigh numbers, by proving rigorously that the linearized problem can be recast into a maximum variational formulation.

2. Mathematical formulation of the problem

Let O be the origin of the coordinate system Oxyz, so that negative Oz axis is the direction in which the buoyancy force acts. Let B be a bounded Lipschitz open set in $\mathbb{R}^2$, connected and locally located on one side of its boundary ∂B.

As mathematical model for the steady convection in a fluid-saturated porous medium, confined in a vertical cylinder of cross-section B and thickness $L>0$, we consider the DOB system; denoting by e the versor of the Oz axis, by V the velocity, by P the pressure and by T the temperature, it is given by:

(2.1) $\qquad \operatorname{div} V = 0 \qquad \text{in } D=B\times(0,L)$

(2.2) $\qquad \frac{\mu}{K}V + \nabla P = -\rho_f g(1-\alpha(T-T_r))e \quad \text{in } D$

(2.3) $\qquad \rho_f c_f V\,\nabla T = \chi \Delta T \quad \text{in } D$

where α, c_f, μ and ρ_f are the coefficient of thermal expansion, the specific heat at constant volume, the viscosity and the density of the saturating fluid, g is the acceleration of gravity, while K, χ and T_r stand for the permeability, the thermal conductivity and the reference temperature of the porous medium.

Supposing that the domain is heated from below, we have the boundary conditions of the corresponding Bénard problem:

(2.4) $\qquad V.n = 0 \quad \text{and} \quad \partial_n T=0 \qquad \text{on } \partial B\times(0,L)$

(2.5) $\qquad V.n=0 \quad \text{and} \quad T=T_0 \qquad \text{for } z = 0$

(2.6) $\qquad V.n=0 \text{ and } T=T_L \quad \text{for } z=L$

where T_o and T_L are constants with $T_o > T_L$, n is the unit outward normal on ∂D and $\partial_n T$ denotes the corresponding normal derivative. In this work we also use ∂_x, ∂_y and ∂_z as partial derivative notations.

Denoting the Rayleigh number by $Ra = \mu^{-1} \chi^{-1} \rho_f^2 c_f K L g \alpha (T_o - T_L)$, a dimensionless form of the system (2.1)-(2.6) can be obtained by defining

(2.7) $a^* = Ra^{1/2}$, $(x^*, y^*, z^*) = L^{-1}(x, y, z)$

(2.8) $V^* = \rho_f c_f L \chi^{-1} V$, $T^* = Ra^{1/2}(T_o - T_L)^{-1}(T - T_L)$

(2.9) $P^* = \mu^{-1} \chi^{-1} \rho_f c_f K (P + \rho_f g (1 + \alpha z (T_r - T_L)))$.

The conduction solution (corresponding to the case when the fluid remains at rest and the heat is transported only by conduction) exists for any value of the parameter $a^* > 0$. Omitting the asterisks in order to simplify the notations, the conduction solution is given by

(2.10) $V_c = 0$, $T_c = a(1-z)$, $P_c = a^2(2z - z^2)/2$ + const.

Introducing the following change of dependent variables

(2.11) $U = V - V_c$, $Q = P - P_c$, $S = T - T_c$

the system (2.1)-(2.6) finally becomes

(2.12)	$\text{div } U = 0$	in D
(2.13)	$U + \nabla Q = aSe$	in D
(2.14)	$-\Delta S + U \nabla S = aUe$	in D
(2.15)	$U.n = \partial_n S = 0$	on $\partial B \times (0,1)$
(2.16)	$U.n = S = 0$	for z=0 and z=1.

The so-called convective solutions are the eigenfunctions of the previous system; obviously, of major physical interest are those which appear when the buoyancy force balance the viscosity of the darcean fluid.

