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by

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Dilations in H-cones

by N. Boboc and Gh. Bucur

In potential theory there are situations when starting with a cone of potentials we construct others which are bigger. For instance if S is a cone of potentials and B is a balayage (or only pseudobalayage) on S then the subset S_B of $S - S$ given by $S_B := \{s - Bs \,/\, s \in S\}$ is also a cone of potentials ([1]). If the base of B has his fine interior empty then we can say that S is a subcone of the cone of potentials generated by S_B.

Suppose now that $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a resolvent of kernels on a measurable space $(X, \mathcal{B})$ such that its initial kernel $V = V_0$ is bounded and let P be a bounded kernel on $(X, \mathcal{B})$ such that $V - PV$ is also a kernel on $(X, \mathcal{B})$ and such that, for any positive $\mathcal{B}$-measurable function f on X, we have

$$\inf(s, Ps + u - Pu + Pf) \in \overline{\mathcal{E}}_{\mathcal{V}}$$

for any $s, u \in \mathcal{E}_{\mathcal{V}}$. (Here $\overline{\mathcal{E}}_{\mathcal{V}}$ (resp. $\mathcal{E}_{\mathcal{V}}$) is the cone of strongly $\mathcal{V}$-supermedian (resp. $\mathcal{V}$-excessive) functions on X). Mokobodzki shows ([2]) that the set $\mathcal{E}_P$ of all functions on X of the forme $s - Ps$ where $s \in \mathcal{E}_{\mathcal{V}}$ is a solide subcone of the cone of all excessive functions with respect to the kernel $V - PV$. This assertion is proved when X is a compact space, $\mathcal{V}$ is a Ray resolvent on X and P does not charge the set of branching points of $\mathcal{V}$. One can say that this generalize the above situation when P is a balayage. The kernel P is called <u>subordination kernel with respect to $\mathcal{V}$</u>.

In this paper we develop a general procedure as above in the frame of the theory of H-cones.

If S is an H-cone, a map $P : S \to S$ which is additive, increasing and contractive ($Ps \le s$, $s \in S$) is called a <u>pseudo-dilation</u> if we have $s \wedge (Ps + t - Pt) \in S$ for all $s, t \in S$. A pseudo-dilation is termed <u>localizable</u> if $s \wedge (Ps + Pf) \in S$ for all $s \in S$

and $f \in (S - S)_+$. A pseudo-dilation is called dilation if we have

$$s - Ps \leq t - Pt \Rightarrow s \leq t \quad .$$

We remember that a map $B : \overset{\bullet}{S} \to S$ is a pseudobalayage if it is additive, increasing contractive and idempotent ($B^2 s = Bs, (\forall)\ s \in S$).

It is proved : 1) any pseudo-balayage is a localizable pseudo-dilation ; 2) for any pseudodilation P on S the subset $S_p = \{ s - Ps \mid s \in S \}$ is an H-cone ; 3) If P is a pseudo-dilation on S then there exists a pseudo-balayage B on S and a dilation Q on S_B uniquely determined such that

$$1 - P = (1 - Q)(1 - B).$$

We have also P localisable iff Q is localizable. 4) If P is a localizable pseudo-dilation on S and Q is a pseudo-dilation on S_P then the map

$$L := P + Q(1 - P)$$

is a pseudo-dilation on S and we have

$$(1 - L) = (1 - Q)(1 - P)$$

Moreover if Q is localizable then L is also localizable.

Definition Let S be an H-cone. A map

$$P : S \to S$$

is called pseudo-dilation on S if it is additive , increasing, contractive (i.e. $Ps \leq s$ for any $s \in S$) and

$$s, t \in S \Rightarrow s \wedge (Ps + t - Pt) \in S.$$

A pseudo-dilation P on S is called dilation if for any $s, t \in S$ we have

$$s - Ps \leq t - Pt \Rightarrow s \leq t$$

A pseudo-dilation P on S is called localizable if the following relation holds

$$f \in (S - S)_+ \ , s \in S \Rightarrow s \wedge (Ps + Pf) \in S \ .$$

Remark We remember that a map

$$B : S \to S$$

is called a pseudobalayage if it is additive, increasing, contractive and idempotent (i.e. $B^2 s = Bs$ for any $s \in S$).

Proposition 1 Let S be an H-cone and B be a pseudobalayage on S. Then B is a localizable pseudodilation on S and the convex cone

$$S_B := \{ s - Bs \ / \ s \in S \}$$

is an H-cone with respect to the natural order from S - S.

