THE SECOND BRAUER-THRALL CONJECTURE FOR ISOLATED SINGULARITIES OF EXCELLENT HYPERSURFACES

by

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0. Introduction

Throughout, let R be an excellent local Henselian k-algebra over an algebraically closed field k, (R,m) an isolated hypersurface singularity (i.e. the completion R^ of R is isomorphic $k[[X_o,\dots,X_n]]/(f)$). The aim of this paper is to give a generalization and a slightly different proof of the following two theorems of Dieterich (cf. [Di]).

Theorem I: Let Γ be the Auslander-Reiten quiver of R (for the notations, cf. 1.), C its connected component containing the class of R. Then

$$\Gamma - C = \coprod_{i \in I} \mathbb{Z}A_{\infty}/\langle\tau^{r(i)}\rangle$$

for a certain set I, and $r(i) \in \{ 1,2 \}$ for $i \in I$ .Further, n(i) = 1 for all i if dim R is even, and $C = \Gamma$ if R is simple.

Theorem II: The second Brauer- Thrall conjecture is true for R, i.e. if R is not of finite representation type, then there is a strictly increasing sequence (n_i) of positive integers, such that for all i , there are infinitely many isomorphic classes of indecomposable maximal Cohen Macaulay modules having rank n_i.

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Both results are known by [Di] for the case of $R = k[[X_o,\dots,X_n]]/(f)$ and (in theorem II) char k different from 2. We supress the condition on char k using a recent result of Greuel and Kröning [GK] , while the base change from R to R^ was considered by the authors in [PR].
A combinatorial remark replaces the application of [HPR].

1. *The AR-quiver of an isolated singularity*

The results in this section are known and only included for convenience of the reader. In the following, the word "quiver" denotes a couple $\Gamma = (\Gamma_o,\Gamma_1)$, where Γ_o is a set (of "vertices"), $\Gamma_1 \subset \Gamma_o \times \Gamma_o$ (a set of "arrows"). If all arrows appear in pairs (i.e. $(i,j) \in \Gamma_1$ iff $(j,i) \in \Gamma_1$), the quiver is determined by its underlying graph.
Consider the graph

$$A_\infty: \quad \underset{1}{o} \text{———} \underset{2}{o} \text{———} \underset{3}{o} \text{—} \quad \dots$$

We define a quiver $\Gamma = \mathbb{Z} A_\infty$ by

$$\Gamma_o = (\mathbb{N} - \{O\}) \times \mathbb{Z}$$
Γ_1 = set of all $(\alpha,\beta) \in \Gamma_o \times \Gamma_o$ such that $\beta = (i,j)$ and

$$\beta-\alpha = \begin{cases} (\pm 1, 1) & \text{if } i \text{ is even,} \\ (\pm 1, O) & \text{if } i \text{ is odd, i.e.} \end{cases}$$

Γ is the quiver

$$\begin{tikzcd}
 & & j-1 & j & j+1 & \\
 & & \vdots & \vdots & \vdots & \\
1 & \cdots & o \arrow[dr] & o \arrow[r, dashed, "\tau"] \arrow[dr] & o & \cdots \\
2 & \cdots & o \arrow[u] \arrow[d] & o \arrow[u] \arrow[d] & o \arrow[u] \arrow[d] & \cdots \\
3 & \cdots & o \arrow[ur] \arrow[dr] & o \arrow[ur] \arrow[dr] & o & \cdots \\
4 & \cdots & o \arrow[u] & o \arrow[u] & o \arrow[u] & \cdots \\
 & & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Further, let $\tau: \Gamma_o \longrightarrow \Gamma_o$ be the map $\tau(i,j) = (i,j+1)$. This is an automorphism of Γ, i.e. τ is a bijection of Γ_o with the property $(i,j) \in \Gamma_1$ iff $(\tau i,\tau j) \in \Gamma_1$. Thus, for an arbitrary $r \in \mathbb{N}$, we may consider the subgroup $G = \langle \tau^r \rangle$ of the automorphism group and define in a natural way the quotiont quiver Γ/G, identifying vertices, resp. arrows mod G. Consider e.g. the case $r = 1$: $\Gamma = \mathbb{Z} A_\infty/\langle\tau\rangle$ is the quiver determined by the graph A_∞.

