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A REMARK ON RUNGE DOMAINS

by Viorel Vajaitu

It is a well-known fact that for any open Runge domain D of any Stein manifold X (i.e. the natural restriction map $\mathcal{O}(X) \longrightarrow \mathcal{O}(D)$ has dense image) such that D itself is Stein and any coherent sheaf $\mathcal{F}$ on X, the natural restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(D)$ has dense image.

In this paper we drop the assumption made on D to be a Stein open subspace of X and we shall impose some conditions on the sheaf $\mathcal{F}$ so that the conclusion remains valid.

From now on X denotes a complex connected manifold of complex dimension n ($n \geq 2$) and by a "domain" of X we understand any open and connected subset of X. We would like to prove the following:

<u>Theorem:</u> Let X be a Stein manifold and $\mathcal{F}$ a torsion free coherent sheaf on X such that $\mathcal{F}^{[n-2]} = \mathcal{F}$. Then, for any Runge domain D of X (not necessarily Stein), the natural restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(D)$ has dense image, for the canonical topology.

In order to establish this statement we need some preliminary ingredients.

<u>Lemma 1:</u> Let D be a Runge domain of a Stein manifold X. Then the envelope of holomorphy $\tilde{D}$ of D exists and is an open subset of X.

<u>Proof:</u> From [5] it follows that there exists the envelope of holomorphy $\tilde{D}$ of D and $\tilde{D}$ is a Riemann domain over X, i.e. we have the following comutative diagram:

$$\begin{tikzcd} & \tilde{D} \arrow[dr, "\varphi"] & \\ D \arrow[ur, hook] \arrow[rr] & & X \end{tikzcd}$$

with φ locally biholomorphic. It remains to prove that φ is injective. Using Lemma 5.4.1 from [3] it follows that for every compact subset $\tilde{K}$ of $\tilde{D}$ we can find a compact subset K of D such that the $\mathcal{O}(D)$ - hull of K contains $\tilde{K}$. Now, let suppose that φ is not injective. Then there exist $x_1 \neq x_2 \in \tilde{D}$, $\varphi(x_1)=\varphi(x_2)=y_0 \in X$. Let $\tilde{K}=\{x_1, x_2\}$ and let K be a compact subset of D stated as above. Since $\tilde{D}$ is a Stein manifold there exists a $f \in \mathcal{O}(\tilde{D})$ holomorphic function such that $f(x_1)=0$, $f(x_2)=1$. By restriction to D, f induces a holomorphic map on D which can be approximated sufficiently close on K by holomorphic functions on X.

Thus, there exists $h \in \mathcal{O}(X)$, $\| h \circ \varphi - f \|_K \leq \frac{1}{3}$.

It follows that $\|h\circ\varphi - f\|_{\tilde{K}} \leq \frac{1}{3}$. Since $\tilde{K} = \{x_1, x_2\}$ we get $|h(y_0) - f(x_1)| \leq \frac{1}{3}$ and $|h(y_0) - f(x_2)| \leq \frac{1}{3}$ which can easily led to a contradiction .

Remark 1 : Under the preceding Lemma conditions it follows that $\tilde{D}$ is the smallest open Stein subspace of X which contains D , i.e. $\tilde{D}$ = the interior of $\bigcap D_\alpha$, where the intersection is performed over all open Stein subspaces D_α of X that contain D .

Lemma 2 : Suppose D is a Runge domain of a Stein manifold X . Then the envelope of holomorphy $\tilde{D}$ is also a Runge domain of X .

Proof: Let $\tilde{K}$ be a compact subset of $\tilde{D}$, $\varepsilon > 0$ and $f \in \mathcal{O}(D)$. As in Lemma 1 , there exists a compact subset K of D such that the $\mathcal{O}(\tilde{D})$ -hull of K contains $\tilde{K}$. Because D is a Runge domain of X , there exists $g \in \mathcal{O}(X)$, $\|g - f\|_K \leq \varepsilon$. It follows $\|g - f\|_{\hat{K}} \leq \varepsilon$. Since $\tilde{K} \subseteq \hat{K}$ we have the desired approximation , $\|g - f\|_{\tilde{K}} \leq \varepsilon$.

Remark 2: Let D be a Runge domain (not necessarily Stein) of a Stein manifold X , $\tilde{D}$ its envelope of holomorphy and $\mathcal{F}$ a coherent sheaf on X . Suppose that the restriction map:

$$\mathcal{F}(\tilde{D}) \longrightarrow \mathcal{F}(D)$$

is surjective . Then the restriction map:

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(D)$$

has dense image .