Proof. Since B is a pseudo-balayage on S then, from ([1], Theorem 5.1.5) it follows that S_B is an H-cone. If s, $t \in S$ then we have

$$(s - Bs) \wedge (t - Bt) = u - Bu$$

for a suitable $u \in S$. Since

$$u - Bu \leq s - Bs$$

it follows, from ([1], Proposition 5.1.2) that

$$u - Bu + Bs \in S$$

or equivalently

$$s \wedge (Bs + t - Bt) \in S.$$

Hence B is a pseudo-dilation on S. The fact that B is localizable follows as in ([1], Theorem 5.1.6). Indeed if s, $t \in S$ are such that $s \leq t$ and if $u \in S$ then we have

$$v := (u + Bs) \wedge (Bu + Bt) \in S,$$

$$Bv = Bu + Bs$$

and therefore

$$u \wedge (Bu + B(t - s)) + Bs = v \ .$$

Since

$$v - Bv \leq u - Bu$$

we deduce, from ([1], Proposition 5.1.2),

$$v - Bv + Bu \in S,$$

or equivalently

$$u \wedge (Bu + B(t - s)) \in S.$$

Definition. Let P be a pseudodilation on an H-cone S. We denote by B_P the map

$$B_P : S \rightarrow S$$

defined by

$$B_P\, s = \bigvee \{ t \in S \mid t \leq s\ ,\ Pt = t \}$$

The following proposition shows that B_P is a pseudobalayage on S. It is termed the pseudobalayage associated with P.

Proposition 2. For any pseudodilation P on S the map B_P is a pseudobalayage on S such that

$$B_P s = P(B_P s) \leq Ps \quad .$$

and such that if B is a pseudo-balayage on S with

$$Bs \leq Ps \qquad (\forall)\ s \in S$$

then we have

$$Bs \leq B_P s \qquad (\forall)\ s \in S.$$

Moreover for any $s \in S$ we have

$$B_P\, s = \bigwedge_{\alpha \in \Omega_S} P^{\alpha} s$$

where Ω_S is the set of all ordinals α on S with card $\alpha \leq$ card S and where $(P^{\alpha})_{\alpha \in \Omega_S}$ is the family of maps $P^{\alpha} : S \rightarrow S$ defined inductively by $P^1 = P$ and if $\alpha > 1$,

$$P^{\alpha}s = P(\wedge\{P^{\beta}s \mid \beta \in \Omega_S,\ \beta < \alpha\}) .$$

Proof. From the definition of B_P it follows that B_P is increasing and contractive Let $s_1, s_2 \in S$ and $t_1, t_2 \in S$ be such that

$$t_1 \leq s_1 ,\ t_2 \leq s_2,\quad Pt_1 = t_1,\quad Pt_2 = t_2 .$$

We have

$$P(t_1 + t_2) = t_1 + t_2$$

and therefore

$$t_1 + t_2 \leq B_P(s_1 + s_2) .$$

Since for any $s \in S$ and any $t', t'' \in S$ with

$$t' \leq s,\ t'' \leq s,\quad Pt' = t',\ Pt'' = t''$$

we have

$$t' \vee t'' \lesssim t' + t'' ,\quad t' \vee t'' \leq s$$

it follows that

$$P(t' \vee t'') = t' \vee t''$$

and therefore the set

$$\{t \in S \ / \ t \leq s,\ Pt = t\}$$

is upper directed. From the above considerations we get

$$B_P\, s_1 + B_P\, s_2 \leq B_P(s_1 + s_2) .$$

On the other hand if $t \in S$ is such that

$$t \leq s_1 + s_2 ,\ Pt = t$$

we deduce that there exist $t_1, t_2 \in S$, such that $t_1 + t_2 = t$, $t_1 \leq s_1$, $t_2 \leq s_2$. Obviously we have $Pt_1 = t_1$, $Pt_2 = t_2$ and therefore

