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LATTICE PROPERTIES IN THE SET OF RESOLVENTS AND SUBORDINATIONS IN EXCESIVE STRUCTURES

N.Boboc and Gh.Bucur

Let $(X,\mathcal{B})$ be a measurable space and let $\mathcal{R}(X)$ (resp. $\mathcal{R}(\mu)$) be the set of all resolvents of kernels on X which are proper (resp.proper and absolutely continuous with respect to a finite measure μ).

We endow $\mathcal{R}(X)$ with the pointwise order relation i.e. $\mathcal{V} \leq \mathcal{W}$ if $V_\alpha \leq W_\alpha$ for any $\alpha > 0$ and we deal with the study of lattice properties of the ordered set $(\mathcal{R}(X), \leq)$. In a very general case ($\mathcal{B}$ countable generated) we show that the set $(\mathcal{R}(X), \leq)$ is a conditionally σ-complete lattice. If, instead of pointwise order relation on $\mathcal{R}(\mu)$, we consider the following order relation.

$$\mathcal{V} \circledleq \mathcal{W} \iff (\mathcal{V} \leq \mathcal{W} \text{ and } \mathcal{E}_{\mathcal{W}} \subset \mathcal{E}_{\mathcal{V}})$$

where $\mathcal{E}_{\mathcal{V}}$ means the set of all $\mathcal{V}$-excessive functions. We show that the set

$$\{\mathcal{W} \in \mathcal{R}(\mu) \,/\, \mathcal{V} \circledleq \mathcal{W}\}$$

is a conditionally complete lattice.

We develop also a theory of perturbation in the set $\mathcal{R}(X)$. If $\mathcal{V} \in \mathcal{R}(X)$ then a proper kernel P on X is called

$\mathcal{V}$-compression operator if $Vf \in \mathcal{E}_{\mathcal{V}}V$ for any positive measurable function such that $Vf<\infty$. For any $\mathcal{V}$-compression operators P there exists an unique resolvent $\mathcal{V}^P$ such that $V^P=(\sum_{n=0}^{\infty} P^n)V$. We have $\mathcal{V} \textcircled{\le} \mathcal{V}^P$ and

$$s \in \mathcal{E}_{\mathcal{V}^P} \iff s \in \mathcal{E}_{\mathcal{V}} \quad \text{and } s - Ps \in \mathcal{E}_{\mathcal{V}}.$$

Moreover, if $\mathcal{V} \in \mathcal{R}(\mu)$ and P is absolutely continuous with respect to μ then for any $\mathcal{W} \in \mathcal{R}(\mu)$ such that

$$\mathcal{V} \textcircled{\le} \mathcal{W} \textcircled{\le} \mathcal{V}^P$$

there exists a $\mathcal{V}$-compression operator Q such that $\mathcal{W}=\mathcal{V}^Q$.

If P,Q are two $\mathcal{V}$-compression operators and $\mathcal{B}$ is generated by $\mathcal{E}_{\mathcal{V}}$ then $\mathcal{V}^P \textcircled{\le} \mathcal{V}^Q$ iff $Qf-Pf \in \mathcal{E}_{\mathcal{V}}$ for any positive measurable function f such that $Qf<\infty$.

1. Subordinations in excessive structures

This part is devoted to the study of perturbations of a given resolvent on a measurable space $(X,\mathcal{B})$ in the sense developed in [2], [3], [4], [5], [6], [7].

In the sequel $(X,\mathcal{B})$ will be a measurable space. We denote by $\mathcal{F}$ (resp. $\mathcal{F}_b$) the set of all positive (resp. positive and bounded) $\mathcal{B}$-measurable functions on X.

If $\mathcal{V}=(V_\alpha)_{\alpha>0}$ is a resolvent of kernels on $(X,\mathcal{B})$ then we denote by $\mathcal{S}_{\mathcal{V}}$ the set of all $\mathcal{V}$-supermedian functions on X and by $\mathcal{E}_{\mathcal{V}}$ the set of all $\mathcal{V}$- excessive functions on X which are finite $\mathcal{V}$ - a.s.

<u>Definition</u>. Let $\mathcal{V}=(V_\alpha)_{\alpha>0}$ be a resolvent of kernels

on a measurable space $(X,\mathcal{B})$. A family $\mathcal{P}=(P_\alpha)_{\alpha>0}$ of kernels on X will be called $\mathcal{V}$-compression if for any $\alpha,\beta\in\mathbb{R}_+$, $\alpha<\beta$ we have

$$P_\alpha V_\beta = P_\beta V_\alpha, \qquad P_\alpha = P_\beta + (\beta-\alpha) V_\alpha P_\beta$$

The kernel $P = P_0 = \sup_{\alpha>0} P_\alpha$ is termed <u>the initial kernel of $\mathcal{P}$</u>. It is easy to see that for any $\alpha>0$ we have

$$PV_\alpha = P_\alpha V, \qquad P = P_\alpha + \alpha V P_\alpha$$

where V is the initial kernel of the resolvent $\mathcal{V}$ ($V = V_0 = \sup_{\alpha>0} V_\alpha$).

For any $\alpha\geq 0$ we denote

$$V_\alpha^{\mathcal{P}} := \sum_{n=0}^{\infty} P_\alpha^n V_\alpha \quad ;$$

The $\mathcal{V}$-compression $\mathcal{P}=(P_\alpha)_{\alpha>0}$ is called <u>bounded</u> (resp. proper) if the kernel P is bounded (resp. proper).

We show that the family $\mathcal{V}^{\mathcal{P}} = (V_\alpha^{\mathcal{P}})_{\alpha>0}$ is a resolvent of kernels on X which satisfies the following relation

$$V_\alpha^{\mathcal{P}} = V_\alpha + P_\alpha V_\alpha^{\mathcal{P}} \qquad (\forall)\ \alpha\geq 0$$

One can see that $V^{\mathcal{P}}$ is the initial kernel of the resolvent $\mathcal{V}^{\mathcal{P}}$.

The resolvent $\mathcal{V}^{\mathcal{P}}$ will be termed <u>the $\mathcal{P}$-perturbation of $\mathcal{V}$</u>.

<u>Remarks</u>. For any real number θ, $\theta>0$ the family

$\theta\mathcal{V} := (\theta V_\alpha)_{\alpha>0}$ is a $\mathcal{V}$-compression and we have

$$V^{\theta\mathcal{V}}_{\alpha+\theta} = V_\alpha \qquad (\forall)\quad \alpha > 0.$$

Indeed, the assertion follows from the relations:

$$\alpha, \beta > 0 \Rightarrow V_\beta(\theta V_\alpha) = V_\alpha(\theta V_\beta)$$

$$0 < \alpha < \beta \Rightarrow \cancel{\theta} V_\alpha = \theta V_\beta + (\beta - \alpha) V_\alpha(\theta V_\beta)$$

$$V^{\theta\mathcal{V}}_{\alpha+\theta} = \sum_{n=0}^{\infty} (\theta V_{\alpha+\theta})^n V_{\alpha+\theta} = \sum_{n=0} \theta^n V^{n+1}_{\alpha+\theta} = V_\alpha$$

2. The family $\mathcal{O} = (P_\alpha)_{\alpha>0}$ where $P =0$ for any $\alpha > 0$ is a $\mathcal{V}$-compression and $\mathcal{V}^{\mathcal{O}} = \mathcal{V}$.

3. If A is a kernel on $(X,\mathcal{B})$ then the family $\mathcal{A} = (P_\alpha)_{\alpha>0}$ where for any $\alpha > 0$, $P_\alpha = V_\alpha A$, is a $\mathcal{V}$-compression. This particular case of $\mathcal{V}$-compression was considered in ([2]) .

4. If P is a bounded kernel on $(X,\mathcal{B})$ such that for any $f \in \mathcal{F}$ the function Pf is $\mathcal{V}$-supermedian, then the family $\mathcal{P} = (P_\alpha)_{\alpha>0}$ where

$$P_\alpha f := (I - \alpha V_\alpha) Pf \qquad (\forall)\ f \in \mathcal{F}_b$$

is a $\mathcal{V}$-compression. This case extends the previous one when the kernel A is bounded since in this case for any $f \in \mathcal{F}$, $Pf = VAf$ and therefore Pf is $\mathcal{V}$-supermedian for any bounded, positive, Borel function f and moreover

$$P_\alpha f = (I - \alpha V_\alpha) Pf = (I - \alpha V_\alpha) VAf = V_\alpha Af.$$

This case was considered in [3]

5. If $\mathcal{P} = (P_\alpha)_{\alpha>0}$ is a $\mathcal{V}$-compression and P is the initial kernel of $\mathcal{P}$ then

$$P = P_\alpha + \alpha V_\alpha P \quad (\forall)\ \alpha > 0$$

and therefore it follows that $Pf \in \mathcal{S}_{\mathcal{V}}$ for any $f \in \mathcal{F}$. If, moreover, P is bounded then we have

$$P_\alpha = (I - \alpha V_\alpha) P \quad (\forall)\ \alpha > 0$$

6. If $\mathcal{V}$ is a bounded resolvent of kernels and P is a proper kernel such that $Pf \in \mathcal{S}_{\mathcal{V}}$ for any positive, Borel function f on X then there exists a $\mathcal{V}$-compression $\mathcal{P} = (P_\alpha)_{\alpha > 0}$ uniquely determined by $P_0 = P$. In this case we put $\mathcal{V}^P$ instead of $\mathcal{V}^{\mathcal{P}}$, and $\mathcal{V}^P$ will be called the <u>P-perturbation</u> of $\mathcal{V}$.

<u>Lemma 1.</u> If $\mathcal{P} = (P_\alpha)_{\alpha > 0}$, $Q = (Q_\alpha)_{\alpha > 0}$ are $\mathcal{V}$-compressions then we have

$$1)\ \alpha < \beta \Rightarrow P_\alpha^n Q_\alpha = P_\beta^n Q_\beta + (\beta - \alpha) \sum_{i+j=n} P_\beta^i V P_\alpha^j Q_\alpha$$

$$2)\ \alpha, \beta > 0 \Rightarrow \sum_{i+j=n} P_\alpha^i V_\alpha P_\beta^j Q_\beta = \sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha$$

for any $n \in N$.

<u>Proof.</u> We prove inductively the stated assertions. For n=0 they follow directly from the definition. Suppose that the relation 1) holds for n. We get

$$P_\alpha^{n+1} Q_\alpha = P_\alpha \left(P_\beta^n Q_\beta + (\beta - \alpha) \sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha\right) =$$

$$= (P_\beta + (\beta - \alpha) V_\alpha P_\beta)\left(P_\beta^n Q_\beta + (\beta - \alpha) \sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha\right) =$$

$$= P_\beta^{n+1} Q_\beta + (\beta - \alpha) V_\alpha P_\beta (P_\alpha^n Q_\alpha) + (\beta - \alpha) \sum_{i+j=n} P_\beta^{i+1} V_\beta P_\alpha^j Q_\alpha .$$

Since $V_\alpha P_\beta = V_\beta P_\alpha$ we deduce

$$P_\alpha^{n+1} Q_\alpha = P_\beta^{n+1} Q_\beta + (\beta-\alpha) \sum_{i+j=n+1} P_\beta^i V_\beta P_\alpha^j Q_\alpha$$

Suppose now that the relation 2) holds for n. We have

$$P_\alpha \Big(\sum_{i+j=n} P_\alpha^i V_\alpha P_\beta^j Q_\beta\Big) + V_\alpha P_\beta^{n+1} Q_\beta = P_\alpha \Big(\sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha\Big) + V_\alpha P_\beta^{n+1} Q_\beta =$$

$$= (P_\beta + (\beta-\alpha) V_\alpha P_\beta)\Big(\sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha\Big) + V_\alpha P_\beta^{n+1} Q_\beta =$$

$$= \sum_{i+j=n} P_\beta^{i+1} V_\beta P_\alpha^i Q_\alpha + V_\alpha P_\beta \Big(P_\beta^n Q_\beta + (\beta-\alpha) \sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha\Big) =$$

$$= \sum_{i+j=n} P_\beta^{i+1} V_\beta P_\alpha^i Q_\alpha + V_\alpha P_\beta P_\alpha^n Q_\alpha =$$

$$= \sum_{i+j=n} P_\beta^{i+1} V_\beta P_\alpha^i Q_\alpha + V_\beta P_\alpha^{n+1} Q_\alpha = \sum_{i+j=n+1} P_\beta^i V_\beta P_\alpha^j Q_\alpha .$$

<u>Definition.</u> If $\mathcal{V} = (V_\alpha)_{\alpha>0}$, $\mathcal{W} = (W_\alpha)_{\alpha>0}$ are two resolvents on X we put $\mathcal{V} \leq \mathcal{W}$ iff

$$V_\alpha \leq W_\alpha \qquad (\forall)\ \alpha > 0$$

and we denote by $\vee$, $\wedge$ the lattice operations in the set $\mathcal{R}(X)$ of all resolvents on X endowed with the above order relation $\leq$.

<u>Theorem 2.</u> If $\mathcal{P} = (P_\alpha)_{\alpha>0}$ is a $\mathcal{V}$-compression then the family $\mathcal{V}^{\mathcal{P}} = (V_\alpha^{\mathcal{P}})_{\alpha>0}$ is the smalest resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ verifying the relation

$$W_\alpha = V_\alpha + P_\alpha W_\alpha \qquad (\forall)\ \alpha > 0$$

<u>Proof.</u> We have

$$V_\alpha^{\mathcal{P}} V_\beta^{\mathcal{P}} = \left(\sum_{n=0}^{\infty} P_\alpha^n\right) V_\alpha \left(\sum_{n=0}^{\infty} P_\beta^n V_\beta\right) = \sum_{n,m=0}^{\infty} P_\alpha^n V_\alpha P_\beta^m V_\beta$$

Using now the above lemma for the $\mathcal{V}$ -[illegible] $\mathcal{P} = (P_\alpha)_{\alpha > 0}$ and $Q = (V_\alpha)_{\alpha > 0}$ we get

$$V_\alpha^{\mathcal{P}} V_\beta^{\mathcal{P}} = V_\beta^{\mathcal{P}} V_\alpha^{\mathcal{P}}$$

Using again lemma 1 we get

$$\alpha < \beta \implies \qquad V_\alpha^{\mathcal{P}} = V_\beta^{\mathcal{P}} + (\beta - \alpha) V_\alpha^{\mathcal{P}} V_\beta^{\mathcal{P}}$$

From the definition of $V_\alpha^{\mathcal{P}}$ we deduce

$$V_\alpha^{\mathcal{P}} = V_\alpha + P_\alpha V_\alpha^{\mathcal{P}}$$

Let now $\mathcal{W} = (W_\alpha)_{\alpha > 0}$ be a resolvent such that

$$W_\alpha = V_\alpha + P_\alpha W_\alpha \qquad (\forall)\ \alpha > 0$$

Obviously we have, inductively,

$$W_\alpha = V_\alpha + P_\alpha V_\alpha + P_\alpha^2 V_\alpha + \ldots + P_\alpha^n V_\alpha + P_\alpha^{n+1} W_\alpha$$

and therefore

$$W_\alpha \geq \left(\sum_{n=0}^{\infty} P_\alpha^n\right) V_\alpha = V_\alpha^{\mathcal{P}} .$$

<u>Definition</u>. A $\mathcal{V}$ -compression $\mathcal{P} = (P_\alpha)_{\alpha > 0}$ is called <u>exact</u> if for any $f \in \mathcal{F}$ we have

$$\lim_{\beta\to\infty} \beta V_\beta P_\alpha f = P_\alpha f \qquad (\forall)\ \alpha>0$$

Remark. If the initial kernel P of $\mathcal{P}$ is proper and for any $f\in\mathcal{F}$ we have

$$\lim_{\beta\to\infty} \beta V_\beta Pf = Pf$$

then $\mathcal{P}$ is an exact $\mathcal{V}$- compression.