Introducing the Hilbert spaces

(2.17) $H=\{V\in(L^2(D))^3 \mid \text{div } V=0 \text{ in } D,\ V.n=0 \text{ on } \partial D\}$

(2.18) $W=\{T\in H^1(D) \mid T=0 \text{ for } z=0 \text{ and } z=1\}$

with the corresponding scalar products and norms

(2.19) $(U,V)=\int_D UV\, dD\ , \qquad |V|=(V,V)^{1/2}$

(2.20) $((S,T))=(\nabla S, \nabla T)\ , \quad \|T\|=((T,T))^{1/2}$

the variational formulation of (2.12)-(2.16) is the following:

To find $a>0$ and non-trivial $U\in H, S\in W$ satisfying

(2.21) $(U,V)=a\ (Se,V),\ (\forall) V\in H$

(2.22) $((S,T))-(U,S\nabla T)=a(U,Te),\ (\forall) T\in W.$

Supposing that the increments of the velocity and of the temperature have small amplitude (i.e. $|U|^2+\|S\|^2 \ll 1$), we obtain to first order the linearized (about the conduction solution) problem:

To find $a>0$ and non-trivial $V\in H$, $S\in W$ satisfying

(2.23) $(U,V) = a(Se,V),\ \ (\forall)\ V\in H$

(2.24) $((S,T))=a(U,Te)\ ,\ \ (\forall)\ T\in W.$

3. The smallest eigenvalue of the linearized problem

Let $\{\lambda_i\}_{i\in N}$ be the eigenvalues and let $\{\varphi_i\}_{i\in N}$

be a complete set in $L^2(B)$ of orthonormal eigenfunctions of $(-\Delta)$ with homogeneous Neumann boundary conditions on ∂B (i.e. the plan-form problem). We have $\lambda_0=0$, $\varphi_0=$const. and

(3.1) $$0<\lambda_1\leqslant\lambda_2\leqslant\ldots\to+\infty$$

For any $k\in\mathbb{N}^*$ we define $a_k:\mathbb{R}_+\to\mathbb{R}_+$ by

(3.2) $$a_k(\lambda)=\lambda^{1/2}+k^2\pi^2\lambda^{-1/2}$$

Proposition 3.1. The set of all eigenvalues of the linearized problem (2.23)-(2.24) is given by

(3.3) $$\sigma=\{a_k(\lambda_1)\ |\ k,1\in\mathbb{N}^*\}$$

Proof. It is easy to verify that

(3.4) $$U_{k1}=(k\pi\partial_x\varphi_1\cos k\pi z,k\pi\partial_y\varphi_1\cos k\pi z,\ \lambda_1\varphi_1\sin k\pi z),$$
$$S_{k1}=\sqrt{\lambda_1}\varphi_1\sin k\pi z$$

are eigenfunctions of (2.23)-(2.24), corresponding to the eigenvalue $a_k(\lambda_1)$.

Conversely, let $a>0$ be an eigenvalue of (2.23)-(2.24) with $U=(u,v,w)\in H$ and $S\in W$ as corresponding eigenfunctions. As the orthogonal complement of H in $(L^2(D))^3$ is

(3.5) $$H^{\perp}=\{V\in(L^2(D))^3\ |\ (\exists)\,q\in H^1(D)\text{ such that }V=\nabla q\}$$

then there exists $q\in H^1(D)$ such that the following system is satisfied in some (weak) sense:

(3.6) $\partial_x u + \partial_y v + \partial_z w = 0$ in D

(3.7) $u+\partial_x q=0$, $v+\partial_y q=0$, $w+\partial_z q=aS$ in D

(3.8) $-\Delta S=aw$ in D

(3.9) $U.n=\partial_n S=0$ on $\partial B\times(0,1)$

(3.10) $w=S=0$ for $z=0$ and $z=1$.

Moreover, one can prove like in [4] Ch.2 that in fact $q\in H^2(D)$ and consequently $U\in(H^1(D))^3$ and $S\in H^2(D)$.

