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Key words and phrases : variational problem, generalized solution, quasi-variational inequality, optimal control problem

1. INTRODUCTION

The study of linear and nonlinear operator equations was widly enlarged on the variational methods. The classical result of Friedrichs allowed the definition of the generalized solution of equation $Au = f$ with A a linear, symmetrical and positive definite operator. The class of operators for which the generalized solution of equation $Au = f$ can be defined, was enlarged to the linear operators with positive definite differentiable [1, 2], then to nonlinear operators with positive definite differentiable [3] and to nonlinear operators with K-positive definite differentiable [4].

Subsequent generalizations was obtained by considering operator equations for multivalue operators, equations suggested by practical problems from continuum mechanics. So, in [5] is studied the equation $Au + \partial\beta(u) \ni f$ with A a linear operator with symmetrical and positive definite differentiable, in [6] the equation $Pu + \partial\beta(u) \ni f$ with P a non-linear operator with symmetrical and positive definite differentiable and in [7] the equation $Pu \ni f$ with P a nonlinear multivalue operator.

In the framwork of variational theory an important place is played by the variational inequalities and this is due to the caracterization of a classical solution for the equation $Pu + \partial\beta(u) \ni f$ as solution of the variational inequality:

$$(Pu, v - u) + \beta(v) - \beta(u) \geq (f, v - u) .$$

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1. INTRODUCTION

The study of linear and nonlinear operator equations was widely developed on the variational methods. The classical result of Friedrichs allowed the definition of the generalized solution of equation $Au = f$ with A a linear, symmetrical and positive definite operator. The class of operators for which the generalized solution of equation $Au = f$ can be defined, was extended to the linear operators with positive definite differentials [1,2], then to nonlinear operators with positive definite differentials [3] and to nonlinear operators with a negative definite differentials [4].

Subsequently generalizations were obtained by considering the Euler equations for multivalued operators, problems suggested by practical problems from continuum mechanics. So, in [5] is studied the equation $Au + \partial\varphi(u) \ni f$ with A a linear operator which is symmetric and positive definite differentials, in [6] the equation $Pu + \partial\varphi(u) \ni f$ with P a nonlinear operator with symmetric and positive definite differentials and in [7] the equation $Pu \ni f$ with P a nonlinear multivalued operator.

In the framework of variational theory an important place is played by the variational inequalities and this is due to the characterization of generalized solution for the equation $Pu + \partial\varphi(u) \ni f$ as solution of the variational inequality:

$$(Pu, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u)$$

A class of variational inequalities which can't be set under the form of an operator equation is the class of so-called quasi-variational inequalities which have been used by many authors to study nonlinear problems in continuum mechanics including a wide variety of free - boundary problems .

In this paper we introduce the concept of generalized solution for nonlinear quasi-variational inequalities .

In Section 2 of this paper we recall some results concerning the generalized solution for nonlinear variational inequalities of second kind .

In Section 3 we define the generalized solution for a nonlinear quasi-variational inequality and we justify this definition (Theorem 3.1).

In Section 4 we consider a controled quasi-variational inequality.

Finnaly, in Section 5 we give am example from mechanics of a contact problem with friction which leads to a nonlinear quasi-variational inequality .

2. PRELIMINARY RESULTS

Let $(H, (\cdot,\cdot))$ be a real Hilbert space and $D(P) \subset H$ a dense linear subspace in H. Let us consider $P : D(P) \to H$ a nonlinear operator and $\beta : D(P) \to (-\infty, +\infty]$ a functional which satisfy the following assumptions :

(P1) The operator P is a potential operator, i.e. there exists a functional $\varphi : D(P) \to R$ such that $D\varphi(u)\cdot v = (Pu, v)$, $\forall\, u \in D(P)$, $\forall\, v \in H$ ($D\varphi(u)$ represents the Gateaux differential of φ) .

(P2) The operator P is monotone, i.e.

$$(Pu - Pv, u - v) \geqslant 0, \quad \forall\, u, v \in D(P).$$

(β_1) The functional β is convex lower semicontinuous and proper ($\beta \neq +\infty$)

A class of variational inequalities which can't be set under the form of an operator equation is the class of so-called quasi-variational inequalities which have been used by many authors to study nonlinear problems in continuum mechanics including a wide variety of free - boundary problems.

In this paper we introduce the concept of penalized solution for nonlinear quasi-variational inequalities.

In Section 2 of this paper we recall some results concerning the penalized solution for nonlinear variational inequalities of second kind.

In Section 3 we define the penalized solution for a nonlinear quasi-variational inequality and we justify this definition (Theorem 3.1).

In Section 4 we consider a controlled quasi-variational inequality.

Finally, in Section 5 we give an example from mechanics of a contact problem with friction which leads to a nonlinear quasi-variational inequality.

2. PRELIMINARY RESULTS

Let $(H, (\cdot,\cdot))$ be a real Hilbert space and $K \subset H$ a closed linear subspace in H. Let us consider $F : D(F) \to H$ a nonlinear operator and $\varphi : D(F) \to (-\infty, +\infty]$ a functional which satisfy the following assumptions :

(P1) : The operator F is a potential operator, i.e. there exists a functional $\psi : D(F) \to R$, such that $\psi'(u)v = (Fu, v)$, $\forall u \in D(F)$, $\forall v \in H$ ($\psi'(u)$ represents the Gateaux differential of ψ).

(P2) : The operator F is monotone, i.e.

$$(Fu - Fv, u - v) \geq 0, \quad \forall u, v \in D(F).$$

(P_3) : The functional φ is convex lower semicontinuous and proper ($\varphi \not\equiv +\infty$)

Remark 2.1. Using hypothesis (P1) and (P2) one can prove that

$$\varphi(v) = \int_0^1 (P(t\,v),\ v)\,dt + \text{const}. \qquad \text{(see [4]).}$$

We recall some results (see, for example, [7], [6]) concerning to the following problem :

$$(P + \partial\beta)(u) \ni f \tag{2.1}$$

with $f \in H$ given.

By definition, a classical solution of (2.1) is an element $u \in D(P)$ such that

$$(Pu,\ v - u) + \beta(v) - \beta(u) \geq (f,\ v - u), \qquad \forall\ v \in D(P)$$

i.e. u verifies a variational inequality of second kind.