$$t \leq B_P\, s_1 + B_P\, s_2,\ B_P(s_1 + s_2) \leq B_P\, s_1 + B_p\, s_2 .$$

Let now $s \in S$. Since $P\, B_P\, s \leq B_P\, s$ and since

$$t \in S,\ Pt = t,\ t \leq s \Rightarrow t \leq B_P\, s$$

we deduce

$$t \in S,\ Pt = t,\ t \leq s \Rightarrow \quad t \leq P(B_p\, s)$$

and therefore

$$B_p\, s \leq P\, B_p\, s \quad .$$

Hence $P\, B_p\, s = B_p\, s$. From the definition of B_p we get

$$B_p^2\, s \geq B_p\, s \quad , B_p^2\, s = B_p\, s$$

Let now B be a pseudo-balayage on S such that

$$Bs \leq Ps \qquad (\forall) \quad s \in S.$$

We deduce

$$B^2 s \leq PBs \leq Bs = B^2 s \qquad (\forall)\ s \in S$$

and therefore

$$Bs = PBs \qquad (\forall)\ s \in S.$$

From this fact and from the relation $Bs \leq s$ we get

$$Bs \leq B_p\, s \qquad (\forall)\ s \in S \quad .$$

It is easy to see that for any $\alpha \in \Omega_S$ the map P^α is additive, increasing and contractive and that

$$\alpha \leq \beta \Rightarrow P^\alpha s \geq P^\beta s, \qquad (\forall)\ s \in S.$$

Hence the map

$$s \longrightarrow Ts := \bigwedge_{\alpha \in \Omega_S} P^\alpha s$$

is also additive, increasing and contractive .

Since

$$\text{card}\ \Omega_S > \text{card}\ S$$

it follows that for any $s \in S$ there exists $\alpha_0 \in \Omega_S$ such that

$$P^\alpha s = P^{\alpha_0} s \qquad\qquad (\forall) \quad \alpha \geq \alpha_0$$

and therefore

$$Ts \geq P(Ts) = P(P^{\alpha_0} s) = P^{\alpha_0 + 1} s \geq Ts,\ Ts = P\ Ts \ ..$$

From this relation and from $Ts \leq s$ we get

$$Ts \leq B_P\, s \quad .$$

On the other hand we get inductively

$$B_P\, s \leq P^{\alpha} s \qquad (\forall) \qquad \alpha \in \Omega_S$$

and therefore

$$B_P\, s \leq Ts \ .$$

Corollary 3 . Let S be an H-cone and P be a pseudodilation on S. Then P will be a dilation iff any pseudo-balayage B on S with PB = B (or equivalent $Bs \leq Ps$ $(\forall)$ $s \in S$) is equal zero.

Proof. Suppose that P is a dilation on S and let B be a pseudo-balayage on S such that PB = B. We have

$$Bs - PBs = 0 \leq 0 - P0 \ .$$

and therefore $Bs \leq 0$. Conversely let $s, t \in S$ be such that

$$s - Ps \leq t - Pt.$$

We deduce, inductively

$$s - P^{\alpha} s \leq t - P^{\alpha} t \qquad\qquad (\forall) \ \alpha \in \Omega_S$$

and therefore

$$s - \bigwedge_{\alpha \in \Omega_S} P^{\alpha} s \leq t - \bigwedge_{\alpha \in \Omega_S} P^{\alpha} t \ .$$

Since B_P is a pseudo-balayage on S with

$$B_P s = \bigwedge_{\alpha \in \Omega_S} P^{\alpha} s \qquad , \ P\, B_P = B_P$$

we get $B_P\, s = 0$, $B_P\, t = 0$. Hence $s \leq t$.

<u>Theorem 4</u> Let S be an H-cone and P be a pseudodilation on S. Then there exists a pseudo-balayage B' on S and a dilation P' on $S_{B'}$ such that

$$1 - P = (1 - P')(1 - B')$$

The cuple (B', P') with the above properties is uniquely determined, B' is the pseudo-balayage associated with P and

$$P'(s - B's) = Ps - B's \qquad (\forall)\ s \in S .$$

Moreover if P is localizable then P' is also localizable.