To see this we observe that the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(D)$ is obtained by composing the two restriction maps: $\mathcal{F}(X) \longrightarrow \mathcal{F}(\tilde{D})$ which has dense image and the surjection $\mathcal{F}(\tilde{D}) \longrightarrow \mathcal{F}(D)$ (by the assumption just made). Therefore to conclude the proof of the Theorem it is enough to prove that the restriction map :

$$\mathcal{F}(\tilde{D}) \longrightarrow \mathcal{F}(D)$$

is surjective .

There is a natural construction which assigns to every (pre -) sheaf $\mathcal{G}$ over X a unique topological space $|\mathcal{G}|$ over X called its sheaf-space (or " espace etale") . One provides each $\mathcal{G}(U)$ with the discrete topology and defines an equivalence relation on the disjoint union

$$\bigcup U \times \mathcal{G}(U)$$

performed over all open subsets of X (viewed as a topological sum)

$$(x,f) \sim (y,g) \Leftrightarrow x=y \text{ and } f=g \text{ near } x$$

The quotient space $|\mathcal{G}|$ with the projection $\pi : |\mathcal{G}| \longrightarrow X$ onto the first component has as fibers $\pi^{-1}(x)$ the stalks $\mathcal{G}_x$ (with the discrete topology); furthermore , π is surjective and a local homeomorphism (hence an open- map) and all sections $\sigma \in \Gamma(U, |\mathcal{G}|)$ are open mappings . The canonical maps

$$\mathcal{G}(U) \longrightarrow \Gamma(U, |\mathcal{G}|)$$

are bijective if $\mathcal{G}$ is a sheaf .

For each element $f \in |\mathcal{G}|$ we denote by $C(f)$ the connected component of $|\mathcal{G}|$ containing f , which we call the existence domain of f .

We call a sheaf $\mathcal{G}$ over X a Hausdorff sheaf if its associated sheaf-space $|\mathcal{G}|$ is a Hausdorff space (e.g. the Identity theorem for holomorphic functions merely state that $\mathcal{O}_X$ is a Hausdorff sheaf).

Since every torsion-free coherent sheaf over X can be injected , locally , in an $\mathcal{O}_X^p$, we get easily the following:

<u>Remark 3</u>: Any coherent torsion free sheaf on X is a Hausdorff sheaf .

In this way , for the sheaf $\mathcal{F}$ as in the Theorem , $|\mathcal{F}|$ becomes a complex manifold of the same dimension as X . Now , let $D = \{z \in \mathbb{C}^n : |z_1| < 1, \ldots |z_n| < 1\}$ and $\delta D = \{z \in \overline{D} ; |z_n| = 1\}$

We say that the Axiom of Continuity is fulfilled for $\mathcal{F}$ when the following holds:

" For any biholomorphic map ψ of an open neighborhood of $\overline{D}$ in $\mathbb{C}^n$ onto an open subset of X , the restriction map

$$\mathcal{F}(\psi(\overline{D})) \longrightarrow \mathcal{F}(\psi(\delta D \cup D))$$

is surjective" .

Now, if d is any positive integer , we define the sheaf $\mathcal{F}^{[d]}$ on X to be the sheaf associated to the following presheaf ,

$$U \longmapsto \varinjlim \mathcal{F}(U \setminus A)$$

where the inductive limit is performed over all analytic subsets of U of dimension $\leq d$. We call $\mathcal{F}^{[d]}$ the d^{th} absolute gap-sheaf of $\mathcal{F}$. See also [6] We write $\mathcal{F}^{[d]} = \mathcal{F}$ if the natural sheaf homomorphism $\mathcal{F} \longrightarrow \mathcal{F}^{[d]}$ is an isomorphism (e.g. any locally free sheaf on X or $\mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{F})$ where $\mathcal{F}^{[d]} = \mathcal{F}$, see [6] , Prop. 3.15)

Therefore , the Proposition 3.14 from [6] gives us this :

<u>Lemma 3:</u> If $\mathcal{F}$ is any coherent sheaf on X such that $\mathcal{F}^{[n-2]} = \mathcal{F}$ then the Axiom of Continuity holds for the sheaf $\mathcal{F}$.

In this way we obtain the following

<u>Lemma 4:</u> Let X be a Stein manifold and $\mathcal{F}$ a coherent torsion free sheaf such that $\mathcal{F}^{[n-2]} = \mathcal{F}$. Then , for each $f \in |\mathcal{F}|$, the existence domain $C(f)$ of f is Stein .

To conclude the proof of the Theorem we state the next:

<u>Lemma 5 :</u> Let $\tilde{D}$ be the envelope of holomorphy of a Runge domain D of a Stein manifold X and $\mathcal{F}$ a coherent torsion free sheaf on X such that

the Axiom of Continuity holds for $\mathcal{F}$.