2) A $\mathcal{V}$-compression is exact iff for any $\alpha>0$ the sequence $(nV_{n+\alpha}P_\alpha f)_n$ increases to $P_\alpha f$.

3) Suppose that P is a proper kernel and let $\mathcal{V},\mathcal{W}$ be two resolvents to of kernels on X such that $\mathcal{E}_{\mathcal{V}}=\mathcal{E}_{\mathcal{W}}$. Then P is the initial kernel of an exact $\mathcal{V}$-compression iff it is the initial kernel of an exact $\mathcal{W}$-compression.

Definition. Let $\mathcal{V}$ be a resolvent on $(X,\mathcal{B})$. A proper kernel P on $(X,\mathcal{B})$ is called a $\mathcal{V}$ -compression operator if $Pf\in\mathcal{E}_{\mathcal{V}}$ whenever $f\in\mathcal{F}$ and $Pf<\infty$.

Obviously from the above considerations it follows that a proper kernel P on $(X,\mathcal{B})$ is a $\mathcal{V}$-compression operator iff it is the initial kernel of an exact $\mathcal{V}$-compression.

Theorem 3. Let $\mathcal{P}=(P_\alpha)_{\alpha>0}$, $Q=(Q_\alpha)_{\alpha>0}$ be two $\mathcal{V}$-compressions and let $(R_\alpha)_{\alpha>0}$ be the family of kernels on X defined by

$$R_\alpha=\left(\sum_{n=0}^{\infty} P_\alpha^n\right)Q_\alpha$$

If we denote

$$\mathcal{P}+Q := (P_\alpha + Q_\alpha)_{\alpha>0}; \quad Q(\mathcal{P}) := (R_\alpha)_{\alpha>0}$$

then $\mathcal{P}+Q$ is a $\mathcal{V}$-compression, $Q(\mathcal{P})$ is a $\mathcal{V}^{\mathcal{P}}$- compression and we have

$$\mathcal{V}^{(\mathcal{P}+Q)} = (\mathcal{V}^{\mathcal{P}})^{Q(\mathcal{P})}$$

If moreover $\mathcal{P}$ and Q are exact $\mathcal{V}$-compressions then $\mathcal{P}+Q$ (resp. $Q(\mathcal{P})$) is an exact $\mathcal{V}$-compression (resp. $\mathcal{V}^{\mathcal{P}}$) -- compressions.

<u>Proof</u>. One can see immediately that $\mathcal{P}+Q$ is a $\mathcal{V}$-compression. Using the definition of $\mathcal{V}^{\mathcal{P}}$ and $Q(\mathcal{P})$ for any $\alpha, \beta > 0$ we have

$$V_\beta^{\mathcal{P}} R_\alpha = \Big(\sum_{n=0}^{\infty} P_\beta^n V_\beta\Big)\Big(\sum_{m=0}^{\infty} P_\alpha^m Q_\alpha\Big) = \sum_{n=0}^{\infty} \sum_{i+j=n} P_\beta^i V_\beta P_\alpha^j Q_\alpha$$

and from Lemma 1 we get

$$V_\beta^{\mathcal{P}} R_\alpha = V_\alpha^{\mathcal{P}} R_\beta$$

Using again Lemma 1 we deduce, for any $\alpha < \beta$,

$$R_\beta + (\beta-\alpha) V_\alpha^{\mathcal{P}} R_\beta = \sum_{n=0}^{\infty} P_\beta^n Q_\beta + (\beta-\alpha)\Big(\sum_{n=0} P_\alpha^n V_\alpha\Big)\Big(\sum_{m=0} P_\beta^m Q_\beta\Big) =$$

$$= \sum_{n=0}^{\infty} P_\beta^n Q_\beta + (\beta-\alpha)\sum_{n=0}^{\infty}\Big(\sum_{i+j=n} P_\alpha^i V_\alpha P_\beta^i Q_\beta\Big) =$$

$$= \sum_{n=0} \Big(P_\beta^n Q_\beta + (\beta-\alpha)\sum_{i+j=n} P_\alpha^i V_\alpha P_\beta^j Q_\beta\Big) = \sum_{n=0} P_\alpha^n Q_\alpha = R_\alpha$$

Hence the family $Q(\mathcal{P})$ is a $\mathcal{V}^{\mathcal{P}}$-compression.

If we denote $(\mathcal{V}^{\mathcal{P}})^{\mathcal{Q}(\mathcal{P})} = (W_\alpha)_{\alpha > 0}$ we have

$$W_\alpha = \sum_{n=0}^{\infty} R_\alpha^n V_\alpha^{\mathcal{P}} = \Big(\sum_{n=0}^{\infty} R_\alpha^n\Big)\Big(\sum_{m=0}^{\infty} P_\alpha^m\Big) V_\alpha =$$

$$= \Big(\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\infty} P_\alpha^k Q_\alpha\Big)^n\Big)\Big(\sum_{m=0}^{\infty} P_\alpha^m\Big) V_\alpha .$$

To finish the proof it well be sufficient to show that

$$\sum_{n=0}^{\infty} (P_\alpha + Q_\alpha)^n = \Big(\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\infty} P_\alpha^k Q_\alpha\Big)^n\Big)\Big(\sum_{m=0}^{\infty} P_\alpha^m\Big).$$

Obviously for any $n \in N$ we have

$$(P_\alpha + Q_\alpha)^n = \sum_{k_1 + k_2 + \dots + k_{m+1} + m = n} P_\alpha^{k_1} Q_\alpha P_\alpha^{k_2} Q_\alpha + \dots + P_\alpha^{h_m} Q_\alpha P_\alpha^{h_{m+1}}$$

and therefore

$$\sum_{n=0}^{\infty} (P_\alpha + Q_\alpha)^n = \sum_{n=0}^{\infty} \Big(\sum_{k_1 + k_2 + \dots + k_{m+1} + m = n} P_\alpha^{k_1} Q_\alpha \dots P_\alpha^{k_m} Q_\alpha P_\alpha^{k_{m+1}} \Big) =$$

$$= \sum_{m=0}^{\infty} \Big(\sum_{k_1 + k_2 + \dots + k_{m+1} = 0}^{\infty} P_\alpha^{k_1} Q_\alpha P_\alpha^{k_2} Q_\alpha \dots P_\alpha^{k_m} Q_\alpha P_\alpha^{k_{m+1}} \Big) =$$

$$= \sum_{m=0}^{\infty} \Big(\Big(\sum_{k=0}^{\infty} P_\alpha^k Q_\alpha\Big)^m \Big(\sum_{n=0}^{\infty} P_\alpha^n\Big)\Big).$$

If $\mathcal{P}$ and Q are exact $\mathcal{V}$- compressions then obviously $\mathcal{P}+Q$ is an exact $\mathcal{V}$-compression. To finish the proof it is sufficient to remark that for any $\alpha > 0$, any $f \in \mathcal{F}$ and any $k \in N$ we have

$$(nV_{\alpha+n}(Q_\alpha f))_n \uparrow Q_\alpha f,$$

$$(nV_{\alpha+n}(P_\alpha^k Q_\alpha f))_n \uparrow P_\alpha^k Q_\alpha f$$

and therefore

$$(nV_{\alpha+n}(R_\alpha f))_n \uparrow R_\alpha f,$$

$$nV_{\alpha+n}(R_\alpha f) \leq nV^{\mathcal{P}}_{\alpha+n}(R_\alpha f) \leq R_\alpha f,$$

$$(nV^{\mathcal{P}}_{\alpha+n}(R_\alpha f))_n \uparrow R_\alpha f.$$

<u>Definition</u>. If $\mathcal{P}=(P_\alpha)_{\alpha>0}$, $Q=(Q_\alpha)_{\alpha>0}$ are two $\mathcal{V}$-compressions we put $\mathcal{P}\leq Q$ if $P_\alpha \leq Q_\alpha$ for any $\alpha>0$. We denote by $\curlyvee$, $\wedge$ the lattice operations on the set of all $\mathcal{V}$-compressions endowed with the above order relation $\leq$.

<u>Theorem 4.</u> Let $\mathcal{P}_n=(P_\alpha^{(n)})_{\alpha>0}$ be an increasing sequence of $\mathcal{V}$-compressions and let $\mathcal{P}:=(P_\alpha)_{\alpha>0}$ be the family of kernels defined by $P_\alpha=\sup\limits_n P_\alpha^{(n)}$. Then $\mathcal{P}$ is a $\mathcal{V}$-compression, $\mathcal{P}=\bigvee\limits_n \mathcal{P}_n$. Moreover the sequence $(\mathcal{V}^{\mathcal{P}_n})_n$ increases to $\mathcal{V}^{\mathcal{P}}$ and if $\mathcal{P}_n$ is an exact $\mathcal{V}$-compression for any $n\in N$ then $\mathcal{P}$ is also an exact $\mathcal{V}$- compression.

<u>Proof</u>. Let $\alpha, \beta \in \mathbb{R}_+$, $\alpha<\beta$ and $n\in N$. We have

$$V_\beta P_\alpha^{(n)} = V_\alpha P_\beta^{(n)}$$

$$P_\alpha^{(n)} = P_\beta^{(n)} + (\beta-\alpha) V_\alpha P_\beta^{(n)}$$

$$V_\alpha^{\mathcal{P}_n} = \sum_{k=0}^{\infty} (P_\alpha^{(n)})^k V_\alpha$$

and therefore, passing to the limit we get

$$V_\beta P_\alpha = V_\alpha P_\beta$$
$$P_\alpha = P_\beta + (\beta - \alpha) V_\alpha P_\beta$$
$$V_\alpha^{\mathcal{P}} = \sup_n V_\alpha^{\mathcal{P}_n}$$

If $\mathcal{P}_k$ is an exact $\mathcal{V}$- compression for any $k \in N$ then, using the relations,

$$nV_{\alpha+n}(P_\alpha^{(k)} f) \leq nV_{\alpha+n}(P_\alpha f) \leq P_\alpha f$$

we get

$$P_\alpha^{(k)} f \leq \lim_{n\to\infty} nV_{\alpha+n}(P_\alpha f) \leq P_\alpha f \quad ,$$
$$\lim_{n\to\infty} nV_{\alpha+n}(P_\alpha f) = P_\alpha f \quad .$$

Theorem 5. Let $\mathcal{P}$ be a $\mathcal{V}$-compression which is bounded and exact. If $\mathcal{V}$ is a bounded resolvent then there exists a sequence $(A_n)_n$ of bounded kernels on X such that the sequence $(\mathcal{P}_n)_n$, of $\mathcal{V}$-compression $\mathcal{P}_n = (P_\alpha^{(n)})_{\alpha > 0}$ defined by

$$P_\alpha^{(n)} = V_\alpha A_n \qquad (\forall)\ \alpha > 0, \quad n \in N$$

is increasing and $\bigvee_n \mathcal{P}_n = \mathcal{P}$.

Proof. If P is the initial kernel of $\mathcal{P}$ then for any $f \in \mathcal{F}_b$ we put

$$A_n f := n(Pf - nV_n Pf)$$

Since $Pf\in\mathcal{S}_{\mathcal{V}}$ we have $A_n f\geq 0$, $A_n f\leq nPf$, i.e. A_n is a bounded kernel on X. From the relation

$$V_\alpha A_n f = nV_\alpha(Pf-nV_n Pf) = nV_n(Pf-\alpha V_\alpha Pf)$$

we deduce that the sequence $(V_\alpha A_n)_n$ is increasing and we have

$$\lim_{n\to\infty} V_\alpha A_n f = Pf-\alpha V_\alpha Pf = (I-\alpha V_\alpha)Pf = P_\alpha f;\ \sup_n V_\alpha A_n = P_\alpha .$$

Theorem 6. Let $(\mathcal{P}_n)_n$ be a decreasing sequence of $\mathcal{V}$-compressions such that $\mathcal{P}_1$ is bounded and suppose that $\mathcal{V}$ is a bounded resolvent. Then the family $\mathcal{P}=(P_\alpha)_{\alpha>0}$ defined by

$$P_\alpha = \inf_n P_\alpha^{(n)}$$

is a $\mathcal{V}$-compression and it is the greatest lower bound of the sequence $(\mathcal{P}_n)_n$ in the ordered set of all $\mathcal{V}$-compressions. If moreover $\mathcal{V}^{\mathcal{P}_1}$ is a bounded (or only proper) resolvent then

$$\bigwedge_n \mathcal{V}^{\mathcal{P}_n} = \mathcal{V}^{\mathcal{P}} .$$

Proof. For any $\alpha, \beta > 0$ and any $n\in N$ we have

$$V_\alpha P_\beta^{(n)} = V_\beta P_\alpha^{(n)},\quad P_\alpha^{(n)} = P_\beta^{(n)} + (\beta-\alpha)V_\alpha P_\beta^{(n)}$$

we deduce

$$V_\alpha P_\beta = V_\beta P_\alpha,\qquad P_\alpha = P_\beta + (\beta-\alpha)V_\alpha P_\beta$$

From the relations

$$V_\alpha^{\mathcal{P}_n} = \sum_{m=0}^{\infty} (P_\alpha^{(n)})^m V_\alpha$$

It follows that for any $f \in \mathcal{F}_b$, $x \in X$ and $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that

$$\sum_{m \geq n_\varepsilon}' (P_\alpha^{(1)})^m V_\alpha f(x) < \varepsilon$$

Hence

$$\inf_n V_\alpha^{\mathcal{P}_n} f \geq \sum_{m=0}^{\infty} (P_\alpha)^m V_\alpha f = V_\alpha^{\mathcal{P}}(f)$$

and on the other hand

$$\sum_{m=0}^{\infty} P_\alpha^m V_\alpha f(x) \geq \sum_{k=0}^{m}{}' P_\alpha^k V_\alpha f(x) = \inf_n \left(\sum (P_\alpha^{(n)})^k V_\alpha f \right)(x) \geq$$

$$\inf_n (V_\alpha^{\mathcal{P}_n} f(x) - \varepsilon) = \inf_n V_\alpha^{\mathcal{P}_n} (f)\ (x) - = \inf_n (V_\alpha^{\mathcal{P}_n}) f(x) - \varepsilon$$

The number ε being arbitrary we get

$$\sum_{m=0}^{\infty} P_\alpha^m V_\alpha f = \inf_n V_\alpha^{\mathcal{P}_n} f .$$

Definition. The $\mathcal{V}$-compression $\mathcal{P} = (P_\alpha)_{\alpha > 0}$ is called absolutely continuous with respect to $\mathcal{V}$ if $P(f)=0$ whenever $Vf=0$, $f \in \mathcal{F}$.

