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HOMOTOPY LIE ALGEBRAS AND FUNDAMENTAL GROUPS VIA DEFORMATION THEORY

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Introduction and Statement of Main Results

The aim of our work is a parallel study of the two natural graded Lie algebra objects associated to topological spaces S in traditional homotopy theory: $\mathrm{gr}^*\pi_1 S$ (the graded Lie algebra obtained from the lower central series of the fundamental group of a connected S, with bracket induced by the group commutator) and $\pi_*\Omega S$ (the connected graded homotopy Lie algebra of a 1-connected S, with Lie bracket given by the Samelson product); in the second case it turns out, as suggested by the analysis of the first case, that more accessible, and still valuable information may be gained on the bigraded homotopy Lie algebra $\mathrm{gr}^*\pi_*\Omega S$, associated to the lower central series of $\pi_*\Omega S$. The main problem one immediately faces here is related to the big difficulties raised by the concrete computation of these invariants, even in rational form. For example, the complete knowledge of the relevant homological information would not be of much help: one knows that $H^*(Sp(5)/SU(5);\mathbb{Q}) \cong H^*((S^6{\times}S^{25})\#(S^{10}{\times}S^{21});\mathbb{Q})$ as algebras [St], while the rational homotopy Lie algebras have a quite contrasting behavior, the first one being finite dimensional ([St],[GHV]), hence nilpotent and the second one being infinite dimensional and not even solvable ([HL]); similarly the nilmanifold $N_{\mathbb{R}}/N_{\mathbb{Z}}$ ($N_{\mathbb{K}}$ being the group of upper triangular unipotent 3×3 matrices with entries in $\mathbb{K}$) and the connected sum

$(S^1 \times S^2)\#(S^1 \times S^2)$ have the same rational multiplication table in low dimensions (i.e. the same $\mu\colon H^1 \wedge H^1 \to H^2$) and still their fundamental groups are strongly different, the first one being two-stage nilpotent and the other one being free on two generators (see [GM]).

By contrast to these examples we are going to look at some fixed graded-commutative unital $\mathbb{Q}$-algebra A^*, supposed to be connected ($A^0 \cong \mathbb{Q}$) and finite dimensional in each degree, and use the deformation-theoretic methods of rational homotopy theory, which provide various convenient algebraic parametrizations of the spaces S with $H^*(S;\mathbb{Q}) \cong A^*$ as algebras (see e.g. [HS],[LS],[MP], and many others), in order to exhibit conditions on A^* guaranteeing that the (bi)graded Lie algebra invariants of S described above are constant within A^*.

Various such so called intrinsic properties of A^* have been considered in the literature. For example S is called formal if its rational homotopy type is entirely determined by its rational cohomology algebra and a basic result of the theory says that any (1-connected) algebra (i.e. $A^1 = 0$) is realized by exactly one formal homotopy type [S]; S is called spherically generated if the image of its rational Hurewicz morphism coincides with the primitives of its rational homology coalgebra and a formal space is always spherically generated [HS; 8.13]. Accordingly A^* is said to be intrinsically formal (spherically generated) if any S with $H^*(S;\mathbb{Q}) \cong A^*$ is formal (spherically generated). Various sufficient conditions for intrinsic formality (spherical generation) are known (see e.g. [F],[T1],[MP]), via deformation theory.

Let us say that A^* is graded (respectively 1-graded) intrinsically formal if the bigraded (resp. graded) Lie algebra $\mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q}$ (resp. $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$) is constant within A^*.

Our first goal is to produce examples (both general and concrete classes of them) of natural sufficient conditions for the above mentioned intrinsic properties (and at the same time weaker that those already known for the other two mentioned intrinsic properties, which are generally very restrictive). Secondly we will give explicit computations of homotopy Lie algebras, in the presence of these conditions, and also give bounds for the size of homotopy Lie algebras, some of them quite generally (Propositions 3.3 and 3.4). A unitary frame will be provided by what we call "a deformation method for the fundamental group". As a consequence of our method the results will be mostly of a rational nature; however, we ought to point out that in the last section, which is devoted to link groups and represents the main application of our ideas, we also succeed to obtain honest integral results, as we shall soon see.

Our hypotheses on A^* are related to the Hopf algebra $\mathrm{Ext}^{*,*}_A(\mathbb{Q},\mathbb{Q})$, where the first degree is the resolution degree and the second is the total degree, as usual, more precisely to its indecomposables $\mathcal{Q}\mathrm{Ext}^{*,*}_A(\mathbb{Q},\mathbb{Q})$ and to its primitives $\mathcal{P}\mathrm{Ext}^{*,*}_A(\mathbb{Q},\mathbb{Q})$. The explicit description of rational homotopy Lie algebras involves: if A^* is 1-connected, there is the (minimal) Quillen model of the formal space S_A associated to A^*, to be denoted by $\mathcal{L}_A$, which is a bigraded differential Lie algebra ([Q],[T; III.3.(1)], see also the next section), and thus $H^*_*\mathcal{L}_A$ becomes

$(S^3 \times S^3)\#(S^2 \times S^5)$ have the same rational multiplication table in low dimensions (i.e. the same $\mu : H^1 \wedge H^1 \to H^2$) and still their fundamental groups are strongly different, the first one being two-stage nilpotent and the other one being free on two generators (see [GM]).

By contrast to these examples we are going to look at some fixed graded commutative unital $\mathbb{Q}$-algebra A^*, supposed to be connected ($A^0 = \mathbb{Q}$) and finite dimensional in each degree, and use the deformation theoretic methods of rational homotopy theory, which provide various convenient algebraic parametrizations of the spaces S with $H^*(S;\mathbb{Q}) \cong A^*$ as algebras (see e.g. [HS],[LS],[Mf], and many others), in order to exhibit conditions on A^* guaranteeing that the (bi)graded Lie algebra invariants of S described above are constant within A^*.

Various such so called *intrinsic* properties of A^* have been considered in the literature. For example S is called formal if its rational homotopy type is entirely determined by its rational cohomology algebra and a basic result of the theory says that any (1-connected) algebra (i.e. $A^1 = 0$) is realised by exactly one formal homotopy type [S]; S is called spherically generated if the image of its rational Hurewicz morphism coincides with the primitives of its rational homology coalgebra and a formal space is always spherically generated [HS, 8.13]. Accordingly A^* is said to be intrinsically formal (spherically generated) if any S with $H^*(S;\mathbb{Q}) \cong A^*$ is formal (spherically generated). Various sufficient conditions for intrinsic formality (spherical generation) are known (see e.g. [F],[TH],[MP]), via deformation theory.

Let us say that A^* is graded (respectively 1-graded) intrinsically formal if the bigraded (resp. graded) Lie algebra $gr^*_* \pi_1 S \otimes \mathbb{Q}$ (resp. $gr^*_1 \pi_1 S \otimes \mathbb{Q}$) is constant within A^*.

Our first goal is to produce examples (both general and concrete classes of them) of natural sufficient conditions for the above new intrinsic properties (and at the same time weaker that those already known for the other two mentioned intrinsic properties, which are generally very restrictive). Secondly, we will give explicit computations of homotopy Lie algebras, in the presence of these conditions, and also give bounds for the size of homotopy Lie algebras, some of them quite sharp (Propositions 8.3 and 8.4). A unitary frame will be provided by what we call "a deformation method for the fundamental group". As a consequence of our method the results will be mostly of a rational nature; however, we ought to point out that in the last section, which is devoted to link groups and represents the main application of our ideas, we also succeed to obtain honest integral results, as we shall soon see.

Our hypotheses on A^* are related to the Hopf algebra $\mathrm{Ext}^*_{A^*}(\mathbb{Q},\mathbb{Q})$, where the first degree is the resolution degree and the second is the total degree, as usual, more precisely to its indecomposables $Q\mathrm{Ext}^*_{A^*}(\mathbb{Q},\mathbb{Q})$ and to its primitives $P\mathrm{Ext}^*_{A^*}(\mathbb{Q},\mathbb{Q})$. The explicit description of rational homotopy Lie algebras involves, if A^* is 1-connected, there is the (minimal) Quillen model of the formal space S_A associated to A^*, to be denoted by L_A, which is a bigraded differential Lie algebra ([Q],[T1],[Tn2]); see also the next section), and thus H_*L_A becomes a bigraded Lie algebra (for an even more precise computation, see Theorem B(i) below); to a connected A^* one may associate the dual of the cup product pairing, $\mu\colon A^1 \wedge A^1 \to A^2$, to be denoted by $\partial\colon Y \to X \wedge X$, and then the graded Lie algebra L_A^*, defined as the quotient of the free Lie algebra on X, $\mathbb{L}^* X$, graded by the bracket length, by the ideal generated by ∂Y - under the obvious identification $\mathbb{L}^2 X \cong X \wedge X$ (see also Lemma 1.8(i) for a further construction, related to the explicit computation of the rational nilpotent completion of $\pi_1 S_A$).

Before stating our first results, let us make one more definition: the <u>cup-genus</u> of A^*, to be denoted by $\mathrm{cg}(A)$, is defined to be the maximal dimension of the graded vector subspaces $N \subset A^+$, having the property that $N \cdot N = 0$ and $N \cap (A^+ \cdot A^+) = 0$; the same definition obviously applies to a vector valued 2-form, $\mu\colon A^1 \wedge A^1 \to A^2$, where $\mathrm{cg}(\mu)$ equals the maximal dimension of the vector subspaces $N \subset A^1$ for which $\mu(N \wedge N) = 0$. In the classical case of the cohomology of a closed oriented surface, the two definitions coincide, their common value being just the genus of the surface, hence the terminology.

THEOREM A. *Let A^* be a 1-connected graded algebra.*

(i) If $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q}) = 0$ and if A^* is intrinsically spherically generated then A^* is graded intrinsically formal and the constant value of the bigraded homotopy Lie algebra equals $H_*^* \mathcal{L}_A$.

(ii) Suppose A^* is graded intrinsically formal. If $\mathrm{cg}(A) > 1$ then, for any 1-connected S with $H^*(S; \mathbb{Q}) \cong A^*$, the graded rational homotopy Lie algebra $\pi_* \Omega S \otimes \mathbb{Q}$ contains a free graded Lie algebra on two generators.

THEOREM A'. *Let A^* be a connected graded algebra, with associated vector-valued 2-form μ.*

(i) If $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q}, \mathbb{Q}) = 0$ then A^* is 1-graded intrinsically formal and the constant value of the rational graded Lie algebra associated to the fundamental group equals L_A^*.

(ii) Suppose A^* is 1-graded intrinsically formal. If $\mathrm{cg}(\mu) > 1$ then, for any connected S with $H^*(S; \mathbb{Q}) \cong A^*$ and with $H_1(S; \mathbb{Z})$ finitely generated, $\pi_1 S$ contains a free group on two generators.

We must point out that the condition of intrinsic spherical generation is necessary in Theorem A(i), see 2.3. As for the vanishing of $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q})$, this condition, while not really necessary (see again 2.3), seems to be a most natural one, as it follows for example from the proof given in Section 2. On the other hand, it is both a familiar condition, being first considered in [P] in connection with the cohomology of the Steenrod algebra and then intensively studied, see [Lo] for the connection with the cohomology of local rings, and there are many other interesting examples (see the next theorems, and also [Pa], as explained in 2.3). Theorem A(ii) should be related to the (yet unsolved) Félix-Avramov conjecture [FT], claiming that, for a space S of finite rational Lusternik-Schnirelmann category, $\pi_* \Omega S \otimes \mathbb{Q}$ contains a bigraded Lie algebra and an even more precise computation, see Theorem B(i) below), to a connected X one may associate the dual of the cup product pairing, $\mu: A^1 \wedge A^1 \to A^2$, to be denoted by $\partial: Y \to X \wedge X$, and then the graded Lie algebra L_μ, defined as the quotient of the free Lie algebra on X, $\mathbb{L}^* X$, graded by the bracket length, by the ideal generated by ∂Y — under the obvious identification $\mathbb{L}^2 X = X \wedge X$ (see also Lemma 1.2(i) for a further consequence related to the explicit computation of the rational nilpotent completion of $\pi_1 S$).

Before stating our first results, let us make one more definition: the cup-genus of A^*, to be denoted by $g(A)$, is defined to be the minimal dimension of the graded vector subspace $N^* \subset A^+$ having the property that $A^+ N^+ = 0$ and $N^+ \cap (A^+ \cdot A^+) \neq 0$; the same definition obviously applies to a vector-valued 2-form, $\mu: A^1 \wedge A^1 \to A^2$, where $g(\mu)$ equals the minimal dimension of the vector subspace $N \subset A^1$ for which $\mu(N \wedge N) \neq 0$. In the classical case of the cohomology of a closed oriented surface, the two definitions coincide, their common value being just the genus of the surface; hence the terminology.

THEOREM A. *Let A^* be a 1-connected graded algebra.*

(i) *If $\mathrm{TExt}_{A}^{**}(\mathbb{Q},\mathbb{Q}) = 0$ and if A^* is intrinsically sufficiently generated then A^* is graded intrinsically formal and the constant value of the bigraded homotopy Lie algebra equals $H_* L_A$.*

(ii) *Suppose A^* is graded intrinsically formal. If $g(A) > 1$ then, for any 1-connected S with $H^*(S,\mathbb{Q}) \cong A^*$, the graded rational homotopy Lie algebra $\pi_* \Omega S \otimes \mathbb{Q}$ contains a free graded Lie algebra on two generators.*

THEOREM A'. *Let A be a connected graded algebra with associated vector-valued 2-form μ.*

(i) *If $\mathrm{TExt}_{A}^{**}(\mathbb{Q},\mathbb{Q}) = 0$ then A is 1-graded intrinsically formal and the constant value of the rational graded Lie algebra associated to the fundamental group equals L_μ.*

(ii) *Suppose A^* is 1-graded intrinsically formal. If $g(\mu) > 1$ then, for any connected S with $H^*(S,\mathbb{Q}) \cong A^*$ and with $\pi_1 S$ finitely generated, $\pi_1 S$ contains a free group on two generators.*

We must point out that the condition of intrinsic sufficient generation is necessary in Theorem A(i), see 2.3. As for the vanishing of $\mathrm{TExt}_{A}^{**}(\mathbb{Q},\mathbb{Q})$, this condition, while not really necessary (see again 2.3), seems to be a most natural one, as it follows for example from the proof given in Section 2. On the other hand, it is both a familiar condition, being first considered in [ ] in connection with the cohomology of the Steenrod algebra, and then extensively studied, see [Lö] for the connection with the cohomology of local rings, and there are many other interesting examples, see the next Theorem, and also [P], as explained in 2.8). Theorem A(ii) should be related to the (yet unsolved) Félix-Avramov conjecture [FT], claiming that, for a space S of finite rational Lusternik-Schnirelmann category, $\pi_* \Omega S \otimes \mathbb{Q}$ contains a free graded Lie algebra on two generators as soon as it is infinite dimensional. Similarly, the vanishing of $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})$ in A'(i) above is not necessary (see 2.3) but again most natural (see the proof), and there are also many examples, as we shall see below, including link cohomologies. Theorem A'(ii) should be compared to similar results from [C1],[C2]; the methods are however entirely different and the hypotheses are rather complementary (see the remarks made after the proof of A'(ii) given in 3.2, and Proposition 3.3). Our Theorems A and A' emerged from our belief that the main results of [KS], namely that the cohomology algebras $H^*(\vee S^1;\mathbb{Q})$ and $H^*((\vee S^1)\times S^1;\mathbb{Q})$ are 1-graded intrinsically formal - in our terminology (which were proved there "by hand"), ask for a proper generalization in a deformation theoretic framework.