<u>Proof.</u> Let B_0 be the pseudo-balayage associated with P and P_0 be the map defined on S_{B_0} by

$$P_0(s - B_0 s) = Ps - B_0 s .$$

Obviously we have

$$B_0 s \geqslant B_0 Ps \geqslant B_0 P B_0 s = B_0 s ,$$

$$B_0 Ps = B_0 s = P B_0 s .$$

From the definition of P_0 it follows that P_0 is additive. Let now s, $t \in S$ be such that

$$s - B_0 s \leqslant t - B_0 t$$

From ([1] , Proposition 5.1.2) we deduce that

$$s' := s - B_0 s + B_0 t \in S \quad , s' \leqslant t$$

and therefore

$$Ps' \leqslant Pt,\ Ps - P B_0 s + P(B_0 t) \leqslant Pt .$$

Since $P(B_0 s) = B_0 s$, $P(B_0 t) = B_0 t$ we get

$$Ps - B_0 s + B_0 t \leqslant Pt$$

$$P_0(s - B_0 s) \leqslant P_0(t - B_0 t) .$$

Hence P_0 is increasing. On the other hand we have

$$P_0(s - B_0 s) = Ps - B_0 s \leq s - B_0 s$$

and therefore P_0 is contractive.

Let now $s, t \in S$. We show now that

$$(s - B_0 s) \wedge (P_0(s - B_0 s) + (t - B_0 t) - P_0(t - B_0 t)) \in S_{B_0} .$$

Indeed we have

$$(s - B_0 s) \wedge (P_0(s - B_0 s) + (t - B_0 t) - P_0(t - B_0 t)) =$$

$$= (s - B_0 s) \wedge (Ps - B_0 s + t - Pt) = s \wedge (Ps + t - Pt) - B_0 s .$$

Since

$$s' := s \wedge (Ps + t - Pt) \in S$$

and

$$Ps \leq s' \leq s$$

we get

$$B_0 s' = B_0 s$$

and therefore

$$(s - B_0 s) \wedge (P_0(s - B_0 s) + (t - B_0 t) - P_0(t - B_0 t)) = s' - B_0 s' .$$

Hence P_0 is a pseudodilation on S_{B_0} . It is easy to see that we have, inductively,

$$P_0^{\alpha}(s - B_0 s) = P^{\alpha} s - B_0 s \ , (\forall)\ s \in S, \quad \alpha \in \Omega_S$$

and therefore

$$\bigwedge_{\alpha \in \Omega_S} P_0^{\alpha}(s - B_0 s) = B_0 s - B_0 s = 0$$

Using Corollary 3 we deduce that P_0 is a dilation on S_{B_0} .

Let now B' be a pseudo-balayage on S and P' be a dilation on $S_{B'}$ such that

$$1 - P = (1 - P')(1 - B')$$

We remark that

$$P = B' + P'(1 - B')$$

and therefore

$$P\,B's = B's \qquad (\forall)\ s \in S.$$

Hence

$$B's \leq B_0 s \qquad (\forall)\ s \in S.$$

We denote by B'' the map

$$B'' : S_{B'} \longrightarrow S_{B'}$$

defined by

$$B''(s - B's) = B_0(s - B's) = B_0 s - B'B_0 s \quad .$$

It is easy to see that B'' is a pseudo-balayage on $S_{B'}$. We have

$$(1 - P)(B_0 s) = 0 \qquad (\forall)\ s \in S,$$

$$(1 - P')(B''(s - B's)) = (1 - P')(1 - B')(B_0 s) =$$

$$= (1 - P)(B_0 s) = 0 \qquad (\forall) \quad s \in S.$$

Since P' is a dilation on $S_{B'}$, we deduce

$$B''(s - B's) = 0 \qquad (\forall)\ s \in S,$$

$$B'' = 0 \quad .$$

Hence

$$B_0 s = B'\, B_0 s = B's \qquad (\forall)\ s \in S,$$

$$B_0 = B'.$$

Suppose now that P is localizable and let s, t ∈ S be such that

$$s - B_0 s \leq t - B_0 t \ .$$

Replaceing s and t by

$$s' := s - s \curlywedge B_0 s \ , \ t' = t - t \curlywedge B_0 t$$

where $\curlywedge$ is the imfimum in S with respect to the specific order we get

$$s - B_0 s = s' - B_0 s' \leqslant t' - B_0 t' \quad ,$$

$$s' \curlywedge B_0 s' = t' \curlywedge B_0 t' = 0$$

and therefore

$$s' \leqslant t', \ B_0 s' \lesssim B_0 t' \quad .$$

We have for any $u \in S$,

$$(u - B_0 u) \wedge (P_0(u - B_0 u) + P_0((t' - B_0 t') - (s' - B_0 s'))) =$$

$$= (u + B_0(t'-s')) \wedge (P(u + B_0(t'-s')) + P(t'-B_0 t' - (s'-B_0 S'))) - B_0(u + t'-s').$$

Since $B_0(t' - s') \in S$ and using the fact that P is localizable it follows that the element

$$v := (u + B_0(t' - s')) \wedge (P(u + B_0(t' - s')) + P(t' - B_0 t' - (s' - B_0 s')))$$

belongs to S and moreover ,

$$B_0 \ v = B_0(u + t' - s') \ .$$

We deduce that

$$(u - B_0 u) \wedge (P_0(u - B_0 u) + P_0((t - B_0 t) - (s - B_0 s)) = v - B_0 v \in S_{B_0}$$

and therefore P_0 is localizable dilation on S_{B_0} .