Now, let us remark that for any $(x,y)\in B$, the eigenfunctions of (3.6)-(3.10) belong to $L^2(0,1)$ and hence they can be developed in Fourier series using any orthogonal complete set in $L^2(0,1)$. Therefore we consider

(3.11) $S=\pi\sum_{j\geq 1} jf_j(x,y)\sin j\pi z$, $w=\pi^2\sum_{j\geq 1} jg_j(x,y)\sin j\pi z$

which satisfy (3.10) also.

Using (3.11), the equations (3.7) yield:

(3.12) $q=-\sum_{j\geq 1} h_j\cos j\pi z$, $u=\sum_{j\geq 1}\partial_x h_j\cos j\pi z$, $v=\sum_{j\geq 1}\partial_y h_j\cos j\pi z$

where $h_j(x,y)=af_j(x,y)-\pi g_j(x,y)$.

Finally, introducing (3.11)-(3.12) in (3.6) and (3.8)-(3.9) we find that for any $j\in\mathbb{N}^*$ it holds:

(3.13) $-\Delta\Psi_j = A_j\Psi_j$ in B

(3.14) $\partial_\nu\Psi_j = 0$ on ∂B

where ν is the unit outward normal on ∂B, Ψ_j is the vector of components f_j and g_j, while A_j is the (2x2) matrix given by

(3.15) $$A_j = \begin{pmatrix} -j^2\pi^2 & a\pi \\ -aj^2\pi & a^2-j^2\pi^2 \end{pmatrix}$$

If A_j has no positive eigenvalues then the problem (3.13)-(3.14) has only the trivial solution (see [3]). Thus we are lead to consider the first index for which the problem (3.13)-(3.14) has a non-trivial solution; let us denote it by k. The characteristic equation of A_k yields

(3.16) $$a = a_k(\mu)$$

where μ is eigenvalue of A_k.

If A_k has distinct eigenvalues, then it is diagonizable and using the corresponding transformation upon (3.13)-(3.14) we find that at least one of the eigenvalues of A_k belongs to $\{\lambda_l\}_{l\in N^*}$. Thus (3.16) implies $a \in \sigma$.

If A_k has a double eigenvalue then it is $k^2\pi^2$. Consequently $a=2k\pi$ and (3.13) can be put into the form

(3.17) $$-\Delta f_k = -\pi^2 k^2 f_k + 2\pi^2 k g_k \quad \text{in } B$$

(3.18) $$-\Delta(2kf_k - g_k) = \pi^2 k^2 g_k \quad \text{in } B.$$

We make the scalar products of (3.17)-(3.18) by $(2kf_k-g_k)$ and respectively f_k; equating the right hand sides we obtain

(3.19) $$g_k = kf_k \quad \text{in } L^2(B).$$

Introducing (3.19) in (3.17) we find that this case holds only if $k^2\pi^2$ is an eigenvalue of the planform problem also; again (3.16) implies $a \in \sigma$. □

In the following we denote by $\lambda_c \in \{\lambda_l\}_{l\in\mathbb{N}^*}$ a minimizer of the function a_1 and by $\varphi_c \in \{\varphi_l\}_{l\in\mathbb{N}^*}$ any eigenfunction corresponding to the same eigenvalue λ_c. It is obvious that λ_c is not unique only when $\lambda_c \neq \pi^2$ and $\tilde{\lambda}_c = \pi^4 \lambda_c^{-1} \in \{\lambda_l\}_{l\in\mathbb{N}^*}$; in this case we denote by $\tilde{\varphi}_c \in \{\varphi_l\}_{l\in\mathbb{N}^*}$ the eigenfunctions corresponding to $\tilde{\lambda}_c$. In order to cover both situations we define $\tilde{\varphi}_c = 0$ when $\lambda_c = \pi^2$ or when $\tilde{\lambda}_c \notin \{\lambda_l\}_{l\in\mathbb{N}^*}$.