Moving to our more concrete classes of examples of (finitely generated) graded algebras A^*, we shall focus here our attention to two parallel types, namely: <u>2-skeletal</u> algebras defined by the requirement that $A^{>2}=0$ (which are plainly uniquely described by a vector-valued 2-form, $\mu\colon A^1\wedge A^1\to A^2$, and here we first have in mind the cohomology algebras arising in link theory), and on the other hand <u>2-stage</u> (1-connected) algebras, defined by the condition $(A^+)^3=0$ (which correspond, by [FH; Corollary 4.10], to formal spaces with rational Lusternik-Schnirelmann category ≤ 2, an intensively studied case). The resemblance is quite clear; to be more precise, there is a one-to-one onto correspondence between 2-skeletal algebras with $\mu=$ onto and 2-stage algebras generated in dimension 3, given by just tripling degrees. Given a 2-skeletal algebra A^*, we first associate to it by duality, as above, the map ∂, then construct a bigraded connected Lie algebra as follows: $E_*^*=\mathbb{L}X/\mathrm{ideal}(\partial Y)$, where the upper degree comes from bracket length and the lower degree is obtained by assigning to X the (lower) degree 2; at the same time we may pick bases $\{x_1,\dots,x_m\}$ for X and $\{y_1,\dots,y_n\}$ for Y and consider the sequence of elements $\partial y_j\in\mathbb{T}^2X$ ($\mathbb{T}^*X=\bigoplus\mathbb{T}^pX$, $\mathbb{T}^pX=X^{\otimes p}$ is the tensor algebra). Similarly, a 2-stage A^* gives rise to a Quillen model of the form $\mathcal{L}_A=(\mathbb{L}(X_*\oplus Y_*),\partial)$, where $\partial X_*=0$ and $\partial|_{Y_*}\colon Y_*\rightarrowtail(\mathbb{L}^2X)_{*-1}$ (see §1 and §4); we may thus define, exactly as before, a bigraded Lie algebra E_*^* and elements $\partial y_j\in\mathbb{T}^2X_*$ (note that in the correspondence given by tripling degrees these objects are <u>the</u> <u>same</u>).

In the results below we shall make heavy use of Anick's notion of <u>strongly</u> <u>free</u> set of elements in a connected graded associative algebra [A; page 127] and of Halperin and Lemaire's natural specialization to <u>inert</u> <u>sequences</u> of elements of a connected graded Lie algebra [HL; Définition 3.1].

THEOREM B. *Let A^* be a 2-stage algebra.*

(i) The following conditions are equivalent:

- $\mathcal{Q}\mathrm{Ext}_A^{\geq 1,*}(\mathbb{Q},\mathbb{Q})=0$

a free graded Lie algebra on two generators as soon as it is infinite dimensional. Similarly the vanishing of $H^{>1}(\mathcal{L}_{\mathfrak{g}}(\epsilon); K)$ above is not necessary (see 2.8) but again most natural (see the proof), and there are also many examples, as we shall see below, including link cohomologies. Theorem A (ii) should be compared to similar results from [C1],[C2]; the methods are however entirely different and the hypotheses are rather complementary (see the remarks made after the proof of A(ii) given in 3.2, and Proposition 3.3). Our Theorems A and A′ emerged from our belief that the main results of [KS], namely that the cohomology algebras $H^*(\vee S^1; \mathbb{Q})$ and $H^*((\vee S^1) \vee S^2; \mathbb{Q})$ are 1-graded intrinsically formal (in knot terminology) (which were proved there "by hand"), ask for a proper generalization in a deformation theoretic framework.

Moving to our more concrete classes of examples of (finitely generated) graded algebras A^*, we shall focus here our attention to two parallel types, namely: 2-skeletal algebras defined by the requirement that $A^{>2} = 0$ (which are plainly uniquely described by a vector-valued 2-form, $\mu: \wedge^2 A^1 \to A^2$, and after we first have in mind the cohomology algebras arising in link theory), and on the other hand 2-step (1-connected) algebras defined by the condition $(A^+)^3 = 0$ (which correspond, by [HH, Corollary 4.10], to formal spaces with rational Lusternik-Schnirelmann category ≤ 2, an intensively studied case). The resemblance is quite close; to be more precise, there is a one-to-one correspondence between 2-skeletal algebras with μ = onto and 2-step algebras generated in dimension 2, given by just tripling degrees. Given a 2-skeletal algebra A^*, we first associate to it, by duality, as above, the map ∂. Then construct a bigraded connected Lie algebra as follows: $E_\mu^* = \mathbb{L}^*(X)/\text{ideal}(\partial Y)$, where the upper degree comes from bracket length and the lower degree is obtained by assigning to X the (lower) degree 2. Assuming μ is onto, we may pick bases $\{x_1, \ldots, x_n\}$ for X and $\{y_1, \ldots, y_m\}$ for Y and consider the sequence of elements $\partial y_j \in T^2X$ ($T^*X = \bigoplus T^nX$, $T^nX = X^{\otimes n}$ is the tensor algebra). Similarly, a 2-stage A^* gives rise to a Quillen model of the form $C_\mu = (\mathbb{L}(X \oplus Y), \partial)$, where $\partial X = 0$ and $\partial y_j \in \mathbb{L}^2_* = (\mathbb{L}^2 X)_{\leq 4}$ (see [T] and [Q]), we may thus define, exactly as before, a bigraded Lie algebra E_μ^* and elements $\partial y_j \in T^2X$ (note that in the current context the upper and lower degrees of these objects are [illegible]).

In the results below we shall make heavy use of Anick's notion of strongly free set of elements in a connected graded associative algebra [A, page 421], and of Halperin and Lemaire's natural specialization to inert sequences of elements of a connected graded Lie algebra [HL, Definition 3.1].

THEOREM B. *Let A^* be a 2-step algebra.*

(i) *The following conditions are equivalent:*

$(\mathcal{C}1)^1_A$: $H^{>1}(\mathcal{L}_A; \mathbb{Q}) = 0$;

- gl dim $E_* \leq 2$

- $\{\partial y_j\}_{1\leq j\leq n}$ *is strongly free in* $\mathbb{T}X$

Any of them implies that $H^*_* \mathcal{L}_A \cong E^*_*$ *(as bigraded algebras).*

(ii) Assume $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q}) = 0$. *If* $\mathcal{Q}A^*$ *is concentrated in odd degrees or in an interval of degrees of the form* $[l-1, 3l-1]$, *then* A^* *is also intrinsically spherically generated.*

THEOREM B'. *Let* A^* *be a 2-skeletal algebra.*

(i) The following are equivalent:

- $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$
- $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$
- gl dim $E_* \leq 2$ *and* μ *is onto*
- $\{\partial y_j\}_{1\leq j\leq n}$ *is strongly free in* $\mathbb{T}X$ *(automatically* μ *must be onto).*

(ii) Any of the above implies the equality of formal power series

$$\prod_{p=1}^{\infty}(1-z^p)^{\dim L_A^p} = 1 - mz + nz^2, \quad \textit{where } m = \dim X$$

The main source of examples for B(i) and B'(i) is provided by Anick's combinatorial criterion [A; Theorem 3.2] for strong freeness in tensor algebras (see 4.3, 5.2). We should also mention that in establishing the formula in B'(ii), where we tackle the difficult problem of computing the ranks rk $\mathrm{gr}^p\pi$ ($\pi = \pi_1 S$), we are not bound by our methods to restrict ourselves to 2-skeletal algebras (see the discussion around the Kohno example [KT] in 4.7).

We finally come to our main application. It is devoted to the computation of the graded Lie algebra $\mathrm{gr}^*\pi$ of link groups π. Unfortunately (as the example of Borromean rings examined in [GM] shows) in general even the ranks rk $\mathrm{gr}^p\pi$ are not a formal consequence of the cohomology algebra of the link. We find out here a possible explanation of this phenomenon, related to the richness of the linking numbers structure.

Consider then $\ell = (l_{ij})_{(i,j)\in I\times I}$, a symmetric matrix with zero on the diagonal, indexed by an m-element set I and with entries in $\mathbb{Q}$ (or $\mathbb{Z}$), and construct an associated 2-skeletal <u>link-algebra</u> $A^* = A^*_\ell$ as follows: set A^1 = $\mathbb{Q}$-vector space with basis $\{e_1, \ldots, e_m\}$, $A^2 = A^1 \wedge A^1$, modulo the relations $e_i \wedge e_j + e_j \wedge e_k = e_i \wedge e_k$, $A^{>2} = 0$, and define $\mu(e_i \wedge e_j) =$ class of $l_{ij}(e_i \wedge e_j)$ in A^2 (of course, if $K \subset S^3$ is an m-component link and ℓ is the matrix of its linking numbers, then $A^*_\ell \cong H^*(S^3 \setminus K; \mathbb{Q})$). Construct then a finite unoriented graph Γ_0, with vertices labeled by $v_1, \ldots, v_m$ and arrows $\{v_i, v_j\}$ introduced if and only if $l_{ij} \neq 0$; for a given prime p we may similarly construct (if all the entries lie in $\mathbb{Z}$) a graph Γ_p, where $\{v_i, v_j\}$ is an arrow if and only if $l_{ij} \not\equiv 0$ mod p. We may now state our result.

THEOREM C.

(i) For a link algebra A^*, the following are equivalent:

- A^* *is 1-graded intrinsically formal*
- $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})=0$
- Γ_0 *is connected*

Any of them implies that the value of the graded Lie algebra $gr^*\pi_1 S\otimes\mathbb{Q}$ is constant as soon as $H^*(S;\mathbb{Q})\cong A^*$ and equal to $\mathbb{L}^*_{\mathbb{Q}}(x_1,\dots,x_m)$ modulo the relations $\sum_{j\in I} l_{ij}[x_i,x_j]=0$, $i\in I$, where $\deg x_i=1$; moreover, its Hilbert series $\sum rk\ gr^p\pi_1 S\cdot z^p$ depends only on m, being equal to $\sum rk\ gr^p(\mathbb{F}_{m-1}\times\mathbb{F}_1)\cdot z^p$, where $\mathbb{F}_k$ stands for the free group on k generators.

(ii) Given an m-component link group π, the connectedness of all graphs Γ_p (p-prime) is equivalent to the fact that the abelian group $gr^2\pi$ is free of rank $(m-1)(m-2)/2$ and in this case the graded Lie algebra $gr^*\pi$ is isomorphic to $\mathbb{L}^*_{\mathbb{Z}}(x_1,\dots,x_m)$ modulo the above mentioned relations, and it is torsion free as a graded $\mathbb{Z}$-module.

Similar results for link groups were obtained, by using a completely different method, in [H]. Hain's geometric method used there enables one, in principle, to deal also with links with poor linking numbers structure. As far as the concrete examples are concerned, he is forced to use a theorem in combinatorial group theory due to Labute [La1], which involves the highly nontrivial verification of a certain independence property for the defining relators of a finitely presented group; consequently the concrete examples do not abound, and thus our work may be viewed as complementary to Hain's. (At this point it is interesting to notice the perfect resemblance between Labute's independence property and Halperin-Lemaire's criterion for inertia in a free Lie algebra [HL]!) Our method of proof of Theorem C also gives a (partial) answer to a question in combinatorial group theory raised by Labute in connection with his independence property ([La2; Problem 5])–see 5.7.

Finally, we ought to mention that in general given a connected algebra A^* with associated graded Lie algebra L^*_A, the vanishing of $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q})$ provides one with a very convenient framework for "the deformation theory of the fundamental group". In particular, one has the following (stronger) rigidity result: if moreover $H^{2,\geq 1}(L^*_A;L^*_A)=0$ (where the first degree of the above Lie algebra cohomology is the resolution degree and the other comes from the grading of L^*_A as usual) then the rational nilpotent completion of $\pi_1 S$ is constant within A^* (for instance Kojima's [KS] rigidity result for $A^*\cong H^*(\vee S^1;\mathbb{Q})$ immediately follows, since in this case L^*_A is free). We shall undertake this further step of our theory in a subsequent paper.

Here is the plan of our paper:

1. Algebraic models and deformation theory

2. Rigidity results (proofs of A(i) and A'(i))
3. Bounds for homotopy Lie algebras (proofs of A(ii) and A'(ii))
4. Rigid examples and inert sequences (proofs of B and B')
5. Link groups (proof of C).

A preliminary version of our results was announced by the second author in a lecture given at the Conference OATE 2, September 1989, Craiova, Romania.

1. Algebraic Models and Deformation Theory

In this first section we will purely algebraically reformulate theorems A and A' and prove a deformation-theoretic result which will be the key step in the proofs of A(i) and A'(i) to be given in the next section.

We shall deal with bigraded Lie algebras (bglie) L^*_*, $L = \bigoplus L^p_n$, $n \geq 0$ and $p > 0$; ignoring upper degrees, we are thus considering just an usual graded Lie algebra (glie) L_*, with the standard sign conventions related to the skew-commutativity and the Jacobi identity [T; 0.4.(1)]; the upper degrees are only required to be compatible with the bracket, i.e. $[L^p, L^q] \subset L^{p+q}$; we shall also frequently have the occasion to meet bigraded Lie algebras whose lower degrees are concentrated in dimension zero; by just ignoring them, we shall speak of a Lie algebra with grading (grlie) L^* (no extra signs !). We shall also consider the lower central series of a (graded) Lie algebra L, the descending chain of (graded) ideals inductively defined by $L^{(1)} = L$ and $L^{(p+1)} = [L, L^{(p)}]$, and the associated (bi)graded Lie algebra $\mathrm{gr}^*L = \bigoplus_{p\geq 1} \mathrm{gr}^p L$, $\mathrm{gr}^p L = L^{(p)}/L^{(p+1)}$; the topological examples we have in mind are the homotopy Lie algebra $\pi_*\Omega S$ of a 1-connected space S, and its associated graded, $\mathrm{gr}^*\pi_*\Omega S$. Similarly one may consider the lower central series of a group π, denoted by $\pi^{(p)}$, $p \geq 1$, and the associated Lie algebra with grading $\mathrm{gr}^*\pi = \bigoplus_{p\geq 1} \mathrm{gr}^p\pi$, $\mathrm{gr}^p = \pi^{(p)}/\pi^{(p+1)}$, see e.g. [Se; II.2], for example the homotopy grlie algebra of a connected space S, $\mathrm{gr}^*\pi_1 S$.