Now let R be an excellent local henselian Cohen-Macaulay k-algebra with residue field k, having an isolated singularity (i.e. $R_{\mathfrak{p}}$ is regular for any nonmaximal prime $\mathfrak{p}$). Let MCM(R) be the category of maximal Cohen-Macaulay modules, i.e. of finitely generated R-modules M with $\mathrm{depth}_R M = \dim R$. For $M \in \mathrm{MCM}(R)$, M indecomposable, consider the set

$$S(M) = \left\{ s \in \mathrm{Ext}^1(M,N) \mid s \neq 0,\ s: 0 \to N_s \to E_s \to M \to 0,\ N_s = N \text{ indecomposable} \right\}$$

with $s \leq t$ in S(M) iff there is a morphism f such that $\mathrm{Ext}^1(M,f)(t) = s$. The (unique) minimal element of S(M) (if it exists) is the so called AR-sequence of M:

$$0 \longrightarrow \tau(M) \longrightarrow E_M \longrightarrow M \longrightarrow 0\ ,$$

the map $M \longmapsto \tau(M)$ is the "AR - translation" . Isolated singularities are characterized by
the following

1.1. *Theorem* (Auslander): If $M \in \mathrm{MCM}(R)$ is indecomposable and not isomorphic to R, then there is an AR-sequence ending in M.

1.1. *Definition*: The AR- quiver $\Gamma = \Gamma(R)$ is given by

Γ_0 : = set of classes [M] (= R-modules isomorphic to M) of indecomposable modules $M \in \mathrm{MCM}(R)$

$\Gamma_1 := \left\{ ([N],[M]) \in \Gamma_0 \times \Gamma_0 \mid \text{there is an irreducible morphism } f: N \to M \right\}$.

Here f is said to be irreducible, if it is not an isomorphism, and given any factorization $f = g \cdot h$ in MCM(R), then g is a split epimorphism or h is a split monomorphism.

1.3. *Lemma* (Reiten): Let $M,N \in \mathrm{MCM}(R)$ be indecomposable, M not isomorphic to R and $0 \to \tau(M) \to E_M \to M \to 0$ the AR-sequence ending in M. The following conditions are equivalent

(i) There is an irreducible map $N \to M$.

(ii) There is an irreducible map $\tau(M) \to N$.

(iii) N is a direct summand of E_M.

For the henselian ring R , by uniqueness of the Krull-Remak-Schmidt decomposition (and a duality argument), we obtain: There are only finitely many arrows in $\Gamma(R)$ starting (resp. ending) in N ("Γ is locally finite"). Further, $\tau(M)$ can be computed using syzygies [Au1]; for the hypersurface R there are the simple formulas

1.4. *Proposition* : $\tau^2(M) \cong M$, especially

(i) $\tau(M) \cong M$ if n is even, and

(ii) $\tau(M) \cong \Omega_R(M)$ if n is odd.

Here $\Omega_R(M)$ is the first syzygy module of M: Let $\mu(M)$ be the minimal number of generators of M, then $\Omega_R(M)$ is the (uniquely determined) kernel of any epimorphism $R^{\mu(M)} \longrightarrow M$. From the preceding, we deduce

1.5. *Remark*: Let ν: MCM(R) $\longrightarrow \mathbb{N}$ be a map with the properties

(i) $\nu(N_1 \oplus N_2) = \nu(N_1) + \nu(N_2)$;

(ii) $\nu(N') + \nu(N'') \geq \nu(N)$ for any exact sequence $O \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow O$.

Then for all $[M] \in \Gamma_0(R)$, $\nu(M) + \nu(\tau(M)) \geq \sum \nu(N)$, where the sum is taken over all [N] such that there exists an irreducible map $N \longrightarrow M$ in $\Gamma_1(R)$.

This follows by 1.3., since $\nu(\tau(M)) + \nu(M) \geq \nu(E_M) = \sum \nu(N)$, where the sum is taken over the Krull - Remak - Schmidt - decomposition of E_M.

A function ν which satisfies 1.5. is said to be subadditive on Γ. Especially, this is true for the functions

$$\rho = \rho_R , \qquad \rho_R(M) = \dim_{Q(R)}(M \otimes_R Q(R))$$

$$\mu = \mu_R , \qquad \mu_R(M) = \dim_{R/m}(M/mM)$$

(Q(R) is the field of fractions if R is a domain).

1.6. *Remark*: $\mu(\tau(M)) = \mu(M)$.

(Consider the exact sequences

$$O \longrightarrow \Omega_R(M) \longrightarrow R^{\mu(M)} \longrightarrow M \longrightarrow O$$

$$O \longrightarrow M \longrightarrow R^{\mu(\Omega_R(M))} \longrightarrow \Omega_R(M) \longrightarrow O$$

and tensorize by Q(R); we obtain equal ranks for the middle terms.)