We have the following variational caracterization of the classical solution of the equation (2.1).

Proposition 2.1. An element $u \in D(P)$ is a classical solution of (2.1) iff u minimizes on D(P) the functional $F_f : H \to (-\infty, +\infty]$ defined by

$$F_f(v) = \varphi(v) + \beta(v) - (f,\ v).$$

In the following we consider a stronger assumption than hypothese (P2), namely :

(P2') The operator P is a strongly monotone operator i.e. there exists $\gamma^2 > 0$ such that for every $u, v \in D(P)$ we have :

$$(Pu - Pv,\ u - v) \geq \gamma^2\ \|u - v\|^2,$$

where $\|\cdot\|$ is the norm on H.

Lemma 2.1. (1) For any element $f \in H$, functional F_f is lower bounded.

(2) Any minimizing sequence for F_f on D(P) is a Cauchy sequence in H,

Remark 2.1. [illegible] Using hypothesis (P1) [illegible] can prove that

$\varphi(y) = \int_{\Omega} [illegible]$ [illegible]

We [illegible] some [illegible], [illegible] for example [illegible]) either one of the following problem

$$[illegible] \qquad (2.12)$$

with $f \in H^{-1}(\Omega)$.

By definition [illegible] there exists an element $u \in H^1_0$ such that

$$(Au, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u) \qquad \forall v \in D(P)$$

i.e. u verifies a variational inequality of [illegible].

We have the following [illegible] of the [illegible] of the equation (2.12).

Proposition 2.1. [illegible] of [illegible] minimum on $D(P)$ the functional [illegible] defined by

$$F(v) = [illegible]$$

In the following we [illegible] consider the hypothesis (P2), namely :

(P2) : The operator P is [illegible], and [illegible] there exists [illegible] such that [illegible]

$$[illegible]$$

where [illegible]

Remark 2.2. [illegible] lower [illegible]

(2) [illegible]

(3) All the minimizing sequences for F_f have the same limit in H.

Lemma 1.1 suggests the following definition : the limit in H of any minimizing sequence for the functional F_f will be called the generalized solution of the equation (2 . 1) .

<u>Proposition 2.2.</u> For any element $f \in H$, the generalized solution of the equation $(P + \partial\beta)(v) \ni f$, exists and is unique .

The name of "generalized solution" is justified by :

<u>Proposition 2.3.</u> (1) The classical solution of the equation (2.1) (if there exists) is a generalized one .

(2) If the generalized solution of (2.1) belongs to D(P) then it is the classical solution .

<u>Corollary 2.1.</u> If D(P) = H, then for any $f \in H$, the problem (2.1) has a unique classical solution .

(ii) All the minimizing sequences for F_f have the same limit in H.

Lemma 1.1 suggests the following definition: the limit in H of the minimizing sequences for the functional F_f will be called the generalized solution of the equation (2.1).

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The notion of "generalized solution" is justified by:

Proposition 2.3. (1) The classical solution of the equation (2.1), (if there exists) is a generalized one.

(2) If the generalized solution of (2.1) belongs to D(F) then it is the classical solution.

Corollary 2.1. If $D(F) = H$, then for any $f \in H$ the equation (2.1) has a unique classical solution.

3. QUASIVARIATIONAL INEQUALITIES AND GENERALIZED SOLUTIONS

Let $(H, (\cdot,\cdot))$ be a real Hilbert space, $D(P) \subset H$ a linear dense subspace in H and $P : D(P) \to H$ a nonlinear operator which satisfies the hypothesis (P1) and (P2') of Section 2.

Let $j : H \times D(P) \to (-\infty, +\infty]$ be a functional which satisfies the following assumptions :

(j1) For every $u \in H$, the functional $j_u = j(u,\cdot) : D(P) \to (-\infty, +\infty]$ is convex lower semicontinuous and proper.

(j2) There exists $0 < k < \gamma^2$ such that for every $u_1, u_2 \in H$, $v_1, v_2 \in D(P)$ we have :

$$|j(u_1,v_2) + j(u_2,v_1) - j(u_1,v_1) - j(u_2,v_2)| \leq k \|u_1-u_2\| \cdot \|v_1-v_2\|.$$

We are interested in the following, by the quasivariational inequality :

$$(Pu, v - u) + j(u, v) - j(u, u) \geq (f, v - u), \qquad \forall v \in D(P) \tag{3.1}$$

where $f \in H$ is given.

A classical solution of (3.1) is an element $u \in D(P)$ which satisfies this inequality.

Remark 3.1. Using hypothese (P2'), it results that the classical solution of (3.1), if there exists, it is unique.

We shall introduce the concept of generalized solution for (3.1) and we shall derive some properties of this.

First, let us remark that the quasi-variational inequality (3.1) can not be writen as an operator equation $Tu \ni f$ so that the standard technique used in the definition of generalized solution is not applicable.

The crucial point in the definition of generalized solution for the quasivariational inequality (3.1) is the approximation of (3.1) by a sequence of

3. QUASI-VARIATIONAL INEQUALITIES AND GENERALIZED SOLUTIONS

Let $(H, (\cdot,\cdot), |\cdot|)$ be a real Hilbert space, $D(P) \subset H$ a linear dense subspace in H and $P : D(P) \to H$ a nonlinear operator which satisfies the hypotheses (P1) and (P2) of Section 1.

Let $j : H \times D(P) \to]-\infty, +\infty]$ be a functional which satisfies the following assumptions :

(j1) For every $u \in H$, the functional $j_u = j(u, \cdot) : D(P) \to (-\infty, +\infty]$ is convex lower semicontinuous and proper.

(j2) There exists $0 < k < 1$ such that for every $u_1, u_2 \in H$, $v_1, v_2 \in D(P)$ we have :

$$|j(u_1, v_2) + j(u_2, v_1) - j(u_1, v_1) - j(u_2, v_2)| \leq k\, |u_1 - u_2|\, |v_1 - v_2| .$$

We are interested in the following , by the quasi-variational inequality :

$$(Pu, v - u) + j(u, v) - j(u, u) \geq (f, v - u), \quad \forall\, v \in D(P) \qquad (3.1)$$

where $f \in H$ is given.

A classical solution of (3.1) is an element $u \in D(P)$ which satisfies this inequality.