<u>Proposition 3.2.</u> The smallest eigenvalue of the linearized problem (2.23)-(2.24) is given by

$$a_c = \sqrt{\lambda_c} + \frac{\pi^2}{\sqrt{\lambda_c}} \tag{3.20}$$

<u>Proof.</u> Obviously, for any $k,l \in \mathbb{N}^*$, we have $a_k(\lambda_l) \geqslant a_1(\lambda_l) \geqslant a_1(\lambda_c)$ and the proof is completed by Proposition 3.1. □

<u>Proposition 3.3.</u> The eigenfunctions of the linearized problem (2.23)-(2.24) corresponding to the smallest eigenvalue a_c are generated by

$$U_c = (\pi \partial_x \varphi_c \cos \pi z, \pi \partial_y \varphi_c \cos \pi z, \lambda_c \varphi_c \sin \pi z), \quad S_c = \sqrt{\lambda_c}\, \varphi_c \sin \pi z \tag{3.21}$$

and by

$$\tilde{U}_c = (\pi \partial_x \tilde{\varphi}_c \cos \pi z, \pi \partial_y \tilde{\varphi}_c \cos \pi z, \tilde{\lambda}_c \tilde{\varphi}_c \sin \pi z), \quad \tilde{S}_c = \sqrt{\tilde{\lambda}_c}\, \tilde{\varphi}_c \sin \pi z \tag{3.22}$$

<u>Proof.</u> If $\lambda_c = \pi^2$, then $a_c = 2\pi$ and from (3.19) we obtain $g_1 = f_1$. Thus problem (3.13)-(3.14) reduces to

(3.23) $$-\Delta f_1 = \pi^2 f_1 \quad \text{in } B, \quad \partial_\nu f_1 = 0 \text{ on } \partial B,$$

and hence f_1 is generated by the set of all φ_c.

When $\lambda_c \neq \pi^2$, if we define

(3.24) $$f = \pi f_1 - \sqrt{\lambda_c}\, g_1 \quad \text{and} \quad g = -\sqrt{\lambda_c}\, f_1 + \pi g_1$$

then the problem (3.13)-(3.14) becomes

(3.25) $$-\Delta f = \lambda_c f \text{ in } B, \quad \partial_\nu f = 0 \text{ on } \partial B$$

(3.26) $$-\Delta g = \tilde{\lambda}_c g \text{ in } B, \quad \partial_\nu g = 0 \text{ on } \partial B$$

It follows that f an g are generated by the set of all φ_c and $\tilde{\varphi}_c$.

In both cases the proof is completed by straightforward computations. □

Remark 3.1. For any $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$ we have

(3.27) $$\frac{2(\alpha U_c + \beta \tilde{U}_c, (\alpha S_c + \beta \tilde{S}_c)e)}{|\alpha U_c + \beta \tilde{U}_c|^2 + \|\alpha S_c + \beta \tilde{S}_c\|^2} = \frac{1}{a_c}.$$

4. Non-existence of subcritical convective solutions

As in the most of thermoconvective studies, the linearized problem can be recast in a maximum formulation. In the present case we have

Proposition 4.1. If a_c is the smallest eigenvalue of the linearized problem (2.23)-(2.24), then

$$(4.1)\qquad \sup\left\{\frac{2(V,Te)}{|V|^2+\|T\|^2}\,\middle|\, V\in H, T\in W,\ |V|^2+\|T\|^2\neq 0\right\}=\frac{1}{a_c}$$

Moreover, any maximizing function satisfy the linearized problem for $a=a_c$.

<u>Proof.</u> Let us denote by $\mu\in\mathbb{R}$ the left hand side of (4.1); Remark 3.1 implies $\mu\geq a_c^{-1}$.