Theorem 7. Let $\mathcal{P} = (P_\alpha)_{\alpha > 0}$ be a $\mathcal{V}$-compression and let s be an element of $\mathcal{F}$. Then we have the following assertions:

$$(s \in \mathcal{S}_{\mathcal{V}} \text{ and } Ps \underset{\mathcal{S}_{\mathcal{V}}}{\precsim} s) \Rightarrow (\alpha V_\alpha s + P_\alpha s \leq s \ (\forall) \alpha > 0) \Rightarrow s \in \mathcal{S}_{\mathcal{V}-\mathcal{P}} .$$

Proof. We suppose $s\in\mathcal{S}_{\mathcal{V}}$, $Ps\preceq_{\mathcal{S}_{\mathcal{V}}} s$ and let $t\in\mathcal{S}_{\mathcal{V}}$ be such that $Ps+t=s$. Since $\alpha V_\alpha t\leq t$ $(\forall)\alpha>0$ we get

$$\alpha>0\Rightarrow\alpha V_\alpha s=\alpha V_\alpha Ps+\alpha V_\alpha t\leq\alpha V_\alpha Ps+t,$$

$$\alpha>0\Rightarrow P_\alpha s+\alpha V_\alpha s\leq P_\alpha s+\alpha V_\alpha Ps+t=P_\alpha s+\alpha VP_\alpha s+t=$$

$$= Ps+t=s,\quad P_\alpha s+\alpha V_\alpha s\leq s.$$

We suppose now that for any $\alpha>0$ we have

$$\alpha V_\alpha s+P_\alpha s\leq s$$

We want to show that $s\in\mathcal{S}_{\mathcal{V}^{\mathcal{P}}}$. Using the definition of $V_\alpha^{\mathcal{P}}$ we have to prove, inductively, the relation

$$\sum_{k=0}^{n} P_\alpha^k V_\alpha s\leq s \qquad \text{for any } n\in N$$

From hypothesis it follows that it holds for $n=0$. Suppose now that it is true for the natural number n i.e.

$$\sum_{k=0} P_\alpha^k V_\alpha s\leq s$$

and therefore we get

$$\alpha V_\alpha s+P_\alpha\Big(\sum_{k=0}^{n} P_\alpha^k V_\alpha s\Big)\leq\alpha V_\alpha s+P_\alpha s\leq s,$$

$$\sum_{k=0}^{n+1} P_\alpha^k V_\alpha s\leq s.$$

Theorem 8. Let $\mathcal{P}$ be an exact $\mathcal{V}$-compression and $s\in\mathcal{F}$. Then the following relations are equivalent:

a) $s \in \mathcal{E}_{\mathcal{V}^{\mathcal{P}}}$

b) $s \in \mathcal{E}_{\mathcal{V}}$ and $Ps \prec_{\mathcal{E}_{\mathcal{V}}} s$

c) $s \in \mathcal{E}_{\mathcal{V}}$ and $\alpha V_\alpha s + P_\alpha s \leq s \quad (\forall)\ \alpha > 0.$

<u>Proof.</u> The relation b) $\Rightarrow$ c) and c) $\Rightarrow$ a) follow directly from the previous theorem and using the obvious relations

$$\alpha V_\alpha \leq \alpha V_\alpha^{\mathcal{P}} \quad (\forall)\ \alpha > 0$$

a) $\Rightarrow$ b) Let $s \in \mathcal{E}_{\mathcal{V}^{\mathcal{P}}}$. If $\mathcal{V}_\alpha$ is the resolvent $\mathcal{V}_\alpha = (V_{\alpha+\beta})_{\beta > 0}$ we have:

$$\mathcal{E}_{\mathcal{V}} = \bigcap_{\alpha > 0} \mathcal{E}_{\mathcal{V}_\alpha}$$

Since $\mathcal{P}$ is an exact $\mathcal{V}$-compression we get

$$\lim_{n \to \infty} nV_{\alpha+n}(P_\alpha f) = P_\alpha f \qquad (\forall)\ f \in \mathcal{F}$$

If moreover $V_\alpha^{\mathcal{P}} f < \infty$ $\mathcal{V}^{\mathcal{P}}$- a.s. then using the relation

$$V_\alpha^{\mathcal{P}} f = V_\alpha f + P_\alpha (V_\alpha^{\mathcal{P}} f)$$

we deduce that $V_\alpha f$ and $P_\alpha (V_\alpha^{\mathcal{P}} f)$ are $\mathcal{V}_\alpha$- excessive functions and therefore

$$V_\alpha^{\mathcal{P}} f \in \mathcal{E}_{\mathcal{V}_\alpha} \quad \text{and} \quad P_\alpha (V_\alpha^{\mathcal{P}} f) \prec_{\mathcal{E}_{\mathcal{V}_\alpha}} V_\alpha^{\mathcal{P}} f.$$

For any $n \in N$ we denote by f_n the function

$$f_n(x) = \begin{cases} s(x) - nV_{\alpha+n}^{\mathcal{P}} s(x) & \text{if } s(x) < \infty \\ +\infty & \text{if } s(x) = +\infty \end{cases}$$

Obviously we have

$$f_n + nV^{\mathcal{P}}_{\alpha+n} s = s,$$

$$V^{\mathcal{P}}_{\alpha} f_n + nV^{\mathcal{P}}_{\alpha} V^{\mathcal{P}}_{\alpha+n} s = V^{\mathcal{P}}_{\alpha} s$$

$$V^{\mathcal{P}}_{\alpha+n} s + nV^{\mathcal{P}}_{\alpha} V^{\mathcal{P}}_{\alpha+n} s = V^{\mathcal{P}}_{\alpha} s$$

and therefore $V^{\mathcal{P}}_{\alpha} f_n < \infty$ $\vartheta^{\mathcal{P}}$-a.s and

$$V^{\mathcal{P}}_{\alpha} f_n = V^{\mathcal{P}}_{\alpha+n} s$$

From the preceding considerations we deduce

$$V^{\mathcal{P}}_{\alpha} f_n \in \mathcal{E}_{\vartheta_\alpha}, \quad P_\alpha (V^{\mathcal{P}}_{\alpha} f_n) \in \mathcal{E}_{\vartheta_\alpha}, \quad P_\alpha (V^{\mathcal{P}}_{\alpha} f_n) \preceq_{\mathcal{E}_{\vartheta_\alpha}} V^{\mathcal{P}}_{\alpha} f_n$$

Hence

$$nV^{\mathcal{P}}_{\alpha+n} s \in \mathcal{E}_{\vartheta_\alpha}, \quad P_\alpha (nV^{\mathcal{P}}_{\alpha+n} s) \in \mathcal{E}_{\vartheta_\alpha}, \quad P_\alpha (nV^{\mathcal{P}}_{\alpha+n} s) \preceq_{\mathcal{E}_{\vartheta_\alpha}} nV^{\mathcal{P}}_{\alpha+n} s$$

and therefore,

$$s \in \mathcal{E}_{\vartheta_\alpha}, \quad P_\alpha s \in \mathcal{E}_{\vartheta_\alpha}, \quad P_\alpha s \preceq_{\mathcal{E}_{\vartheta_\alpha}} s$$

From the previous considerations we get

$$s \in \mathcal{E}_{\vartheta}, \quad Ps \in \mathcal{E}_{\vartheta}, \quad Ps \preceq_{\mathcal{E}_{\vartheta}} s.$$

<u>Remark</u>. If $\mathcal{P}$ is absolutely continuous with respect to ϑ then for any $s \in \mathcal{S}_\vartheta$ we have $Ps = P(\hat{s})$ where

$$\hat{s} = \lim_{\alpha \to \infty} \alpha V_\alpha s$$

Theorem 9. If $\mathcal{P}$ is an exact $\mathcal{V}$-compression which is absolutely continuous with respect to $\mathcal{V}$ and $s \in \mathcal{F}$ is finite $\mathcal{V}$-a.s. then the following assertions are equivalent

a) $s \in \mathcal{S}_{\mathcal{V}\mathcal{P}}$

b) $s \in \mathcal{S}_{\mathcal{V}}$ and $Ps \precsim_{\mathcal{S}_{\mathcal{V}}} s$

c) $\alpha V_\alpha s + P_\alpha s \leq s \qquad (\forall)\ \alpha > 0$

Proof. Using Theorem 7 it remains only to show that a)$\Rightarrow$b).

First we remark that if u is an excessive function with respect to a resolvent $\mathcal{W} = (W_\alpha)_{\alpha > 0}$ and $v \in \mathcal{F}$ is such that $u \leq v$ on X, $u = v$ $\mathcal{V}$-a.s. then $v \in \mathcal{S}_{\mathcal{W}}$. Let now $s \in \mathcal{S}_{\mathcal{V}\mathcal{P}}$. Obviously the function

$$\hat{s} = \lim \alpha V_\alpha^{\mathcal{P}} s$$

belongs to $\mathcal{E}_{\mathcal{V}\mathcal{P}}$ and therefore by Theorem 8 we get $\hat{s} \in \mathcal{E}_{\mathcal{V}}$, $Ps = P\hat{s} \precsim_{\mathcal{E}_{\mathcal{V}}} s$ so $Ps \precsim_{\mathcal{S}_{\mathcal{V}}} s$.

2. The order relation in the set of resolvents

In this section $(X, \mathcal{B})$ will be a measurable space. We denote by $\mathcal{R}(X)$ the set of all resolvents of kernels on X which are proper and for any finite measure μ on $(X, \mathcal{B})$ we denote by $\mathcal{R}(\mu)$ the set of all resolvents from $\mathcal{R}(X)$ which are absolutely continuous with respect to μ. We remember that in $\mathcal{R}(X)$ was given an order relation $\precsim$ defined by

$$\mathcal{V} \leq \mathcal{W} \iff V_\alpha \leq W_\alpha\ (\forall)\ \alpha > 0$$

where $\mathcal{V} = (V_\alpha)_{\alpha > 0}$, $\mathcal{W} = (W_\alpha)_{\alpha > 0}$. We remember also that if $\mathcal{V} \in \mathcal{R}(X)$

we have denoted by $\mathcal{S}_{\mathcal{V}}$ (resp. $\mathcal{E}_{\mathcal{V}}$) the set of all $\mathcal{V}$-supermedian function (resp. $\mathcal{V}$- excessive functions).

Definition. A family $(V_\alpha)_{\alpha>0}$ of kernels on $(X,\mathcal{B})$ is called sub-resolvent (resp. super-resolvent) if

$$V_\beta \le V_\alpha \le V_\beta + (\beta-\alpha)V_\alpha V_\beta \qquad (\forall)\ \alpha<\beta$$

(resp. $V_\alpha \ge V_\beta + (\beta-\alpha)V_\alpha V_\beta \quad (\forall)\ \alpha<\beta$

Proposition 1. Let $(V_\alpha)_{\alpha>0}$ be a family of kernels such that there exists $s\in\mathcal{F}$, $0<s<\infty$, such that

$$\alpha V_\alpha s \le s \qquad (\forall)\ \alpha>0$$

$$V_\alpha = V_\beta + (\beta-\alpha)V_\alpha V_\beta \quad (\forall)\ \alpha,\beta\in\mathbb{R}_+, \alpha<\beta$$

Then $(V_\alpha)_{\alpha>0}$ is a resolvent family on X i.e.

$$V_\alpha V_\beta = V_\beta V_\alpha \qquad (\forall)\ \alpha,\beta\in\mathbb{R}_+ .$$

Proof. If $\alpha<\beta$ we have, inductively,

$$V_\alpha = V_\beta + (\beta-\alpha)V_\beta^2 + (\beta-\alpha)^2 V_\beta^3 + \dots + (\beta-\alpha)^n V_\beta^{n+1} + (\beta-\alpha)^{n+1} V_\alpha V_\beta^{n+1}$$

Since

$$(\beta-\alpha)^{n+1} V_\beta^{n+1} s \le \left(\frac{\beta-\alpha}{\beta}\right)^{n+1} s$$

we deduce

$$V_\alpha f = V_\beta f + \sum_{i=1} (\beta-\alpha)^i V_\beta^{i+1} f$$

for any $\alpha, \beta, \alpha<\beta$ and any $f \in \mathcal{F}$ such that $f \leq rs$ for a suitable $r \in R$, $r>0$. Hence

$$V_\alpha V_\beta f = V_\beta V_\alpha f, \quad V_\alpha V_\beta = V_\beta V_\alpha \quad (\forall)\ \alpha, \beta \in \mathbb{R}_+$$

From now on, at this point, $\mathcal{V} = (V_\alpha)_{\alpha>0}$ will be a fixed sub-resolvent on $(X, \mathcal{B})$ such that there exists $s \in \mathcal{F}$, $0<s<\infty$, with $\alpha V_\alpha s \leq s$ for any $\alpha > 0$. We shall denote by $\mathcal{S}_\mathcal{V}$ the set

$$\mathcal{S}_\mathcal{V} := \{ t \in \mathcal{F} \mid \alpha V_\alpha t \leq t \quad (\forall)\ \alpha > 0 \}$$

The elements of $\mathcal{S}_\mathcal{V}$ are called <u>$\mathcal{V}$-supermedian functions.</u>

<u>Notation.</u> For any $\alpha > 0$ we denote by d_α the set of all finite subsets $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset R$ with $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. For any $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \in d_\alpha$ we put

$$V_\alpha^\Delta = (I + (\alpha_1 - \alpha_0) V_{\alpha_0})(I + (\alpha_2 - \alpha_1) V_{\alpha_1}) \ldots (I + (\alpha_n - \alpha_{n-1}) V_{\alpha_{n-1}}) V_{\alpha_n}$$

<u>Proposition 2.</u> If $\Delta, \Delta' \in d_\alpha$, $\Delta \subset \Delta'$ we have

$$\alpha V_\alpha^\Delta s \leq s, \quad V_\alpha \leq V_\alpha^\Delta \leq V_\alpha^{\Delta'} \quad (\forall)\ s \in \mathcal{S}_\mathcal{V}, \ (\forall)\ \alpha > 0$$

<u>Proof.</u> Let $s \in \mathcal{S}_\mathcal{V}$ and let $\Delta \in d_\alpha$, $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. Since $(I + (\alpha_{k+1} - \alpha_k) V_{\alpha_k}) s \leq s + \dfrac{\alpha_{k+1} - \alpha_k}{\alpha_k} s = \dfrac{\alpha_{k+1}}{\alpha_k} s$ for any $k = 0, 1, 2, \ldots, n-1$ we deduce

$$V_\alpha^\Delta s \leq \frac{\alpha_1}{\alpha_0} \cdot \frac{\alpha_2}{\alpha_1} \cdots \frac{\alpha_n}{\alpha_{n-1}} \frac{1}{\alpha_n} s = \frac{1}{\alpha_0} s; \quad \alpha V_\alpha^\Delta s \leq s$$

For the second part we may suppose that the set $\Delta' \setminus \Delta$ is a singleton i.e.

$$\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n\}, \quad \Delta' = \{\alpha_0, \alpha_1, \dots, \alpha_k, \beta, \alpha_{k+1}, \dots, \alpha_n\}$$

Since

$$V_{\alpha_k} \le (1+(\beta-\alpha_k)V_{\alpha_k})V_\beta$$

we deduce

$$(\alpha_{k+1}-\beta)V_{\alpha_k} \le (\alpha_{k+1}-\beta)(1+(\beta-\alpha_k)V_{\alpha_k})V_\beta$$

and therefore

$$1+(\alpha_{k+1}-\alpha_k)V_{\alpha_k} = 1+(\beta-\alpha_k)V_{\alpha_k}+(\alpha_{k+1}-\beta)V_{\alpha_k} \le$$

$$\le 1+(\beta-\alpha_k)V_{\alpha_k}+(\alpha_{k+1}-\beta)(1+(\beta-\alpha_k)V_{\alpha_k})V_\beta = (1+(\beta-\alpha_k)V_{\alpha_k})(1+(\alpha_{k+1}-\beta)V_\beta).$$