<u>Remark 3.1.</u> Using hypotheses (P2'), it results that the classical solution of (3.1), if there exists, is unique.

We shall introduce the concept of generalized solution for (3.1) and we shall derive some properties of this.

First, let us remark that the quasi-variational inequality (3.1) can not be written as an operator equation in u, f so that the standard techniques used in the definition of generalized solution is not applicable.

The crucial point in the definition of generalized solution for the quasi-variational inequality (3.1) is the approximation of (3.1) by a sequence of variational inequalities of second kind as it is precisely shown in the sequel.

Let us denote by S the mapping $S : H \to H$ which associates to every $w \in H$ the generalized solution, that there exists and is unique (cf Proposition 2.2), of the following variational inequality of second kind :

$$(Pu, v - u) + j(w, v) - j(w, u) \geq (f, v - u), \quad \forall v \in D(P)$$

or, equivalently,

$$(P + \partial j_w)(u) \ni f$$

where $j_w(v) = j(w, v)$, $\forall v \in D(P)$.

Lemma 3.1. The mapping $S : H \to H$ is a contraction.

Proof Let w_1 $w_2 \in H$ be arbitrarily and let Sw_1, Sw_2 be the corresponding generalized solutions i.e. Sw_i (i = 1, 2) is the limit in H of any minimizing sequence for the functional $F_f^i : D(P) \to (-\infty, +\infty]$ (i = 1, 2) where :

$$F_f^i(v) = \varphi(v) + j(w_i, v) - (f, v).$$

Let us consider the minimizing sequences $(w_n^1)_n (w_n^2)_n \subset D(P)$ for F_f^1 and F_f^2, respectively. So, we have $w_n^i \xrightarrow[n]{} Sw_i$ (i = 1, 2) strongly on H.

It is easy to verify that the hypothese (P2') of strongly monotony of P implies the uniforms convexity of φ hence the uniform convexity of F_f^i i.e.

$$\lambda F_f^i(v) + (1-\lambda)F_f^i(u) - F_f^i(\lambda v + (1-\lambda)u) \geq \gamma^2 \lambda(1-\lambda)\|v - u\|^2, \; i = 1,2,$$

$\forall \lambda \in (0, 1)$, $\forall u, v \in D(P)$.

Substituting $v = w_n^i$ and $u = w_n^{3-i}$ (i = 1, 2) into the last inequality, we get :

$$\gamma^2 \lambda(1-\lambda)\|w_n^1 - w_n^2\|^2 \leq \lambda F_f^i(w_n^i) + (1-\lambda)F_f^i(w_n^{3-i}) -$$

$$- F_f^i(\lambda w_n^i + (1-\lambda) w_n^{3-i}) \leq \lambda F_f^i(w_n^i) + (1-\lambda)F_f^i(w_n^{3-i}) - d_i, \; i = (1,2) \quad (3.2)$$

variational inequalities of [illegible] shown in the sequel.

Let us denote by S the mapping [illegible] $u \in H$

the generalized solution, that is [illegible] of Proposition 2.2),

of the following variational inequality of monotone kind:

$$(F(u), v - u) + j(u, v) - j(u, u) \geq [illegible], \quad \forall v \in D(F)$$

or, equivalently,

$$0 \in F(u) + \partial j_u(u) [illegible]$$

where $j_u(v) = j(u, v)$, $\forall v \in D(F)$.

Lemma 3.1. The mapping S [illegible] continuous.

Proof. Let $u_1, u_2 \in H$ [illegible] the corresponding generalized solutions [illegible] in H of [illegible] satisfying [illegible] $(i = 1, 2)$ [illegible]

$$[illegible]$$

Let us consider [illegible] and [illegible]

respectively. [illegible] we have [illegible]

It is easy to verify [illegible] the hypotheses [illegible] strong monotonicity of F implies the following [illegible] monotonicity of [illegible]

$$[illegible] \quad i = 1, 2,$$

$\forall \lambda \in (0, 1)$, $\forall$ [illegible] $\in D(F)$.

Substituting [illegible] into [illegible] take the last inequality,

we get

$$[illegible] \qquad (3.3)$$

where we have used :

$$d_i = \inf_{v \in D(P)} F_f^i(v) \leq F_f^i(\lambda w_n^i + (1-\lambda) w_n^{3-i}).$$

By adding the inequalities (3.2) for i = 1 and i = 2, we get :

$$2\gamma^2 \lambda(1-\lambda) \|w_n^1 - w_n^2\|^2 \leq \lambda\left[(F_f^1(w_n^1) - d_1) + (F_f^2(w_n^2) - d_2)\right] +$$

$$+ (1-\lambda)\left(F_f^1(w_n^2) + F_f^2(w_n^1) - d_1 - d_2 \right). \qquad (3.3)$$

From the definition of F_f^i we have :

$$F_f^i(w_n^{3-i}) = F_f^{3-i}(w_n^{3-i}) + j(w_i, w_n^{3-i}) - j(w_{3-i}, w_n^{3-i}), \ i = 1, 2$$

hence, by using (j2), we obtain :

$$F_f^1(w_n^2) + F_f^2(w_n^1) - d_1 - d_2 \leq \left(F_f^2(w_n^2) - d_2\right) + \left(F_f^1(w_n^1) - d_1\right) +$$

$$+ k \|w_1 - w_2\| \, \|w_n^1 - w_n^2\|. \qquad (3.4)$$

On the other hand, from the definition of generalized solution we have :

$$\lim_{n\to\infty} \| w_n^1 - w_n^2 \| = \|Sw_1 - Sw_2\| \quad , \qquad (3.5)$$

$$\lim_{n\to\infty} F_f^i(w_n^i) = d_i \, . \qquad (3.6)$$

Now, passing to the limit in (3.3) for $n \to \infty$ and taking into account (3.4) - (3.6) we get :

$$2\gamma^2 \lambda(1-\lambda) \|Sw_1 - Sw_2\| \leq (1-\lambda)\lambda \|w_1 - w_2\|.$$

Finally, taking $\lambda = \frac{1}{2}$, we obtain :

$$\|Sw_1 - Sw_2\| \leq C \, \|w_1 - w_2\|$$

with $C = \frac{k}{\gamma^2} < 1$. Therefore, the lemma is proved. Hence the mapping S has a unique fixed point which will be noted by u .