On the other side, let us notice first that

$$(4.2)\qquad \mu=\sup\left\{2(V,Te)\,\middle|\, V\in H,\ T\in W,\ |V|^2+\|T\|^2=1\right\}$$

It follows that for any $\varepsilon>0$ there exist $V_\varepsilon\in H$, $T_\varepsilon\in W$ with $|V_\varepsilon|^2+\|T_\varepsilon\|^2=1$ and such that

$$(4.3)\qquad 2(V_\varepsilon,\ T_\varepsilon e)>\mu-\varepsilon$$

Let $V\in H$ and $T\in W$; then for sufficiently small $\eta>0$ we have

$$(4.4)\qquad |V_\varepsilon+\eta V|^2+\|T_\varepsilon+\eta T\|^2\neq 0.$$

Using (4.4) in the definition of μ and taking in account (4.3) we obtain

$$(4.5)\qquad 2\eta(V_\varepsilon,\ Te)+2\eta(V,T_\varepsilon e)+2\eta^2(V,Te)\leq\varepsilon+2\mu\eta(V_\varepsilon,V)+ +2\mu\eta((T_\varepsilon,T))+\mu\eta^2|V|^2+\mu\eta^2\|T\|^2$$

As the sequences $\{V_\varepsilon\}_\varepsilon$ and $\{T_\varepsilon\}_\varepsilon$ are bounded in H and respectively W, we find sub-sequences (still denoted by ε) for which we have

(4.6) $\quad V_\varepsilon \to U$ weakly in H

(4.7) $\quad T_\varepsilon \to S$ weakly in W (strongly in $L^2(D)$).

Therefore, if we let $\varepsilon \to 0$, (4.5) reduces to

(4.8)
$$2(U,Te)+2(V,Se)+2\eta(V,Te)\leq 2\mu(U,V)+ +2\mu((S,T))+\mu\eta|V|^2+\mu\eta\|T\|^2$$

For $\eta \to 0$ and for a proper choice of the test functions, (4.8) yields

(4.9) $\quad (U,V)=\mu^{-1}(Se,V) \qquad (\forall) V\in H$

(4.10) $\quad ((S,T))=\mu^{-1}(U,Te) \qquad (\forall) T\in W$

Moreover, using the Friedrich's inequality in V and the inequality (4.2), we obtain

(4.11)
$$|U|^2+\|S\|^2 \geq \frac{2}{c_F}(U,Se)=\frac{2}{c_F}\lim_{\varepsilon\to 0}(V_\varepsilon,T_\varepsilon e)\geq \frac{\mu}{c_F}\geq \frac{1}{c_F a_c}$$

where c_F is some positive constant.

Hence $U\in H$, $S\in W$ are eigenfunctions of the linearized problem (2.23)-(2.24) for the eigenvalue μ^{-1}; then Proposition 3.2 implies $\mu^{-1}\geq a_c$ and (4.1) is proved.

Let us finally remark that the previous considerations contain also the proof that any maximizing function of (4.1) must satisfy the linearized problem for $a=a_c$. □

The result concerning the non-existence of subcritical convective solutions is the following:

Proposition 4.2. If a is an eigenvalue of the nonlinear problem (2.21)-(2.22) then

(4.12) $$a > a_c$$

<u>Proof.</u> If $a>0$, $U\in H$ and $S\in W$ is an eigensolution of the nonlinear problem (2.21)-(2.22) then

(4.13) $$(U, S\nabla S) = \frac{1}{2}\int_D \operatorname{div}(S^2 U)\,dD = \frac{1}{2}\int_{\partial D} S^2 (U\cdot n)\,d\sigma = 0$$

Consequently, adding (2.21) and (2.22) for V=U and T=S, we get

(4.14) $$|U|^2 + \|S\|^2 = 2a\,(U, Se)$$

Obviously, Proposition 4.1 implies $a \geqslant a_c$.

let us assume that $a=a_c$; from (4.14) it follows that U and S are maximizing functions of (4.1). Then Proposition 4.1 implies that U and S are eigenfunctions of the linearized problem. Comparing (2.22) with (2.24) we obtain

(4.15) $$(U, S\nabla T) = 0 \quad (\forall)\, T\in W.$$

If we set $T=\sin 2\pi z$ in (4.15), then by using Proposition 3.3 we easily find a contradiction. □

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