Since for any $s \in S$ we have

$$P_0((1 - B_0)s) = Ps - B_0 s$$

we get

$$(1 - P_0)(1 - B_0)s = s - B_0 s - (Ps - B_0 s) = s - Ps$$

Remark The following theorem shows that if P is a pseudodilation on a given H-cone S then the convex cone $S_P := \{ s - Ps \,/\, s \in S \}$ is also an H-cone with respect to the natural order from S - S.

The preceding proposition represent a way to obtain S_P in two particular steps ; first we pass from S to S_{B_0} where B_0 is a pseudo-balayage (the pseudo-balayage associated with P) and then we pass from S_{B_0} to $S_P = (S_{B_0})_{P_0}$ where P_0 is at this time a dilation on S_{B_0} .

Lemma 5 . Let S be an H-cone, P be a pseudodilation on S and let M be a subset of S - S which is dominated by an element of the form s - Ps with $s \in S$. Then the set

$$\mathcal{A} := \{ u \in S \,/\, f \leq u - Pu \quad (\forall)\, f \in M \}$$

has a smallest element u_0 . Moreover we have

$$u_0 - Pu_0 = \wedge \{ u - Pu \,/\, u \in \mathcal{A} \} .$$

Proof. If we denote

$$u_0 := \wedge \mathcal{A}$$

we get

$$f + Pu_0 < f + Pu \leq u, (\forall)\, f \in M,\ u \in \mathcal{A}$$

and therefore

$$f \leq u_0 - Pu_0 \qquad (\forall)\ f \in M,$$

$$u_0 \in \mathcal{A} .$$

Let now $u \in \mathcal{A}$. If we put

$$u_1 := u_0 \wedge (Pu_0 + u - Pu)$$

we get $u_1 \in S$ and $u_1 \leq u_0$. On the other hand we have, for any $f \in M$,

$$f \leq u - Pu, \ f \leq u_0 - Pu_0 ,$$

$$f \leq (u_0 - P u_0) \wedge (u - Pu) = u_1 - P u_0 \leq u_1 - P u_1$$

and therefore $u_1 \in \mathcal{A}$, $u_1 \geq u_0$. Hence $u_1 = u_0$,

$$P u_0 + (u - Pu) \geq u_0,$$
$$u - Pu \geq u_0 - P u_0 .$$

<u>Corollary 6</u> For any $s \in S$ there exists an element $Ms \in S$ such that $s - Ps = Ms - PMs$ and such that for any $t \in S$ for which $s - Ps \leq t - Pt$ we have $Ms \leq t$.

Moreover we have

$$Ms = \wedge \{ t \in S \ / \ s - Ps \leq t - Pt \} = \wedge \{ t \in S \ / \ s - Ps = t - Pt \} .$$

<u>Proof</u> If we put

$$Ms = \wedge \{ t \in S \ / \ s - Ps \leq t - Pt \}$$

then we have, using Lemma 5 ,

$$Ms - PMs = \wedge \{ t - Pt \mid t \in S, \ s - Ps \leq t - Pt \} = s - Ps$$

Hence Ms is the smallest element of the set

$$\{ t \in S \mid s - Ps = t - Pt \} .$$

<u>Theorem 7</u> Let S be an H-cone and P be a pseudodilation on S. Then the convex cone

$$S_p := \{ s - Ps \mid s \in S \}$$

is an H-cone with respect to the natural order in S - S and

$$\bigwedge_{S_p} A = \bigwedge_{S-S} A \text{ for any } A \subset S_p ,$$

$$\bigvee_{S_p} A = \bigvee_{S-S} A \text{ for any upper directed and dominated } A \subset S_p .$$