Lemma 3.1 suggests the definition of the following sequence : for $u_0 \in H$ chosen arbitrarily, we put $u^n = Su^{n-1}$ i.e. u^n is the generalized solution of

where we have used :

$$d_i = \inf_{v \in D(A)} \left(F_i^1(v) + F_i^2\left(\frac{1}{\lambda}(v - \lambda) x_n^{2i}\right) \right).$$

By adding the inequalities (3.2) for $i=1$ and $i=2$, we get :

$$2\gamma^{-1}\lambda(1-\lambda)\|w_n^2 - w_n^1\|^2 \le \ldots$$

$$+ (1-\lambda)\left(F_i^1(u_n^{1i}) + F_i^2(u_n^{2i}) - \varepsilon_i - d_i\right) \qquad (3.3)$$

From the definition of ε_n^i we have :

$$F_i^1(u_n^{1i}) - F_i^1(u_n^{2i}) = \ldots, \quad i = 1, 2$$

Hence, by using (3.3), we obtain :

$$\ldots$$

$$+ \lambda\|w_1 - w_2\|\,\|w_n^1 - w_n^2\| \qquad (3.4)$$

On the other hand, from the definition of generalized solution we have :

$$\lim_{n\to\infty} \|w_n^1 - w_n^2\| = \|Sw_1 - Sw_2\| \qquad (3.5)$$

$$\lim_{n\to\infty} F_i^1(u_n^{1i}) = d_i \qquad (3.6)$$

Now, passing to the limit in (3.4) for $n \to \infty$ and taking into account (3.4) – (3.6) we get

$$2\gamma^{-1}\lambda(1-\lambda)\|Sw_1 - Sw_2\|^2 \le (1-\lambda)\|w_1 - w_2\|\,\|Sw_1 - Sw_2\|$$

Finally, taking $\lambda = \frac{\gamma}{2}$ we obtain :

$$\|Sw_1 - Sw_2\| \le c_\gamma \|w_1 - w_2\|$$

with $0 < c_\gamma < 1$. Therefore, the lemma is proved. Hence the mapping S has a unique fixed point which will be denoted by u.

Lemma 3.1 suggests the introduction of the following sequence : for $u_0 \in H$ chosen arbitrarily, we put $u^n = Su^{n-1}$; u^n is the generalized solution of the problem :

$$(P + \partial j_n)(v) \ni f$$

where $j_n(v) = j(u_{n-1}, v)$. It is immediatly that the sequence $(u_n)_n$ such defined is a Cauchy sequence hence it is convergent .

Remark 3.2. The sequence $(u_n)_n$ converges to the unique fixed point of the mapping S . Indeed, we have :

$$\|u^n - u\| = \|Su^{n-1} - Su\| \leq \frac{k}{\gamma^2}\|u^n - u\| \leq \ldots \leq \left(\frac{k}{\gamma^2}\right)^n \|u^o - u\| .$$

Definition 3.1. The limit in H of the sequence $(u_n)_n$ of generalized solutions for the problems (3.7) will be called the generalized solution of the quasi-variational inequality (3.1).

This definition is justified by the following result .

Theorem 3.1. (1) For any element $f \in H$, the generalized solution of the quasi-variational inequality (3.1) exists and is unique .

(2) The classical solution of the quasi-variational inequality (3.1) (if there exists) is a generalized one .

(3) If the generalized solution of (3.1) belongs to D(P) then it is a classical solution .

Proof (1) As it is shown, the generalized solution of (3.1) is the fixed point of the mapping S which, by lemma 3.1 , there exists and is unique .

(2) If u is the classical solution for (3.1) then u is the classical solution of the problem :

$$(P + \partial j_n)(v) \ni f \tag{3.8}$$

where $j_u(v) = j(u,v)$, $\forall\ v \in D(P)$. Hence, by proposition 2.3(1), u is the generalized solution for (3.8). On the other side, by the definition of the mapping S, the unique generalized solution of (3.8) is Su. Therefore, u = Su i.e. u is the generalized solution of the quasi-variational inequality (3.1).

(3) Let $u \in D(P)$ be the generalized solution of the quasi-variational inequality (3.1). By remark 3.2, u = Su hence the generalized solution Su of a variational inequality of second kind belongs to D(P). By applying Proposition 2.3 (2) we obtain that Su is also the classical solution i.e.

$$(P(Su),\ v - Su) + j_u(v) - j_u(Su) \geqslant (f,\ v - Su), \qquad \forall\ v \in D(P).$$

Bearing in mind that Su = u we find from the last inequality that u is the classical solution for the quasi-variational inequality (3.1)

Corollary 3.1. If D(P) = H then, for any element $f \in H$, the quasi-variational inequality (3.1) has a unique (classical) solution.

Remark 3.3. The Corollary 3.1 can be obtained directlly. Indeed, let $T : H \to H$ be the mapping which associates to every $w \in D(P) = H$ the classical solution (that there exists and is unique by corollary 2.1) of the problem :

$$(P + \partial j_w)\ (v) \ni f.$$

By using (P2') and (j2) it results that the mapping T is a contraction and the unique fixed point of T is the classical solution of (3.1) (see [8]).

Remark 3 4 By taking $j(u,v) = \beta(v)$, $\forall\ u \in H$, $\forall\ v \in D(P)$ we refind the results obtained by Dincă [6].

where $j_u(v) = j(u,v)$, $\forall\, v \in D(P)$. Hence, by proposition 2.3(1), u is the generalized solution for (3.8). On the other side, by the definition of the mapping S, the unique generalized solution of (3.8) is Su. Therefore u = Su i.e., u is the generalized solution of the quasi-variational inequality (3.1).

(3). Let $u \in D(P)$ be the generalized solution of the quasi-variational inequality (3.1). By remark 3.2, $u = Su$ hence the generalized solution Su of a variational inequality of seqond kind belongs to D(P). By applying Proposition 2.3 (2) we obtain that Su is also the classical solution, i.e.

$$(P(Su), v - Su) + j_u(v) - j_u(Su) \geq (f, v - Su), \qquad \forall\, v \in D(P).$$

Bearing in mind that Su = u we find from the last inequality that u is the classical solution for the quasi-variational inequality (3.1).

