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BOUNDS FOR HOMOGENISED COEFFICIENTS IN THE NON-SYMMETRIC CASE

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RESUME

Nous obtenons certains estimations pour les coefficients homogénéises d'un material composite periodique, quand les directions de periodicité sont pas les mêmes avec les axes de coordonées. Les formules obtenus contains pas l'inverse des matrices d'elasticite et homogénéisée.

ABSTRACT

We obtain some new estimations foe the effective coefficients of a periodic composite material, when the periodicity directions do not coincide with the coordonate axes. The obtained formulas do not contains the inverse of the elastic and effective matrices.

INTRODUCTION

We consider a periodic composite material, whose coefficients A_{ijkh} are Y periodic, Y being a parallelepiped in R^3; the dimension of Y is very small compared with that of the entire medium. In a suitable topology, the coefficients are convergent, and the periodic material is replaced with another one with constant coefficients. We study the problem of the a priori estimates for this constant (homogenized) coefficients.

The problem was studied in [1], [2], using the minimum principle for the potential energy.

Estimates in the case of thermic conductivity are given in [3], using the homogenization method.

Tartar's result were generalized in [4] to the case of linear elasticity and for isotropic materials.

In [5] was given a different generalization of the Tartar's result: the case of linear elasticity without symmetry condition $A_{ijkh} = A_{khij}$ and for non-isotropic materials. The non-symmetry may appear in the case when the periodicity directions do not coincide with coordonate axes, as in [2], where it is considered a hexagonal periodicity. Only a particular coefficient it is estimated in [5], using a condition of a "partial" symmetry.

In this paper, we obtain some general estimates in the non-symmetric case, in which are not present the inverse of elastic and "homogenized" matrices (which, on the contrary, are present in the estimates given in [3]). For this, are used some properties of the equations with fast oscillating right hand side.

RECALL OF BASIC RESULTS

We consider the problem in the case of thermic conductivity. The temperature θ satisfies the equation

$$(1) \qquad -\mathrm{div}(A^{\varepsilon} \operatorname{grad} \theta^{\varepsilon}) = b \quad \text{in} \quad D \subset R^3 ,$$

where $A = a_{ij}$, $a^{\varepsilon}_{ij}(x) = a_{ij}(x/\varepsilon)$ and a_{ij} are Y periodic, bounded, positive definite, $a_{ij} \neq a_{ji}$. Here ε represents the ratio between the dimension of Y and those of the medium.

The homogenization theory (see [7], [8]) give us the existence of A^H and θ^H such that $\theta \longrightarrow \theta^H$ in the weak topology of the Sobolev space $H^1_0(D)$ when $\varepsilon \longrightarrow 0$ and

$$-\mathrm{div}(A^H \mathrm{grad}\ \theta^H) = b \quad \text{in} \quad D \subset R^3$$

For the study of the above convergence are considered the local problems

$$\int_Y a_{ij}\partial P^m/\partial y_j \partial h/\partial y_i = -\int_Y a_{pm}\partial h/\partial y_p \ , (\forall) h \in V_Y$$

$$\int_Y a_{ji}\partial \hat{P}^k/\partial y_j \partial h/\partial y_i = -\int_Y a_{kp}\partial h/\partial y_p \ , (\forall) h \in V_Y$$

where $\quad V_Y = \left\{ u \in H^1(Y) \ , \ u = Y \text{ periodic} \right\}$

It is proved in [7], [8] that

$$\left\langle a_{kp}\partial P^m / \partial y_p \right\rangle = \left\langle a_{pm}\partial \hat{P}^k / \partial y_p \right\rangle$$

where $\quad \langle \ . \ \rangle = \int_Y . dy / |Y| \ .$

Moreover, in this case it is possible to obtain a formula for the effective coefficients:

$$a^H_{ij} = \left\langle a_{ij} + a_{ip}\partial P^j / \partial y_p \right\rangle$$

To obtain estimation for A^H, in the paper of Tartar [3] it is used the following construction: we search for a t^ε such that, for given $c \in R^3$ we have

$$\mathrm{grad}\ t^\varepsilon \longrightarrow c \text{ weakly in } L_2(D)$$

$$A^\varepsilon \mathrm{grad}\ t^\varepsilon \longrightarrow A^H c \text{ weakly in } L_2(D)$$