Hence

$$V_\alpha^{\Delta} \le V_\alpha^{\Delta'}$$

Simmilar proofs for the case

$$\Delta' = \{\alpha_0, \alpha_1, \dots, \alpha_n, \beta\}$$

Proposition 3. For any $f \in \mathcal{F}$ and any $\alpha > 0$ the function

$$x \longrightarrow \sup\left\{ V_\alpha^{\Delta} f(x) \,\middle|\, \Delta \in d_\alpha \right\}$$

is $\mathcal{B}$ - measurable. More precisely we have

$$\sup_{\Delta\in d_\alpha} V f = \sup\left\{ V_\alpha^{\Delta} f \,\middle|\, \Delta\in d_\alpha ,\ \Delta\setminus\{\alpha\}\subset Q \right\}.$$

Proof. Let $\alpha>0$ and let $f\in\mathcal{F}$ be such that there exists $s\in\mathcal{S}_{\mathcal{V}}$, $f\leq s<\infty$. We show that for any $\Delta\in d_\alpha$, $\Delta=\{\alpha_0,\alpha_1,\ldots,\alpha_n\}$ we have

$$V_\alpha^{\Delta} f\leq \sup\left\{ V_\alpha^{\Delta'} f \,\middle|\, \Delta'\setminus\{\alpha\}\subset Q \right\}$$

For any $k\in N$, $k>0$ such that $\frac{1}{k}\leq\alpha_{i+1}-\alpha_i$ we choose $\beta_i^k\in Q\cap(\alpha_i,\alpha_i+\frac{1}{k})$, $i=1,2,\ldots n$. If we denote

$$\Delta^k=\{\beta^k,\beta_1^k,\ldots,\beta_n^k\}\quad\text{where } \beta_o^k=\alpha_o=\alpha$$

we have

$$V_{\beta_i^k}\leq V_{\alpha_i}\leq V_{\beta_i^k}+(\beta_i^k-\alpha_i)V_{\alpha_i}V_{\beta_i^k}$$

and therefore

$$\lim_{k\to\infty} V_{\beta_i^k} f = V_{\alpha_i} f$$

$$\lim_{k\to\infty} V_\alpha^{\Delta^k} f = V_\alpha^{\Delta} f,$$

$$\sup\left\{ V_\alpha^{\Delta'} f \,\middle|\, \Delta'\in d_\alpha ,\ \Delta'-\{\alpha\}\subset Q\right\}\geq V_\alpha^{\Delta} f$$

For an arbitrary $f\in\mathcal{F}$ the assertion follows using the fact that V_α^s is a kernel on $(X,\mathcal{B})$ for any $\alpha\in R_+$ and any $\Delta\in d_\alpha$.

Notation. For any $\alpha>0$ and any $f\in\mathcal{F}$ we put

$$V_\alpha^{\square} f := \sup\{ V_\alpha^{\Delta} f \mid \Delta \in d_\alpha \}$$

Remark. From Propositions 2 and 3 it follows that the map

$$f \to V_\alpha^{\square} f$$

is a proper kernel on $(X,\mathcal{B})$ for any $\alpha > 0$ and

$$\alpha V_\alpha^{\square} s \leq s \qquad (\forall) s \in \mathcal{S}_{\mathcal{V}}$$

Proposition 4. The family $(V_\alpha^{\square})_{\alpha>0}$ of kernels on $(X,\mathcal{B})$ is a sub-resolvent such that

$$V_\alpha \leq V_\alpha^{\square} \qquad (\forall)\alpha > 0$$
$$(I+(\beta-\alpha)V_\alpha)V_\beta^{\square} \leq V_\alpha^{\square} \qquad (\forall)\ \alpha,\beta > 0,\ \alpha < \beta.$$

Moreover, if $(W_\alpha)_{\alpha>0}$ is a sub-resolvent (of kernels) on $(X,\mathcal{B})$ such that

$$V_\alpha \leq W_\alpha \qquad (\forall)\alpha > 0$$
$$(I+(\beta-\alpha)V_\alpha)W_\beta \leq W_\alpha \qquad (\forall)\ \alpha,\beta > 0,\ \alpha < \beta$$

then we have

$$V_\alpha^{\square} \leq W_\alpha \ (\forall)\ \alpha > 0.$$

Proof. For any $\alpha,\beta > 0, \alpha < \beta$ we have

$$V_\alpha \leq V_\beta + (\beta-\alpha)V_\alpha V_\beta \leq V_\alpha^{\square}$$

Let now $\alpha,\beta > 0$, $\alpha < \beta$ and let $\Delta \in d_\alpha$, $\Delta = \{\alpha, \beta, \beta_1, \beta_2, \dots \beta_n\}$

If we denote by Δ' the element of d_β given by

$$\Delta' = \{\beta, \beta_1, \dots, \beta_n\}$$

we have

$$V_\alpha^\Delta = (1+(\beta-\alpha)V_\alpha)V_\beta^{\Delta'}$$

and therefore, $\Delta \in d_\alpha$ being arbitrary,

$$(I+(\beta-\alpha)V_\alpha)V_\beta^\square \leq V_\alpha^\square$$

We suppose that $(W_\alpha)_{\alpha>0}$ is a sub-resolvent on $(X,\mathcal{B})$ such that

$$V_\alpha \leq W_\alpha \quad (\forall)\ \alpha > 0$$

$$(1+(\beta-\alpha)V_\alpha)W_\beta \leq W_\alpha \qquad (\forall)\ \alpha, \beta > 0,\ \alpha < \beta$$

From these relation it follows, inductively, that for any $\Delta \in d_\alpha$, $\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$, $\alpha_0 = \alpha$, $\alpha_n = \beta$; we have

$$V_\alpha^\Delta = (1+(\alpha_1-\alpha_0)V_{\alpha_0})(1+(\alpha_2-\alpha_1)V_{\alpha_1})\dots(1+(\alpha_n-\alpha_{n-1})V_{\alpha_{n-1}})V_{\alpha_n} \leq$$

$$\leq (1+(\alpha_1-\alpha_0)V_{\alpha_0})\dots(1+(\alpha_n-\alpha_{n-1})V_{\alpha_{n-1}})W_{\alpha_n} \leq (1+(\alpha_1-\alpha_0)V_{\alpha_0})\dots(1+(\alpha_{n-1}-\alpha_{n-2})V_{\alpha_{n-2}})W_{\alpha_{n-1}}$$

$$\leq \dots \dots \leq (1+(\alpha_1-\alpha_0)V_{\alpha_0})W_{\alpha_1} \leq W_{\alpha_0},$$

$$V_\alpha^\square \leq W_\alpha \quad (\forall)\ \alpha > 0.$$

We show now that the family $(V_\alpha^\square)_{\alpha>0}$ is a sub-resolvent on $(X,\mathcal{B})$. If $0<\alpha<\beta$ then for any $\Delta \in d_\alpha$, with $\beta \in \Delta$,

$$\Delta = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}, \ldots, \alpha_n\}, \quad \alpha_p = \beta$$

we denote by Δ' the element of d_α given by

$$\Delta' = \{\alpha_p, \alpha_{p+1}, \ldots, \alpha_n\}$$

then we obviously have

$$V_\alpha^\Delta = (1+(\alpha_1-\alpha_0)V_\alpha)\ldots(1+(\alpha_p-\alpha_{p-1})V_{\alpha_{p-1}})V_\beta^{\Delta'}$$

Hence

$$V_\alpha^\Delta \geq V_\beta^{\Delta'}, \quad V_\alpha^\square \geq V_\beta^{\Delta'}, \quad V_\alpha^\square \geq V_\beta^\square$$
$$V_\alpha^\Delta \leq (1+(\alpha_1-\alpha)V_\alpha)\ldots(1+(\alpha_p-\alpha_{p-1})V_{\alpha_{p-1}})V_\beta^\square$$

On the other hand we show inductively that

$$(1+(\alpha_1-\alpha)V_\alpha)(1+(\alpha_2-\alpha_1)V_{\alpha_1})\ldots(1+(\alpha_p-\alpha_{p-1})V_{\alpha_{p-1}})\leq 1+(\alpha_p-\alpha)V_\alpha^\square$$

Indeed, for p=1 the relation follows from the inequality $V_\alpha \leq V_\alpha^\square$. We suppose that the assertion is valid for p=k and let $\alpha_i \in R_+$, $i=0,1,2,\ldots,p+1$, $\alpha=\alpha_0<\alpha_1<\alpha_2 \ldots <\alpha_p<\alpha_{p+1}$.
We have

$$(1+(\alpha_2-\alpha_1)V_{\alpha_1})(1+(\alpha_3-\alpha_2)V_{\alpha_2})\ldots(1+(\alpha_{p+1}-\alpha_p)V_{\alpha_p})\leq 1+(\alpha_{p+1}-\alpha_1)V_\alpha^\square$$

and therefore, using the relation

$$(1+(\alpha_1-\alpha)V_\alpha)V_{\alpha_1}^\square \leq V_\alpha^\square$$

we get

$$(I+(\alpha_1-\alpha)V_\alpha)(I+(\alpha_2-\alpha_1)V_{\alpha_1})\dots(I+(\alpha_{p+1}-\alpha_p)V_{\alpha_p}) \le$$

$$(I+(\alpha_1-\alpha)V_\alpha)(I+(\alpha_{p+1}-\alpha_1)V^{\Box}_{\alpha_1}) =$$

$$= I+(\alpha_1-\alpha)V_\alpha+(\alpha_{p+1}-\alpha_1)(I+(\alpha_1-\alpha)V_\alpha)V^{\Box}_{\alpha_1} \le I+(\alpha_1-\alpha)V_\alpha+$$

$$+(\alpha_{p+1}-\alpha_1)V^{\Box}_\alpha \le I+(\alpha_1-\alpha)V^{\Box}_\alpha+(\alpha_{p+1}-\alpha_1)V^{\Box}_\alpha = I+(\alpha_{p+1}-\alpha)V^{\Box}_\alpha$$

<u>Notation.</u> We define inductively the families $(\overset{(n)}{V}_\alpha)_{\alpha>0}$ of sub-resolvents on $(X,\mathcal{B})$ by

$$\overset{(1)}{V}_\alpha = V^{\Box}_\alpha$$
$$\overset{(n+1)}{V}_\alpha = \overset{(n)}{V}{}^{\Box}_\alpha$$

and we put for any $\alpha > 0$,

$$\tilde{V}_\alpha = \sup_n \overset{(n)}{V}_\alpha$$

<u>Theorem 5.</u> The family by $\tilde{\mathcal{V}} = (\tilde{V}_\alpha)_{\alpha>0}$ is a resolvent on $(X,\mathcal{B})$ such that

$$V_\alpha \le \tilde{V}_\alpha \quad (\forall)\,\alpha > 0$$

and such that $\mathcal{S}_{\tilde{\mathcal{V}}} = \mathcal{S}_{\mathcal{V}}$

Moreover, for any resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ of kernels on $(X,\mathcal{B})$ such that

$$V_\alpha \le W_\alpha \qquad (\forall) \quad \alpha > 0$$

we have

$$\tilde{V}_\alpha \le W_\alpha$$

Proof. Obviously for any $\alpha > 0$ the sequence $(V_\alpha^{(n)})_n$ is increasing and

$$\alpha V_\alpha^{(n)} s \leq s \qquad (\forall)\ s \in \mathcal{S}_{\mathcal{V}}$$

Hence $s \in \mathcal{S}_{\tilde{\mathcal{V}}}$. The relation $\mathcal{S}_{\tilde{\mathcal{V}}} \subset \mathcal{S}_{\mathcal{V}}$ is obvious.

If $\alpha < \beta$ we have, using the Proposition 4

$$(1+(\beta-\alpha)V_\alpha^{(n)})V_\beta^{(n)} \leq (1+(\beta-\alpha)V_\alpha^{(n)})V_\beta^{(n+1)} \leq V_\alpha^{(n+1)} \leq (1+(\beta-\alpha)V_\alpha^{(n+1)})V_\beta^{(n+1)}$$

and therefore

$$(1+(\beta-\alpha)\tilde{V}_\alpha)\tilde{V}_\beta = \tilde{V}_\alpha$$

If $\mathcal{W} = (W_\alpha)_{\alpha>0}$ is a resolvent on $(X, \mathcal{B})$ with $V_\alpha \leq W_\alpha$ for any $\alpha > 0$ then we have by Proposition 4.

$$V_\alpha^{0} \leq W_\alpha^{0} = W_\alpha \qquad (\forall)\ \alpha > 0$$
$$V_\alpha^{(n)} \leq W_\alpha\,, \quad \tilde{V}_\alpha \leq W_\alpha \qquad (\forall)\ \alpha > 0.$$

From now on, at this point, $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ will be a fixed superresolvent on $(X, \mathcal{B})$ such that (V_α) is bounded for $\alpha > 0$.

Notation. For any $\alpha \in R$, $\alpha > 0$ and any $\Delta \in d_\alpha$

$$\Delta = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n\}\qquad \alpha = \alpha_0$$

we denote

$$V_\alpha^{\Delta} = (1+(\alpha_1-\alpha_0)V_{\alpha_0})(1+(\alpha_2-\alpha_1)V_{\alpha_1})\dots(1+(\alpha_n-\alpha_{n-1})V_{\alpha_{n-1}})V_{\alpha_n}$$

Proposition 6. If $\Delta, \Delta' \in d_\alpha$, $\Delta \subset \Delta'$ we have

$$V_\alpha \geq V_\alpha^\Delta \geq V_\alpha^{\Delta'}$$

Proof. For the inequality $V_\alpha^{\Delta'} \leq V_\alpha^\Delta$ it will be sufficient to suppose that the set $\Delta' \setminus \Delta$ is a singleton i. e.

$$\Delta = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n\}, \Delta' = \{\alpha_0, \alpha_1, \dots, \alpha_k, \beta, \alpha_{k+1}, \dots, \alpha_n\}$$

In this case to show the inequality $V_\alpha^{\Delta'} \leq V_\alpha^\Delta$ is equivalent to show the inequality

$$(1+(\alpha_{k+1}-\alpha_k)V_{\alpha_k}) \geq (1+(\beta-\alpha_k)V_{\alpha_k})(1+(\alpha_{k+1}-\beta)V_\beta)$$

which may be drown from the fact that $\mathcal{V}$ is a superresolvent. A similar proof for the case where $\beta > \alpha_n$.