Corollary 3.1. If D(P) = H then, for any element $f \in H$, the quasi-variational inequality (3.1) has a unique (classical) solution.

Remark 3.3. The Corollary 3.1 can be obtained directly. Indeed, let $T : H \to H$ be the mapping which associates to every $u \in D(P) = H$ the classical solution (that there exists and is unique by corollary 2.1) of the problem :

$$(P + \partial j_u)(v) \ni f.$$

By using (3.1) and (3.2) it results that the mapping T is a contraction and the unique fixed point of T is the classical solution of (3.1) (see [3]).

Remark 3.4. By taking $j(u,v) = \varphi(v)$, $\forall\, u \in H$, $\forall\, v \in D(P)$, we refind the results obtained by Dinca [6].

4. CONTROLED QUASI-VARIATIONAL INEQUALITIES

Let $(W, \langle\cdot,\cdot\rangle)$ and $(H,(\cdot,\cdot))$ be a pair of real Hilbert spaces such that H is a dense subset of W and $H \subset W \subset H'$ where H' is the dual of H. We denote by $|\cdot|$ and $\|\cdot\|$ the norms on W and H, respectively.

Let $B : W \to H'$ be a linear and completely continuous operator i.e.

$$w_n \longrightarrow w \quad \text{weakly in } W \Rightarrow Bw_n \longrightarrow Bw \text{ strongly in } H. \tag{4.1}$$

For every $w \in W$ we consider the quasi-variational inequality :

$$(Pu, v-u) + j(u,v) - j(u,u) \geq (f + Bw, v-u) \quad , \forall\, v \in D(P) \tag{4.2}$$

where P , D(P), j and f are defined as in Section 3.

<u>Definition 4.1.</u> The parameter $w \in W$ will be called the control and the corresponding generalized solution of (4.2), denoted by u^w (that there exists by theorem 3.1) will be called the generalized state of (4.2).

The optimal control problem to be studied in this section can be set in the following form :

$$\min_{w \in W} \mathcal{L}(w, u^w)$$

where $\mathcal{L} : W \times H \to R$ is defined by

$$\mathcal{L}(w, v) = h(w) + g(v)$$

where $g : H \to R$ and $h : W \to R$ are given functions satisfying the following assumptions :

(i) g is locally Lipschitz and non-negative on H ;

(ii) h is convex and lower-semicontinuous on W ;

(iii) there exists the constants $C_1 > 0$, $C_2 \in R$ such that :

$$h(w) > C_1 |w| + C_2, \quad \forall\, w \in W .$$

<u>Definition 4.2.</u> A pair $(w^*, u^{w^*}) \in W \times H$ for which :

$$\mathcal{L}(w^*, u^{w^*}) = \min_{w \in W} \mathcal{L}(w, u^w)$$

will be called a generelized optimal pair and the corresponding control w^* will be called the generalized optimal control.

<u>Lemma 4.1.</u> The map $w \longmapsto u^w$ is weakly-strongly continuous from W to H.

<u>Proof</u> Let $(w_n)_n \subset W$ be a weakly convergente sequence in W to w. Let $n \in N$ be arbitrarly.

Let $(u^w_m)_m$, $(u^n_m)_m \subset D(P)$ be two minimizing sequences for the functional F_w and F_{w_n}, respectively, on D(P) where :

$$F_s(v) = \varphi(v) + j(u^s, v) - (f + Bs, v), \quad \forall\, v \in D(P),\ \forall\, s \in W. \tag{4.4}$$

By the remark 3.2 we have $u^w = Su^w$ and $u^{w_n} = Su^{w_n}$ hence $u^w_m \longrightarrow u^w$ and $u^n_m \longrightarrow u^{w_n}$ strongly in H when $m \longrightarrow \infty$.

The hypothese (P2') of P implies the uniform convexity of the functional F_s defined by (4.4) i.e.

$$\gamma^2 \lambda(1-\lambda)\|u-v\|^2 \leq \lambda F_s(u) + (1-\lambda)F_s(v) - F_s(\lambda u + (1-\lambda)v),\ \forall\, u,v \in D(P),$$
$$\forall\, \lambda \in (0,1),\ \forall\, s \in W.$$

Substituting $u = u^w_m$, $v = u^n_m$, $s = w$, $\lambda = \frac{1}{2}$ and $u = u^n_m$, $v = u^w_m$, $s = w_n$, $\lambda = \frac{1}{2}$ onto the last inequality and adding these two inequalities, we obtain :

$$\gamma^2 \|u^w_m - u^n_m\|^2 \leq \left(F_{w_n}(u^n_m) - d_{w_n}\right) + \left(F_w(u^n_m) + F_{w_n}(u^w_m) - d_w - d_{w_n}\right) + \tag{4,5}$$
$$+ \left(F_w(u^w_m) - d_w\right).$$

<u>Definition 4.2</u>. A pair $(w^*, u^{w^*}) \in W \times U$ for which :

$$\mathcal{L}(w^*, u^{w^*}) = \min_{w \in W} \mathcal{L}(w, u^w)$$

will be called a generalized optimal pair and the corresponding control w^* will be called the generalized optimal control.

<u>Lemma 4.1</u>. The map $w \mapsto u^w$ is weakly-strongly continuous from W to H.

<u>Proof</u>. Let $(w_n)_n \subset W$ be a weakly convergente sequence in W to w. Let $n \in N$ be arbitrarily.

Let $(u_m^w)_m$, $(u_m^n)_m \subset D(P)$ be two minimizing sequences for the functional F_w and F_{w_n}, respectively, on $D(P)$ where :

$$F_s(v) = \varphi(v) + j(\mathcal{E}^s, v) - (f + Bs, v) \quad , \quad \forall v \in D(P), \forall s \in W. \qquad (4.4)$$

By the remark 3.2 we have $u^w = \partial F^w$ and $u^{w_n} = \partial u^{w_n}$. Hence $u_m^w \longrightarrow u^w$ and $u_m^n \longrightarrow u^{w_n}$ strongly in H when $m \to \infty$.