We are going to exploit the duality between Lie algebras and (quadratic free) differential graded commutative algebras (dga's)–see [T; Proposition I.1.(5)] (in particular we shall follow the notation of [T] and constantly denote vector space duals by $\#$). We thus recall that there is a (categorial) equivalence between bigraded Lie algebras, which are of finite type with respect to the lower degree, and free dga's of the form $(\wedge Z^*_*, D)$, where $Z = \bigoplus Z^n_p$, $n > 0$ and $p \geq 0$, $\dim Z^n < \infty$ for all n and the differential D is quadratic and bihomogeneous, i.e. $DZ^n_p \subset (\wedge^2 Z)^{n+1}_{p-1}$. The equivalence is described by $L \longmapsto C^*(L) = (\wedge Z, D)$ [T; I.1], where $Z^n_p = \# L^{p+1}_{n-1}$ and $D\colon Z \to Z \wedge Z$ is dual to the Lie bracket.

Let $(\wedge V^*, D)$ be a free dga, which is of finite type and with quadratic differential, and let

2. Rigidity results (proofs of A(i) and A'(i))
3. Bounds for homotopy Lie algebras (proofs of A(ii) and A'(ii))
4. Rigid examples and large sequences (proofs of B and B')
5. Link groups (proof of C)

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In this first section we will purely algebraically reformulate theorems A and A' and prove a deformation-theoretic result which will be the key step in the proofs of A(i) and A'(i) to be given in the next section.

We shall deal with bigraded Lie algebras (briefly L), $L = \bigoplus L_n^p$, $n \geq 0$ and $p \geq 0$; ignoring upper degrees, we are thus considering just an usual graded Lie algebra (glie) L_* with the standard sign conventions related to the skew commutativity and the Jacobi identity [T; 0.4.(1)]; the upper degrees are only required to be compatible with the brackets, i.e. $[L^p, L^q] \subset L^{p+q}$. We shall also frequently have the occasion to meet ungraded Lie algebras whose lower degrees are concentrated in dimension zero; by just ignoring them, we shall speak of a Lie algebra with filtration (briefly L^*) (no extra signs!). We shall also consider the lower central series of a (graded) Lie algebra L, the descending chain of (graded) ideals inductively defined by $L^{(1)} = L$ and $L^{(p+1)} = [L, L^{(p)}]$, and the associated (bigraded) Lie algebra $\mathrm{gr}^* L = \bigoplus_{p \geq 1} \mathrm{gr}^p L$, $\mathrm{gr}^p L = L^{(p)}/L^{(p+1)}$. The topological examples we have in mind are the homotopy Lie algebra $\pi_*^{\mathbb{Q}}$ of a 1-connected space S, and its associated graded, $\mathrm{gr}^*\pi_* S$. Similarly one may consider the lower central series of a group π, denoted by $\pi^{(p)}$, $p \geq 1$, and the associated Lie algebra with gradings $\mathrm{gr}^*\pi = \bigoplus_{p \geq 1} \mathrm{gr}^p \pi$, $\mathrm{gr}^p \pi = \pi^{(p)}/\pi^{(p+1)}$, see e.g. [Se; II.2], for example the fundamental group of a connected space S, $\mathrm{gr}^*\pi_1 S$.

We are going to exploit the duality between Lie algebras and (quadratic free) differential graded commutative algebras (briefly cdga), see [T; Proposition I.1.(5)] (in particular we shall follow the notation of [T] and consistently denote vector space duals by #). We thus recall that there is a (categorical) equivalence between bigraded Lie algebras, which are of finite type with respect to the lower degrees, and cdga's of the form $(\Lambda Z_*^*, D)$, where $Z = \bigoplus Z_n^p$, $n > 0$ and $p \geq 0$, the Z^p's are finite, and the differential D is quadratic and homogeneous, i.e. $DZ_n^p \subset (\Lambda^2 Z)_{n-1}^p$. The equivalence is described by $L \mapsto C^*(L) = (\Lambda Z, D)$ [T; I.1], where $Z_n^p = \# L_{n-1}^p$, and $D: Z \to \Lambda^2 Z$ is dual to the Lie bracket.

Let $(\Lambda Z^*, D)$ be a cdga which is of finite type and with quadratic differential, and let

L_* be the dual glie. By a nilpotent filtration on $(\wedge V, D)$ we shall mean an ascending filtration on V^*, $\{F_p\}_{p\geq 0}$, with $F_0 = 0$ and $DF_p \subset \wedge^2 F_{p-1}$, any $p \geq 1$, which will be called exhaustive if $V = \bigcup F_p$. The canonical filtration $\{F_pV\}_{p\geq 0}$ is defined by $F_0V = 0$ and, inductively, $F_pV = (D|_V)^{-1} \wedge^2 F_{p-1}V$, for $p \geq 1$. By construction, it is nilpotent.

1.1. LEMMA. *For any $p \geq 0$, F_pV coincides, by duality, with the space orthogonal to $L^{(p+1)}$, $L^{(p+1)^\perp}$.*

PROOF: This lemma is both elementary and perhaps well known. For the reader's convenience we are going to sketch a proof.

For $p = 0, 1$, this is obvious. For $v \in V$ and $f, g \in L$ recall the basic duality equation [T; page 26]:

$$\langle v, [f,g]\rangle = (-1)^{\deg g} \cdot \langle Dv, f \wedge g\rangle. \tag{1.2}$$

The induction step will be based on the equation $N^\perp \wedge N^\perp \cong (M \wedge N)^\perp$, where M is a (graded) vector space and $N \subset M$ a (graded) subspace, which follows from elementary multilinear algebra.

Suppose then $F_{p-1}V \cong L^{(p)^\perp}$. By definition, $v \in F_pV$ if and only if $Dv \in \wedge^2 F_{p-1}V$. Invoking the above equation (with $M = L$ and $N = L^{(p)}$), we have $\wedge^2 F_{p-1}V \cong \wedge^2 L^{(p)^\perp} \cong (L \wedge L^{(p)})^\perp$. But $Dv \in (L \wedge L^{(p)})^\perp$ means, by (1.2), exactly $v \in [L, L^{(p)}]^\perp \cong L^{(p+1)^\perp}$, which completes the proof. ∎

We are now going to briefly review the algebraic models of rational homotopy Lie algebras. Any connected space S has a so-called (Sullivan) minimal model $\mathcal{M}_S$: it is a free dga $(\wedge Z^*, d)$, which is both nilpotent, i.e. Z is increasingly filtered by subspaces $\{F_p\}_{p\geq 0}$, with $F_0 = 0$ and $dF_p \subset \wedge F_{p-1}$ for $p > 0$, and minimal, that is $d|_Z = d_2 + d_3 + \ldots$, where each d_i takes values in $\wedge^i Z$, where i = monomial length; when S is 1-connected with finite Betti numbers, Z^* is of finite type. In this latter case there is also the Quillen model of S, $\mathcal{L}_S$; this is a free dgLie $(\mathbb{L}W_*, \partial)$, which is also minimal, i.e. $\partial W \subset [\mathbb{L}W, \mathbb{L}W]$. The first basic result reads

THEOREM ([Q],[S],[T]). *$\pi_*\Omega S \otimes \mathbb{Q} \cong \mathcal{C}^{*^{-1}}(\wedge Z^*, d_2) \cong H_*\mathcal{L}_S$ as graded algebras.*

For a general connected S one may also consider the 1-minimal model, namely the sub dga $\mathcal{M}_1 \subset \mathcal{M}$ given by $\mathcal{M}_1 = (\wedge V, d)$ (where we put $Z^1 = V$); this will be a 1-minimal algebra, i.e. a free nilpotent dga generated in degree one. For such algebras, one may still define the canonical filtration exactly as above: this will be an exhaustive nilpotent filtration on $(\wedge V, d)$, $\mathcal{F} = \{F_pV\}_{p\geq 0}$. If moreover the first Betti number of S is finite, then it easily follows that

3. In the final step [illegible] we shall regard as [illegible] filtration [illegible] which will be called exhaustive [illegible]. The associated filtration [illegible] is defined by [illegible] $= 0$ and, inductively, [illegible]. By construction, it is [illegible].

Lemma. For any $p \geq 0$, [illegible] coincides, by duality, with the space orthogonal to [illegible].

Proof. The steps [illegible] are well known. For the reader's convenience we are going to sketch a proof.

Let $p = 1$. In this case [illegible] recall the basic duality equation [illegible]

$$(3.2) \quad [illegible]$$

The induction step will be based on the equation [illegible], where V is a (graded) vector space and [illegible] (graded) subspace, which follows from elementary multilinear algebra.

Suppose that [illegible]. By definition, [illegible] if and only if [illegible]. Inserting the above equation [illegible], we have [illegible]. But [illegible]. Hence, by (3.2), [illegible], which completes the proof.

We are now going to apply this to the algebraic models of rational homotopy. [illegible] connected space X has associated [illegible] minimal model [illegible] with [illegible]. [illegible] whenever X is connected with finite Betti numbers, [illegible] of finite type. In this latter case [illegible] the Quillen model of X [illegible]. The first main result reads

Theorem 1. [illegible] as graded algebras.

For a general connected X one may also consider [illegible] model, namely the sub-dga [illegible] generated by [illegible]; this will be a [illegible] algebra [illegible] free [illegible] generated in degree one. For such algebras, one may still define the canonical filtration [illegible]. If moreover the first Betti number of X is finite, then it easily follows that

$\dim F_pV < \infty$, for any p, and we may safely dualize. We thus define the associated Lie algebra with grading of a 1-minimal algebra $(\wedge V, d)$, to be denoted by $\mathrm{gr}^*(\wedge V, d)$, by

$$\mathrm{gr}^*(\wedge V, d) = \varprojlim_p \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge F_pV, d). \tag{1.3}$$

THEOREM ([S],[CP],[Q]). *$gr^*\pi_1 S \otimes \mathbb{Q} \cong gr^*(\wedge V, d)$ as Lie algebras with grading.*

Given a 1-minimal $(\wedge V, d)$ and assuming $\dim H^1(\wedge V, d) < \infty$ in order to smoothly dualize, we may also start with an arbitrary exhaustive nilpotent filtration $\mathcal{F}' = \{F'_pV\}_{p\geq 0}$. It readily follows by induction that $F'_p \subset F_p$, for any p, hence we have a natural grlie map

$$\mathrm{gr}^*_{\mathcal{F}}(\wedge V, d) \cong \varprojlim_p \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge F_p, d) \to \varprojlim_p \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge F'_p, d) \cong \mathrm{gr}^*_{\mathcal{F}'}(\wedge V, d).$$

1.4 PROPOSITION. *For any exhaustive nilpotent filtration $\mathcal{F}'$ the above map $gr^*_{\mathcal{F}}(\wedge V, d) \to gr^*_{\mathcal{F}'}(\wedge V, d)$ is an isomorphism.*

PROOF: For a fixed n and an arbitrary exhaustive nilpotent filtration $\{\mathcal{F}'_p\}$, we have to evaluate the vector space $\varprojlim_p \mathrm{gr}^n \mathcal{C}^{*^{-1}}(\wedge F'_p, d)$. By Lemma 1.1 this is naturally isomorphic to $\varprojlim_p \#(F_nF'_p/F_{n-1}F'_p) \cong \#(\varinjlim_p F_nF'_p/\varinjlim_p F_{n-1}F'_p)$. It plainly suffices to show that the natural map $\varinjlim_p F_nF'_p \to F_nV$ is isomorphic (to be more precise epic). For $n = 1$ this is obvious (recall that $V = \varinjlim_p F'_p$) and the induction goes well on, by the very definition of the canonical filtration. ∎

We move to the algebraic models of formal spaces. Given a connected algebra of finite type A^*, it is constructed in [HS; pages 242-243] the <u>bigraded</u> <u>model</u>, $\mathcal{B}_A = (\wedge Z^*_*, D)$; it is a minimal dga, which also carries a second (lower) graduation, with respect to which D is homogeneous of (lower) degree -1 (it is a minimal bdga). It is uniquely characterized by the existence of a bdga map $\mathcal{B}_A \to (A^*, 0)$ (where A^* is concentrated in lower degree zero, and is endowed with trivial differential) inducting a cohomology isomorphism; forgetting the lower degrees, it represents the minimal model of the formal space S_A. If A^* is 1-connected, there is also a formal Quillen model (see [T; III.4.(5)]) $\mathcal{L}_A = (\mathbb{L}W_*, \partial_2)$, which corresponds to A^* by duality: $W = \#s^{-1}\overline{A}$, and the quadratic Lie differential ∂_2 is essentially dual to the multiplication; this dg Lie model also carries a second (upper) grading, given by bracket length, and ∂_2 is homogeneous of upper degree $+1$. We have a well-known [T] equality of bglie

$$\mathcal{C}^{*^{-1}}(\wedge Z^*_*, D_2) \cong H^*_*\mathcal{L}_A. \tag{1.5}$$

Given a bglie L^*_* (of finite type with respect to the lower degree) its dual quadratic bdga, $(\wedge Z^*_*, D)$ also carries a third grading, coming from monomial length; the induced grading on cohomology will be denoted by $H^*_*(\wedge Z, D) = \bigoplus_{p\geq 1} {}^pH^*_*(\wedge Z, D)$.

dim $F_p^1 < \infty$ for any p, and we may [illegible] finitely. We thus define the associated Lie algebra with grading of $\mathcal{F}$ [illegible] $(\mathsf{A}^\dagger, d)$, to be denoted by $\mathrm{gr}^\bullet_{\mathcal{F}}$, as by

(1.9) $$\mathrm{gr}^\bullet_{\mathcal{F}}(\mathsf{A}^\dagger, d) = \lim_p C^\bullet(F_p\mathfrak{g}^*, d).$$

Therefore [illegible] $\mathrm{gr}^\bullet_{\mathcal{F}}(\mathsf{A}^\dagger, d)$ is the algebra with grading.