The relation $V_\alpha^\Delta \leq V_\alpha$ follows from the above relations and from the inequality

$$(1+(\alpha_1-\alpha_0)V_{\alpha_1})V_{\alpha_1} \leq V_{\alpha_1} \leq V_{\alpha_0} \quad (v) \qquad \alpha_1 > \alpha$$

Proposition 7. For any $f \in \mathcal{F}_b$ and any $\alpha > 0$ the function

$$\inf_{\Delta \in d_\alpha} V_\alpha^\Delta f$$

is $\mathcal{B}$-measurable. More precisely we have

$$\inf_{\Delta \in d_\alpha} V_\alpha^\Delta f = \inf\{V_\alpha^\Delta f \mid \Delta \in d_\alpha, \Delta \setminus \{\alpha\} \subset \mathbb{Q}\}$$

Proof. Let $\alpha > 0$, $f \in \mathcal{F}_b$. We show that for any $\Delta' \in d_\alpha$ we have

$$V_\alpha^{\Delta'} f \geq \inf \{ V_\alpha^{\Delta} f \mid \Delta \in d_\alpha, \ \Delta \setminus \{\alpha\} \subset Q \}$$

Let $\Delta' = \{\alpha'_0, \alpha'_1, \alpha'_2, \dots, \alpha'_n\}$ be such that $\alpha'_0 = \alpha$. For any $k \in N$, $k \neq 0$ such that

$$\frac{1}{k} \leq \min\{\alpha'_{i+1} - \alpha'_i \ , \ i < n\}$$

we choose the numbers $\beta_i^k \in Q$ such that

$$\alpha'_i < \beta^k_i < \alpha'_i + \frac{1}{k}$$

and we denote

$$\Delta^k = \{\alpha, \beta^k_1, \beta^k_2, \dots, \beta^k_n\}$$

Form the inequalities

$$V_{\alpha'_i} \geq V_{\beta^k_i} \quad (\forall)\ i,\ 1 \leq i \leq n, \quad (\forall)\ k \in N$$

and from the above considerations we deduce

$$I + (\alpha'_{i+1} - \alpha'_i) V_{\alpha'_i} \geq I + (\alpha'_{i+1} - \alpha'_i) V_{\beta^k_i} \ ,$$

$$I + (\alpha'_{i+1} - \alpha'_i) V_{\alpha'_i} \geq \lim_{k \to \infty} I + (\beta^k_{i+1} - \beta^k_i) V_{\beta^k_i}$$

and therefore

$$V_\alpha^{\Delta} f \geq \lim_{k \to \infty} V_\alpha^{\Delta^k} f \geq \inf \{ V_\alpha^{\Delta} f \mid \Delta \in d_\alpha, \Delta \setminus \{\alpha\} \subset Q \}$$

Notation. For any $\alpha \in R$, $\alpha > 0$ and any $f \in \mathcal{F}_b$ we put

$$V_\alpha^\square f := \inf\{V_\alpha^\Delta f \mid \Delta \in d_\alpha\}$$

Remark. From the above considerations we deduce that the map

$$f \to V_\alpha^\square f$$

is a kernel on $(X, \mathcal{B})$ for any $\alpha > 0$.

Proposition 8. The family $(V_\alpha^\square)_{\alpha>0}$ of kernels on $(X, \mathcal{B})$ is a superresolvent such that

$$V_\alpha \geq V_\alpha^\square \qquad (\forall)\ \alpha > 0$$

$$(1+(\beta-\alpha)V_\alpha)V_\beta^\square \geq V_\alpha^\square \qquad (\forall)\ \alpha, \beta > 0,\ \alpha < \beta$$

Moreover, if $(W_\alpha)_{\alpha>0}$ is a superresolvent on $(X, \mathcal{B})$ such that

$$W_\alpha \leq V_\alpha \qquad (\forall)\ \alpha > 0$$

$$(1+(\beta-\alpha)V_\alpha)W_\beta \geq W_\alpha \qquad (\forall)\ \alpha, \beta > 0,\ \alpha < \beta$$

then we have

$$W_\alpha \leq V_\alpha^\square \qquad (\forall)\ \alpha > 0$$

Proof. The assertion follows using Proposition 6 and similar arguments as in the proof of Proposition 4.

Notation. For any superresolvent $\mathcal{V}=(V_\alpha)_{\alpha>0}$ on $(X,\mathcal{B})$ with V_α bounded for any $\alpha>0$ we define inductively $(\overset{(n)}{V_\alpha})_{\alpha>0}$ by

$$\overset{(1)}{V_\alpha} = V_\alpha^{\square}$$

$$\overset{(n+1)}{V_\alpha} = \overset{(n)}{\mathcal{V}}^{\square}$$

$$\widetilde{V}_\alpha = \inf_n \overset{(n)}{V_\alpha}$$

Theorem 9. The family $(\widetilde{V}_\alpha)_{\alpha>0}$ is a resolvent family of kernels on $(X,\mathcal{B})$ such that

$$\widetilde{V}_\alpha \leq V_\alpha \quad (\forall)\ \alpha>0$$

Moreover, for any resolvent $\mathcal{W}=(W_\alpha)_{\alpha>0}$ of kernels on $(X,\mathcal{B})$ with $W_\alpha \leq V_\alpha$, for any $\alpha>0$, we have

$$W_\alpha \leq \widetilde{V}_\alpha \quad (\forall)\ \alpha>0.$$

Proof. The sequence $(\overset{(n)}{V_\alpha})_n$ is decreasing and for any $\alpha, \beta>0$, $\alpha<\beta$, we have from the above considerations

$$(I+(\beta-\alpha)\overset{(n)}{V_\alpha})\overset{(n)}{V_\beta} \geq (I+(\beta-\alpha)\overset{(n)}{V_\alpha})\overset{(n+1)}{V_\beta} \geq \overset{(n+1)}{V_\alpha} \geq (I+(\beta-\alpha)\overset{(n+1)}{V_\alpha})\overset{(n+1)}{V_\beta}$$

Hence

$$(I+(\beta-\alpha)\widetilde{V}_\alpha)\widetilde{V}_\beta = \lim_{n\to\infty}(I+(\beta-\alpha)\overset{(n)}{V_\alpha})\overset{(n)}{V_\beta} \geq$$

$$\lim_{n\to\infty}\overset{(n+1)}{V_\alpha} = \widetilde{V}_\alpha \geq \lim_{n\to\infty}(I+(\beta-\alpha)\overset{(n+1)}{V_\alpha})\overset{(n+1)}{V_\beta} = (I+(\beta-\alpha)\widetilde{V}_\alpha)\widetilde{V}_\beta$$

and therefore the family $(\widetilde{V}_\alpha)_{\alpha>0}$ is a resolvent family of kernels on X such that

$$V_\alpha \geq \overset{(n)}{V}_\alpha \geq \widetilde{V}_\alpha \qquad (\forall)\ \alpha > 0\ , \qquad (\forall)\ n \in N .$$

$\mathrm{Inf}\ (W_\alpha)_{\alpha>0}$ is a resolvent on $(X,\mathcal{B})$ such that

$$W_\alpha \leq V_\alpha \qquad (\forall)\ \alpha > 0$$

then

$$W_\alpha = W_\alpha^\square \leq V_\alpha^\square \qquad (\forall)\ \alpha > 0$$

and therefore

$$\overset{(n)}{W}_\alpha = W_\alpha \leq \overset{(n)}{V}_\alpha \qquad (\forall)\ \alpha > 0,$$

$$W_\alpha \leq \widetilde{V}_\alpha \qquad (\forall)\ \alpha > 0.$$

<u>Remark</u>. Theorem 9 is also valid if we suppose that, for any $\alpha > 0$, the kernel V_α is proper.

<u>Proposition 10</u>. Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$, $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be two resolvents from $\mathcal{R}(X)$ such that for any $\alpha > 0$ there exists the kernel

$$V_\alpha \wedge W_\alpha \quad [illegible]$$

Then there exists $\mathcal{V} \wedge \mathcal{W}$. [illegible].

<u>Proof</u>. Obviously, the family $(V_\alpha \wedge W_\alpha)_{\alpha>0}$ is a super-resolvent on $(X,\mathcal{B})$. Using Theorem 9 there exists a resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(X,\mathcal{B})$ such that

$$U_\alpha \leq V_\alpha \wedge W_\alpha \quad (\forall)\ \alpha > 0$$

and such that for any other resolvent $\mathcal{U}'=(U'_\alpha)_\alpha$ for which

$$U'_\alpha \leq V_\alpha \wedge W_\alpha \qquad (\forall)\ \alpha > 0$$

we have

$$U'_\alpha \leq U_\alpha \qquad (\forall)\ \alpha > 0.$$

Hence, from the above considerations the resolvent $\mathcal{U}$ is the greatest lower bound of the set $\{\mathcal{U}, \mathcal{W}\}$ in the ordered set $(\mathcal{R}(X), \leq)$.

Proposition 11. Let $\mathcal{V}=(V_\alpha)_{\alpha>0}$, $\mathcal{W}=(W_\alpha)_{\alpha>0}$ be two resolvents from $\mathcal{R}(X)$ such that for any $\alpha > 0$ there exists the kernel

$$V_\alpha \vee W_\alpha$$

and such that there exists $\mathcal{U} \in \mathcal{R}(X)$ with $\mathcal{V} \leq \mathcal{U}$, $\mathcal{W} \leq \mathcal{U}$. Then there exists $\mathcal{V} \vee \mathcal{W}$

Proof. Obviously, the family, $(V_\alpha \vee W_\alpha)_{\alpha>0}$ is a subresolvent on $(X, \mathcal{B})$. Since $\mathcal{U}=(U_\alpha)_{\alpha>0}$ is such that

$$V_\alpha \vee W_\alpha \leq U_\alpha \qquad (\forall)\ \alpha > 0$$

and since $\mathcal{U} \in \mathcal{R}(X)$, there exists $s \in \mathcal{S}_{\mathcal{U}}$, $s < \infty$ on X and $s > 0$ $\mathcal{U}$ - a.s. and we have

$$\alpha (V_\alpha \vee W_\alpha)(s) \leq s \qquad (\forall)\ \alpha > 0.$$

Using Theorem 5 , we deduce the existence of a resolvent $\mathcal{U}'=(U'_\alpha)_\alpha$ on $(X,\mathcal{B})$ such that

$$V_\alpha \vee W_\alpha \le U'_\alpha \quad (\forall)\ \alpha > 0$$

and such that for any other resolvent $\mathcal{U}''=(U''_\alpha)_{\alpha>0}$ on $(X,\mathcal{B})$ for which

$$V_\alpha \vee W_\alpha \le U''_\alpha \quad (\forall)\ \alpha > 0$$

we have

$$U'_\alpha \le U''_\alpha \quad (\forall)\ \alpha > 0.$$

Hence

$$U'_\alpha \le U_\alpha \quad (\forall)\ \alpha > 0$$

and therefore the resolvent $\mathcal{U}'$ belongs to $\mathcal{R}(X)$ and we have

$$\mathcal{U}' = \mathcal{V} \vee \mathcal{W}$$

<u>Proposition 12.</u> If $(\mathcal{U}_n)_n$, $\mathcal{U}_n=(U^{(n)}_\alpha)_{\alpha>0}$ is an increasing (resp. decreasing) sequence from $\mathcal{R}(X)$ - which is dominated in $\mathcal{R}(X)$ than there exists

$$\bigvee_n \mathcal{U}_n \quad (\text{resp.} \bigwedge_n \mathcal{U}_n).$$

Moreover we have

$$\bigvee_n \mathcal{U}_n = (\bigvee_n U^{(n)}_\alpha)_{\alpha>0}$$

$$(\text{resp.} \bigwedge_n \mathcal{U}_n = (\bigwedge_n U^{(n)}_\alpha)_{\alpha>0})$$

Proof. If we put, for any $\alpha > 0$,

$$U_\alpha = \bigvee_n U_\alpha^{(n)} \quad (\text{resp.}\, U_\alpha = \bigwedge_n U_\alpha^{(n)}$$

one can see that $\mathcal{U} := (U_\alpha)_{\alpha > 0}$ is a resolvent on $(X, \mathcal{B})$, $\mathcal{U} \in \mathcal{R}(X)$ and

$$\mathcal{U} = \bigvee_n \mathcal{U}_n \quad (\text{resp.}\ \mathcal{U} = \bigwedge_n \mathcal{U}_n\,)$$

Theorem 13. Suppose that $(X, \mathcal{B})$ is such that $\mathcal{B}$ is countable generated. Then $\mathcal{R}(X)$ is a conditionally σ-complete lattice.

Proof. Since $\mathcal{B}$ is countable generated then for any two proper kernels V, W there exists $V \wedge W$ and $V \vee W$ in the ordered set of all kernels on $(X, \mathcal{B})$. The assertion from theorem follows now using Propositions 10,11,12.

In the sequel μ will be a finite measure on X and $\mathcal{R}(\mu)$ denotes the set of all resolvents $\mathcal{V} \in \mathcal{R}(X)$ which are absolutely continuous with respect to μ.

Theorem 14. The ordered set $(\mathcal{R}(\mu), \leq)$ is a conditionally σ- complete lattice. Moreover for any sequence $(\mathcal{V}^n)_n$ from $\mathcal{R}(\mu)$ dominated in $\mathcal{R}(X)$

there exists $\bigvee_{(\mathcal{R}(X), \leq)} \{\mathcal{V}^n \mid n \in N\}$ and it is equal with $\bigvee_{(\mathcal{R}(\mu), \leq)} \{\mathcal{V}^n \mid n \in N\}$

Proof. Since $\mathcal{R}(\mu)$ is a solid part of $\mathcal{R}(X)$ and using Proposition 10 and Theorem 13 it will be sufficient to show that if $\mathcal{V} = (V_\alpha)_\alpha$, $\mathcal{W} = (W_\alpha)_\alpha$ are two resolvents from $\mathcal{R}(\mu)$ then there exists

$$V_\alpha \vee W_\alpha \quad (\forall)\ \alpha > 0$$

$$V_\alpha \wedge W_\alpha \quad (\forall)\ \alpha > 0$$

in the set of all kernels on $(X,\mathcal{B})$. Because $\mathcal{V}$ (resp. $\mathcal{W}$) is a proper resolvent which is absolutely continuous with respect to the finite measure μ then, for any $\alpha > 0$, there exists a measurable function G_α (resp. Γ_α) on $X \times X$ such that

$$V_\alpha f(x) = \int G_\alpha(x,y) f(y)\, d\mu(y)$$
$$(\text{resp. } W_\alpha f(x) = \int \Gamma_\alpha(x,y) f(y)\, d\mu(y)$$

for any $x \in X$ and any $f \in \mathcal{F}$. (See [8] H.Kunita, T.Watanabe, Markov processes and Martin Boundary).

Obviously there exists $V_\alpha \vee W_\alpha$ and $V_\alpha \wedge W_\alpha$ and we have

$$(V_\alpha \vee W_\alpha)\ f(x) = \int \sup(G_\alpha, \Gamma_\alpha)(x,y) f(y)\, d\mu(y)$$
$$(V_\alpha \wedge W_\alpha) f(x) = \int \inf(G_\alpha, \Gamma_\alpha)(x,y) f(y)\, d\mu(y)$$

for any $x \in X$, $f \in \mathcal{F}$.

To finish the proof we remark that if $\mathcal{V}, \mathcal{W} \in \mathcal{R}(\mu)$ are such that there exists $\mathcal{U} \in \mathcal{R}(X)$ with

$$\mathcal{V} \leq \mathcal{U} \quad , \mathcal{W} \leq \mathcal{U}$$

then the element $\underset{(\mathcal{R}(X),\leq)}{\mathcal{V} \vee \mathcal{W}}$ belongs to $\mathcal{R}(\mu)$

<u>Definition.</u> In $\mathcal{R}(\mu)$ we consider the following order relation $\circledleq$ given by

$$\mathcal{V} \circledleq \mathcal{W} \iff \mathcal{V} \leq \mathcal{W} \quad \text{and} \quad \mathcal{E}_{\mathcal{W}} \subset \mathcal{E}_{\mathcal{V}}$$

<u>Definition.</u> Let $\mathcal{V}, \mathcal{W} \in \mathcal{R}(X)$ be such that $\mathcal{V} \leq \mathcal{W}$. We denote by $\widehat{\mathcal{W}}^{\mathcal{V}} = (\widehat{W}_\alpha^{\mathcal{V}})_{\alpha > 0}$ the resolvent on X given by

$$\widehat{W}_\alpha^{\mathcal{V}} f = \sup_\beta \beta V_{\alpha+\beta} W_\alpha f, \qquad f \in \mathcal{F}$$

This resolvent is called <u>Meyer-regularized</u> of $\mathcal{W}$ with respect to $\mathcal{V}$.