The hypothese (P') of φ implies the uniform convexity of the functional F_s defined by (4.4), i.e.

$$\frac{\zeta}{2}\lambda(1-\lambda)\|u-v\|^2 \le \lambda F_s(u) + (1-\lambda)F_s(v) - F_s(\lambda u + (1-\lambda)v), \ \forall u, v \in D(P),$$
$$\forall \lambda \in (0,1), \ \forall s \in W.$$

Substituting $u = u_m^{w_n}$, $v = u_m^n$, $s = w_n$, $\lambda = \frac{1}{2}$ and $u = u_m^n$, $v = u_m^w$, $s = w_n$, $\lambda = \frac{1}{2}$ onto the last inequality and adding these two inequalities, we obtain :

$$\frac{\zeta}{4}\|u_m^w - u_m^n\|^2 \le \left(F_{w_m}(u_m^n) - d_{w_n}\right) + \left(F_w(u_m^n) + F_{w_n}(u_m^w) - d_w - d_{w_n}\right) + \qquad (4.5)$$

$$+\left(F_w(u_m^w) - d_w\right).$$

where we have used :

$$d_s = \inf_{v\in D(P)} F_s(v) \leqslant F_s\left(\frac{u_m^n + u_m^w}{2}\right) \quad , \forall s \in W .$$

On the other side, by using (j2), we have :

$$F_w(u_m^n) + F_{w_n}(u_m^w) - d_w - d_{w_n} \leqslant$$

$$\leqslant \left(F_w(u_m^n) - d_w\right) + \left(F_{w_n}(u_m^n) - d_{w_n}\right) + k\|u^w - u^{w_n}\|\cdot\|u_m^w - u_m^{w_n}\| \quad +$$

$$+ \|Bw_n - Bw\|\,\|u_m^n - u_m^w\| . \tag{4 6}$$

Bearing in the mind that :

$$\lim_{m\to\infty}\|u_m^n - u_m^w\| = \|u^n - u^w\| \quad ,$$

$$\lim_{m\to\infty} F_w(u_m^w) = d_w \;,\; \lim_{m\to\infty} F_{w_n}(u_m^n) = d_{w_n} \;,$$

and using (4.6) we obtain, by passing to the limit with $m\to\infty$ in (4.5) :

$$\|u^w - u^n\| \leqslant C\|Bw_n - Bw\|$$

where $C = \dfrac{1}{\gamma^2 - k} > 0$

Passing to the limit with $n\to\infty$ in the last inequality and using (4.1), the lemma follows.

<u>Theorem 4.1.</u> The optimal control problem (4.3) has at least one generalized optimal pair.

The proof of theorem 4.1 follows by using lemma 4.1, the hypothesis (i) - (iii) and the same techniques as in [9].

In the following we consider the case $D(P) = H$. From corollary 3.1 it results that, for every $w \in W$, the inequality (4.2) has a unique classical solution which will be noted by u_c^w.

where we have used :

$$d_w = \inf_{v \in U(P)} F_w(v) \le F_w\left(\frac{u_n^n + u_m^w}{2}\right), \quad \forall\, s \in W.$$

On the other side, by using (32), we have :

$$F_w(u_m^n) - F_{w_n}(u_n^w) - d_w - d_{w_n} \le$$

$$\le \left(F_w(u_m^n) - d_w\right) + \left(F_{w_n}(u_m^n) - d_{w_n}\right) + k\|u^w - \varepsilon^{n}(u_m^w - u_m^w)\| +$$

$$+ \|Bw_n - Bw\| \|u_m^n - u_m^w\| \qquad (4.6)$$

Bearing in the mind that :

$$\lim_{m\to\infty} \|u_m^n - u_m^w\| = \|u^n - u^w\|,$$

$$\lim_{m\to\infty} F_w(u_m^w) = d_w, \quad \lim_{m\to\infty} F_{w_n}(u_m^n) = d_{w_n},$$

and using (4.6) we obtain, by passing to the limit with $m \to \infty$ in (4.5) :

$$\|u^w - u^n\| \le C \|Bw_n - Bw\|$$

where $C = \frac{1}{1-k} > 0$

Passing to the limit with $n \to \infty$ in the last inequality and using (4.1), the lemma follows.

<u>Theorem 4.1.</u> The optimal control problem (4.3) has at least one generalized optimal pair.

The proof of theorem 4.1 follows by using lemma 4.1, the hypothesis (i) - (iii) and the same techniques as in [5].

In the following we consider the case $U(P) = H$. From corollary 3.1 it results that, for every $w \in W$, the inequality (4.2) has a unique classical solution which will be noted by u_c^w.

In the following we consider the case $D(P) = H$. From corollary 3.1 it results that, for every $w \in W$, the inequality (4.2) has a unique classical solution which will be noted by u_c^w.

We can set the following optimal control problem :

$$\min_{w \in W} \mathcal{L}(w, u_c^w) \tag{4.7}$$

and a pair $(w^*, u_c^{w^*}) \in W \times H$ which realizes the minimum in (4.7) will be called an optimal pair.

By theorem 3.1 we obtain :

$$\min_{w \in W} \mathcal{L}(w, u_c^w) = \min_{w \in W} \mathcal{L}(w, u^w)$$

from which, by using theorem 4.1 , we conclude :

<u>Proposition 4.1.</u> Let $D(P) = H$. Then

(1) A pair (w^*, u^{w^*}) is a generalized optimal pair iff it is an optimal one.

(2) The optimal problem (4.7) has at least an optimal pair.

<u>Remark 4.1.</u> By taking $D(P) = H$, P a linear continuous and symmetrical operator and $j(u,v) = \beta(v)$, $\forall u,v \in H$ with β as in Section 2, we refind some results obtained in [9] about controled variational inequalities.

5. APPLICATION TO A CONTACT PROBLEM WITH FRICTION

We shall consider a contact problem with friction in the theory of small elastic-plastic deformation whose variational formulation is a nonlinear quasi-variational inequality as formulated in Section 3.

Let Ω be an open bounded Lipschitz domain in R^p ($p = 2, 3$), occupied by the interior of an elastic-plastic body in the initial unstressed state. Let $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ be a descomposition of the boundary of the domain Ω where Γ_0, Γ_1, Γ_2 are open, disjoint and non-empty parts of $\partial\Omega$, Γ_2 beying the part of the boundary which is in contact with a rigid fix support. The body Ω is subject to body forces $f = (f_i)$ and to surface tractions $t = (t_i)$ on Γ_1. On Γ_0 we consider the displacements given. We also suppose that the support don't permit a detachment of Ω on Γ_2 so that the normal component of the displacement is zero, while the tangential component on Γ_2 is a displacement with friction.