Given a 1-minimal $(\mathsf{A}^\dagger, d)$ and any filtration $\mathcal{F}$ [illegible] in order to smoothly dualize we may also start with [illegible], consider the subjacent filtration $\mathcal{F}' = (F'_p)_{p\geq 0}$. It readily follows by induction that $F'_p \subseteq F_p$ for every p, hence we have a natural lattice map:

$$\mathrm{gr}^\bullet_{\mathcal{F}'}(\mathsf{A}^\dagger, d) = \lim_p C^\bullet(F'_p\mathfrak{g}^*, d) \to \lim_p C^\bullet(F_p\mathfrak{g}^*, d) = \mathrm{gr}^\bullet_{\mathcal{F}}(\mathsf{A}^\dagger, d).$$

1 PROPOSITION. *For any [illegible] subjacent filtration $\mathcal{F}'$ the above map $\mathrm{gr}^\bullet_{\mathcal{F}'}(\mathsf{A}^\dagger, d) \to \mathrm{gr}^\bullet_{\mathcal{F}}(\mathsf{A}^\dagger, d)$ is an isomorphism.*

PROOF. For a fixed n and an arbitrary [illegible] filtration $\{F_p\}_p$ we have to [illegible]. By Lemma 3.1 this is [illegible] isomorphic [illegible]. It suffices to show that the natural map [illegible] is [illegible]. For $n = 1$ this is obvious (recall that $F'_1 = F_1$) and induction goes well on, by the very definition of the subjacent filtration.

We move to the algebraic models of formal spaces. Let M a connected manifold of finite type [illegible]. It is [illegible] the 1-minimal model, $\mathfrak{M} = (\mathsf{A}^\dagger, D)$. It is a minimal algebra which [illegible] second (lower) gradation, with respect to which D is homogeneous of (lower) degree -1 [illegible]. It is uniquely characterized by the existence of a [illegible] map $\rho\colon \mathfrak{M} \to (\mathsf{A}^\dagger, 0)$ [illegible] concentrated in lower degree zero, and is endowed with [illegible] cohomology isomorphism, respecting the lower degrees [illegible] formal space M. If M is 1-connected, there is [illegible], which corresponds to [illegible] and the quadratic [illegible] is canonically dual to the [illegible] carries a second (upper) grading, given by bracket length, and D is homogeneous of upper degree $+1$. We have a well-known [?] equality of [illegible]:

(1.10) $$H^\bullet(\mathfrak{M}_1, D_1) \cong [illegible]$$

Given a [illegible] with respect to the lower degree [illegible] cohomology will be [illegible].

1.6. LEMMA. *Consider $C^*(L^*_*) = (\wedge Z^*_*, D)$. The following are equivalent:*

(i) $L^{(p)} \cong L^{\geq p}$, *any* p

(ii) $F_p Z \cong Z_{<p}$, *any* p

(iii) L^* *is generated by* L^1

(iv) ${}^1H^*_+(\wedge Z, D) = 0$

(v) $gr^* L_* \cong L^*_*$

(vi) $L^* \cong \varprojlim_p gr^*(L/L^{(p)})$

PROOF: By Lemma 1.1, $F_p Z \cong Z_{<p}$ is equivalent to $L^{(p+1)} \cong L^{\geq p+1}$. Since L^* is strictly positively graded, we have in general an inclusion $L^{(p)} \subset L^{\geq p}$, for any p; an easy homogeneity argument shows that the equality is in fact equivalent to $L^1 \to L/[L, L]$ being onto, hence, by duality, to the fact that the canonical projection $\mathrm{Ker}(D|_{Z_*}) \to Z_0$ is monic; this last condition precisely says that ${}^1H_+(\wedge Z_*, D) = 0$. In general, $\mathrm{gr}^* L$ is always generated by $\mathrm{gr}^1 L$; conversely, assuming L^* is generated by L^1, it easily follows from (i) that $\mathrm{gr}^* L \cong L^*$. Finally, by an obvious stability argument, one always knows that $\varprojlim_p \mathrm{gr}^*(L/L^{(p)}) \cong \mathrm{gr}^* L$. ∎

A convenient set-up for the description of formal 1-minimal models is provided by considering Lie algebras with grading, L^*, which are required to be generated by L^1, $\dim L^1 < \infty$. Given such L^*, consider the inverse system $\cdots \twoheadrightarrow L/L^{(p+1)} \twoheadrightarrow L/L^{(p)} \twoheadrightarrow \cdots$ of central extensions of finite-dimensional grlie algebras (considered as bglie concentrated in lower degree zero). Set

$$(1.7) \qquad (\wedge V_*, d) = \varinjlim_p C^{*^{-1}}(L^*/L^{(p)*}).$$

It is a 1-minimal dga, by the well-known duality between central extensions of Lie algebras and elementary extensions of dga's [GM], which is also a bdga. It carries a natural (nilpotent exhaustive) filtration $\mathcal{F}$ given by $F_p = V_{<p}$, for which one has by construction and the preceding lemma $\mathrm{gr}^*_{\mathcal{F}}(\wedge V, d) \cong L^*$. Starting with an algebra A^* (connected and of finite type as usual), first construct a gr Lie algebra L^*_A as in the introduction, namely $L^*_A = \mathbb{L}^*_X/\mathrm{ideal}(\partial Y)$, where $\partial \colon Y \to X \wedge X \cong \#(\mu \colon A^1 \wedge A^1 \to A^2)$ (also noticing that L^*_A depends only on $\mu \colon A^1 \wedge A^1 \to \mathrm{Im}\mu \subset A^2$), and then associate to L^*_A the 1-bigraded model $(\wedge V_*, d)$ as in (1.7). The next lemma seems to be folklore, but we chose to include a proof, being unable to find a reference (not to speak of the fact that the construction (1.7) will be again useful later on, see the proof of Proposition 5.3).

1.8. LEMMA. *Let A^*, L^*_A and $(\wedge V_*, d)$ be as above. Then:*

(i) *$(\wedge V_*, d)$ is the 1-bigraded model of the formal space S_A*

(ii) $gr^* \pi_1 S_A \otimes \mathbb{Q} \cong L^*_A$.

1.6. LEMMA. Consider $E^*(\cdot)_q$ on $(A\mathscr{Z}, D)$. The following are equivalent:

(i) $E_q^{(\tau)} \cong E_q^{(\tau)}(\mathrm{pt})$ [illegible]

(ii) ${}^{\sigma}L_q E \simeq E_q$, [illegible]

(iii) E^* is [illegible] by $\mathfrak{p}^*$

(iv) $H_*(A\mathscr{Z}, D) = 0$

$C_*\ \pi^* L_q \cong C_*$

(vi) $E^* \cong \lim\_{\to} \pi^*(C, E^{(\tau)})$

PROOF. By Lemma 1.4, 1.3 [illegible] is equivalent to $E^{(\tau)} \cong L_q^{(\tau)}$. Since E is strictly positively graded, as the [illegible] inclusion $\Gamma^{\sigma} \subset A\mathscr{Z}$, for any p, an easy homogeneity argument shows that the equality is in fact equivalent to $E_q = L_q/I_q$ [illegible] being onto, hence by duality, to the fact that the canonical projection [illegible] is [illegible]; this last is precisely [illegible] $H_*(A\mathscr{Z}, D) = 0$. In general, $\pi^* L$ is always generated by $\mathfrak{p}^* L$; conversely, assuming E is generated by E, it easily follows from (i) that $\pi^* E \cong C$. Finally by an obvious stability argument we always know that $\lim\_{\to} \pi(\mathbb{Q}/\mathbb{Z}^{(\tau)}) \cong \pi^* L$. ∎

A convenient set-up for the description of actual [illegible] models is provided by considering [illegible] algebras with gradings L_q, which are required to be generated by L_q^1, that is, $L_q^1 = \ldots$ Given such L_q, consider the [illegible] system [illegible] of central extensions of finite dimensional Lie algebras (considered as being concentrated in lower degree zero), [illegible]

(1.7) $$(\Lambda\mathscr{K}, d) = \lim\_{\to} C^*(L/L^{(\tau)}).$$

If $\mathscr{Z}$ is finite-dimensional, by [?] well known duality between central extensions of Lie algebras and elementary extensions of [illegible], which we have already discussed, it corresponds in turn to a minimal (although infinite) exhaustive filtration $\mathscr{F}$ given by [illegible]. For the other one has by construction and the preceding lemma $\pi(\Lambda\mathscr{K}, d) \cong L_q$. Starting with an algebra $\mathscr{A}$ (connected and of finite type as usual), first [illegible] as in the introduction, namely $E_q = E_q(\mathrm{nil}(\Omega))$, where [illegible] ($\mathscr{A}$ [illegible]) (also noticing that E_q depends only on $\mathscr{A}$) and then associate to E_q the 1-minimal model $(\Lambda\mathscr{K}, d)$ as in (1.7). The next lemma seems to be familiar, but we choose to include a proof, being unable to find a reference for it. In the last part the construction (1.7) will be again useful later on, see the proof of Proposition 6.2.

1.8. LEMMA. Let $\mathscr{A}$ [illegible] and $(\Lambda\mathscr{K}, d)$ be as above. Then:

(i) $(\Lambda\mathscr{K}, d)$ is the 1-minimal model of the formal space $\mathscr{A}$,

(ii) $E^* \cong \lim\_{\to} C^*(L_q/L_q^{(\tau)})$.

PROOF: Given the general theory, Proposition 1.4 and the above remarks, (ii) will follow at once from (i).

As far as (i) is concerned, we start by constructing a bdga map $f\colon (\wedge V_*, d) \to (A^*, 0)$. We set $f|_{V_+} = 0$, notice that $V_0 \cong \# L_A^1 \cong \# X \cong A^1$, and put $f|_{V_0} = \mathrm{id}$; due to the homogeneity property of d with respect to lower degrees, checking that f commutes with the differentials is reduced to showing that $fdV_1 = 0$, i.e. the composition $V_1 \xrightarrow{d} V_0 \wedge V_0 \xrightarrow{f \wedge f} A^1 \wedge A^1 \xrightarrow{\mu} A^2$ equals zero. Taking duals, this amounts to seeing that $L_A^2 \xleftarrow{[\,,\,]} L_A^1 \wedge L_A^1 = X \wedge X \xleftarrow{\partial} Y$ equals zero, which is obvious by the construction of L_A^*. By the uniqueness of 1-minimal models we must only verify that $H^1 f$ is an isomorphism and $H^2 f$ is monic [S]. But we know that $H^1(\wedge V_*, d) \cong \varinjlim_p {}^1H(\mathcal{C}^{*^{-1}}(L_A^*/L_A^{(p)*}))$ and, by Lemma 1.6, ${}^1H(\mathcal{C}^{*^{-1}}(L_A^*/L_A^{(p)*})) \cong {}^1H_0(\mathcal{C}^{*^{-1}}(L_A^*/L_A^{(p)*})) \cong V_0$, which takes care of the condition on $H^1 f$. On the other hand $H^2(\wedge V_*, d) \cong H_0^2(\wedge V_*, d) \oplus H_+^2(\wedge V_*, d)$, and $\mathrm{Im} H_0^2 f \cong \mathrm{Im}\mu$, while $\mathrm{Im} H_+^2 f = 0$, by the construction of f. We may thus use a dimension argument: $H^2 f$ is monic is equivalent to $\dim H^2(\wedge V, d) = \dim \mathrm{Im}\, \mu$. We may notice again that $H^k(\wedge V, d) \cong {}^kH(\wedge V, d)$, and this in turn equals $H^k(L_A; \mathbb{Q})$ - classical Lie algebra cohomology with trivial coefficients via the Koszul resolution, see e.g. [T] - for any k. We are thus led to compute $\dim H_2(L_A; \mathbb{Q})$, and we may use for this purpose the description of the second homology group of a Lie algebra of the form $\mathfrak{f}/\mathfrak{r}$, where $\mathfrak{f}$ is a free Lie algebra and $\mathfrak{r}$ an ideal, given in [HlS; page 238, Exercise 3.2]: $H_2(\mathfrak{f}/\mathfrak{r}; \mathbb{Q}) \cong [\mathfrak{f}, \mathfrak{f}] \cap \mathfrak{r}/[\mathfrak{f}, \mathfrak{r}]$. We infer that $H_2(L_A; \mathbb{Q}) \cong I/[\mathbb{L}X, I]$, where I is the ideal generated by ∂Y. The dimension of the last object plainly equals $\dim \mathrm{Im}\, \partial$, and finally $\dim \mathrm{Im}\, \partial = \dim \mathrm{Im}\, \mu$, by duality. ∎

Next we are going to conveniently rephrase the conditions on the Hopf algebra $\mathrm{Ext}_A^{*,*}(\mathbb{Q}, \mathbb{Q})$ stated in the introduction.

1.9. LEMMA. *Let A^* be a connected algebra, with bigraded model $\mathcal{B}_A = (\wedge Z_*^*, D)$ and Quillen model $\mathcal{L}_A$ (in the 1-connected case).*

(i) $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q}) = 0$ if and only if $H^*\mathcal{L}_A$ is generated (as a Lie algebra) by $H^1\mathcal{L}_A$.

(ii) $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q}, \mathbb{Q}) = 0$ if and only if $Z_*^2 = 0$.

PROOF: (i) The condition $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q}) = 0$ simply means that the Yoneda Ext-algebra of A is generated (as an algebra) by $\mathrm{Ext}_A^1(\mathbb{Q}, \mathbb{Q})$. Consider then the formal space S_A and its formal Quillen dgrlie model $\mathcal{L}_A^*$, graded by the bracket length. In [A1; Theorem 2] is established a graded isomorphism $\bigoplus_{i \geq 0} \mathrm{Ext}_A^{i,*} \cong \bigoplus_{i \geq 0} H_*^i \mathcal{U}\mathcal{L}_A$ (where $\mathcal{U}$ = universal enveloping algebra functor), which is also compatible with the algebra structures, up to sign. Since $H^*\mathcal{U}\mathcal{L}_A \cong \mathcal{U}H^*\mathcal{L}_A$ as algebras and $\mathcal{Q}\mathcal{U}H^*\mathcal{L}_A \cong \mathcal{Q}H^*\mathcal{L}_A \cong H^*\mathcal{L}_A/[H^*\mathcal{L}_A, H^*\mathcal{L}_A]$ as graded vector spaces [Q], our assertion follows.

PROOF. Given the proof of the Proposition 1.4 and the above remarks (ii) will follow at once from (i).