1.7. *Definition* : Let $\eta : N \longrightarrow X$ be a homomorphism of MCM - R-modules and N indecomposable. Then η is called minimal left almost split, if it is not a split monomorphism and the following conditions are satisfied:

(i) For all $\eta' : N \longrightarrow X'$ such that η' is not a split monomorphism, there is a $\rho : X \longrightarrow X'$, $\rho \cdot \eta = \eta'$.

(ii) All $\varepsilon \in End_R(X)$ such that $\varepsilon \cdot \eta = \eta$ are automorphisms.

1.8. *Theorem* (Auslander, Reiten): For all indecomposable $N \in$ MCM(R) there is a minimal almost split morphism $\eta : N \longrightarrow X$, unique up to isomorphism. Further, let $X = \oplus X_i$ be a decomposition into indecomposable objects. Then the induced morphisms $\eta_i: N \longrightarrow X_i$ are irreducible.

Note that there is a dual definition ("right almost spliIt") and statement to 1.7. and 1.8., respectively.

Further, we use the following version of the Harada - Sai - lemma (the proof uses the existence of a "reduction ideal", cf. [Di 1], [Po]):

1.9. *Lemma* : There is a function $\rho : \mathbb{N} \longrightarrow \mathbb{N} - \{ O \}$ with the following property: Let b > O be an integer and $M_1 \longrightarrow M_2 \longrightarrow \ldots \longrightarrow M_{\rho(b)+1}$ sequence of $\rho(b)+1$ indecomposable R-modules M_i and irreducible morphisms $\psi_i: M_i \longrightarrow M_{i+1}$ ($i = 1,\dots,\rho(b)$) such that $\mu(M_i) \leq b$ ($i=1,\dots,\rho(b)+1$). Then $\mathrm{im}\,(\psi_{\rho(b)} \cdot \dots \cdot \psi_1) \subset m_R \cdot M_{\rho(b)+1}$.

From 1.8. and 1.9. we deduce ([Di 1], [Yo] 1.1., [Po] 5.4.)

1.10. ***Proposition*** : Let $C = (C_o, C_1)$ be a connected component of the AR-quiver Γ of R. If μ is bounded on C_o, then C_o is finite and $C = \Gamma$.

Proof: Let $[N] \in \Gamma_o$ be such that $[N] \neq [R]$ and choose any $\varphi: R \longrightarrow N$ such that $\mathrm{im}\,\varphi$ is not contained in $m_R \cdot N$. By the dual statement of 1.8. there is a right almost split map $\varphi': Y_N \longrightarrow N$, $Y_N = Y_1 \oplus \dots \oplus Y_n$, inducing irrreducible maps $\varphi_i: Y_i \longrightarrow N$ and $\zeta_i: R \longrightarrow Y_i$ such that $\varphi = \sum \varphi_i \cdot \zeta_i$, since φ is not a split epimorphism. Let b be an upper bound for μ on C_o.

case a):If $[R] \notin C_o$,$[N] \in C_o$, the procedure can be continued inductively to obtain a decomposition $\varphi = \sum \gamma_i \cdot \xi_i$, where $\gamma_i: M_1^{(i)} \longrightarrow \dots \longrightarrow M_{c+1}^{(i)}$ is a composite of c irreducible maps, $c \in \mathbb{N}$ arbitrary. Since $[M_i] \in C_o$ for all i, lemma 1.9. implies $\mathrm{im}\,\varphi \subset m_R \cdot N$ if we choose $c \geq \rho(b)$, contradiction.

case b): Let $[R] \in C_o$, $[N] \notin C_o$; apply similarly 1.8. and 1.9. to obtain a contradiction.

Consequently, $C = \Gamma$. Further, by a similar reasoning, all $[N] \in \Gamma_o$ are such that there is a chain

$$R \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_c$$

of irreducible maps with $c \leq \mu(b)$. Since Γ is locally finite, this implies Γ_o is finite.

1.11. ***Proposition*** : Let C be the connected component of [R] and C^s be the subquiver, obtained by removing [R]. If C' is a connected component of C^s, then C' is finite or μ is unbounded on C'.