It is easy to see that we have

$$\mathcal{V} \textcircled{\le} \widehat{\mathcal{W}}^{\mathcal{V}} \le \mathcal{W}$$

and that for any resolvent $\mathcal{W}' \in \mathcal{R}(X)$ we have

$$\mathcal{V} \textcircled{\le} \mathcal{W}' \le \mathcal{W} \Longrightarrow \mathcal{W}' \le \widehat{\mathcal{W}}^{\mathcal{V}}$$

<u>Proposition 15</u> . Let $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{R}(\mu)$ be such that

$$\mathcal{V} \le \mathcal{W} \le \mathcal{U}$$

Then we have

$$\mathcal{V} \textcircled{\le} \mathcal{U} \Longrightarrow \mathcal{W} \textcircled{\le} \mathcal{U}$$

$$\widehat{\mathcal{W}}^{\mathcal{V}} \textcircled{\le} \widehat{\mathcal{U}}^{\mathcal{V}}$$

<u>Proof</u> . Suppose that $\mathcal{V} \textcircled{\le} \mathcal{U}$. Then if $f \in \mathcal{F}$ we have

$$Uf = \sup_{\alpha} \alpha V_{\alpha} Uf \le \sup_{\alpha} \alpha W_{\alpha} Uf \le Uf,$$

and therefore $Uf \in \mathcal{E}_{\mathcal{W}}$. Hence

$$\mathcal{E}_{\mathcal{U}} \subset \mathcal{E}_{\mathcal{W}}, \quad \mathcal{W} \textcircled{\le} \mathcal{U}$$

On the other hand

$$\widehat{W}^{\mathcal{V}}_{\alpha} f = \sup_{\beta} \beta V_{\alpha+\beta} W_{\alpha} f \le \sup_{\beta} \beta V_{\alpha+\beta} U_{\alpha} f = \widehat{U}^{\mathcal{V}}_{\alpha} f$$

$$\widehat{\mathcal{W}}^{\vartheta} \leq \widehat{\mathcal{U}}^{\vartheta}$$

From this relation and from the relation

$$\vartheta \leq \widehat{\mathcal{W}}^{\vartheta} \leq \widehat{\mathcal{U}}^{\vartheta}$$

we get, using the first part of the proof,

$$\widehat{\mathcal{W}}^{\vartheta} \leq \widehat{\mathcal{U}}^{\vartheta}$$

Theorem 16. For any $\vartheta \in \mathcal{R}(\mu)$ the set

$$\mathcal{A} := \{\vartheta' \in \mathcal{R}(\mu) \mid \vartheta \leq \vartheta'\}$$

is a conditionally complete lattice with respect to the order relation $\leq$. Moreover

a) for any family $(\vartheta^i)_{i\in I}$ from $\mathcal{A}$ which is dominated in $(\mathcal{R}(X);\leq)$ there exists $\bigvee_{(\mathcal{R}(\mu),\leq)} \{\vartheta^i \mid i \in I\}$ and we have

$$\bigvee_{(\mathcal{R}(\mu),\leq)} \{\vartheta^i \mid i\in I\} = \bigvee_{(\mathcal{R}(X),\leq)} \{\vartheta^i \mid i\in I\};$$

b) for any $\vartheta', \vartheta'' \in \mathcal{A}$ we have

$$\vartheta' \underset{(\mathcal{R}(\mu),\leq)}{\wedge} \vartheta'' = \widehat{\left(\vartheta' \underset{(\mathcal{R}(X),\leq)}{\wedge} \vartheta''\right)}^{\vartheta}$$

c) for any increasing (resp. decreasing) family $(\vartheta^i)_{i\in I}$ in $\mathcal{A}$ there exists an increasing sequence (ι_n) in I such that

$$\bigvee_{(\mathcal{R}(\mu),\leq)} \{\vartheta^i \mid i\in I\} = \bigvee_{(\mathcal{R}(X),\leq)} \{\vartheta^{\iota_n} \mid n\in \mathbb{N}\} = \bigvee_{(\mathcal{R}(\mu),\leq)} \{\vartheta^{\iota_n} \mid n \in \mathbb{N}\}$$

resp. $$\bigwedge_{(\mathcal{R}(\mu),\leq)} \{\vartheta^i \mid i\in I\} = \widehat{\left(\bigwedge_{(\mathcal{R}(X),\leq)} \{\vartheta^{\iota_n} \mid n\in \mathbb{N}\}\right)}^{\vartheta}.$$

a)+c) Let $\mathcal{V}', \mathcal{V}'' \in \mathcal{A}$ be such that there exist $\mathcal{W} \in \mathcal{R}(X)$ with

$$\mathcal{V}' \le \mathcal{W}, \quad \mathcal{V}'' \le \mathcal{W}$$

From Theorem 14 there exists $\mathcal{V}' \vee_{(\mathcal{R}(X),\le)} \mathcal{V}''$ and we have

$$\mathcal{V} \le \mathcal{V}' \vee_{(\mathcal{R}(X),\le)} \mathcal{V}'' \le \mathcal{W}$$

If we put

$$\mathcal{U} = \widehat{\mathcal{V}' \vee_{(\mathcal{R}(X),\le)} \mathcal{V}''}^{\mathcal{V}}$$

we have from Proposition 15

$$\mathcal{V}' = \widehat{\mathcal{V}'}^{\mathcal{V}} \le \mathcal{U} \le \widehat{\mathcal{W}}^{\mathcal{V}}, \quad \mathcal{V}'' = \widehat{\mathcal{V}''}^{\mathcal{V}} \le \mathcal{U} \le \widehat{\mathcal{W}}^{\mathcal{V}}$$

and therefore

$$\mathcal{U} = \mathcal{V}' \vee_{(\mathcal{R}(X),\le)} \mathcal{V}'' \le \widetilde{\mathcal{W}}$$

Hence

$$\mathcal{U} = \mathcal{V}' \vee_{(\mathcal{R}(\mu),\le)} \mathcal{V}'' = \mathcal{V}' \vee_{(\mathcal{R}(X),\le)} \mathcal{V}''$$

Let now $(\mathcal{V}^i)_{i\in I}$ be an increasing family from $\mathcal{A}$ which is dominated in $\mathcal{R}(X)$ by an element $\mathcal{W}$. If

$$\mathcal{U} := (U_\alpha)_{\alpha > 0}$$

is such that

$$U_\alpha f = \bigvee_{\mathcal{E}_{\mathcal{V}_\alpha}} V^i_\alpha f \qquad (\forall) f \in \mathcal{F}$$

then $\mathcal{U}$ is a resolvent on X, $\mathcal{U} \in \mathcal{R}(\mu)$ and

$$\mathcal{V}^i \leq \mathcal{U} \ , \ \mathcal{U} = \widehat{\mathcal{U}}^{\mathcal{V}}$$

From these relations we deduce

$$\mathcal{V}^i \ (\leq) \ \mathcal{U} \quad (\forall) \ i \in I$$

and if $\mathcal{W} \in \mathcal{R}(X)$, $\mathcal{W} \geq \mathcal{V}^i \ (\forall) \ i \in I$ then

$$\mathcal{U} \leq \mathcal{W}$$

and therefore

$$\mathcal{U} = \bigvee_{(\mathcal{R}(X), \leq)} \{\mathcal{V}^i \mid i \in I\}$$

If $\mathcal{W} \in \mathcal{R}(\mu)$, $\mathcal{W} \ (\geq) \ \mathcal{V}^i \ (\forall) \ i \in I$ then we have

$$\mathcal{V} \leq \mathcal{U} \leq \mathcal{W} \ , \ \mathcal{V} \ (\leq) \ \mathcal{W}$$

and therefore

$$\mathcal{U} \ (\leq) \ \mathcal{W}$$

Hence

$$\mathcal{U} = \bigvee_{(\mathcal{R}(\mu), (\leq))} \{\mathcal{V}^i \mid i \in I\}$$

We choose $f \in \mathcal{F}, f > 0$ such that $Wf < \infty$ where $\mathcal{W} = (W_\alpha)_{\alpha > 0} \in \mathcal{R}(X)$

is such that

$$\mathcal{V}^{i} \leq \mathcal{W} \quad (\forall) \quad i \in I$$

Since $\mathcal{V}$ is absolutely continuous with respect to μ then there exists an increasing sequence $(i_n)_n$ in I such that

$$\bigvee_{\mathcal{E}_{\mathcal{V}}} \{ V^{i} f \mid i \in I \} = \bigvee_{\mathcal{E}_{\mathcal{V}}} \{ V^{i_n} f \mid n \in N \}$$

From this fact it follows that for any $g \in \mathcal{F}$, $0 \leq g \leq f$ we have also,

$$\bigvee_{\mathcal{E}_{\mathcal{V}}} \{ V^{i} g \mid i \in I \} = \bigvee_{\mathcal{E}_{\mathcal{V}}} \{ V^{i_n} g \mid n \in N \}$$

or equivalently

$$\bigvee_{(\mathcal{R}(X), \leq)} \{ \mathcal{V}^{i} \mid i \in I \} = \bigvee_{(\mathcal{R}(X), \leq)} \{ \mathcal{V}^{i_n} \mid n \in N \}$$

Suppose now that $(\mathcal{V}^{i})_{i \in I}$ is a decreasing family from $\mathcal{A}$ and let $f \in \mathcal{F}$, $f > 0$ be such that $V^{i_0} f < \infty$ for a fixed $i_0 \in I$. For any $g \in \mathcal{F}$, $g \leq f$ and any $\alpha \geq 0$ we put

$$U'_{\alpha} g = \bigwedge_{\mathcal{E}_{V_{\alpha}}} \{ V^{i}_{\alpha} g \mid i \in I, \; i \geq i_0 \}$$

There exists an unique kernel U_{α} on X such that

$$U_{\alpha} g = U'_{\alpha} g \qquad (\forall) g \in \mathcal{F}, \quad g \leq f$$

It is easy to see that $\mathcal{U} = (U_{\alpha})_{\alpha > 0}$ is a resolvent on X, $\mathcal{U} \in \mathcal{A}$,

$$\mathcal{U} \leq \mathcal{V}^{i} \qquad (\forall) \quad i \in I$$

Since

$$\mathcal{V} \leq \mathcal{U} \leq \mathcal{V}^i;\quad \mathcal{V} \circledleq \mathcal{V}^i$$

it follows from Proposition 15, that

$$\mathcal{U} \circledleq \mathcal{V}^i \quad (\forall)\ i \in I$$

If $\mathcal{W} \in \mathcal{A}$ is such that

$$\mathcal{W} \circledleq \mathcal{V}^i \quad (\forall)\ i \in I$$

then

$$\mathcal{W} \leq \mathcal{V}^i \quad (\forall)\ i \in I$$

and therefore

$$\mathcal{W} \leq \mathcal{U}$$

Since

$$\mathcal{V} \leq \mathcal{W} \leq \mathcal{U},\quad \mathcal{V} \circledleq \mathcal{U}$$

we deduce

$$\mathcal{W} \circledleq \mathcal{U}$$

Hence

$$\mathcal{U} = \bigwedge_{(\mathcal{R}(\mu), \circledleq)} \{\mathcal{V}^i \mid i \in I\}$$

Since $\mathcal{V}$ is absolutely continuous with respect to μ then there exists an increasing sequence $(i_n)_n$ in I such that

$$\bigwedge_{\mathcal{E}_{\mathcal{V}}} \{ \mathcal{V}^{i} f \mid i \in I \} = \bigwedge_{\mathcal{E}_{\mathcal{V}}} \{ \mathcal{V}^{i_n} f \mid n \in N \}$$

and therefore

$$\bigwedge_{\mathcal{E}_{\mathcal{V}}} \{ \mathcal{V}^{i} g \mid i \in I \} = \bigwedge_{\mathcal{E}_{\mathcal{V}}} \{ \mathcal{V}^{i_n} g \mid n \in N \}$$

for any $g \in \mathcal{F}$, $g \leq f$. If we put

$$\mathcal{W} := \bigwedge_{\mathcal{R}(X)} \{ \mathcal{V}^{i_n} \mid n \in N \}, \quad \mathcal{W} = (W_\alpha)_{\alpha > 0}$$

we have $\mathcal{W} \in \mathcal{R}(\mu)$,

$$\mathcal{U} \leq \mathcal{W} \leq \mathcal{V}^{i_n} \quad (\forall)\, n \in N$$

and therefore

$$\mathcal{U} \circledS \widehat{\mathcal{W}}^{\mathcal{V}} \leq \mathcal{V}^{i_n} \qquad (\forall)\ n \in N$$

Since

$$Ug \leq \widehat{W}^{\mathcal{V}} g \leq Wg = Ug \qquad (\forall) \qquad g \leq f$$

it follows that

$$\widehat{W}^{\mathcal{V}} = U$$

and therefore

$$\widehat{\mathcal{W}}^{\mathcal{V}} = \mathcal{U}$$

b) Let now $\mathcal{V}', \mathcal{V}'' \in \mathcal{A}$. If we put

$$\mathcal{W} := \mathcal{V}' \wedge_{(\mathcal{R}(X),\leq)} \mathcal{V}''$$

we have

$$\mathcal{V} \leq \mathcal{W} \leq \mathcal{V}', \quad \mathcal{V} \leq \mathcal{W} \leq \mathcal{V}''$$

From Proposition 15 we deduce

$$\mathcal{V} (\leq) \widehat{\mathcal{W}}^{\mathcal{V}} (\leq) \mathcal{V}', \quad \mathcal{V} (\leq) \widehat{\mathcal{W}}^{\mathcal{V}} (\leq) \mathcal{V}''$$

On the other hand if $\mathcal{W} \in \mathcal{A}$ is such that

$$\mathcal{W}' (\leq) \mathcal{V}', \quad \mathcal{W}' (\leq) \mathcal{V}''$$

then we have

$$\mathcal{W}' \leq \mathcal{V}' \wedge_{(\mathcal{R}(X),\leq)} \mathcal{V}'' = \mathcal{W}$$

$$\mathcal{W}' = \widehat{\mathcal{W}'}^{\mathcal{V}} \leq \widehat{\mathcal{W}}^{\mathcal{V}}$$

Hence

$$\widehat{\mathcal{W}}^{\mathcal{V}} \leq \mathcal{V}' \wedge_{(\mathcal{R}(\mu),(\leq))} \mathcal{V}''$$

<u>Lemma 17</u>. Let $\mathcal{V}, \mathcal{W} \in \mathcal{R}(\mu)$ be such that

$$\mathcal{V} (\leq) \mathcal{W}$$

and let $\preceq$ the specific order given by $\mathcal{E}_{\mathcal{V}}$.