For this, it is considered the problem

$$(2) \qquad \begin{cases} \mathrm{div}(A^\varepsilon . \mathrm{grad}\ t^\varepsilon) = 0 \\ t^\varepsilon - c.x \in H^1_0(D) \end{cases}$$

and, denoting $u^\varepsilon = t^\varepsilon - c.x$, it is obtained the following problem for u^ε :

$$(3)\qquad \begin{cases} \int_D a^{\varepsilon}_{ij}(\partial u^{\varepsilon}/\partial x_j + c_j)\partial f/\partial x_i = 0, (\forall) f \in H^1_0(D) \\ u^{\varepsilon} \in H^1_0(D) \end{cases}$$

In the following we need :

Proposition 1.

Let u be the solution of (3) and $p^{\varepsilon}_i = a^{\varepsilon}_{ij}\partial u/\partial x_j$.

Then

1) $p^{\varepsilon}_i \rightharpoonup p^*_i = a^H_{im}\, c_m - \langle a_{im}\rangle c_m$ weakly in $L_2(D)$

2) $u^{\varepsilon} \rightharpoonup u^H = 0$ weakly in $H^1_0(D)$.

Proof.

As in [7] ,[8],we use the energetic method of L.Tartar,considering the function

$$v^{\varepsilon} = x_k + \varepsilon\cdot\hat{P}^k(x/\varepsilon) ,$$

the corresponding problem with integral formulation and also the problem (3).First,we obtain for the limit of p^{ε}_i the expression

$$p^*_k = a^H_{ki}\,\partial u^H/\partial x_i + a^H_{kq}\, c_q - \langle a_{kq}\rangle c_q ;$$

passing to the limit in (3) we have

$$\operatorname{div} p^* = 0.$$

These last two relations and the fact that $u^H \in H^1_0(D)$ give us the desired result .In the above relations we use the fact that a periodic function $f(x/\varepsilon)$ converges weakly in $L_2(D)$ to his mean value.

ESTIMATIONS

Using the Proposition 1,we can compute some limits,which are needed in the sequel.We have

(4) $a^{\varepsilon}_{ij}\partial u^{\varepsilon}/\partial x_j \partial u^{\varepsilon}/\partial x_i \longrightarrow -\langle a_{ik}\partial P^m/\partial y_i\rangle c_k c_m$ weakly L^1_{loc}

(5) $a^{\varepsilon}_{ij}\, c_j \partial u^{\varepsilon}/\partial x_i \longrightarrow \langle a_{ik}\partial P^m/\partial y_i\rangle c_k c_m$ weakly in L^1.

First, we prove (4), considering, as in [6], the "non-standard" problem

(6) $\operatorname{div}_y(\, A \operatorname{grad}_y z\,) = \operatorname{div} F\ ,\ y = x/\varepsilon\ ,\ z \in V_Y$.

We can see that $z^{\varepsilon}(x) = z(x/\varepsilon) \longrightarrow 0$ strongly in $L_2(D)$ and using the energetic method for the equation (6) and (3), we obtain (because $u^{\varepsilon} \longrightarrow 0$)

$$\lim_{\varepsilon\to 0} \int_D F^{\varepsilon}_i \partial u^{\varepsilon}/\partial x_i\, g = \lim_{\varepsilon\to 0} \int_D F^{\varepsilon}_i \partial z^{\varepsilon}/\partial x_i\, g = \langle F_i\, \partial z/\partial y_i\rangle \int_D g$$

where

$$F^{\varepsilon}_i = -\, a^{\varepsilon}_{ik}\, c_k\ .$$

But we have also

$$\int_D a^{\varepsilon}_{ij}\, \partial u^{\varepsilon}/\partial x_j (\partial u^{\varepsilon}/\partial x_i \cdot g + u^{\varepsilon}\cdot \partial g/\partial x_i) = \int_D F^{\varepsilon}_i (\partial u^{\varepsilon}/\partial x_i g + u^{\varepsilon}\, \partial g/\partial x_i\ )\ ,$$

and these relations give us

$$a^{\varepsilon}_{ij}\partial u^{\varepsilon}/\partial x_j \partial u^{\varepsilon}/\partial x_i \longrightarrow \langle F_i \partial z/\partial y_i\rangle\ .$$

The comparison between z and P^m give the desired result.