Proof: Let b be an upper bound for μ on C' and assume C' is infinite. Choose $[N] \in C'$ such that there is no sequence $R \longrightarrow N_1 \longrightarrow \dots \longrightarrow N_c = N$ of irreducible maps with $c \leq \rho(b)$ (if there is one for each N, C' is finite as in the preceeding proof). Now there is a map $\varphi: R \longrightarrow N'$, $\mathrm{im}\,\varphi \not\subset m_R \cdot N$, and this is in contradiction with 1.9. and the above factorization property.

2. MCM - *modules over nonsimple singularities*

The aim of this section is to prove the following

2.1. ***Proposition*** : Suppose that R is not a simple singularity. Then there exists a positive integer d such that there are infinitely many isomorphism classes of indecomposable MCM-R-modules of rank $\leq$ d.

By [PR], (3.10) we may suppose R is complete (from now on). For the proof we need some preparations. The following two lemmas can be deduced from [BGS] (1.7.) and (2.5.) 3), resp. [BGS] (3.1.).

2.2. ***Lemma*** : If R is a complete domain and there are infinitely many different ideals I in

$k[[X]]$, $X = (X_1,\dots,X_n)$ such that

(i) $f \in I^2$,

(ii) I is generated by r elements,

then there are infinitely many isomorphism classes of indecomposable MCM-R-modules of rank $\le 2^{r-1}$.

2.3. *Lemma* : Proposition 2.1. is true if R is a complete domain and either

a) $f \in (X^4)$ and $n \ge 1$, or

b) $f \in (X^3)$ and $n \ge 2$.

2.4. *Lemma* : Our proposition holds for n=1.

Proof : Suppose that R is a (complete) domain. Then we follow [BGS] (3.5.). Note that in this case there are infinitely many different ideals $I \subset k[[X]]$ generated by 4 elements such that $f \in I^2$ and we are ready by (2.2.). If R is not a domain, we have R reduced because it is an isolated singularity. Let g be a prime divisor of f. Then as above, our proposition holds for $R' = k[[X]]/(g)$. As MCM-R' -modules are still MCM-R-modules, we are ready (here, for reduced rings we put $\operatorname{rank}_R M = \sup\{\operatorname{rank}_{R/\mathfrak{q}}(M/\mathfrak{q}M) \mid \mathfrak{q} \in \operatorname{Min}(R)\}$).

2.5. *Lemma* : Our Proposition holds if $f = X_0^2 + h$, $h \in (X)^3$, $n \ge 3$.

Proof : We follow [GK] (3.6.) ; let

$$C = \{ \lambda = (0,\lambda_1,\dots,\lambda_n) \in \mathbb{P}^n_k \mid h^{(3)}(\lambda) = 0 \},$$

where $h^{(3)}$ is the 3-form of h. Clearly, C contains infinitely many points because k is algebraically closed.

For $\lambda \in C$ let $I(\lambda) \subset k[[X]]$ be the ideal generated by all linear polynomials $\lambda_i X_j - \lambda_j X_i$, where $0 \le i,j \le n$, and $I_\lambda = I(\lambda) + (X^2)$. As in [GK] (3.6.) we have $I_\lambda \ne I_\mu$ for $\lambda \ne \mu$ and $f \in I_\lambda^2$. Note, that all I_λ are generated by (n^2+n) elements and so we are ready by lemma 2.2. (R is a domain because $n \ge 2$ and f is an isolated singularity).

2.6. *Lemma* : Our proposition holds if char $k \ne 2$.

Proof : (after [BGS] (3.6.)) Apply induction on n, the case n=1 already being done in 2.4.; we suppose $n \ge 2$. By 2.3.b) we can assume mult $f = 2$. Then, changing variables, we may suppose that $f = X_n^2 + g$ for $g \in P_{n-1} := k[[X_0,\dots,X_{n-1}]]$.Clearly, g defines an isolated singularity which is not simple, because f does so. By induction hypothesis we can suppose that $R' := P_{n-1}/(g) \cong R/(X_n)$ satisfies our Proposition. Thus there exists an infinite set $\{M'_j \mid j \in J\}$ of indecomposable nonisomorphic MCM-R'-modules of rank smaller than a positive integer $d' \ge 1$. Let $\Omega_R(M'_j)$ (resp. $\Omega_{R'}(M'_j)$) be the first syzygy of M'_j as an R-module (resp. R' - module), i.e. there exists an exact sequence

1) $$0 \longrightarrow \Omega_R(M'_j) \longrightarrow G_j \longrightarrow M' \longrightarrow 0$$

(resp. $$0 \longrightarrow \Omega_{R'}(M'_j) \longrightarrow G'_j \longrightarrow M'_j \longrightarrow 0 \;)$$

of R-modules (resp. R'-modules) with G_j (resp. G'_j) free of rank $\mu_{R'}(M'_j)$. Since M'_j as an R-module is killed by X_n we obtain from 1):

2) $\qquad \operatorname{rank}_R(\Omega_R(M'_j) = \mu_{R'}(M'_j)$.