<u>Proof</u>. If $A \subset S_p$ we denote

$$M := \{ f \in S - S \mid f \leq t \ (\forall) \ t \in A \}$$

and

$$\mathcal{A} = \{u \in S \mid f \leq u - Pu \quad (\forall)\, f \in M\} .$$

Obviously

$$A \subset \{u - Pu \mid u \in \mathcal{A}\}$$

and from Lemma 5 there exists $u_0 \in \mathcal{A}$ such that

$$\wedge \mathcal{A} = u_0, \; u_0 - \mathcal{P} u_0 = \wedge \{u - \mathcal{P}u \mid u \in \mathcal{A}\},$$

$$u_0 - \mathcal{P} u_0 = \bigvee_{S-S} M = \bigwedge_{S-S} A,$$

$$u_0 - P\, u_0 = \bigwedge_{S_{\mathcal{P}}} A \; .$$

Suppose now that $A = \{s_i - Ps_i \mid i \in I\}$ is upper directed and dominated in $S_{\mathcal{P}}$.

We denote

$$\mathcal{B} = \{u \in S \mid f \leq u - Pu \;, \quad (\forall\, f \in A\} .$$

From Lemma 5 the smallest element u_0 of $\mathcal{B}$ is such that

$$u_0 - P\, u_0 \leq u - Pu \qquad (\forall)\, u \in \mathcal{B}.$$

Using Corollary 6 we may suppose that $s_i = MS_i$ for any $i \in I$ and therefore we deduce that the family $(s_i)_{i \in I}$ is upper directed and dominated by u for any $u \in \mathcal{B}$. Hence $s_i \leq u_0$ for any $i \in I$. We have

$$s_i - Ps_i \leq u_0 - Pu_0 \qquad (\forall)\, i \in I,$$

$$s_i - Ps_i + Pu_0 \in S, \; s_i - Ps_i + Pu_0 \leq u_0, \; (\forall) \; i \in I$$

and therefore the element

$$t := \bigvee_S \{s_i - Ps_i + Pu_0 \mid i \in I\} = Pu_0 + \bigvee_{S-S} \{s_i - Ps_i \mid i \in I\}$$

belongs to S and $t \leq u_0$. Since $s_i \leq t$ for any $i \in I$ we have $t \leq u_0$. On the other hand $s_i - Ps_i \leq t - Pu_0 \leq t - Pt$ for any $i \in I$ and therefore $u_0 \leq t$, $u_0 = t$. From the above considerations it follows

$$t - Pt = u_0 - P\,u_0 = \bigvee_{S-S}\{s_i - Ps_i \mid i \in I\}.$$

It remains only to show that for any f, g $\in S_P$ we have

$$g - R^{S_P}(g-f) \in S_P \quad .$$

From the preceding considerations we have $R^{S_P}(g - f) \in S_P$ and

$$g - f \leq R^{S_P}(g - f) \quad .$$

Let u, v, w $\in$ S be such that $g = u - Pu$, $f = v - Pv$ and $R^{S_P}(g - f) = w - Pw$.

From Lemma 6 we may suppose that

$$w = \bigwedge_S \{w' \in S \mid g - f \leq w' - Pw'\} \quad .$$

We show that we have

$$(u - Pu) - (w - Pw) = R^{S_P}((u - Pu) - (w - Pw)) \in S_P \quad .$$

Indeed we have, using again Lemma 6,

$$R^{S_P}((u - Pu) - (w - Pw)) = v' - Pv'$$

where

$$v' = \bigwedge\{s \in S \mid (u - Pu) - (w - Pw) \leq s - Ps\}$$

and

$$v' - Pv' \leq v - Pv \quad .$$

Since

$$(u - Pu) - (w - Pw) \leq v' - Pv' \leq v - Pv$$

we deduce that the element

$$t := u - Pu + Pw + Pv'$$

belongs to S and $t \leq v' + w$.