Using (4) and (5) we obtain also the corresponding convergence relation for t^{ε} :

i) $$a^{\varepsilon}_{ij}\, c_j\, \partial t^{\varepsilon}/\partial x_i \longrightarrow \langle a_{ik}\, \partial p^m/\partial y_i \rangle c_k \cdot c_m + \langle a_{ij} \rangle c_i c_j \;,$$

ii) $$a^{\varepsilon}_{ij}\, \partial t^{\varepsilon}/\partial x_j \longrightarrow a^{H}_{ik} \cdot c_k \;,$$

iii) $$a^{\varepsilon}_{ij}\, \partial t^{\varepsilon}/\partial x_i\, \partial t^{\varepsilon}/\partial x_j \longrightarrow a^{H}_{ij} \cdot c_i \cdot c_j \;.$$

We start with the definition of t^{ε} and u^{ε} :

$$a^{\varepsilon}_{ij}\, c_j\, \partial t^{\varepsilon}/\partial x_i = a^{\varepsilon}_{ij}\, (\partial u^{\varepsilon}/\partial x_i + c_i) \cdot c_j \;,$$

from which we obtain the relation i), using (5). The second relation it is obtained using the properties of p^{ε}_i , which was proved in Proposition 1 :

$$p^{\varepsilon}_i = a^{\varepsilon}_{ij}(\partial t^{\varepsilon}/\partial x_j - c_j)$$

$$p^{\varepsilon}_i \longrightarrow a^{H}_{ik} \cdot c_k - \langle a_{ik} \rangle \cdot c_k \;.$$

The last relation concerning t^{ε} may be obtained using the compensated compactness, introduced by L. Tartar and F. Murat; but in this case, it is possible to obtain it from straight calculation and the above convergence relations:

$$A^{\varepsilon} \nabla t^{\varepsilon} \nabla t^{\varepsilon} = A^{\varepsilon} \nabla u^{\varepsilon} \nabla u^{\varepsilon} + A^{\varepsilon} c \nabla u^{\varepsilon} + A^{\varepsilon} \nabla u^{\varepsilon} c + A^{\varepsilon} \cdot c \cdot c \;;$$

it is possible to see that the first two terms of the above relation reduce to zero, by the equation of u^{ε} . In the last relation, ∇ denotes the operator grad_x .

We emphasize that the compensated compactness is not applicable to obtain (4).

We consider in the sequel the function:

(7) $$w^{\varepsilon} = x_s + \varepsilon P^s (x/\varepsilon)$$

Proposition 2

We have the following convergence

(8) $$a^{\varepsilon}_{ij} \cdot \partial w^{\varepsilon}/\partial x_i \cdot \partial w^{\varepsilon}/\partial x_j \rightharpoonup a^H_{ss} \quad \text{weakly } L^1_{loc}$$

Proof

As v^{ε}, w^{ε} verifies a global equation :

$$\int_D a^{\varepsilon}_{ij} \partial w^{\varepsilon}/\partial x_j \cdot \partial f/\partial x_i = 0 \quad (\forall) f \in H^1_0(D)$$

Taking $f = w^{\varepsilon} g$ with $g \in H^1_0(D)$, using the fact that $w^{\varepsilon} \longrightarrow x_s$ strongly in $L_2(D)$ and the definition of effective coefficients, we obtain the result.

Proposition 3

With the above notations, we have

(9) $$a^{\varepsilon}_{ij} \partial u^{\varepsilon}/\partial x_j \cdot \partial w^{\varepsilon}/\partial x_i \rightharpoonup a^H_{sj} c_j - \left\langle a_{sj} + a_{ij} \partial P^s/\partial y_i \right\rangle c_j$$

(10) $$a^{\varepsilon}_{ij} \partial w^{\varepsilon}/\partial x_j \cdot \partial u^{\varepsilon}/\partial x_i \rightharpoonup 0,$$

weakly in L^1_{loc}.