This mechanical problem can be formulated as : find the field of displacements $u = (u_i)$ which satisfies the equilibrum equations :

$$\sigma_{ij,j}(u) + f_i = 0 \qquad \text{in } \Omega \tag{5.1}$$

and the boundary conditions :

$$u = 0 \qquad \text{on } \Gamma_0 \tag{5.2}$$

$$\sigma_{ij} n_j = t_i \text{ on } \Gamma_1 \tag{5.3}$$

$$\left.\begin{array}{l} u_n = 0, \\ |\sigma_t(u)| \leq \eta(x)\, |\sigma_n^*(u)| \text{ and} \\ \text{if } |\sigma_t(u)| < \eta(x)\, |\sigma_n^*(u)| \text{ then } u_t = 0, \\ \text{if } |\sigma_t(u)| = \eta(x)\, |\sigma_n^*(u)| \text{ then there exists } \lambda > 0 \\ \qquad\qquad \text{such that } u_t = -\lambda \sigma_t \end{array}\right\} \text{ on } \Gamma_2 \tag{5.4}$$

where $\sigma = (\sigma_{ij})$ is the stress tensor related to the strain tensor $\varepsilon = (\varepsilon_{ij})$ by means of generalized nonlinear Hooke's law :

$$\sigma_{ij}(u) = 2\, g(\gamma(u))\, \varepsilon_{ij}(u) + \left[k - \frac{2}{3} g(\gamma(u))\right] \varepsilon_{kk}(u)\, \delta_{ij} \qquad (5.5)$$

where $k = \lambda + \frac{2}{3}\mu$, λ and μ being the Lamé's coefficients of the body, g is a given function and

$$\gamma(u) = \gamma(u, u)$$

$$\gamma(u,v) = 2\, \varepsilon_{ij}(u)\, \varepsilon_{ij}(v) - \frac{2}{3}\, \varepsilon_{ii}(u)\, \varepsilon_{jj}(v).$$

In (5.4) η is the coefficient of friction and $\sigma_n^* u$ represents a regularized stress . We also have denote by u_n, u_t, σ_n, σ_t the normal and tangential components of the displacements and of the stress vector, respectively, $v_n = v_i n_i$, $v_{t_i} = v_i - v_n n_i$, where $n = (n_i)$ is the outward normal unit vector to the boundary of Ω. Throughout the paper, summation convention is used .

<u>Remark 5.1.</u> The problem (5.1) - (5.4) differs from the Signorini problem with non-local friction (see, e.g. [10]) only by the nonlinear constitutive equation (5.5).

In order to give a variational formulation of this problem let us consider the following assumptions and notations :

$$H = \left\{ v \in [H^1(\Omega)]^P \ ;\ v = 0 \text{ on } \Gamma_0,\ v_n = 0 \text{ on } \Gamma_2 \right\}$$

$$D(P) = H \cap [C^2(\Omega)]^P \cap [C(\bar{\Omega})]^P$$

$$\sigma_n^*(u) \in C^1(\bar{\Gamma}_2), \qquad \forall\, u \in H, \qquad (5.6)$$

$$\eta \in L^\infty(\Gamma_2) \text{ such that } \eta \geq 0 \text{ on } \Gamma_2, \qquad (5.7)$$

$$g \in C^2\, [0, \infty) \qquad (5.8).$$

$$(Pu, v) = \int_\Omega \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, dx, \quad \forall\, u \in D(P),\ \forall\, v \in H$$

where $\sigma = (\sigma_{ij})$ is the stress tensor related to the strain tensor $\varepsilon = \varepsilon(u)$ by means of generalized Hencky-von Mises's law :

$$\sigma_{ij}(u) = 2\mu(\gamma(u))\,\varepsilon_{ij}(u) + \left(k - \frac{2}{3}\mu(\gamma(u))\right)\varepsilon_{kk}(u)\,\delta_{ij} \qquad (5.5)$$

where $k = \lambda + \frac{2}{3}\mu$, λ and μ being the Lamé's coefficients of the body, μ is a given function and

$$\gamma(u) = \gamma(u,u),$$

$$\gamma(u,v) = 2\,\varepsilon_{ij}(u)\,\varepsilon_{ij}(v) - \frac{2}{3}\,\varepsilon_{ii}(u)\,\varepsilon_{jj}(v).$$

In (5.4) $\mathcal{F}$ is the coefficient of friction and σ_n^* represents a regularized stress. We also have denote by $u_n, u_t, \sigma_n, \sigma_t$ the normal and tangential components of the displacements and of the stress vector, respectively, $v_n = v_i n_i$, $v_{t_i} = v_i - v_n n_i$, where $n = (n_i)$ is the outward normal unit vector to the boundary of Ω. Throughout the paper, summation convention is used.

Remark 5.1. The problem (5.1) – (5.4) differs from the Signorini problem with non-local friction (see, e.g. [12]) only by the nonlinear constitutive equation (5.5).

In order to give a variational formulation of this problem let us consider the following assumptions and notations :

$$H = \{v \in [H^1(\Omega)]^p,\ v = 0 \text{ on } \Gamma_U,\ v_n \le 0 \text{ on } \Gamma_C\}$$

$$D(P) = \{u \in [C^2(\Omega)]^p \cap [C^1(\bar{\Omega})]^p\}$$

$$\sigma_n^*(\cdot) \in C^0(\Gamma_C), \quad \forall u \in P \qquad (5.6)$$

$$\mathcal{F} \in C^0(\Gamma_C) \text{ such that } \mathcal{F} \ge 0 \text{ on } \Gamma_C \qquad (5.7)$$

$$f \in [L^2(\Omega)]^p \qquad (5.8)$$

$$(Pu,v) = \int_\Omega \sigma_{ij}(u)\,\varepsilon_{ij}(v)\,dx, \quad \forall u \in D(P), \ \forall v \in H.$$

$$j(u,v) = \begin{cases} \int_{\Gamma_2} \eta(x)\, |\sigma_n^*(u)|\, |v_t|\, ds, & \text{if } u \in H,\ v \in D(P), \\ +\infty, & \text{if } v \in H \setminus D(P). \end{cases}$$