Let

$$r := R^S(t - w) \quad .$$

We have $r \lesssim t$, $r \leq v'$. On the other hand, from

$$(u - Pu) - (w - Pw) + Pr \leqslant (u - Pu) - (w - Pw) + Pv' \lesssim r$$

we deduce

$$(u - Pu) - (w - Pw) \lesssim r - Pr$$

and therefore $r = v'$. Hence

$$v' = r \lesseqgtr t$$

We put $v_1 := t - v'$. We have $v_1 \in S$, $v_1 \leqslant w$, $t \leqslant w + v$ and we deduce

$$(u - Pu) - (v - Pv) < (u - Pu) - (v' - Pv') =$$

$$t - v' - Pw = v_1 - Pw \lesssim v_1 - Pv_1$$

From the definition of w it follows that

$$v_1 = w, \; w = t - v'$$

and therefore

$$(u - Pu) - (w - Pw) = v' - Pv' \quad .$$

Theorem 8 . Let P be a pseudodilation on S and $f \in (S - S)_+$. Then the following assertions are equivalent :

a) $f \wedge u \in S_P$ for any $u \in S_P$

b) $s \wedge (Ps + f) \in S$ for any $s \in S$

c) $s \wedge (Ps + t - Pt + f) \in S$ for any $s, t \in S$

where $\wedge$ means the infimum in S - S *with* respect to the natural order .

Proof. a) $\Rightarrow$ c) Let $s, t \in S$. From a) there exists $s' \in S$ such that

$$f \wedge (s - Ps) = s' - Ps' .$$

Since

$$(s - Ps) \wedge (t - Pt + f) = (s - Ps) \wedge (t - Pt + f \wedge (s - Ps)) =$$
$$= (s - Ps) \wedge (t - Pt + s' - Ps') .$$

we get

$$(s - Ps) \wedge (t - Pt + f) \in S_P \quad \text{and}$$
$$(s - Ps) \wedge (t - Pt + f) \leq s - Ps$$

and therefore

$$(s - Ps) \wedge (t - Pt + f) + Ps \in S,$$
$$s \wedge (Ps + t - Pt + f) \in S \quad .$$

c) $\Rightarrow$ b) is trivial

b) $\Rightarrow$ a) . Let $s \in S$. We consider the set

$$\mathcal{A} = \{ t \in S \mid f \wedge (s - Ps) \leq t - Pt \} .$$

From Lemma 5 it follows that posseses a smallest element t_0 and moreover

$$t_0 - Pt_0 = \wedge \{ t - Pt \mid t \in \mathcal{A} \} .$$

We show that

$$f \wedge (s - Ps) = t_0 - Pt_0 .$$

Obviously $t_0 \leq s$ and

$$f \wedge (s - Ps) \leq t_0 - Pt_0, \; t_0 - Pt_0 \leq s - Ps .$$

Hence

$$f \wedge (s - Ps) = f \wedge (t_0 - Pt_0) \quad .$$

From the assertion b) it follows that .

$$t_1 := t_0 \wedge (Pt_0 + f) \in S.$$

Since , $t_1 \leq t_0$ and

$$f \wedge (s - Ps) = f \wedge (t_0 - Pt_0) = t_1 - Pt_0 \leq t_1 - Pt_1$$

we deduce, from the definition of $\mathcal{A}$, that

$$t_1 \in \mathcal{A} \quad , t_1 = t_0 \quad ,$$

$$f \wedge (t_0 - Pt_0) = t_0 - Pt_0 \quad ,$$

$$f \wedge (s - Ps) = t_0 - Pt_0 \quad .$$

<u>Theorem 9</u> . Let S be an H-cone and let P be a localizable pseudodilation on S. Then for any pseudodilation Q on S_P the map

$$L := P + Q(1 - P)$$

is a pseudodilation on S and we have .

$$1 - L = (1 - Q)(1 - P)$$

Morover L will be a localizable pseudodilation if Q is a localizable pseudodilation on S_P

<u>Proof</u>. We show that L is a pseudodilation on S. Let $s \in S$ and let $u \in S$ be such that

$$Q(s - Ps) = u - Pu \leq s - Ps \quad .$$

Since P is a pseudodilation on S it follows that

$$u - Pu + Ps \in S, \quad u - Pu + Ps \leq s$$

and therefore

$$Ls = Q(s - Ps) + Ps = u - Pu + Ps \in S,$$

$$Ls \leq s \quad .$$

We consider now $f \in (S - S)_+$. Since P is localizable it follows that

$$s \wedge (Ps + Pf) \in S \quad , (\forall)\ s \in S$$

or equivalently (see Theorem 8)

$$Pf \wedge (s - Ps) \in S_P \quad , \quad (\forall)\ s \in S.$$

Now we show that

$$f \in (S - S)_+ \Rightarrow Lf \geq 0 \quad .$$

Let $s, t \in S$ be such that $f = s - t$. We have

$$t - Pt \leq s - Ps + Pf \quad ,$$

and therefore

$$t - Pt \leq s - Ps + Pf \wedge (t - Pt)$$

Since P is localizable we have $Pf \wedge (t - Pt) \in S_P$ and therefore

$$Q(t - Pt) \leq Q(s - Ps) + Q(Pf \wedge (t - Pt)) \leq$$

$$\leq Q(s - Ps) + Pf \wedge (t - Pt) \leq Q(s - Ps) + Pf \quad .$$

Hence

$$Pt + Q(t - Pt) \leq Q(s - Ps) + Ps$$

or equivalently

$$Lt \leq Ls, \ Lf \geq 0$$

and therefore L is additive, increasing and contractive.