So far as (i) is concerned, we start by constructing a bijection map $f: (\Lambda V, d) \to (\Lambda^* \mathfrak{g}, 0)$. We set $f(y_i) = 0$ and note that [illegible], and that $f(x_i) = \ldots$ [illegible] due to the homogeneity property of d with respect to lower degrees, checking that f commutes with the differentials is reduced to showing that $f d(y_i) = 0$, i.e. the composition $V_1 \to V_0 \wedge V_0 \to \Lambda^2 \wedge \Lambda^2 \to \Lambda^*$ equals zero. Taking duals, this amounts to seeing that $\mathfrak{L}_2 \leftarrow \mathfrak{L}_1 \wedge \mathfrak{L}_1 = X \wedge X \to \mathfrak{L}$ equals zero, which is obvious by the construction of $\mathfrak{L}$. By the uniqueness of 1-minimal models we must only verify that $H^1 f$ is an isomorphism and $H^2 f$ is mono [5]. But we know that $H^1(\Lambda V, d) \cong Hom_{\mathbb{Z}}(\Gamma, \mathbb{Q})$ and, by Lemma 1.6, $H^2(G^*, \mathbb{Q}) \cong H_2(\Gamma; \mathbb{Q})$, which is the kernel of the bracket on H^1. On the other hand $H^*(\Lambda V, d) \cong H^*(\Lambda V, d) \cong H_*(\Gamma, \mathbb{Q})$, and $\dim H^1 = \dim H^1$, while $\dim H^2 f = 0$, by the construction of $\mathfrak{L}$. We may thus use a dimension argument: $H^2 f$ is mono is equivalent to $\dim H^2(\Lambda V, d) = \dim \ker d$. We may notice at this point that $H^*(\Lambda V, d) \cong H^*(\Lambda V, \delta)$, and this in turn equals $H^*(\mathfrak{L}; \mathbb{Q})$, classical Lie algebra cohomology with trivial coefficients via the Koszul resolution, see e.g. [?] for $\mathfrak{L}$. We are thus led to compute $\dim H_2(\mathfrak{L};\mathbb{Q})$, and we may use for this purpose the description of the second homology group of a Lie algebra of the form $\mathfrak{L}/I$, where $\mathfrak{L}$ is free, due to Hopf, as explained in [HS, page 238, Exercise 3.7]: $H_2(\mathfrak{L}/I; \mathbb{Q}) \cong (I \cap [\mathfrak{L},\mathfrak{L}])/[\mathfrak{L}, I]$. We infer that $H_2(\mathfrak{L}, \mathbb{Q}) \cong I/[\mathfrak{L}, I]$, where I is the ideal generated by R. The dimension of the last object plainly equals $\dim \mathfrak{n}_2$, and finally $\dim \ker = \dim \mathfrak{n}_2$. ∎

Next we are going to investigate in terms the conditions on the Hopf algebra $Ext_A^*(\mathbb{Q},\mathbb{Q})$ found in the Introduction.

1.9. LEMMA. Let G be a connected nilpotent Lie group, $G_A = (V^*, D)$ and Quillen model $\mathcal{L}_A$ of G. Then the following are true:

(i) $Ext_A^*(\mathbb{Q}, \mathbb{Q})$ is finitely generated if and only if $H_*\mathcal{L}_A$ is generated (as a Lie algebra) by $H^1 G_A$.

(ii) $Ext_A^{2,*}(\mathbb{Q}, \mathbb{Q}) = 0$ if and only if $H^2_* = 0$.

PROOF. (i) The condition $Ext_A^{1,*}(\mathbb{Q},\mathbb{Q}) = 0$ simply means that the Yoneda Ext algebra of A is generated (as an algebra) by $Ext_A^1(\mathbb{Q},\mathbb{Q})$. Consider then the formal space S_A and its formal Quillen model $\mathcal{L}_A$ graded by the bracket length, as in [?]. In [?] Lemaire established a graded isomorphism $Ext_A(\mathbb{Q},\mathbb{Q}) \cong U H_*(\mathcal{L}_A)$ (where U = universal enveloping algebra functor), which is also compatible with the Hopf structures, up to sign. Since $H_*(\mathcal{L}_A)$ is generated as Lie algebras and $U(\ldots) \cong \ldots$ of graded vector spaces, [illegible] assertion (ii) follows.

(ii) It is proven in [HS; Corollary 7.17] that one has $Z_*^n \cong \# \mathcal{P}\mathrm{Ext}_A^{*,n-1}(\mathbb{Q},\mathbb{Q})$ for any n. ∎

We describe now the algebraic parametrization of rational homotopy types with fixed cohomology algebra A^* in terms of deformation theory, following [HS] and [LS] (see also [MP]). In the dga setting of [HS], one starts with the bigraded model $\mathcal{B}_A = (\wedge Z_*^*, D)$; it is convenient to set $D = D^1$. Then any space S with $H^*(S;\mathbb{Q}) \cong A^*$ has a free dga model of the form $(\wedge Z_*^*, D^1 + p)$, where the algebraic parameter, the perturbation p, may be written $p = p^2 + p^3 + \dots$, each p^i being homogeneous of lower degree $-i$; the trouble comes from the fact that this (nilpotent) model may fail to be minimal (this is geometrically related to the collapsing of the Eilenberg-Moore spectral sequence of S, see the next section). If A^* is 1-connected, there is the alternative dglie setting of [LS], where one starts with the Quillen formal model $\mathcal{L}_A = (\mathbb{L}^* W_*, \partial)$; here we set $\partial = \partial^1$. One may represent any S within A^* by a (minimal) dglie model of the form $(\mathbb{L}W_*, \partial^1 + p)$, where, again, $p = p^2 + p^3 + \dots$, each p^i being homogeneous of upper degree i. Finally, here comes our basic deformation-theoretic result.

1.10. Proposition. *Consider a bigraded Lie algebra L_*^* (of finite type with respect to lower degrees) and its quadratic bdga dual, $(\wedge Z_*^*, D)$. Suppose that L^* is generated by L^1. Then for any quadratic dga of the form $(\wedge Z^*, d)$, where $d = D^1 + p^2 + p^3 + \dots$, $D^1 = D$ and each p^i is homogeneous of degree $-i$ with respect to lower degrees, we have an isomorphism of bigraded Lie algebras:*

$$gr^*\mathcal{C}^{*^{-1}}(\wedge Z^*, d) \cong gr^*\mathcal{C}^{*^{-1}}(\wedge Z^*, D).$$

Proof: Set $\mathcal{C}^{*^{-1}}(\wedge Z^*, d) = (L_*, [\ ,\ ]_p)$. By just dualizing the decomposition of the perturbed differential, $d = D + p$, one infers that the perturbed Lie bracket $[\ ,\ ]_p$ has the following property: for any $x \in L^m$, $y \in L^n$, $[x,y]_p = [x,y]$ modulo $L^{>m+n}$ (and of course $[x,y] \in L^{m+n}$). By Lemma 1.6 one may precisely describe the lower central series of the original graded Lie algebra $(L_*, [\ ,\ ])$, in terms of the upper graduation. Our assertion will immediately follow, as soon as we prove that the lower central series of the perturbed Lie algebra is the same , or equivalently (by Lemma 1.1) the canonical filtration of $(\wedge Z, d)$, denoted by ${}^p\mathcal{F} = \{{}^pF_n Z\}_{n\geq 0}$ coincides with the canonical filtration of $(\wedge Z, D)$, which is, again by 1.6, $\mathcal{F} = \{F_n Z = Z_{<n}\}_{n\geq 0}$. By (lower) degree inspection, $dZ_n \subset \wedge^2 Z_{<n}$, for any n, and this inductively implies that $F_n Z \subset {}^pF_n Z$, for any n. The remaining inclusion will also be proven by induction, trivially starting with $n = 0$. Assume then $z \in {}^pF_n Z$ and write $z = z_0 + \dots + z_m$, where $z_i \in Z_i$ and $z_m \neq 0$. By the definition of the canonical filtration and by the induction hypothesis, we know that $dz \in \wedge^2 Z_{<n-1}$; writing $d = D + p$ and examining the top component of dz with respect to lower degree, we infer that $Dz_m \in \wedge^2 Z_{<n-1}$, hence $z_m \in F_n Z = Z_{<n}$, therefore $m < n$ and $z \in Z_{<n} = F_n Z$, as desired.

(iii) It is proven in [HS, Theorem 4.17] that one has $\mathcal{Z}_2 \cong \mathcal{X}^{(0)}(\pi_\mathcal{O}^{-1} \otimes_R \Omega)$ for any π. ∎

We again denote the algebraic parametrization of rational homotopy types with fixed cohomology algebra H in terms of deformation theory, following [HS] and [Ha] (see also [SS]). In the first sections of [HS], one works with the bigraded model $\mathcal{M}_H = (\Lambda Z, D)$. It is convenient to set $D = D'$ [illegible]. The moduli space $\mathcal{M}$ with $H^*(S; \mathbb{Q}) \cong H$ has a free dga model of the form $(\Lambda Z, D)$ supplied with the obstruction parameters; the derivation for p may be written $p + p' + p'' + \ldots$, where p' means homogeneous of lower degree. As the tangible classes from the fact that (the [illegible]) may fail to be unfolded (this is geometrically related to the collapsing of the [illegible] spectral sequence of S, cf. the next section). If A is [illegible], there is an alternative description of [1.1], where one starts with the Quillen formal model $\mathcal{L}_H = (\mathbb{L}^{H}, \partial)$, where ∂ is the H. One may represent any S with A by a (minimal) dgl model of the form $(\mathbb{L}^{H}, \partial + p)$, where, again, $p = p' + p'' + \ldots$, each p' being homogeneous of upper degree. Finally, here comes our basic deformation theoretic result.

1.10. Proposition. Consider a bigraded algebra H of finite type with respect to lower degree, and its minimal bigraded model $(\Lambda Z, D)$. Suppose that E is generated by E'. Then for any rational space S with $H^*(S; \mathbb{Q}) \cong H$, whose $D = D' + p' + p'' + \ldots$, $D' = D$ and each p' is homogeneous of degree [illegible] with respect to lower degree, we have an isomorphism of graded Lie algebras

$$\pi^{[illegible]}(\Lambda Z, d) \cong \pi^{[illegible]}(\Lambda Z, D).$$

Proof. Set $C = (\Lambda Z, d)$ [illegible] and denoting the decomposition of the perturbed differential $d = D + p$, we [illegible] the perturbed Lie bracket [illegible] has the following property: for any $\alpha \in L$, $\alpha \in L$, [illegible] modulo L [illegible] of course [illegible]. By Lemma 1.8 (ii), [illegible] the lower central series of the original graded Lie algebra $(\mathcal{L}, [\ ,\ ])$, in terms of the upper gradation. Our assertion will immediately follow, as soon as we prove that the lower central series of the perturbed Lie algebra is the same, or equivalently, by Lemma 1.11 the canonical filtration of $(\Lambda Z, d)$, denoted by $\mathcal{F} = \{\mathcal{F}^n Z\}_{n \geq 1}$, coincides with the canonical filtration of $(\Lambda Z, D)$, which is, again by 1.6, $\mathcal{F}^n = (\Lambda Z^{\leq n})$. Be [illegible] the [illegible] inequality, [illegible] for any n, and this immediately implies that [illegible] for any n. The remaining inclusion will also be proven by induction. Using [illegible]. Assume that $z \in \mathcal{F}^n Z$ and write $z = z_0 + \ldots + z_k$, where $z_i \in Z_i$ and [illegible]. By the definition of the canonical filtration and by the induction hypothesis, we know that $dz \in \Lambda(\mathcal{F}^{n-1}Z)$; writing $dz = Dz + pz$ and considering the top component of dz with respect to lower degree, we infer that [illegible], hence $z_k \in \mathcal{F}^n Z$. Therefore $z - z_k \in \mathcal{F}^n Z$, and $z \in \mathcal{F}^n Z$, as desired. ∎

2. Rigidity results

This section contains the (almost simultaneous) proofs of Theorems A(i) and A'(i), and the first examples.

2.1. PROOF OF THEOREM A(i). Represent any S with $H^*(S;\mathbb{Q}) \cong A^*$ by the free dga model $(\wedge Z^*, D+p)$, as explained before. We claim that it will be enough to show that the Eilenberg-Moore spectral sequence of S collapses at the E_2 term. Indeed, we know from [HS; Theorem 7.20] that this is equivalent to the minimality of $(\wedge Z, D+p)$, and also equivalent to $\dim \pi_k S \otimes \mathbb{Q} = \dim \pi_k S_A \otimes \mathbb{Q}$, for any k. Once we may assume this, we know that $\mathcal{C}^{*^{-1}} \pi_* \Omega S \otimes \mathbb{Q} \cong (\wedge Z^*, D_2 + p_2)$ where the subscript 2 indicates that we have taken the quadratic parts. We may now use the previous proposition, by taking $L^*_* = H^*_* \mathcal{L}_A$, with dual $(\wedge Z^*_*, D_2)$, see (1.5); our hypothesis on $\mathcal{Q}\mathrm{Ext}_A$ guarantees that L^* is generated by L^1 (Lemma 1.9(i)). We deduce a bglie isomorphism $\mathrm{gr}^* \pi_* \Omega S \otimes \mathbb{Q} \cong \mathrm{gr}^* L_*$, the second bglie being isomorphic to L^*_*, again due to the above-mentioned hypothesis (see Lemma 1.6(v)).

In order to establish the EMss collapse property, we are going to use the numerical criterion in terms of ranks of homotopy groups and the dglie approach. Represent then S by a Quillen minimal model of the form $(\mathbb{L}W_*, \partial + p)$, as described before. The hypothesis that A^* is also intrinsically spherically generated comes now into play and allows us to suppose moreover (see [MP; Proposition 1.8]) that $p|_{\mathcal{P}} = 0$, where the primitive subspace $\mathcal{P}$ equals $\mathrm{Ker}(\partial|_W)$. Filtering $\mathbb{L}W$ by bracket length, we obtain a well-known [Q] spectral sequence of graded Lie algebras, converging to $H_*(\mathbb{L}W, \partial + p) \cong \pi_* \Omega S \otimes \mathbb{Q}$ and starting with $E^1_* \cong (\mathbb{L}W_*, \partial)$ and $E^2_* \cong H_* \mathcal{L}_A \cong \pi_* \Omega S_A \otimes \mathbb{Q}$. On the other hand we invoke again our assumption on $\mathcal{Q}\mathrm{Ext}_A$, recalling that $H^* \mathcal{L}_A$ is generated as a Lie algebra by $H^1 \mathcal{L}_A \cong \mathcal{P}$, which consists only of permanent cycles, by the spherical generation property, hence $E^2 \cong E^\infty$ and $\dim \pi_k \Omega S \otimes \mathbb{Q} = \dim \pi_k \Omega S_A \otimes \mathbb{Q}$, as claimed. Our proof is complete. ∎

2.2. PROOF OF THEOREM A'(i). This is similar but simpler. Use again the perturbed free dga model $(\wedge Z^*, D+p)$ of S and set $Z^1_* = V_*$. Since $Z^2 = 0$ (Lemma 1.9(ii)) $(\wedge V, D+p)$ is a subgda of $(\wedge Z, D+p)$, for trivial degree reasons; it is equally trivial to see that the above dga inclusion induces an isomorphism at the H^1 level and is monic at the H^2 level, hence [S] $(\wedge V, D+p)$ represents the 1-minimal model of S (the nilpotence condition is easily checked along the lower degree filtration of V_*). Set then $F_n = V_{<n}$. By Proposition 1.4, $\mathrm{gr}^* \pi_1 S \otimes \mathbb{Q} \cong \varprojlim_n \mathrm{gr}^* \mathcal{C}^{*^{-1}} (\wedge V_{<n}, D+p)$.