By the depth lemma (cf. [EG] (1.1.)) we note that $\Omega_R(M'_j)$ are MCM- R-modules. Now we have an infinite set $\mathcal{M}$ of nonisomorphic indecomposable MCM-R-modules, summands of a certain $\Omega_R(M'_j)$, $j \in J$ (J infinite). Indeed, otherwise all M_j' will be direct summands in a certain MCM-module of the form $\oplus M/X_n(\oplus M)$, the sums taken over $M \in \mathcal{M}$, because

3) $\qquad \Omega_R(M'_j) / X_n\Omega_R(M'_j) \cong M'_j \oplus \Omega_{R'}(M'_j)$

by [Sc] (3.2)a), contradiction! But we have $\operatorname{rank}_R(\Omega_R(M'_j) = \mu_{R'}(M'_j) \leq e_{R'}(M'_j) = e(R')\cdot d' =: d$ for all $j \in J$ by [Ma] (14.8.), where $e_{R'}(M')$ denotes the multiplicity of M'. Thus the modules from $\mathcal{M}$ have their rank bounded by d; note that there is a system of parameters x such that $e_{R'}(M'_j) = e_R((x),M'_j) = \Sigma(-1)^i \operatorname{length}(H_i(x,M'_j)) = \operatorname{length}(M'_j/xM'_j) \geq \mu(M'_j)$ ([Ma] (14.14), (14.13)).

2.7. *Remark* : The assertion 3) in the above proof can be found directly from [Kn] (2.5.),(ii) as follows: Let MF(f) (resp. MF(G)) be the category of matrix factorizations of f (resp. g) over k[[X]] (resp. P_{n-1}) and CM(R) the category of all MCM-R-modules. Let

$$G:\ MF(g) \longrightarrow MF(f)$$

be the functor $\qquad (\varphi,\psi) \longmapsto (\Theta,\Theta) \quad , \quad \Theta := \begin{pmatrix} X_n\cdot id & \psi \\ \varphi & -X_n\cdot id \end{pmatrix}$

(cf. [Kn] p. 156),

$$\text{Rest}:\ MF(f) \longrightarrow MF(g)$$

the functor $\qquad (\Phi,\Psi) \longmapsto (P_{n-1} \otimes_{k[[X]]} \Phi, P_{n-1} \otimes_{k[[X]]} \Psi)$

and $\qquad \operatorname{Cok}_R:\ MF(f) \longrightarrow CM(R) \qquad$ be the functor given by

$$(\varphi,\psi) \longmapsto \operatorname{Coker} \varphi\ .$$

By [Kn] (2.5.) ii) we have: Rest $\cdot$ G is equivalent with $id \oplus T_{P_{n-1}}$, where

$$T_{P_{n-1}}:\ MF(g) \longrightarrow MF(g) \qquad \text{is given by}$$

$$(\varphi,\psi) \longmapsto (\psi,\varphi)\ .$$

But $\operatorname{Cok}_{R'}\cdot T_{P_{n-1}}$ is in fact given by $(\varphi,\psi) \mapsto \Omega_{R'}(M)$, where $M := \operatorname{coker} \varphi$ and the map

$\operatorname{Cok}_{R'} \cdot$ Rest $\cdot$ G is given by $(\varphi,\psi) \mapsto \Omega_R(M)/X_n\Omega_R(M)$. Thus 3) holds.

2.8. *Proof of proposition* 2.1.: It is enough to consider the case $n \geq 2$ and char $k = 2$ by 2.4. and 2.6.; using 2.3. we may suppose that mult $f = 2$. Changing variables we may suppose by the splitting lemma in characteristic 2 (cf. [GK] (3.5), corollary 3)) that either

1) $\qquad f = X_o^2 + X_1X_2 + \ldots + X_{2l-1}X_{2l} + h\ ,\quad$ where $h \in k[[X_o, X_{2l+1}, \ldots, X_n]]$, $h \in (X)^3$ with $0 \leq 2l \leq n$, or

2) $f = X_oX_1 + \ldots + X_{2l}X_{2l+1} + h$, where $h \in k[[X_{2l+2},\ldots,X_n]]$, $h \in (X)^3$ with $1 \le 2l+1 \le n$.