Then the restriction map:

$$\mathcal{F}(\tilde{D}) \longrightarrow \mathcal{F}(D)$$

is surjective.

<u>Proof</u> : Let $f \in \mathcal{F}(D)$ and $C(f)$ be the existence domain of f over $\tilde{D}$ (it is easily checked that $\{f_x ; x \in D\}$ is a connected open subset of $|\mathcal{F}|$ and $C(f_x) = C(f_y)$ for every $x, y \in D$, so we have $C(f) = C(f_{x_0})$, $x_0 \in D$).

First , we show that $\pi|_{C(f)}$ is injective .

Let $x_1 \neq x_2 \in C(f)$ such that $\pi(x_1) = \pi(x_2) = y \in \tilde{D}$.

Since $C(f)$ is holomorphic separable, there exists a holomorphic map:

$$g : C(f) \longrightarrow \mathbb{C} \ , \ g(x_1) \neq g(x_2)$$

We can view D as an open subset of $C(f)$ via the map $x \longmapsto f_x$ and let $h : D \longrightarrow \mathbb{C}$, $h(x) = g(f_x)$, $x \in D$.

It follows $h \in \mathcal{O}(D)$.

Let $H : \tilde{D} \longrightarrow \mathbb{C}$ be the holomorphic extension of h and let $g' : C(f) \longrightarrow \mathbb{C}$, $g' = H \circ \pi$

We get the following comutative diagram:

$$\begin{tikzcd}
 & C(f) \arrow[dl, "\pi"'] \arrow[dd, shift right, "g'"'] \arrow[dd, shift left, "g"] & \\
\tilde{D} \arrow[dr, "H"'] & & D \arrow[ll, hook'] \arrow[ul, bend right] \arrow[dl, "h"] \\
 & \mathbb{C} &
\end{tikzcd}$$

We have: $g' = g$ on D ($g'(f_x) = H(x) = h(x) = g(f_x), x \in D$)
From the Principle of analytic continuation we obtain that $g = g'$ and so $g'(x_1) = H(y) = g'(x_2)$. Thus $g(x_1) = g(x_2)$, which is a contradiction .

It follows that $C(f)$ is an open Stein subspace of $\tilde{D}$ containing D .

Now , we show that $C(f) = \tilde{D}$.
If this were not the case , there would exist a holomorphic function on $C(f)$ (and also on D) that cannot be extended to a holomorphic function on $\tilde{D}$, but this contradicts the fact that $\tilde{D}$ is the envelope of holomorphy of D .

Finally, let $s \in \Gamma(D, |\mathcal{F}|)$. Defining:

$$\tilde{s} = \left(\tilde{\pi}|_{C(f)}\right)^{-1} : \tilde{D} \longrightarrow C(s) \subset |\mathcal{F}|$$

we obtain a section $\tilde{s} \in \Gamma(\tilde{D}, |\mathcal{F}|)$ which extends s .
Therefore , the map $\mathcal{F}(\tilde{D}) \longrightarrow \mathcal{F}(D)$ is surjective.
Proof of Theorem: It follows at once from Remark 2 and Lemma 5 .

We conclude this paper by giving an example of a sheaf $\mathcal{F}$ such that $\mathcal{F}^{[n-2]} \neq \mathcal{F}$ and the Theorem does not hold for $\mathcal{F}$.

Let $X = \mathbb{C}^n$ and $H = \{z \in X ; z_{n-1} = z_n = 0\}$ put $D = X \setminus H$ and $\mathcal{F} = \mathcal{J}_H$ (the ideal defining of H).
Then ,

$$\operatorname{prof} \mathcal{F}_x = \begin{cases} n-1 & \text{if } x \in H \\ n & \text{if } x \in X \setminus H \end{cases}$$

and $S_{n-1}(\mathcal{F}) = \{x \in X ; \operatorname{prof} \mathcal{F}_x \leq n-1\} = H$. So we have $\dim S_{n-1}(\mathcal{F}) = n-2$ and using Prop. 3.13 from [6] we obtain that $\mathcal{F}^{[n-2]} \neq \mathcal{F}$ (and $\mathcal{F}$ is obviously torsion free) . It is obvious that $\widetilde{D} = X$ and the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(D)$ has not dense image (e.g. $\mathbb{1} \in \mathcal{F}(D)$ cannot be approximated sufficiently close on the compact $K = \{ z \in X ;\ |z_j| \leq 1,\ j = 1, \dots, n,\ |z_{n-1}| \geq \frac{1}{2},\ |z_n| \geq \frac{1}{2} \}$ because

$$\hat{K} = \{ z \in X ;\ |z_j| \leq 1,\ j = 1, \dots, n \}$$
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