– $f \in [L^2(\Omega)]^P$, $t \in [L^2(\Gamma_1)]^P$ (5.9)

$$L(v) = \int_\Omega f_i v_i \, dx + \int_{\Gamma_1} t_i v_i \, ds, \qquad \forall\, v \in H$$

Arguiying as in [10] it is easy to prove that a variational formulation of the problem (5 . 1) - (5. 4) is the following nonlinear quasi-variational inequality :

find $u \in D(P)$ such that

$$(Pu, v-u) + j(u,v) - j(u,u) \geq L(v-u), \quad \forall\, v \in D(P). \tag{5.10}$$

Theorem 5.1. Suppose that the conditions (5. 6) - (5. 9) hold and that

$(\exists)\ g_0 = \text{const. i.e. } 0 < g_0 \leq g(s) \leq \frac{3}{2} k, \quad \forall\, s \geq 0,$

$(\exists)\ g_1 = \text{const. i.e. } g(s) + 2sg'(s) \geq g_1 > 0, \quad \forall\, s \geq 0.$

Then there exists a constant $\eta_1 > 0$ such that for every η with $\|\eta\|_{L^\infty(\Gamma_2)} < \eta_1$, the problem (5.10) has a unique generalized solution $u \in H$. Moreover, if the generalized solution u of (5.10) belongs to D(P) then u is the classical solution of the mechanical problem (5.1) - (5. 4).

Proof. The hypothesis of theorem 3.1 are fullfielled . Indeed, if we denote by $\varphi : H \to R$ the functional defined by :

$$\varphi(v) = \int_\Omega \Big[\frac{1}{2} k\, \varepsilon_{ii}^2(v) + \frac{1}{2} \int_0^{\Upsilon(v)} g(\sigma)\, d\sigma \Big] dx,$$

we obtain that the operator P is a potential (see, e.g. [11]) i.e.

$$(Pu, v) = D\varphi(u) \cdot v, \qquad \forall\, u \in D(P),\ \forall\, v \in H.$$

$$j(u,v) = \begin{cases} \int_{\Gamma} g(u)|v_T|\, ds, & \text{if } v \in B(v) \\ +\infty, & \text{if } v \in H \setminus B(v) \end{cases}$$

$$f \in [L^2(\Omega)]^p, \quad t \in [L^2(\Gamma_1)]^p \qquad (5.9)$$

$$L(v) = \int_{\Omega} f_i v_i \, dx + \int_{\Gamma_1} t_i v_i \, ds, \qquad \forall v \in H$$

Arguing as in [10] it is easy to prove that a variational formulation of the problem (5.1) – (5.4) is the following nonlinear quasi-variational inequality :

Find $u \in D(P)$ such that

$$(Pu, v-u) + j(u,v) - j(u,u) \geq L(v-u), \quad \forall v \in D(P), \qquad (5.10)$$

<u>Theorem 5.1.</u> Suppose that the conditions (5.6) – (5.9) hold and that

(3) $g_0 = \text{const.}$ i.e. $0 < g_0 \leq g(s) \leq \frac{?}{?} k, \quad \forall s \geq 0,$

(4) $g_1 = \text{const.}$ i.e. $\mu(s) + 2s\mu'(s) \geq g_1 > 0, \quad \forall s \geq 0$

Then there exists a constant $\gamma_1 > 0$ such that for every g with $\|g\|_{L^\infty(\Gamma_3)} \leq \gamma_1$, the problem (5.10) has a unique generalized solution $u \in H$. Moreover, if the generalized solution u of (5.10) belongs to $D(P)$ then u is the classical solution of the mechanical problem (5.1) – (5.4).

<u>Proof.</u> The hypothesis of Theorem 5.1 are fulfilled. Indeed, if we denote by $\Phi : H \to R$ the functional defined by :

$$\Phi(v) = \int_{\Omega} \Big[\frac{1}{2} k \, I_{1,\varepsilon}(v) + \frac{1}{2} \int_0^{\Gamma(v)} \mu(s)\, ds \Big] dx$$

we obtain that the operator P is a potential (see, e.g. [11]) i.e.

$$(Pu, v) = (\Phi'(u), v), \qquad \forall u \in D(P), \ \forall v \in H.$$

Also, we have :

$$(P(u+v) - Pu,v) \geqslant 2g_1 \int_{\Omega} \varepsilon_{ij}^2 (v)\, dx, \quad \forall\, u,\, v \in D(P),$$

from which, by Korn's inequality , we obtain (P2') with $\gamma^2 = 2g_1$.

On the other side it is easy to verify that the mapping j satisfies (j1) and that :

$$|j(u_1,v_2) + j(u_2,v_1) - j(u_1,v_1) - j(u_2,v_2)| \leqslant C \|\eta\|_{L^\infty(\Gamma_2)} \|u_1-u_2\| \, \|v_1-v_2\|$$

$$\forall\, u_1,\, u_2 \in H,\ \forall\, v_1,\, v_2 \in D(P).$$

Taking $\eta_1 < \frac{2g_1}{C}$ then, for every η with $\|\eta\|_{L^\infty(\Gamma_2)} < \eta_1$, we obtain the hypothese (j2) holds with $k = C\, \|\eta\|_{L^\infty(\Gamma_2)}$.

Now, the theorem follows by applying the theorem 3.1.

Also we have :

$$(F(u),v) = F_0(u,v) \geq 2a_1 \int_{\Omega} [illegible] (v)\,dx, \quad \forall u, v \in D(F),$$

from where, by [illegible], we obtain (F2') with [illegible] $= 2a_1$.

On the other side it is easy to verify that the mapping [illegible] satisfies (G1) and that :

$$|H(u_1,v_1) - H(u_2,v_2)| = |[illegible](u_1,v_1) - [illegible](u_2,v_2)| \leq C(|u_1 - u_2| + |v_1 - v_2|)$$

$$\forall u_1, u_2 \in H, \ \forall v_1, v_2 \in B(F).$$

Taking [illegible] $<$ [illegible], we [illegible] with [illegible], we obtain the hypothesis (H) [illegible].

Now, the theorem follows by applying the theorem 3.1.

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