If $n = 2$ then R is not a rational double point, and we can apply e.g. [PR] (4.10).

Now apply induction on n and suppose $n \ge 3$. Using 2.5. we reduce to the case when (after permutation of variables) f is of the form $f = X_nX_{n-1}+g$, $g \in P_{n-2} := k[[X_o,\ldots,X_{n-2}]]$. We apply [So] Proposition 4 i) (compare with 2.7.): Put $R'' := P_{n-2}/(g) \cong R/(X_{n-1},X_n)R$.

$$F : \quad MF(g) \longrightarrow MF(f)$$

$$(\varphi,\psi) \longmapsto \left(\begin{pmatrix} X_n\cdot id & \varphi \\ \psi & -X_{n-1}\cdot id\end{pmatrix},\begin{pmatrix} X_{n-1}\cdot id & \varphi \\ \psi & -X_n\cdot id\end{pmatrix}\right)$$

and Rest : $MF(f) \longrightarrow MF(g)$

the functor $(\Phi,\Psi) \longmapsto (P_{n-2}\otimes_{k[[X]]}\Phi \, , \, P_{n-2}\otimes_{k[[X]]}\Psi)$.

By [So],proposition 4i) we have $\mathrm{Rest}\cdot F = id \oplus T_{P_{n-2}}$, and thus

3) $$\mathrm{Cok}_R F(\varphi,\psi)/(X_n,X_{n-1})\mathrm{Cok}_R F(\varphi,\psi) \cong \mathrm{Cok}_{R''}\cdot \mathrm{Rest}\cdot F(\varphi,\psi) \cong$$
$$\cong \mathrm{Coker}_{R''}\varphi \oplus \Omega_{R''}(\mathrm{Coker}_{R''}\varphi).$$

By induction hypothesis we can suppose that R" satisfies our Proposition, i.e. there exists an infinite set $\{M_j'' \mid j \in J\}$ of nonisomorphic indecomposable MCM-R"-modules of rank smaller than a positive integer $d'' \ge 2$. By [Ei], Cok_R gives an equivalence of categories, and given $M_j'' \in CM(R'')$, we can find φ_j from a minimal resolution of M_j'' over $k[[X]]$, and the rank of the adress of φ_j is $\mu_{R''}(M_j'')$. Denote $N_j := \mathrm{Coker}_R\, F(\varphi_j,\psi_j)$. By 3) we have

4) $$N_j/(X_n,X_{n-1})N_j \cong M''_j \oplus \Omega_{R''}(M_j'') \quad , \quad j \in J.$$

Then there exists an infinite set $\mathcal{M}$ of nonisomorphic indecomposable MCM-R-modules which are direct summands in a certain N_j. Indeed, otherwise by 4) all M_j'' will be direct summands in a certain MCM-R"-module of the type

$$\bigoplus_{N\in\mathcal{M}} N/(X_n,X_{n-1})N \, ,$$

contradiction.

Now by construction of F we have $\mu_R(N_j) \le 2\mu_{R''}(M_j'')$. But as in 2.6., we obtain

$$\mu_{R''}(M_j'') \le e(R'')\,\mathrm{rank}(M_j'') \le e(R'')\,d'' .$$

Thus all indecomposable R-modules from $\mathcal{M}$ have their rank bounded by $d := 2e(R'')d''$.

3. *Proof of the theorems*

3.1. *Lemma*: Let $\Gamma = (\Gamma_o, \Gamma_1)$ be a quiver with two maps

$$\tau\colon \Gamma_o \longrightarrow \Gamma_o$$
$$\mu\colon \Gamma_o \longrightarrow \mathbb{N} - \{0\}$$

such that

(o) $(\tau(i))^+ = i^-$ for all i (here j^+ denotes the set of all k such that there exists an arrow $j \dashrightarrow k$ in Γ_1, and j^- denotes the set of all k such that there is an arrow $k \longrightarrow j$ in Γ_1).

(i) τ^2 is the identity.

(ii) μ is subadditive, i.e. $2\mu(i) \geq \sum_{j,\exists j\to i} \mu(j)$ for all i.

(iii) μ is unbounded on every connected component of Γ.

(iv) $\mu(\tau(i)) = \mu(i)$ for all $i \in \Gamma_o$.