If we denote by $\mathcal{B}_{\mathcal{W}}$ the set of all $\mathcal{B}$-measurable functions f on X such that there exists $s \in \mathcal{E}_{\mathcal{W}}$ with $|f| < s$ and $W(|f|) < \infty$ then we have

a) $f \in \mathcal{B}_{\mathcal{W}}, \; f \geq 0 \Longrightarrow Wf - Vf \in \mathcal{E}_{\mathcal{W}}$

b) $f \in \mathcal{B}_{\mathcal{W}}, \; Wf \geq 0 \Longrightarrow Wf - Vf \in \mathcal{E}_{\mathcal{V}}$

c) there exists a map $T=T(\mathcal{V};\mathcal{W})$

$$T:\mathcal{E}_{\mathcal{W}}\longrightarrow\mathcal{E}_{\mathcal{W}}$$

such that T is additive, increasing, continuous in order from below,

$$s_1\leq s_2 \Longrightarrow Ts_1 \precsim Ts_2 \precsim s_2$$

and such that

$$f\in B_{\mathcal{W}},\ f\geq 0 \quad T(Wf)=Wf-Vf.$$

d) there exists a $\mathcal{E}_{\mathcal{V}}$-valued map $\tilde{T}=\tilde{T}(\mathcal{V};\mathcal{W})$ defined on a naturally solid convex subcone $D(\tilde{T})$ of $\mathcal{E}_{\mathcal{V}}$ such that $\tilde{T}$ is additive, increasing, continuous in order from below,

$$s_1,s_2\ D(\tilde{T}),\ s_1\leq s_2 \Rightarrow \tilde{T}s_1 \precsim \tilde{T}s_2,$$

$$\mathcal{E}_{\mathcal{W}}\subset D(\tilde{T}),\quad \tilde{T}s=Ts \quad \text{for any} \quad s\in\mathcal{E}_{\mathcal{W}}$$

e) if $\mathcal{U}\in\mathcal{R}(\mu)$ is such that

$$\mathcal{V}\ \textcircled{\leq}\ \mathcal{U}\ \textcircled{\leq}\ \mathcal{W}$$

Then we have

$$\tilde{T}(\mathcal{V};\mathcal{W})=\tilde{T}(\mathcal{V};\mathcal{U})+(1-\tilde{T}(\mathcal{V};\mathcal{U}))\tilde{T}(\mathcal{U};\mathcal{W}).$$

<u>Proof.</u> Using the relation $\mathcal{V}\leq\mathcal{W}$ and similar procedures as in (Meyer P.A. [5]) we get

$$f \in \mathcal{B}_{\mathcal{W}},\quad Wf \geq 0 \Rightarrow Wf - Vf \in \mathcal{S}_{\mathcal{V}}$$

$$f \in \mathcal{B}_{\mathcal{W}},\ f \geq 0 \Rightarrow Wf - Vf \in \mathcal{S}_{\mathcal{W}}$$

Since for any $f \in \mathcal{F}$ we have

$$\lim_{\alpha \to \infty} \alpha V_\alpha Wf = Wf$$

it follows that

$$f \in B_{\mathcal{V}},\quad Wf \geq 0 \Rightarrow \quad Wf - Vf \in \mathcal{E}_{\mathcal{V}}$$

$$f \in B_{\mathcal{W}},\quad f \geq 0 \Longrightarrow \quad Wf - Vf \in \mathcal{E}_{\mathcal{W}}$$

For any $s \in \mathcal{E}_{\mathcal{W}}$ we put

$$Ts := \sup \left\{ Wf - Vf \mid f \in \mathcal{B}_{\mathcal{W}},\ f \geq 0,\ Wf \leq s \right\}$$

We remark, using b, that

$$f \in \mathcal{B}_{\mathcal{W}},\ f \geq 0 \Rightarrow T(Wf) = Wf - Vf \in \mathcal{E}_{\mathcal{W}}$$

and

$$f_1, f_2 \in \mathcal{B}_{\mathcal{W}},\ 0 \leq f_1 \leq f_2 \Longrightarrow T(Wf_1) \preceq_{\mathcal{E}_{\mathcal{W}}} T(Wf_2)$$

We show now that if $(f_n)_n$ is a sequence in $\mathcal{B}_{\mathcal{W}}$, $f_n \geq 0$ and $s \in \mathcal{E}_{\mathcal{W}}$ is such that the such that the sequence $(Wf_n)_n$ increases and

$$\lim_n Wf_n \geq s$$

then the sequence $(TWf_n)_n$ is increasing and

$$\sup_{n} TWf_n \geq Ts$$

Indeed, let $f \in \mathcal{B}_{\mathcal{W}}$ be such that $f \geq 0$, $Wf \leq s$ and let $g \in \mathcal{B}_{\mathcal{W}}$ be such that $0 \leq g \leq f$ and such that Wg is universally continuous in $\mathcal{E}_{\mathcal{W}}([1])$. If we consider $\varphi \in \mathcal{B}_{\mathcal{W}}$, $0 < \varphi$ then, since

$$\sup_{n} Wf_n \geq Wg$$

we deduce that for any $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that

$$n \geq n_\varepsilon \Rightarrow Wg \leq Wf_n + \varepsilon\, W\varphi$$

and therefore, using the assertion b,

$$n \geq n_\varepsilon \Rightarrow Wg - Vg \leq Wf_n - Vf_n + \varepsilon\, W\varphi$$

Hence

$$Wg - Vg \leq \sup_{n} (Wf_n - Vf_n) + \varepsilon\, W\varphi \qquad (\forall)\ \varepsilon > 0,$$

Since $\varepsilon > 0$ any g are arbitrary we get

$$Wf - Vf \leq \sup_{n} (Wf_n - Vf_n)$$

and therefore, f being arbitrary,

$$Ts \leq \sup_{n} (Wf_n - Vf_n)$$

From the above considerations it follows that if $(f_n)_n$ is a sequence in $\mathcal{B}_{\mathcal{W}}$, $f_n \geq 0$ such that the sequence $(Wf_n)_n$ increases to s then

$$(TWf_n)_n \uparrow Ts$$

and therefore $Ts\in\mathcal{E}_{\mathcal{W}}$, T is additive, increasing continuous in order from below and

$$s_1, s_2\in\mathcal{E}_{\mathcal{W}}, \quad s_1\leq s_2 \Rightarrow Ts_1 \lesssim Ts_2$$

For any $f\in\mathcal{B}_{\mathcal{W}}$, $f\geq 0$ we have

$$T(Wf)=Wf-Vf \lesssim Wf.$$

We deduce using the definition that for any $s\in\mathcal{E}_{\mathcal{W}}$ we may consider a sequence $(f_n)_n\subset\mathcal{B}_{\mathcal{W}}$, $f_n\geq 0$ such that the sequence $(Wf_n)_n$ increases to s.

We have, from the above considerations:

$$T(Wf_n) \lesssim Wf_n,$$

$$Ts=\lim_n T(Wf_n)\lesssim \lim_n Wf_n = s.$$

d) For any $f\in\mathcal{B}_{\mathcal{W}}$, $f\geq 0$ we have

$$Vf = Wf - T(Wf)\in\mathcal{E}_{\mathcal{W}}-\mathcal{E}_{\mathcal{W}}$$

We put, by definition

$$\tilde{T}(Vf):=TWf - T^2Wf$$

Using now the properties a), b), c) of T we get that the map $\tilde{T}$ defined on $W(\mathcal{F}\cap\mathcal{B}_{\mathcal{W}})$ with values in $\mathcal{E}_{\mathcal{V}}$ is increasing, additive and positively homogeneous. Using similar arguments as above one can show that if $f_n \in \mathcal{F} \cap \mathcal{B}_{\mathcal{W}}$, $f \in \mathcal{F} \cap \mathcal{B}_{\mathcal{W}}$ are such that the sequence $(Vf_n)_n$ increases and $\sup_n Vf_n \geq Vf$ then $\sup_n \tilde{T}(Vf_n) \geq \tilde{T}(Vf)$.

If for any $s \in \mathcal{E}_{\mathcal{V}}$ we put

$$\tilde{T}s := \sup\{\tilde{T}(Vf)/f \in \mathcal{F} \cap \mathcal{B}_{\mathcal{W}} \quad , Vf \leq s\}$$

then the map

$$\tilde{T}: \mathcal{E}_{\mathcal{V}} \to \tilde{\mathcal{E}}_{\mathcal{V}} \quad , (\tilde{\mathcal{E}}_{\mathcal{V}} = \{f \in \mathcal{S}_{\mathcal{V}} \mid \alpha V_\alpha f \uparrow f\})$$

is additive, increasing, continuous in order from below and moreover

$$s_1, s_2 \in \mathcal{E}_{\mathcal{V}}, \ s_1 \leq s_2 \Rightarrow \tilde{T}s_1 \leq \tilde{T}s_2$$

$$\tilde{T}s = Ts \text{ for any } s \in \mathcal{E}_{\mathcal{W}}$$

If we denote by $D(\tilde{T})$ the set of all elements $s \in \mathcal{E}_{\mathcal{V}}$ for which $\tilde{T}s \in \mathcal{E}_{\mathcal{V}}$ then the restriction of $\tilde{T}$ to $D(\tilde{T})$ satisfies the required conditions

e) For $f \in \mathcal{B}_{\mathcal{W}}$, $f \geq 0$ we have

$$T(\mathcal{V};\mathcal{U})(Wf) = Uf - Vf$$

$$T(\mathcal{U};\mathcal{W})(Wf) = Wf - Uf$$

$$(I - T(\mathcal{V};\mathcal{U}))(T(\mathcal{U};\mathcal{W})(Wf) =$$

$$= (I - T(\mathcal{V};\mathcal{U}))(Wf - Uf) = Wf - Uf - T(\mathcal{V};\mathcal{U})(Wf) + Uf - Vf$$

and therefore

$$T(\mathcal{V};\mathcal{U})(Wf) + (I - T(\mathcal{V};\mathcal{U}))T(\mathcal{U};\mathcal{W})(Wf) = Wf - Vf = T(\mathcal{V};\mathcal{W})(Wf) \ .$$

Theorem 18. Let $\mathcal{V},\mathcal{U}\in\mathcal{R}(\mu)$ be such that there exists an exact $\mathcal{V}$-compression $\mathcal{P}$ which is μ-absolutely continuous, $\mathcal{V}^{\mathcal{P}}\in\mathcal{R}(\mu)$ and

$$\mathcal{V} \circledleq \mathcal{U} \leq \mathcal{V}^{\mathcal{P}}$$

Then, there exists an exact $\mathcal{V}$-compression operator, Q such that $\mathcal{U}=\mathcal{V}^{Q}$ and such that

$$Qs=\tilde{T}(\mathcal{V},\mathcal{U})(s) \qquad (\forall)\ s\in D(\tilde{T})$$

Moreover we have

$$Qf \leqq Pf \quad (\forall) f\in\mathcal{F}, \qquad Pf<\infty$$

where P is the initial kernel associated with $\mathcal{P}$.

Proof. Let $\mathcal{B}_{\mathcal{P}}$ be the set of all $\mathcal{B}$-measurable real functions f on X such that

$$\mathcal{V}^{\mathcal{P}}(|f|)<\infty$$

Since $\mathcal{V}\circledleq\mathcal{V}$ and $\mathcal{U}\circledleq\mathcal{V}^{\mathcal{P}}$ we consider as in the preceding lemma

$$S:=\tilde{T}(\mathcal{V};\mathcal{U}),\ \mathbb{L}:=T(\mathcal{U};\mathcal{V}^{\mathcal{P}})$$

We remember that S is defined on a solid convexe subcone D(S) of $\mathcal{E}_{\mathcal{V}}$ which is dense in order from below in $\mathcal{E}_{\mathcal{V}}$ with values in $\mathcal{E}_{\mathcal{V}}$.

We have, using the preceding lemma

$$P(V^{\mathcal{P}} f)=(S+(I-S).L)(V^{\mathcal{P}} f) \quad (\forall)\ f\in \mathcal{B}_{\mathcal{P}}$$

and

$$f\in\mathcal{B}_{\mathcal{P}},\ V^{\mathcal{P}} f\geq 0 \Rightarrow S(V^{\mathcal{P}} f)\preccurlyeq P(V^{\mathcal{P}} f)$$

where $\preccurlyeq$ means the specific order generated by $\mathcal{E}_{\mathcal{V}}$ and where P is the initial kernel associated with $\mathcal{P}$.

We denote by E the set of all $\mathcal{B}$-measurable real function f on X such that

$$P(|f|) < \infty$$
$$E_o = V^{\mathcal{P}}(\mathcal{B}_{\mathcal{P}}).$$

Obviously E is a solid subspace of the space of all B-measurable real functions on X and E_o is a subspace of E.

Further we denote by $\overline{P}$ the map

$$\overline{P} : E \to \mathcal{E}_{\mathcal{V}}$$

defined by

$$\overline{P}(f) = P(f_+)$$

Obviously we have

$$\overline{P}(f_1+f_2)\preccurlyeq \overline{P}f_1+\overline{P}f_2$$
$$\overline{P}(\alpha f)=\alpha\overline{P}f \quad (\forall)\ \alpha>0$$

and

$$f \in E_0 \Longrightarrow S(f) \precsim \overline{P}(f)$$

Using the fact that $\mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}}$ is a conditionally complete vector lattice with respect to the specific order and Hahn-Banach extension theorem we deduce that there exists a liniar map

$$\tilde{S} : E \longrightarrow \mathcal{E}_{\mathcal{V}} - \mathcal{E}_{\mathcal{V}}$$

such that

$$\tilde{S}/_{E_0} = S, \quad \tilde{S}(f) \precsim \overline{P}(f) \quad (\forall) f \in E.$$

Particularly

$$f \in E, \ f \leq 0 \Longrightarrow \tilde{S}(f) \precsim \overline{P}(f) = 0$$

and therefore

$$f \in E, \ f \geq 0 \quad \Longrightarrow \quad \tilde{S}(f) \in \mathcal{E}_{\mathcal{V}}$$

$$f \in E, \ f \geq 0 \quad \tilde{S}(f) \precsim \overline{P}(f) = Pf.$$

Hence $\tilde{S}$ is the restriction to E of a kernel on X, denoted by Q, such that

$$Q(f) \precsim P(f) \qquad (\forall) f \in \mathcal{F} .$$

From the relation

$$Wf = Vf + QWf \qquad (\forall) f \in \mathcal{B}_{\mathcal{F}}, \ f \geq 0$$

we deduce

$$Wf = \left(\sum_{n=0}^{\infty} Q^n\right)Vf + \lim_{n\to\infty} Q^n Wf$$

Since

$$Q^n Wf \leq P^n Wf \leq P^n V^{\mathcal{P}} f$$

and since

$$P^n V^{\mathcal{P}} f \downarrow 0$$

it follows that

$$Wf = \left(\sum_{n=0}^{\infty} Q^n\right)Vf$$

and therefore the $\mathcal{V}$-compression

$$Q = (Q_\alpha)_{\alpha \geq 0},$$

where

$$Q_\alpha f = (1 - \alpha V_\alpha)Qf, \qquad f \in \mathcal{F}$$

satisfies the required conditions.

<u>Notation.</u> If $\mathcal{V} \in \mathcal{R}(\mu)$ we denote by $\mathcal{B}_{\mathcal{V}}$ the set of all $\mathcal{B}$-measurable function f for which there exists $s \in \mathcal{E}_{\mathcal{V}}$, $s < \infty$ such that

$$|f| \leq s, \qquad V(|f|) < \infty.$$

We denote also by $\precsim$ the specific order generated by the convex cone $\mathscr{C}_{\nu}$.