Let now $s, t \in S$. We have

$$s \wedge (Ls + t - Lt) = s \wedge (Ps + Q(1 - P)s + t - Pt - Q(1 - P)t) =$$

$$(s - Ps) \wedge (Q(s - Ps) + (t - Pt) - Q(t - Pt)) + Ps \quad .$$

Since Q is a pseudodilation on S_P it follows that there exists $u \in S$ such that

$$(s - Ps) \wedge (Q(s - Ps) + (t - Pt) - Q(t - Pt)) = u - Pu \quad .$$

Hence

$$u - Pu \leq s - Ps$$

and therefore

$$u - Pu + Ps \in S \quad ,$$

$$s \wedge (Ls + t - Lt) = u - Pu + Ps \in S.$$

From the above consideration it follows that L is a pseudodilation on S.

Obviously we have

$$(1 - Q)(1 - P) = 1 - L .$$

Suppose now that Q is localizable and let $f = s_1 - s_2$, $s_1, s_2 \in S$, $s_1 \geq s_2$.

We remark that for any $s \in S$ and any $u \in S$ such that

$$u - Pu \geq s - Ps + s_2 - Ps_2$$

we have

$$(s - Ps) \wedge (Q(s - Ps) + Pf \wedge (u - Pu) + Q(f - Pf)) =$$

$$= (s - Ps) \wedge (Q(s - Ps) + Pf + Q(f - Pf)) .$$

Indeed we have

$$(s - Ps + s_2 - Ps_2) \wedge (Q(s - Ps) + Pf \wedge (u - Pu) + Q(s_1 - Ps_1)) \leq$$

$$\leq (s - Ps + s_2 - Ps_2) \wedge (Q(s - Ps) + Pf + Q(s_1 - Ps_1)) \leq$$

$$\leq (s - Ps + s_2 - Ps_2) \wedge (Q(s - Ps) + Pf \wedge [(s - Ps) + s_2 - Ps_2)] + Q(s_1 - Ps_1)) \leq$$

$$\leq (s - Ps + s_2 - Ps_2) \wedge (Q(s - Ps) + Pf \wedge (u - Pu) + Q(s_1 - Ps_1))$$

and therefore

$$((s - Ps) + (s_2 - Ps_2)) \wedge (Q(s - Ps) + Pf \wedge (u - Pu) + Q(s_1 - Ps_1)) =$$

$$= (s - Ps + s_2 - Ps_2) \wedge (Q(s - Ps) + Pf + Q(s_1 - Ps_1)) ,$$

$$(s - Ps) \wedge (Q(s - Ps) + Pf \wedge (u - Pu) + Q(f - Pf)) =$$

$$= (s - Ps) \wedge (Q(s - Ps) + Pf + Q(f - Pf)) .$$

On the other hand if $s \in S$ and $u \in S$ are such that

$$u - Pu \geq s - Ps + s_2 - Ps_2$$

then we have

$$f - Pf + Pf \wedge (u - Pu) \in (S_P - S_P)_+$$

Since P is localizable we get

$$g := Pf \wedge (u - Pu) \in S_P$$

and using the fact that Q is localizable we deduce

$$(s - Ps) \wedge (Q(s - Ps) + g - Qg + Q(f - Pf + g))) \in S_P \quad .$$

From the above considerations it follows

$$(s - Ps) \wedge (Q(s - Ps) + Pf \wedge (u - Pu) + Q(f - Pf) \in S_P \quad ,$$

$$(s - Ps) \wedge (Q(s - Ps) + Pf + Q(f - Pf)) \in S_P$$

$$(s - Ps) \wedge (Q(s - Ps) + Lf) \in S_P$$

for any $s \in S$ and any $f \in (S - S)_+$. Hence L is localizable.

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