Under these assumptions, we have:

(a) $i \to j \in \Gamma_1$ iff there is an arrow $\tau(i) \to \tau(j) \in \Gamma_1$, and for all $i \to j$, there are arrows

$$\begin{tikzcd} i \arrow[r] & j \arrow[d] \\ \tau(j) \arrow[u] & \tau(i) \arrow[l] \end{tikzcd}$$

(b) τ acts on the components of Γ.

(c) Let $\Gamma^\sim = \Gamma/\tau$ be the following quiver: $\Gamma_o^\sim = \Gamma_o \bmod \tau$, and $\Gamma_1^\sim$ denotes the set of arrows $p(i) \longrightarrow p(j)$, where $p\colon \Gamma_o \longrightarrow \Gamma_o^\sim$ is the projection. Then μ is well defined and subadditive on $\Gamma^\sim$, i.e.

$$2\mu(i) \geq \sum_{j,\exists j\to k\in\Gamma_o^\sim} \mu(j) \qquad (\text{ here } k \text{ denotes } p(i)\ ).$$

Proof: (a) and (b) are obvious. Now (c) follows from (ii) and (a).

3.2. *Lemma*: Under the assumptions of lemma 3.1., let Γ be connected and τ the identity map. Then all arrows appear in pairs $i \longleftrightarrow j$, and the underlying undirected graph of Γ is

$$A_\infty: \quad \underset{1}{o} \text{———} \underset{2}{o} \text{———} \underset{3}{o} \text{———} \ldots$$

Further, with these notations, μ is the map

$$\mu(i) = \sum_{j=1}^{i} c_j\,, \qquad c_j \text{ integers with } 1 \leq c_{j+1} \leq c_j \text{ for all } j \geq 1.$$

***Proof*:** By induction, we show:

[*] For all n, there is a subgraph

$$\underset{1}{o} \text{———} \underset{2}{o} \text{———} \ldots \text{———} \underset{n-1}{o} \text{———} \underset{n}{o}$$

of Γ such that

for $1 \leq j \leq n-1$, $d_j = 1$ if $j=1$ and $d_j = 2$ if $j \geq 2$, where $d_j = \#(j^-)$, i.e.

$d_j = \#\{ i\in\Gamma_0 \mid \exists\, i - j \in\Gamma_1\}$, and for $1 \le j \le n$

$\mu(j) = c_1 + \dots + c_j$ with integers c_i, $1 \le i \le n$, such that $1 \le c_i \le c_{i-1}$.

Before proving this, we remark

[**] [monotony]: Let

$$\begin{array}{ccccc} o & — & o & — & o \\ i & & j & & k \end{array}$$

be a subgraph of Γ. Then $\mu(i) \ge \mu(j)$ implies $\mu(j) \ge \mu(k)$.

(Indeed, subadditivity ((ii), lemma 2.1.) implies $2\cdot\mu(j) \ge \sum_{t\in j^-} \mu(t) \ge \mu(i) + \mu(k) \ge \mu(j) + \mu(k)$.)

To prove [*], we choose an element $1 \in \Gamma_0$ with $\mu(1) = \text{minimum } \mu(\Gamma_0)$. Put $\mu(1) = c_1$. Now assume [*] holds for some $n \ge 1$. Then we find a subgraph

$$\begin{array}{cccccccccl} & & & & & & & & & / \; o \;\; i_1 \\ o & — & o & — & \dots & — & o & — & o & — \; o \;\; i_2 \\ 1 & & 2 & & & & n-1 & & n & \setminus \;\; \vdots \\ & & & & & & & & & \quad\; o \;\; i_s \end{array}$$

of Γ with $\qquad s = d_n - 1$ if $n \ge 2$, $\quad s = d_n$ if $n = 1$

and

$c_1, \dots, c_n \in \mathbb{N}-\{0\}$, $\qquad 1 \le c_n \le c_{n-1} \le \dots \le c_1$

such that

$\mu(j) = c_1 + \dots + c_j$, $\qquad 1 \le j \le n$

Subadditivity of μ implies

[***] $\quad 2\mu(n) \ge \delta\cdot\mu(n-1) + \sum_{t=1}^{s} \mu(i_t)$, $\quad$ where $\quad \delta = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n \ge 2 \end{cases}$.

Since μ is not bounded on Γ, [**] implies

[****] $\quad \mu(i_t) \ge \mu(n) + 1 \quad$ for some t.