We may apply Proposition 1.10 to the finitely generated quadratic bdga $(\wedge V_{<n}, D)$. The requirement that the dual Lie algebra L^* be generated by L^1 is now automatically satisfied. Indeed we may check the equivalent condition given by Lemma 1.6(iv) by noticing that

2. Rigidity results.

This section contains the (mostly computational) proofs of Theorem A(i) and A(ii), and the last examples.

2.1. Proof of Theorem A(i). [illegible] with $H^*(S, \mathbb{Q}) \cong A^*$ by the free dga model $(\Lambda Z^*, D + \delta)$ as explained before. We claim that it will be enough to show that the Eilenberg-Moore spectral sequence of S collapses at the E_2-term. Indeed, we know from [HS, Theorem 7.20] that this is equivalent to the formality of $(\Lambda Z, D + \delta)$, and also equivalent to $\dim \pi_*(S) \otimes \mathbb{Q} = \dim [illegible]$ and [illegible]. Once we may assume this, we know that $C^* [illegible] \cong (\Lambda Z^*, D_2) [illegible]$ where the subscript 2 indicates that we have taken the quadratic part. We may now [illegible] the previous proposition, by taking $L^r_* = [illegible]$, with dual $(\Lambda Z^*, D_2)$; see [illegible]. The hypothesis on [illegible] guarantees that L is generated by L^1 (Lemma 1.9[illegible]). [illegible] homomorphism [illegible], the second [illegible] being isomorphic to L_* [illegible] due to [illegible] hypothesis (see Lemma 1.10[illegible]).

In order to establish the EMSS collapse property, we are going to use the numerical criterion in terms of [illegible] Recall that S has a Quillen minimal model of the form $(\mathbb{L}(W), \partial)$, as described before. The hypothesis that A is intrinsically [illegible] comes now into play and allows us to suppose moreover (see [illegible], Proposition [illegible]) that [illegible], where the primitive subspace P equals [illegible]. Filtering [illegible] by bracket length, we obtain a well-known [Q] spectral sequence of graded Lie algebras, converging to [illegible], starting with [illegible] and [illegible]. On the other hand we have [illegible] assumption [illegible] recalling that [illegible] which consists only of permanent cycles [illegible] and [illegible] Our proof is complete. ■

2.2. Proof of Theorem A(ii). This is similar but quicker. Use again the [illegible] dga model $(\Lambda Z^*, D + \delta)$ of [illegible]. [illegible] (Lemma 1.8[illegible]) $(\Lambda Z, D + \delta)$ is a [illegible] of $(\Lambda Z, D + \delta)$ [illegible] it is entirely trivial to see that the [illegible] level [illegible] in the E^2 level. Hence [illegible] $(\Lambda V, D + \delta)$ [illegible] minimal model of a [illegible] condition is easily checked along the lower degree [illegible]. By Proposition 1.4, [illegible] $\otimes \mathbb{Q} \cong [illegible]$.

We may apply Proposition [illegible] to the [illegible] algebra $(\Lambda V, D)$. The requirement that the [illegible] generated by [illegible] is now automatically satisfied. Indeed we may check the equivalent condition given by Lemma 1.8[illegible] by noticing that obviously

$$^{1}H_+(\wedge V^1_{<n}, D) \cong H^1_+(\wedge V^1_{<n}, D) \subset H^1_+(\wedge Z^1_*, D) \cong H^1_+(\wedge Z^*_*, D)$$

and that the cohomology of the bigraded model $(\wedge Z^*_*, D)$ is concentrated by definition in lower degree zero. We infer that $\mathrm{gr}^*\mathcal{C}^{*^{-1}}(\wedge V_{<n}, D+p) \cong \mathrm{gr}^*\mathcal{C}^{*^{-1}}(\wedge V_{<n}, D)$, which is independent of p, for any n. Finally, for the formal space S_A corresponding to $p = 0$, Lemma 1.8(ii) tells us that $\mathrm{gr}^*\pi_1 S_A \otimes \mathbb{Q} \cong L^*_A$, which was the last assertion to be proved. ∎

2.3. REMARKS AND EXAMPLES. First we ought to notice that intrinsic spherical generation is a necessary condition for graded intrinsic formality (in the 1-connected case), indeed, if $H^*(S;\mathbb{Q}) \cong A^*$, the isomorphism $\mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q} \cong \mathrm{gr}^*\pi_*\Omega S_A \otimes \mathbb{Q}$ evidently implies that $\dim \pi_k S \otimes \mathbb{Q} = \dim \pi_k S_A \otimes \mathbb{Q}$, for any k, hence $E_2 \cong E_\infty$ in the EMss, and this in turn forces S to be spherically generated, as shown in [HS; 8.13]. On the other hand the assumption on the Yoneda Ext-algebra of A^* made in A(i), albeit very natural, is not strictly necessary, as shows the following very simple example, namely $A^* = H^*\mathbb{P}^2\mathbb{C}$. This is an intrinsically formal (hence graded intrinsically formal) example – this is very easy, see e.g. [S]. A short direct computation gives that $H^*_*\mathcal{L}_A$ is a 2-dimensional abelian Lie algebra with basis $a \in H^1_1$ and $b \in H^2_4$, therefore (Lemma 1.9(i)) $\mathcal{Q}\mathrm{Ext}^{>1,*}_A(\mathbb{Q};\mathbb{Q}) \neq 0$.

As a first natural series of examples where $\mathcal{Q}\mathrm{Ext}^{>1,*}_A(\mathbb{Q};\mathbb{Q}) = 0$ we may quote $A^* = H^*MG$, where MG is the universal Thom space associated to an arbitrary orthogonal representation of the compact connected Lie group G, see [Pa]. In the other direction, any homogeneously generated algebra A^* is intrinsically spherically generated ([MP; 2.4], see also §4).

We also have to notice tha the hypotheses of A(i) are independent. We have just seen that $H^*\mathbb{P}^2\mathbb{C}$ is intrinsically spherically generated and still the condition on $\mathcal{Q}\mathrm{Ext}_A$ is violated. Let us now define an algebra A^* by describing its Quillen formal model, as in [T]: $\mathcal{L}_A = (\mathbb{L}(x_1, x_2, x_3, x, y), \partial)$, where $\deg x_1 = \deg x_2 = \deg x_3 = 2$, $\deg x = 7$ and $\deg y = 5$, and the only nontrivial action of ∂ is on y, namely $\partial y = [x_1, x_2]$. Anticipating a little (see 4.1 and 4.3), we know that $\mathcal{Q}\mathrm{Ext}^{>1,*}_A(\mathbb{Q}, \mathbb{Q}) = 0$. Defining a perturbation p by requiring that the only nontrivial action is $px = [x_1, [x_2, x_3]]$, we got a space S with $H^*(S;\mathbb{Q}) \cong A^*$, whose minimal Quillen model is $(\mathbb{L}, \partial + p)$ [LS]. Finally, due to the fact that, in $\mathbb{L}(x_1, x_2, x_3)$, $px \notin \mathrm{ideal}(\partial Y)$, a simple computation with the algebraic rational Hurewicz homomorphism as in [T; III.3.(5)] shows that the primitive element of $H_*(S;\mathbb{Q})$ corresponding to x is not spherical, hence A^* is not intrinsically spherically generated.

The first nontrivial examples of 1-graded intrinsically formal algebras are those of [KS], namely $A^*_m = H^*((\vee_{m-1} S^1) \times S^1)$, for $m > 2$; these fit into our theory and satisfy $\mathcal{P}\mathrm{Ext}^{*,1}_A(\mathbb{Q}, \mathbb{Q}) = 0$ (see the remarks made in 5.4).

obviously,

$$H_*(\mathcal{M}\mathbb{Z}_{m_\Omega};\mathcal{D}) = [illegible] \cong H_*(\mathcal{M}\mathbb{Z}_m;\mathcal{D}) \cong H_*^{[illegible]}(\mathcal{M}\mathbb{Z}_m;\mathcal{D}) \cong H_*^{[illegible]}(\mathcal{M}\mathbb{Z};\mathcal{D})$$

and the cohomology of the [illegible] $(\mathcal{M}\mathbb{Z}, \mathcal{D})$ is concentrated by definition in degree zero. We have that $q(\Omega_k) = [illegible](\mathcal{M}\mathbb{Z}_m, \mathcal{D}) = \chi^{[illegible]}(\mathcal{M}\mathbb{Z}_m, \mathcal{D})$, which is independent of p [illegible]. Finally, for the [illegible] space Ω_k [illegible], $k \to \infty$ [illegible], Lemma 1.6(c) tells us that $[illegible] \cong \mathbb{Q} \otimes \mathbb{Z}_{[illegible]}$, [illegible] for [illegible] to be proved. ∎

2.6. [illegible] with [illegible]. [illegible] to realize that [illegible] spherical [illegible] is a necessary condition for [illegible] (in the 1-connected case); indeed, if $H^*(S;\mathbb{Q}) \cong \Lambda^*$, the [illegible] $\otimes \mathbb{Q} \cong [illegible] \otimes \mathbb{Q}$ evidently implies that [illegible] in the [illegible], and therefore this forces S to be spherically generated, as [illegible] [HS, 3.1B]. On the other hand the assumption on the Koszul [illegible] of $S(t)$ [illegible] is not strictly necessary, as shown the following [illegible], namely $H^* = \Lambda \otimes \mathbb{Q}$. This is [illegible] [illegible] group [illegible] example — this is very easy, see e.g. [S]. A short [illegible] that [illegible] is a [illegible] with base $a \in H_1^*$ and $b \in [illegible]$, therefore [illegible] $\neq 0$.

[illegible] $(\Omega\mathbb{Z}^{[illegible]};\mathbb{Q}) = 0$ we may quote $\Lambda^* = H^*M\mathbb{Q}$, where $M(G)$ [illegible]. [illegible] associated to an arbitrary orthogonal representation of the compact [illegible] Lie group G, see [Pa]. In the other direction, [illegible] action of G is [illegible] spherically generated [illegible] (see also [S]).

We [illegible] that the hypotheses of [illegible] are [illegible] independent. We have [illegible] seen that $\Lambda^{[illegible]}$ is [illegible] spherically generated [illegible] the condition on [illegible] is violated. [illegible] we may define [illegible] Koszul module as in [illegible], with $\deg x_1 = \deg x_2 = 2$, $\deg y_1 = 7$ and $\deg y_2 = 3$, and the only [illegible] d is on y_1, namely $dy_1 = [illegible]$. [illegible] (Sec. 4.1 and 6.5), we know that [illegible] is [illegible] by requiring that the only [illegible] action is [illegible] and a space S with $H^*(S;\mathbb{Q}) \cong \Lambda^*$, whose [illegible] is $\mathbb{Z} \otimes [illegible]$ [S]. Finally, due to the fact that in [illegible] [illegible], [illegible] with the [illegible] in [illegible] [illegible] as in [T]. [illegible] [illegible] of $\pi_*(S;\mathbb{Q})$ corresponding to [illegible] is not spherical, hence Λ^* is not [illegible] spherically generated!

The first [illegible] [illegible] [illegible] algebras are those of [HS], namely $\Lambda_n = H^*(M_n;\mathbb{Q})$ [illegible], these fit into our theory and satisfy [illegible] $\otimes \mathbb{Q}) = 0$ [illegible] in [illegible].

Finally consider A^* = the cohomology algebra of the 2-skeleton of the m-torus, $m > 2$. Concerning the bigraded model $\mathcal{B}_A$ of [HS], it may be easily seen that the 1-bigraded model is $(\wedge Z_0^1, D = 0)$, with $\dim Z_0^1 = m$, $Z_0^2 = 0$ and $\dim Z_1^2 = m(m-1)(m-2)/6$. Using deformation theory exactly as in the proof of A'(i), it immediately follows that even π_1"$\otimes$"$\mathbb{Q}$ is constant within A^*, hence A^* is 1-graded intrinsically formal, though $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) \neq 0$. However it seems that this is the right kind of condition for having a reasonable "deformation theory" for the fundamental group.

3. Bounds for Homotopy Lie Algebras

Here we give the proofs of A(ii) and A'(ii) and also exhibit two quite general types of bounds for homotopy Lie algebras directly related to the main ideas of the paper.

3.1. Proof of Theorem A(ii). Assuming $H^*(S;\mathbb{Q}) \cong A^*$, we know that $\mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q} \cong \mathrm{gr}^* H_*\mathcal{L}_A$, by the graded intrinsic formality of A^*. We claim that from $\mathrm{cg}(A) \geq n$ it follows that there exists a bglie onto map $f\colon \mathrm{gr}^* H_*\mathcal{L}_A \to \mathbb{L}_n^*$, where $\mathbb{L}_n^*$ is a free graded Lie algebra on n homogeneous generators of strictly positive degrees, which is bigraded by using the bracket length as upper degree. Postponing for the moment the proof of the claim, we finish by observing that a bglie onto map $f\colon \mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q} \to \mathbb{L}^*(x,y)$ gives rise to a bglie monic section $s\colon \mathbb{L}^*(x,y) \to \mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q}$, which, by freeness of $\mathbb{L}(x,y)$ and by lifting in upper degree one, finally provides a glie map $h\colon \mathbb{L}(x,y) \to \pi_*\Omega S \otimes \mathbb{Q}$. Since the free Lie algebra is generated in upper degree one, we know that $\mathrm{gr}^*\mathbb{L} \cong \mathbb{L}^*$ and thus $\mathrm{gr}^* h = s$, therefore h is also monic.