Theorem 19. Let $\nu \in \mathscr{R}(\mu)$ and let $\mathscr{P}' = (P'_{\alpha})_{\alpha > 0}$ and $\mathscr{P}'' = (P''_{\alpha})_{\alpha > 0}$ be two exact ν-compressions such that $\nu^{\mathscr{P}'}, \nu^{\mathscr{P}''} \in \mathscr{R}(\mu)$. Then the following assertions are equivalent

a) $\nu^{\mathscr{P}'} \le \nu^{\mathscr{P}''}$

b) for any $f \in \mathscr{B}_{\nu^{\mathscr{P}'}} \cap \mathscr{B}_{\nu^{\mathscr{P}''}}$ such that $Vf \geq 0$ we have

$$P'_{o}(Vf) \precsim P''_{o}(Vf)$$

c) for any $s, t \in \mathscr{C}_{\nu}$, $s \geq t$, $P''_{o}(s) < \infty$ we have

$$P'_{o}(s-t) \precsim P''_{o}(s-t)$$

Moreover if $\mathscr{P}', \mathscr{P}''$ are absolutely continuous with respect to ν and $\mathscr{B}$ is generated by $\mathscr{S}_{\nu}$ then each of the assertions a), b), c) is equivalent with the following one

d) for any $f \in \mathscr{F}$ such that $P''_{o}(f) < \infty$ we have

$$P'_{o}(f) \precsim P''_{o}(f).$$

Proof. a) $\Rightarrow$ b) Let us denote $\nu' = \nu^{\mathscr{P}'}$, $\nu'' = \nu^{\mathscr{P}''}$. As in the preceding Lemma we may consider the maps

$$T(\nu, \nu'), \quad T(\nu, \nu'') \quad \text{and} \quad T(\nu', \nu'') \text{ which are}$$

defined on $D(T(\nu,\nu'))$, $D(T(\nu,\nu''))$ and $D(T(\nu',\nu''))$ respectively. We know by the same Lemma that, for any $f \in \mathscr{B}_{\nu'} \cap \mathscr{B}_{\nu''}$ such that $V'' f \geq 0$, we have

$$T_{(\mathcal{V},\mathcal{V}'')}(V''f)=T_{(\mathcal{V},\mathcal{V}')}V''f+(I-T_{(\mathcal{V},\mathcal{V}')})T_{(\mathcal{V}',\mathcal{V}'')}V''f.$$

If we denote by $\precsim$ the specific order generated by $\mathcal{E}_{\mathcal{V}}$ we have

$$T_{(\mathcal{V},\mathcal{V}')}(V''f) \precsim T_{(\mathcal{V},\mathcal{V}'')}(V''f),$$

$$P'(V''f) \precsim P''(V''f),$$

$$P'(u-v) \precsim P''(u-v) \quad (\forall), u,v \in \mathcal{E}_{\mathcal{V}''},\ u-v \geq 0,$$

$$P'(Vf) \precsim P''(Vf) \quad (\forall)\ f \in \mathcal{B}_{\mathcal{V}'} \cap \mathcal{B}_{\mathcal{V}''},\quad Vf \geq 0.$$

Conversely, if for any $f \in \mathcal{B}_{\mathcal{V}'} \cap \mathcal{B}_{\mathcal{V}''}$ with $Vf \geq 0$ we have

$$P'(Vf) \precsim P''(Vf)$$

then if we put, for any $f \in \mathcal{B}_{\mathcal{V}'} \cap \mathcal{B}_{\mathcal{V}''} \cap \mathcal{F}$

$$Qf = P''f-P'f$$

then Q may be naturally extended to a kernel, denoted also by Q, on X for which $Qf \in \mathcal{E}_{\mathcal{V}}$ whenever $f \in \mathcal{B}_{\mathcal{V}'} \cap \mathcal{B}_{\mathcal{V}''} \cap \mathcal{F}$. Obviously the family $Q=(Q_\alpha)_{\alpha>0}$ defined by

$$Q_\alpha f=(I-\alpha V_\alpha)Qf \quad (\forall)\ f \in \mathcal{B}_{\mathcal{V}'} \cap \mathcal{B}_{\mathcal{V}''} \cap \mathcal{F}$$

is an exact $\mathcal{V}$-compression and therefore, from the relation

$$\mathcal{P}''=\mathcal{P}'+Q$$

we deduce, using Theorem 3,

$$\mathcal{V}'' = \mathcal{V}'^{Q(\mathcal{P}')}$$

$$\mathcal{V} \leqq \mathcal{V}' \leq \mathcal{V}'^{Q(\mathcal{P}')}$$

$$\mathcal{V}' \leqq \mathcal{V}''$$

Obviously c)$\Rightarrow$b).

b)$\Rightarrow$c) Let $s,t \in \mathcal{E}_{\mathcal{V}}$ be such that $s \geq t$ and $P''s < \infty$. We consider two sequence $(f_n)_n$, $(g_n)_n$ $P''s$. We consider two sequences $(f_n)_n$, $(g_n)_n$ in $\mathcal{F}$ such that $Vf_n \uparrow s$, $Vg_n \uparrow t$ and such that Vg_n is universally continuous in $\mathcal{E}_{\mathcal{V}}$. Let now $h \in \mathcal{F}$ be such that $h > 0$ and $P''(Vh) < \infty$. Then for $\varepsilon > 0$ any $n \in N$ there exists $m_o > n$ such that

$$Vf_{m_o} + \varepsilon Vh \geq Vg_n$$

Hence

$$P'(Vf_m + \varepsilon Vh - Vg_n) \precsim P''(Vf_m + \varepsilon Vh - Vg_n)$$

for any $m \geq m_o$. We deduce

$$P'(s + \varepsilon Vh - Vg_n) \precsim P''(s + \varepsilon Vh - Vg_n),$$
$$P'(s + \varepsilon Vh - t) \precsim P''(s + \varepsilon Vh - t),$$

and since ε is arbitrary we get

$$P'(s-t) \precsim P''(s-t).$$

Suppose now that $\mathcal{P}'$ and $\mathcal{P}''$ are absolutely continuous with respect to $\mathcal{V}$. Obviously d)$\Rightarrow$ c).

c)$\Rightarrow$ d). follows using standard arguments of monotone classes and the fact that for any $s,t \in \mathcal{S}_{\mathcal{V}}$ $s-t \geq 0$, $P''(\tilde{s}) < \infty$

where

$$\tilde{s} = \sup_{\alpha} \alpha V_{\alpha} s$$

we have

$$P'(s-t)=P'(\tilde{s}-\tilde{t})\geq P''(\tilde{s}-\tilde{t})=P''(s-t).$$

Theorem 20. Let $\mathcal{V}\in\mathcal{R}(\mu)$ be such that $\mathcal{B}$ is generated by $\mathcal{S}_{\mathcal{V}}$ and let $\mathcal{P}', \mathcal{P}''$ be two exact $\mathcal{V}$-compressions which are absolutely continuous with respect to $\mathcal{V}$ and such that $\mathcal{V}^{\mathcal{P}'}, \mathcal{V}^{\mathcal{P}''}\in\mathcal{R}(\mu)$. Then there exists an exact $\mathcal{V}$-compression $\mathcal{P}$ which is absolutely continuous with respect to $\mathcal{V}$ such that

$$\mathcal{V}^{\mathcal{P}'} \underset{(\mathcal{R}(\mu),\subseteq)}{\wedge} \mathcal{V}^{\mathcal{P}''} = \mathcal{V}^{\mathcal{P}}$$

Moreover if P', P'' and P are the initial kernels of $\mathcal{P}', \mathcal{P}''$ and $\mathcal{P}$ respectively, we have

$$Pf = \bigwedge_{\mathcal{E}_{\mathcal{V}}}\{P'f_1+P''f_2 \mid f_1, f_2\in\mathcal{F},\ f_1+f_2=f\}$$

for any $f\in\mathcal{F}$ for which $P'f + P''f < \infty$.

Proof. We denote by $\mathcal{F}_0$ the set

$$\{ f\in\mathcal{F} \mid P'f+P''f < \infty\}$$

Obviously $\mathcal{F}_0$ is a solid convexe subcone of $\mathcal{F}$ and since $\mathcal{V}^{\mathcal{P}'}, \mathcal{V}^{\mathcal{P}''}$ are proper there exists $f_0\in\mathcal{F}_0$, $f_0 > 0$. We consider now the map

$$P:\mathcal{F}_0\longrightarrow\mathcal{E}_{\mathcal{V}}$$

defined by

$$Pf = \bigwedge_{\mathcal{E}_\mathcal{V}} \{ P'f_1 + P''f_2 \mid f_1, f_2 \in \mathcal{F},\ f_1 + f_2 = f \}$$

It is easy to set that P is additive and

$$Pf \lesssim P'f, \quad Pf \lesssim P''f$$

Since P', P'' are kernels then P is the restriction to $\mathcal{F}_0$ of a unique kernel, denoted also by P. Obviously for any $f \in \mathcal{F}$ we have

$$Pf < \infty \implies Pf \in \mathcal{E}_\mathcal{V}$$

Let us denote by $\mathcal{P}$ the exact $\mathcal{V}$-compression on X such that P is its initial kernel. From the proceding considerations and from Theorem 19 it follows

$$\mathcal{V}^{\mathcal{P}} \leq \mathcal{V}^{\mathcal{P}'}, \quad \mathcal{V}^{\mathcal{P}} \leq \mathcal{V}^{\mathcal{P}''}$$

and therefore

$$\mathcal{V}^{\mathcal{P}} \leq \mathcal{V}^{\mathcal{P}'} \underset{(\mathcal{R}(\mu), \leq)}{\wedge} \mathcal{V}^{\mathcal{P}''} =: \mathcal{W}$$

Further, using Theorem 18 and the relations

$$\mathcal{V} \leq \mathcal{W} \leq \mathcal{V}^{\mathcal{P}'}$$

we deduce that there exists an exact $\mathcal{V}$-compression Q such that $\mathcal{V}^Q \in \mathcal{R}(\mu)$ and

$$\mathcal{W} = \mathcal{V}^{\mathcal{Q}}.$$

Since we have

$$\mathcal{W} = \mathcal{V}^{\mathcal{Q}} \circledleq \mathcal{V}^{\mathcal{P}'}$$
$$\mathcal{W} = \mathcal{V}^{\mathcal{Q}} \circledleq \mathcal{V}^{\mathcal{P}''}$$

it follows from Theorem 19 d) that

$$Qf \lesssim P'f \wedge P''f \qquad (\forall) f \in \mathcal{F}_0$$

and therefore

$$Qf \lesssim \mathbb{P}f \qquad (\forall) f, \ f \in \mathcal{F}_0,$$

where Q is the initial kernel associated with $\mathcal{Q}$. Hence, using again Theorem 19, we get

$$\mathcal{W} \circledleq \mathcal{V}^{\mathcal{P}}$$
$$\mathcal{V}^{\mathcal{P}} = \mathcal{V}^{\mathcal{P}'} \underset{(\mathcal{R}(\mu), \circledleq)}{\wedge} \mathcal{V}^{\mathcal{P}''}$$

<u>Theorem 21.</u> Let $\mathcal{V} \in \mathcal{R}(\mu)$ be such that $\mathcal{B}$ is generated by $\mathcal{S}_{\mathcal{V}}$ and let $\mathcal{P}'$, $\mathcal{P}''$, $\mathcal{Q}$ be three exact $\mathcal{V}$-compressions which are absolutely continuous with respect to $\mathcal{V}$ such that $\mathcal{V}^{\mathcal{P}'}$, $\mathcal{V}^{\mathcal{P}''}$, $\mathcal{V}^{\mathcal{Q}} \in \mathcal{R}(\mu)$ and such that $\mathcal{V}^{\mathcal{P}'} \leq \mathcal{V}^{\mathcal{Q}}$, $\mathcal{V}^{\mathcal{P}''} \leq \mathcal{V}^{\mathcal{Q}}$

Then there exists an exact $\mathcal{V}$-compression $\mathcal{P}$ which is absolutely continuous with respect to $\mathcal{V}$ such that

$$\mathcal{V}^{\mathcal{P}'} \underset{(\mathcal{R}(\mu), \circledleq)}{\vee} \mathcal{V}^{\mathcal{P}''} = \mathcal{V}^{\mathcal{P}}$$

Moreover if $\mathbb{P}'$, $\mathbb{P}''$, $\mathbb{P}$ are the initial kernels associated with $\mathcal{P}'$, $\mathcal{P}''$ and $\mathcal{P}$ respectively, then we have

$$Pf = \bigvee_{\mathcal{E}_\nu} \{P'f_1 + P''f_2 \mid f_1, f_2 \in \mathcal{F}, \ f_1+f_2=f\}$$

for any $f \in \mathcal{F}$ for which $P'f+P''f < \infty$.

Proof. We consider the set $\mathcal{F}_0$ of all $f \in \mathcal{F}$ for which $P'f+P''f < \infty$. We denote by P the map

$$P : \mathcal{F}_0 \longrightarrow \mathcal{E}_{\nu}$$

defined by

$$Pf = \bigvee_{\mathcal{E}_\nu} \{P'f_1 + P''f_2 \mid f_1, f_2 \in \mathcal{F}_0, \ f_1+f_2=f\}$$

From the definition it is easy to see that P is additive and

$$Pf \precsim 0f \qquad (\forall) f \in \mathcal{F}_0.$$

Since $\mathcal{F}_0$ is a solide convexe subcone of $\mathcal{F}$ and since $P'+P''$ is a kernel, it follows, that P is the restriction to $\mathcal{F}_0$ of a unique kernel on X which will be denoted also by P. Obiously we have

$$f \in \mathcal{F}_0 \Longrightarrow Pf < \infty \ , \qquad Pf \in \mathcal{E}_{\nu}$$

Let us denote by $\mathcal{P}$ the exact ν-compression such that P is its initial kernel. Obviously $\mathcal{P}$ is absolutely continuous with respect to ν and $\nu^{\mathcal{P}} \in \mathcal{R}(\mu)$. From Theorem 19 we have

$$\mathcal{V}^{\mathcal{P}'} \leq \mathcal{V}^{\mathcal{P}}, \quad \mathcal{V}^{\mathcal{P}''} \leq \mathcal{V}^{\mathcal{P}}$$

$$\mathcal{W} := \mathcal{V}^{\mathcal{P}'} \underset{(\mathcal{R}(\mu), \leq)}{\vee} \mathcal{V}^{\mathcal{P}''} \leq \mathcal{V}^{\mathcal{P}}$$

Since

$$\mathcal{V} \leq \mathcal{W} \leq \mathcal{V}^{\mathcal{P}}$$

we deduce, using Theorem 18, that there exists an exact $\mathcal{V}$-compression $\mathcal{T}$ such that $\mathcal{V}^{\mathcal{T}} \in \mathcal{R}(\mu)$, and such that

$$\mathcal{V}^{\mathcal{T}} = \mathcal{W}$$

From Theorem 19 and from the relations

$$\mathcal{V} \leq \mathcal{V}^{\mathcal{T}} \leq \mathcal{V}^{\mathcal{P}}$$

$$\mathcal{V} \leq \mathcal{V}^{\mathcal{P}_1} \leq \mathcal{V}^{\mathcal{T}}$$

$$\mathcal{V} \leq \mathcal{V}^{\mathcal{P}_2} \leq \mathcal{V}^{\mathcal{T}}$$

we get

$$Tf \precsim Pf \qquad (\forall) f \in \mathcal{F}_0$$

$$P_1 f \precsim Tf, \quad P_2 f \precsim Tf$$

where T is the initial kernel associated with $\mathcal{T}$. Hence

$$Tf = Pf \qquad (\forall)\ f \in \mathcal{F}_0,$$

$$T = P, \qquad \mathcal{V}^{\mathcal{P}} = \mathcal{V}^{\mathcal{P}_1} \underset{(\mathcal{R}(\mu), \leq)}{\vee} \mathcal{V}^{\mathcal{P}_2},$$

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