Now choose any t with that property, and denote $i_t = n+1$. Then $\mu(n+1) = \mu(n) + c_{n+1}$ for some $c_{n+1} \ge 1$. We have to show: $c_n \ge c_{n+1}$ and $s = 1$. Put

$$\Sigma = \sum_{r=1}^{s} \mu(i_r) - \mu(i_t) .$$

Then $\Sigma \ge 0$ with equality iff $s = 1$.

case a) $n = 1$: By [***] and [****] we obtain $2c_1 \ge \Sigma + \mu(2) = \Sigma + c_1 + c_2 \ge c_1 + c_2$, i.e. $c_1 \ge c_2$, and $c_1 = \text{minimum } \mu(\Gamma_0)$ implies $\Sigma = 0$ or $\Sigma \ge c_1$ (impossible, since $c_2 \ge 1$).

case b) $n \ge 2$: We obtain similarly

$$2\mu(n) \ge \mu(n-1) + \mu(n+1) + \Sigma \ge \mu(n-1) + \mu(n+1), \text{ i.e.}$$

$$c_n \ge c_{n+1} \text{ and } \Sigma \le c_n - c_{n+1} \le c_1 - 1, \text{ i.e. } \Sigma = 0.$$

3.3. *Lemma*: Assume Γ is connected and there is an $i \in \Gamma$ with $\tau(i) \neq i$. Then the inverse image of a subquiver

$$\begin{array}{ccc} 0 & \longleftrightarrow & 0 \\ p(i) & & p(j) \end{array}$$

of $\Gamma^{\sim}$ via p has the shape

$$(*) \qquad \begin{tikzcd} i' \arrow[r] & (i+1)' \arrow[dl] \\ i \arrow[r] & (i+1) \arrow[ul] \end{tikzcd}$$

Especially, all τ-orbits have lenght 2, and Γ is the "tube"

$$\begin{tikzcd} 1' \arrow[r] & 2' \arrow[dl] \arrow[dr] & 3' \arrow[l] \arrow[r] & 4' \arrow[dl] & \cdots \\ 1 \arrow[r] & 2 \arrow[ul] \arrow[ur] & 3 \arrow[l] \arrow[r] & 4 \arrow[ul] & \cdots \end{tikzcd}$$

with $\mu(i) = \mu(i') = \sum_{j=1}^{i} c_j \;,\; 1 \leq c_{j+1} \leq c_j$.

Proof: By lemma 3.2., $\Gamma^{\sim}$ is of type A_{∞}. If $i \in \tau(i)$, $i+1 \neq \tau(i+1)$, we obtain easily the configuration (*) by lemma 3.1.(a) (further arrows are excluded by the subadditivity of μ). If τ acts identically on i or i+1, we would obtain one of the "degenerated" subquivers

$$\begin{array}{ccccc} 0 & \longleftrightarrow & 0 & \longleftrightarrow & 0 \\ i+1 & & i = i' & & (i+1)' \end{array}$$

$$\begin{array}{ccccc} 0 & \longleftrightarrow & 0 & \longleftrightarrow & 0 \\ i & & (i+1)' = i+1 & & i' \end{array}$$

$$\begin{array}{ccc} 0 & \longleftrightarrow & 0 \\ i = i' & & i+1 = (i+1)' \end{array}$$

By hypothesis, not all can be of the third type, and continuation of the first two with $p(\Gamma) = A_\infty$ is in contradiction with the subadditivity of μ. Therefore, $\tau(i) \neq i$ for all $i \in \Gamma_o$, and the assertion follows by lemma 3.2.

3.4. *Corollary* : Under the same assumptions as in the introduction, assume that the AR-quiver of R is not finite. Then there is a sequence (n_i) of strictly increasing natural numbers such that for all i, there are infinitely many isomorphic classes of indecomposable maximal Cohen Macaulay modules M with $\mu(M) = n_i$.

Proof : By 2.1. , there is a number $c \in \mathbb{N}-\{0\}$ such that $\mu^{-1}(c) = M \subset \Gamma$ is infinite. Let c be minimal with that property. Put

$$M' = \{ \, i \mid i \in M, \; c = \text{minimum } \mu(\Gamma_i) \, \} ,$$

where Γ_i denotes the connected component of i in Γ. Then $\# M' = \infty$, and each interval $[nc,(n+1)c]$, $n \in \mathbb{N}$, $n \geq 1$, has an infinite inverse image via $\mu\colon \Gamma \longrightarrow \mathbb{N}-\{0\}$ by lemma 3.3.

Thus we obtain theorem II. Theorem I follows if we remove the connected component of the isomorphic class of R from the AR quiver of R and apply 3.3.

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