Coming back to our claim, we recall that we have an n-dimensional graded vector subspace $N \subset A^+$, with $N \cdot N = 0$ and $N \subset A^+ \twoheadrightarrow QA$ monic. We therefore have a graded algebra map $\mathbb{Q}\cdot 1 \oplus N \overset{j}{\hookrightarrow} A^*$, where the multiplication in the first algebra is defined by $(q \oplus n)\cdot(q' \oplus n') = qq' \oplus (qn' + q'n)$. This gives rise by duality to a bdglie map $g\colon \mathcal{L}_A \to \mathcal{L}_{\mathbb{Q}\cdot 1 \oplus N} = (\mathbb{L}^* V_*, \partial = 0)$, where $V_* = \# s^{-1} N^*$ [T]. We may take then $f = \mathrm{gr}^* H_* g$, and it will be plainly enough to show that $H_* g$ is onto; since g is bihomogeneous and $\mathbb{L}$ is generated by $\mathbb{L}^1$, this is equivalent to $H^1 g$ being onto or, by duality, to $Q(j)$ being monic, which is precisely the injectivity condition for $N \subset A^+ \twoheadrightarrow QA$. ∎

3.2. Proof of Theorem A'(ii). Here we know that $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong \mathrm{gr}^*\pi_1 S_A \otimes \mathbb{Q} \cong L_A^*$ (see 1.8.(ii)). We now claim that $\mathrm{cg}(\mu)$ = maximal n for which there is a grlie onto map $f\colon L_A^* \to \mathbb{L}_n^*$. Temporarily taking this for granted, we are going to complete the proof, in a way similar with the preceding one, by first taking a grlie section of f, $s\colon \mathbb{L}^*(x,y) \to L_A^* \cong \mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$. Due to the finite generation property of $H_1(S;\mathbb{Z})$ we may also suppose (possibly

Finally, consider H = the [illegible] of the $\mathbb{Z}$-skeleton of the m terms, $m \geq 2$. [illegible] of [illegible] E_2 of $[H]$; [illegible] sometimes the [illegible] $(\Sigma_{\eta})^{*} = 0$, with $\mathbb{E}_{2}^{*}$ [illegible] $= 0$ and [illegible] $\ldots (m - 3)/2$. Using deformation theory exactly as in the proof of [illegible] immediately follows that even $\pi_{*}\Omega^{*}Q$ is contained within [illegible], [illegible] intrinsically formal, though [illegible] $(C, Q) \neq 0$. However it seems that this is the right kind of condition for having a reasonable "deformation theory" for the [illegible].

8. Bounds for Homotopy Lie Algebras

Here we give the proofs of [illegible] and also exhibit two quite general types of bounds for homotopy Lie algebras directly related to the main ideas of the paper.

8.1. Proof of Theorem [illegible]. Assuming [illegible], we show that $\pi_{*}\Omega Y \otimes \mathbb{Q} \cong$ [illegible]. By the [illegible] we claim that from [illegible] it follows that there exists a [illegible] where [illegible] is a free graded Lie algebra on a homogeneous [illegible] of strictly positive degrees, which is bounded by using the bracket length as [illegible] in the proof of the claim, we finish by observing that [illegible] gives rise to a [illegible] [illegible], and by [illegible] there [illegible]. Since the free Lie algebra is [illegible], and thus [illegible], therefore [illegible] is also [illegible].

Coming back to our [illegible] now [illegible] graded vector subspace [illegible], with [illegible]. We therefore have a graded algebra map [illegible], where the multiplication in the [illegible] is defined by [illegible]. This is [illegible] where [illegible], [illegible] $= 0$, where [illegible]. We [illegible] and it will be plainly enough to show that [illegible] is [illegible], which is [illegible] equivalent to [illegible] being [illegible], [illegible] which is precisely the [illegible] condition for [illegible]. ∎

8.2. Proof of Theorem [illegible]. [illegible] we know that [illegible] $\otimes \mathbb{Q} \neq L_{Q}$ [illegible]. We [illegible] that [illegible] there is a [illegible] map [illegible]. [illegible] we are going to complete the proof, in a [illegible] with the [illegible], by [illegible] a [illegible] section of [illegible]. [illegible] also [illegible] after replacing x and y by suitable nonzero multiples) that sx and sy lift to $\mathrm{gr}^1\pi_1 S$, hence to $\pi_1 S$. We have thus obtained a group homomorphism from the free group on two generators, $h\colon \mathsf{F}_2 = \mathsf{F}(x,y) \to \pi_1 S$, with the property that $\mathrm{gr}^* h \otimes \mathbb{Q} = s$ ($\mathrm{gr}^*\mathsf{F}(x,y) \cong \mathsf{L}^*_{\mathbb{Z}}(x,y)$, [Se; IV.6, Theorem 1]). Consequently $\mathrm{gr}^* h$ is monic. A rather standard argument shows then h to be monic, by working with the nilpotent quotients: for any n consider the induced map $h_n\colon \mathsf{F}_2/\mathsf{F}_2^{(n)} \to \pi_1 S/\pi_1 S^{(n)}$ which will also have the property that $\mathrm{gr}^* h_n$ is monic; use this, and the natural short exact sequences $\mathrm{gr}^m G \to G/G^{(m+1)} \to G/G^{(m)}$ associated to a group G, together with the nilpotence of each $G/G^{(n)}$, to establish inductively the injectivity of h_n; finally infer that $\mathrm{Ker}\, h \subset \bigcap_n \mathsf{F}_2^{(n)} = \{1\}$.

The truth of the claim may be easily seen, by observing first that the graded Lie maps $f\colon \mathsf{L}^* X/\mathrm{ideal}(\partial Y) \to \mathsf{L}^* V$ are in a bijective correspondence, by duality, with the linear maps $g\colon N \to A^1$ with the property that $\mu \circ (g \wedge g) = 0$, and next that f is onto if and only if g is monic. ∎

REMARKS. Chen's method of iterated integrals ([C1],[C2]) allows one to obtain results which are of a similar nature with the above A'(ii), but it requires the presence of conditions imposed at the level of differential forms, and not just at the level of the de Rham cohomology; for example our condition $\mathrm{cg}(\mu) > 1$ is replaced by: there exist closed 1-forms ω_1 and ω_2 on the manifold M, representing independent cohomology classes, and such that $\omega_1 \wedge \omega_2 = 0$ as a form. From this point of view the two approaches are to be considered as complementary: the manifolds $N_{\mathbb{R}}/N_{\mathbb{Z}}$ and $(S^1 \times S^2)\#(S^1 \times S^2)$ mentioned in the introduction have the same cohomology algebra and $\mathrm{cg}(\mu) = 2$, but in the case of the first (nil)manifold it is impossible to find representatives with $\omega_1 \wedge \omega_2 = 0$ (this would imply by [C2] that its fundamental - nilpotent - group contains an F_2!), while in the other case this is easily done geometrically (by taking the two closed 1-forms dual to two disjointly embedded 2-spheres, one in each term of the connected sum), but A'(ii) cannot be used since $Z_0^2 \neq 0$.

The next result is also complementary to Chen's [C1], but this time concerning its conclusion. It should be noted that there is also an integral version of the result below, claiming, for any connected complex S with $H_1(S;\mathbb{Z}) =$ free, the existence of a natural onto grlie map $L^*_{\mathbb{Z}} \twoheadrightarrow \mathrm{gr}^*\pi_1 S$, induced by the isomorphism $H_1(S;\mathbb{Z}) \cong \mathrm{gr}^1\pi_1 S$, where $L^*_{\mathbb{Z}}$ is the quotient of the free integral grlie algebra generated by $H_1(S;\mathbb{Z})$, graded by bracket length, modulo the ideal generated by the image of the reduced diagonal $\overline{\Delta}\colon H_2(S;\mathbb{Z}) \to H_1(S;\mathbb{Z}) \wedge H_1(S;\mathbb{Z})$. Since we are not going to use this more precise version here, except in the case of links (where it was established by Chen [C3]), see 5.5, we omit its proof.

3.3. PROPOSITION. *Let A^* be given, with vector-valued 2-form μ and associated grlie algebra L^*_A. If S is any connected space whose cohomology algebra has μ as associated vector-valued* where [illegible] and [illegible] are [illegible] conditions that [illegible] and [illegible] lift to [illegible], hence in [illegible]. We [illegible] is a [illegible] from the free group on two generators [illegible] with the property that [illegible] ([illegible] IV.6, Theorem 1]). [illegible] A [illegible] argument shows that h to be monic, by [illegible] with the [illegible] the only [illegible] for the [illegible] map [illegible] which will also have the property [illegible]. As [illegible] and the natural [illegible] associated to a group G, together with the [illegible] of each [illegible] to establish [illegible] the [illegible] of h, finally infer that [illegible] (ii).

The truth of the claim may be readily seen by observing first that the [illegible] are in [illegible] correspondence by duality, with the [illegible] with the property [illegible], and next that f is onto if and only if g is monic. ∎

REMARK. Chen's method of iterated integrals ([Ch], [C2]) allows one to obtain results which are [illegible] but it requires the presence of conditions imposed at the level of differential forms, and not just at the level of the de Rham cohomology; for example our condition [illegible] the manifold [illegible] representing [illegible] cohomology classes, and [illegible]. Apart from this [illegible] the two approaches are to be considered as complementary [illegible] the manifolds [illegible] in the [illegible] cohomology algebra [illegible] and it is [illegible] to find [illegible] that its fundamental [illegible] group [illegible] taking the [illegible] in each [illegible] cannot be [illegible].

The next result is [illegible] but this time [illegible]. It should be noted that there is also an [illegible] of the result below, [illegible] for any connected complex S with [illegible], where [illegible] is the quotient of the [illegible] the [illegible] generated by [illegible]. Since we are not going to use [illegible], except in the case of links (where it was established by Chen [illegible]), we omit its proof.

3.3. PROPOSITION. Let Γ be [illegible]. If S is [illegible] *two-form then there exists a grlie epimorphism* $L_A^* \twoheadrightarrow gr^*\pi_1 S \otimes \mathbb{Q}$.

PROOF: Take the 1-minimal model of S, $(\wedge V, d)$, consider the canonical filtration $\{F_n V\}$ and set $L_n \cong \mathcal{C}^{*^{-1}}(\wedge F_n V, d)$; we know that $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong \varprojlim_n \mathrm{gr}^* L_n$. Fixing the 1-minimal dga $(\wedge F_n V, d)$, we obviously have, by naturality, $F_m(F_n V) \subset F_m V$, for any m; a straightforward induction, which only uses the definition of the canonical filtration, shows that we have in fact $F_m(F_n V) \cong F_m V$, for $m \leq n$; the general equality $\#\mathrm{gr}^p \cong F_p/F_{p-1}$ following from Lemma 1.1 and the preceding remark show that the inverse limit $\varprojlim_n \mathrm{gr}^p L_n$ stabilizes for $n \geq p$, for any p. For $p = 1$, L_1 is just the abelian Lie algebra on $\#F_1 V \cong \#H^1(\wedge V, d) \cong \#H^1 S \cong \#A^1 \cong X$. By stability we have a tower of grlie maps $g_n \colon \mathbb{L}^* X \to \mathrm{gr}^* L_n$, given by $g_n|_X = \mathrm{id}$ (whence they are all onto). In order to check that they all factor through L_A^* giving thus rise to a tower $f_n \colon \mathbb{L}_A^* \to \mathrm{gr}^* L_n$ (consisting of epic grlie maps), hence to a grlie epimorphism $f \colon L_A^* \to \varprojlim_n \mathrm{gr}^* L_n \cong \mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$ as desired, it would suffice to check that $g_2 \partial Y = 0$ in $\mathrm{gr}^2 L_2$, again by stability, i.e. that the composition $Y \xrightarrow{\partial} X \wedge X \cong \mathrm{gr}^1 L_2 \wedge \mathrm{gr}^1 L_2 \xrightarrow{[\ ,\ ]} \mathrm{gr}^2 L_2$ equals zero. By duality, this is equivalent to seeing that $F_2/F_1 \xrightarrow{d} F_1 \wedge F_1 \cong A^1 \wedge A^1 \xrightarrow{\mu} A^2$ equals zero. Denoting by $\mathcal{D}$ the decomposable elements of a graded algebra, plainly $dF_2 = 0$ in $\mathcal{D}H^2(\wedge V, d) \cong \mathcal{D}H^2 S \cong \mathcal{D}A^2$, the last equality coming from our assumption on the vector-valued 2-form associated to $H^* S$. The proof is now complete. ∎

The following interesting numerical test for 1-graded intrinsic formality may be immediately deduced.

COROLLARY. *A^* is 1-graded intrinsically formal if and only if rk $gr^p\pi_1 S$ is constant within A^* (and equal to* $\dim L_A^p$*), for any p.*

Our last result in this direction is somewhat surprising, since the other known qualitative numerical results indicate (see [F; Chapitre 7]) that the numerical invariants of the formal space S_A would represent an upper bound for the corresponding numerical invariants of S if $H^*(S; \mathbb{Q}) \cong A^*$.

3.4. PROPOSITION. *Let S be a 1-connected space of finite type, with rational cohomology algebra A^*. If the Eilenberg-Moore spectral sequence of S collapses at the E_2 term, then we have inequalities $rk(\pi_*\Omega S)_n^{(p)} \geq$* $\dim(H_*\mathcal{L}_A)_n^{(p)}$*, for any n, p.*

PROOF: Represent S by a (minimal !) model of the form $(\wedge Z^*, D + p)$, as in 2.1, and note that $\pi_*\Omega S \otimes \mathbb{Q} \cong \mathcal{C}^{*^{-1}}(\wedge Z^*, D_2 + p_2)$, and $H_*^*\mathcal{L}_A \cong \mathcal{C}^{*^{-1}}(\wedge Z_*^*, D_2)$. We thus have a bigraded vector space $L_*^* = \#Z_{*-1}^{*+1}$ and two graded Lie brackets, $[\ ,\ ]_p$ and $[\ ,\ ]$; the second is actually bihomogeneous and the conditions on the perturbation p translate, as in the proof of 1.10, to the fact that, for any $x \in L^m$, $y \in L^n$, $[x, y]_p = [x, y]$ modulo $L^{>m+n}$. We may

also suppose that $\dim L < \infty$. This can be seen as follows: for any fixed lower degree n, as in our statement, the vector space $L_n^{(q)}$ remains unchanged (for any q) after taking the quotient of L by the graded Lie ideal $L_{>n}$.

Having established this framework, let us denote by $\{F_p^m\}_{m\geq 0}$ respectively by $\{F^m\}_{m\geq 0}$, the lower central series corresponding to $[\ ,\ ]_p$, respectively to $[\ ,\ ]$. We have to show that $\dim (F_p^m)_n \geq \dim F_n^m$, for any m, n. Consider then the (decreasing and finite) filtration on F_p^m defined by $G^k F_p^m$ = vector subspace spanned by $[x_1, [\ldots, [x_{m-1}, x_m]_p \ldots]_p]_p$, where $x_i \in L_{s_i}^{r_i}$ and $\sum r_i \geq k$ (and similarly for $G^k F^m$). In the bihomogeneous case we evidently have $G^k F^m \cong F^m \cap L^{\geq k}$.