Proof

We start with :

$$\int_D a^{\varepsilon}_{ij} \partial t^{\varepsilon}/\partial x_j \left(g \partial w^{\varepsilon}/\partial x_i + w^{\varepsilon} \partial g/\partial x_i\right) = 0,$$

obtained from the problem verified by w^{ε}, using the test function $w^{\varepsilon} g$, with $g \in H^1_0(D)$. We have also the relation (11), obtained from (4) and (5) :

(11) $$a^{\varepsilon}_{ij} \partial t^{\varepsilon}/\partial x_j \rightharpoonup a^H_{ik} c_k \quad \text{weakly in } L_2(D).$$

Using the last relation, the strong convergence of w^{ε}, the definition of u^{ε} and of the homogenized coefficients, we obtain (9).

For (10), we start with

$$\int_{D} a^{\varepsilon}_{ij}\, \partial w^{\varepsilon}/\partial x_j \,(g\, \partial u^{\varepsilon}/\partial x_i + u^{\varepsilon}\, \partial g/\partial x_i) = 0,$$

and use the results of Proposition 1.

We have also to remark that the method of compensated compactness, introduced by L.Tartar and F.Murat (see [9]) it is not applicable to obtain (4), because although the norms in L_2 of u^{ε} and p^{ε} are bounded by a constant, div p^{ε} do not belongs to L_2 .

The desired estimations are obtained starting with the following inequalities:

$$(A^{\varepsilon}(\nabla w^{\varepsilon}+c), \nabla w^{\varepsilon}+ c) \geqslant 0,$$

$$(A^{\varepsilon}(\nabla t^{\varepsilon}+c), \nabla t^{\varepsilon}+ c) \geqslant 0,$$

$$(A^{\varepsilon}(\nabla t^{\varepsilon}+ \nabla w^{\varepsilon}), \nabla t^{\varepsilon}+ \nabla w^{\varepsilon}) \geqslant 0,$$

where ∇ denotes the operator grad_x . Here $M \geqslant 0$ means $(M c, c) \geqslant 0$ $(\forall)$ $c \in R^3$, $M \cdot c = m_{ij} c_j$ and $(.,.)$ represents the scalar product in R^3. The above relations give us, taking into account the obtained convergences, the inequalities :

$$(12) \quad a^H_{ss} + a^H_{is} c_i + \langle a_{sj} + a_{ij}\, \partial p^s/\partial y_i \rangle c_j + \langle a_{ij} \rangle c_i c_j \geqslant 0$$

$$(13) \quad 2.\, a^H_{ij}\, c_i c_j + \langle a_{ik}\, \partial p^m/\partial y_i \rangle c_k c_m + 2.\langle a_{ij} \rangle c_i c_j \geqslant 0$$

$$(14) \quad -\langle a_{ik}\, \partial p^m/\partial y_i \rangle c_k c_m + a^H_{sj} c_j - \langle a_{sj} + a_{ij}\, \partial p^s/\partial y_i \rangle c_j + a^H_{ss} \geqslant 0.$$

Adding the last three equations, we obtain

$$(15) \quad a^H_{is}\, c_i + a^H_{sj}\, c_j \leqslant 2 . a^H_{ss} + 2 . a^H_{ij} c_i c_j + 3 \langle a_{ij} \rangle c_i c_j$$

where the summation convention is only for the indices i and j. For comparison, we recall the result of [3] :

Let a_{ij} = elastic coefficients, q_{ij} = homogenized coefficients, $a^T_{ij} = a_{ji}$. If

$$A^\varepsilon ((A^\varepsilon + A^{\varepsilon\, T})/2)^{-1} \rightharpoonup B^o$$

$$((A^\varepsilon + A^{\varepsilon\, T})/2)^{-1} A \rightharpoonup C^o$$

$$((A^\varepsilon + A^{\varepsilon\, T})/2)^{-1} \rightharpoonup D^o \text{ , weakly in } L_2 \text{ ,}$$

then

$$(B^o w, w) + (C^o w, z) + (w, C^o z) + (D^o z, z) \geqslant$$

$$\geqslant \left(\left(\frac{Q + Q^T}{2} \right)^{-1} (Q \cdot w + z), Q \cdot w + z \right),$$

$(\forall)\ w, z \in R^3$.

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