For fixed m and k, define $f\colon G^k F_p^m / G^{k+1} F_p^m \to G^k F^m / G^{k+1} F^m$ by $f(\Sigma_p) = \Sigma$, where Σ_p is a sum of terms of the form $[x_1, [\ldots, [x_{m-1}, x_m]_p \ldots]_p]_p$ in $G^k F_p^m$ (modulo $G^{k+1} F_p^m$) and Σ is the sum (in $G^k F^m / G^{k+1} F^m$) which is obtained by replacing the above monomials by $[x_1, [\ldots, [x_{m-1}, x_m] \ldots]]$. The map f obviously being onto (if well-defined !) and compatible with lower degrees, the desired inequalities will follow (both filtrations being finite).

It remains to be shown that $\Sigma_p = 0$ in $G^k F_p^m$ implies $\Sigma = 0$ in $G^k F^m / G^{k+1} F^m$. By expanding the brackets in Σ_p and replacing them by unperturbed brackets we find out that $\Sigma_p = \Sigma - z$, with $z \in L^{>k}$; if $\Sigma_p = 0$, then $\Sigma = z \in F^m \cap L^{>k} \cong G^{k+1} F^m$ and we are done. ∎

4. Rigid examples and inert sequences

We prove Theorems B and B' and we explain our main source of examples, based on Anick's [A; page 133] notion of combinatorial freeness.

4.1. Proof of Theorem B(i). By duality (see [T; III.4.(5) and I.1.(7)]) the condition $(A^+)^3 = 0$ may be rephrased as follows: $\mathcal{L}_A = (\mathbb{L}(X_* \oplus Y_*), \partial)$, where $X_* \oplus Y_* = \# s^{-1}\overline{A}^*$, $\partial X_* = 0$ and $\partial|_{Y_*}$ is monic and takes values in $(\mathbb{L}^2 X)_{*-1}$. The fact that the ideal generated by ∂Y in $\mathbb{L}X$ is <u>inert</u> (in the sense of [HL; Définition 3.1]) is equivalent to the fact that the sequence $\partial y_1, \ldots, \partial y_n$ is inert in $\mathbb{L}X$ (which means by definition that the sequence is <u>strongly free</u> - in the sense of [A; page 127] - when viewed in $\mathbb{T}X$), and this is also equivalent to $H^*\mathcal{L}_A$ being generated as a Lie algebra by $H^1\mathcal{L}_A$ (i.e. $Q\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q}) = 0$, see Lemma 1.9(i)), and further equivalent to the fact that the natural projection $\mathbb{L}X_* \to E_* = \mathbb{L}X_*/\mathrm{ideal}(\partial Y_*)$ induces a monomorphism on $\mathrm{Tor}_2^{\mathcal{U}(.)}(\mathbb{Q}, \mathbb{Q})$ and an isomorphism on $\mathrm{Tor}_{\geq 3}^{\mathcal{U}(.)}(\mathbb{Q}, \mathbb{Q})$; all these are to be found in [HL; Proposition 3.2, Théorème 1.1]. It is also proven there that if they are fulfilled then $H_*^*\mathcal{L}_A \cong E_*^*$. In our case $\mathrm{Tor}_{\geq 2}^{\mathcal{U}\mathbb{L}X}(\mathbb{Q}, \mathbb{Q}) = 0$, due to the freeness of $\mathbb{L}X$, and thus the above conditions on Tor_2 and $\mathrm{Tor}_{\geq 3}$ simply reduce to $\mathrm{gl}\ \dim E_* \leq 2$. This remark completes our proof. ∎

also suppose that that $k < \infty$. The claim is seen as follows: for any fixed lower degree n, as in our statement the vector space Γ^n remains unchanged (for any ℓ) after taking the quotient of Γ by the ideal $I\Gamma$ [illegible].

Having established this, we denote by $\{\Gamma^n_{[p]}\}$ [illegible] respectively by $\{\Gamma^n\}$ [illegible] respectively to [illegible]. We need to show that $\dim(\Gamma^n_{[p]}) \geq \dim \Gamma^n$ for any n. Consider then the (decreasing and finite) filtration on Γ^n defined by [illegible] vector subspace spanned by [illegible], where $x_i \in \Gamma^1$ and [illegible] (and similarly for [illegible]). In the homogeneous case we evidently have $\Gamma^n = \Gamma^n[\ell]$.

For fixed n and k, define f: [illegible] $\to$ [illegible] by $f(\Sigma_p) = \Sigma$, where Σ_p is a sum of terms of the form [illegible] in [illegible] (modulo [illegible]) and Σ is the sum (in [illegible]) which is obtained by replacing the above monomials by [illegible]. The map f obviously being onto (if well-defined) and compatible with lower degrees, the desired inequalities will follow (both filtrations being finite).

It remains to be shown that [illegible] $= 0$ in [illegible] implies $\Sigma = 0$ in [illegible]. By expanding the brackets in Σ_p and applying them to [illegible] brackets we find out that [illegible] $= \Sigma$ [illegible] with [illegible], and thus [illegible] and we are done. ∎

4. Hecke examples and Koszul sequences

We prove Theorem B and [illegible] on our main class of examples, based on Anick's [A, page 13] notion of combinatorial free-ness.

4.1. Proof of Theorem B(i). By Anick (see [A, Thm. 2.6] and [illegible]) the condition [illegible] $= 0$ may be rephrased as follows: [illegible], where [illegible]. This [illegible] ideal generated by [illegible] is [illegible] (in the sense of [illegible]) is equivalent to the fact that the sequence [illegible] (which means by definition that the sequence is [illegible]) in the sense of [illegible] and this is also equivalent to [illegible] being regarded as a [illegible] (i.e. [illegible] $(\mathbb{Q}, \mathbb{Q}) = 0$, see Lemma 1.8(i)), and further equivalent to the fact that the natural projection [illegible] induces a monomorphism [illegible] $(\mathbb{Q}, \mathbb{Q})$ and an isomorphism on [illegible]; all these are to be found in [illegible] Proposition 3.5, Theorem 4.1. It is also proven there that if they are fulfilled then [illegible]. In our case [illegible], due to the freeness of [illegible], and thus the above condition on [illegible] simply reduces to [illegible]. This remark completes our proof. ∎

4.2. Proof of Theorem B'(i). Consider the bigraded model of A^*, $\mathcal{B}_A = (\wedge Z_*^*, D)$. As remarked in the proof of Lemma 1.9(ii) $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ if and only if $Z^2 = 0$ and $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1} = 0$ if and only if $Z^{\geq 2} = 0$. By the general uniqueness results for k-stage minimal models (i.e. minimal algebras generated in degree $\leq k$ together with modelling dga maps inducing a cohomology isomorphism up to degree k and a cohomology monomorphism in degree $k+1$) $Z^2 = 0$ is equivalent to the fact that the (b)dga 1-modelling canonical map of [HS], $f\colon (\wedge Z_*^1, D) \to (A^*, 0)$ is isomorphic in cohomology in degrees 1 and 2 and monomorphic in degree 3; similarly $Z^{\geq 2} = 0$ is equivalent to H^*f being an isomorphism.

Since f is a 1-modelling map, $H^1 f$ is isomorphic and $H^2 f$ is a monomorphism onto the decomposables $\mathcal{D}A^2$; on the other hand $A^3 = 0$, the algebra A^* being 2-skeletal. These remarks show that $Z^2 = 0$ is equivalent to the surjectivity of μ plus $H^3(\wedge Z_*^1, D) = 0$. As we have already noticed in the proof of Lemma 1.8, $H^k(\wedge Z_*^1, D) \cong H^k(L_A^*; \mathbb{Q})$, for any k. Now use an innocuous but very useful trick (which will enable us to use freely the results obtained in [A] and [HL] for <u>conected graded</u> algebras): we replace the grlie algebra L_A^* by the connected glie algebra E_* constructed in the introduction. We have changed nothing except doubling upper degrees and then transforming them into lower degrees; consequently $H^k(\wedge Z^1, D) \cong \#\mathrm{Tor}_k^{\mathcal{U}E_*}(\mathbb{Q},\mathbb{Q})$, any k; since the $\mathrm{Tor}_k^{\mathcal{U}(\cdot)}$ test of [HL] is enough to be checked only for $k = 2$ and 3 (Proposition 3.2 of [HL]), we conclude that $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ is equivalent to μ being onto and gl $\dim E_* \leq 2$ (which implies $H^{\geq 3}(\wedge Z^1, D) = 0$, therefore H^*f is an isomorphism and $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$) and also equivalent to ∂ being monic and the sequence $\partial y_1, \ldots, \partial y_n$ being inert in $\mathbb{L}X$ (or strongly free in $\mathbb{T}X$), as before. Noting that the strong freeness of a sequence implies the linear independence of its elements, the surjectivity of μ follows, and our proof is complete. ∎

We point out that we have a characterization of the vanishing property for $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})$ valid for general algebras A^* (as usual, connected and of finite type), similar to the one given in Theorem B'(i). We chose not to give it here, because we are not going to use it here.

4.3. Combinatorial conditions for strong freeness. We shall describe now, following Anick [A], a very useful combinatorial test for the strong freeness of sequences of elements in graded tensor algebras. Let then $\mathbb{T}X$ be the (connected graded) tensor algebra on a positively graded vector space X, $\dim X = m$. Pick an <u>ordered</u> homogeneous basis of X, say $\{x_1, \ldots, x_m\}$, and then extend this order to a total order on the monomials $u = x_{i_1} \otimes \cdots \otimes x_{i_r}$ of $\mathbb{T}X$, having the properties: $\deg u < \deg v \Longrightarrow u < v$ and $u < v \Longrightarrow zut < zvt$, for any z and t (we shall explicitely use, on the monomials of the same degree, the <u>lexicographic order from the right</u>). Given any nonzero element $y \in \mathbb{T}X$, write $y = c_1 u_1 + \cdots + c_r u_r$, where c_i are constants and u_i are monomials, and define <u>the highest term</u> of y, to be denoted by $\overline{y}$, by

4.2. Proof of Theorem B (i). Consider the bigraded model of $A^!$, $E_1 = (\Lambda Z_*^*, D)$. As computed in the proof of Lemma 1.5(d), $\mathrm{Ext}_{A}^{*}(\mathbb{Q}, \mathbb{Q}) = 0$ if and only if $Z_*^2 = 0$ and $H(\Lambda Z_0^*)^{\geq 2} = 0$ if and only if $Z^1 = 0$. By the general uniqueness results for bigraded minimal models (i.e. minimal algebras generated in degree $\leq k$ together with modelling dga maps inducing a cohomology isomorphism up to degree k and a cohomology monomorphism in degree $k+1$), $Z^1 = 0$ is equivalent to the fact that the (γ)deg 1-modelling canonical map of [HS], $\gamma : (\Lambda Z_0^1, D) \to (H^*, 0)$ is isomorphic in cohomology in degrees 1 and 2 and monomorphic in degree 3; similarly $Z^2 = 0$ is equivalent to $H^*\gamma$ being an isomorphism.

Since γ is a formalizing map, $H^1\gamma$ is isomorphic and $H^2\gamma$ a monomorphism onto the decomposables $\mathcal{D}H$; on the other hand $A^3 = 0$, the algebra A being 2-related. Thus immediately that $Z^2 = 0$ is equivalent to the surjectivity of γ plus $H^3(\Lambda Z_0^1, D) = 0$. As we have already noticed in the proof of Lemma 1.5, $H^*(\Lambda Z_0^1, D) \cong H^*(L_A, \mathbb{Q})$, for any A. Here is an innocuous but very useful trick (which will enable us to use freely the results obtained in [A] and [PY] for connected graded algebras): replace the graded algebra L_A by the connected graded algebra L constructed in the introduction. We have changed nothing except doubling upper degrees and then transforming them into lower degrees, consequently $H^*(\Lambda Z_0^1, D) \cong \mathrm{Ext}_{UL}^{*}(\mathbb{Q}, \mathbb{Q})$ for any L. Since the "final" part of [A] is known to be checked only for $k = 2$ and 3 (Proposition 5.2.? of [PY]), we conclude that $\mathrm{Ext}_{UL}^{3}(\mathbb{Q}, \mathbb{Q}) = 0$ is equivalent to $\mathfrak{g}$ being onto and gl.dim $UL \leq 2$ (which means $H^{\geq 3}(\Lambda Z_0^1, D) = 0$), therefore $H^*\gamma$ is an isomorphism if and only if $\mathrm{Ext}_{UL}^{3}(\mathbb{Q}, \mathbb{Q}) = 0$ and also equivalent to G being inert and the sequence $\theta_1, \ldots, \theta_r$ being strongly free in $\mathbb{L}$ (as before). Notice that the strong freeness of a sequence implies the linear independence of its elements; the injectivity of γ follows and our proof is complete. ∎

We point out that we have a characterization of the vanishing property for $\mathrm{Ext}_{A}^{3}(\mathbb{Q}, \mathbb{Q})$ valid for general algebras $A^!$ (quadratic, connected and of finite type), similar to the one given in Theorem B (i). We choose not to give it here, because we are not going to use it here.

4.3. Combinatorics of strongly free sequences. We shall describe now, following Anick [A], a very useful combinatorial test for the strong freeness of sequences of elements in graded tensor algebras. Let then TV be the (connected graded) tensor algebra on a positively graded vector space V. Fix a basis X of V. The monomials (monomials of TV, say $x_{i_1} \cdots x_{i_k}$), and the lexicographic order ... from ... the monomials $\mathfrak{m} = x_{i_1} \cdots x_{i_k}$ of TX, having the properties ... ordering ... for any s and t (we shall ... the monomials of the same degree, the lexicographic order from ... (2.1)). ... element $\varphi \in TV$, write $\varphi = c_1 \mathfrak{m}_1 + \cdots + c_k \mathfrak{m}_k$, where the constants $c_i \neq 0$, are nonzero and define the highest term of φ to be denoted by $\hat{\varphi}$, by

$\overline{y} = u_i$, where u_i = the largest u_j for which $c_j \neq 0$. For a given monomial $u = x_{i_1} \otimes \cdots \otimes x_{i_r}$, define its origin by $\mathrm{o}(u) = i_1$ and its end by $\mathrm{e}(u) = i_r$. To simplify matters (having in mind our applications via B(i) and B'(i)) we shall only consider sequences $y_1, \ldots, y_n$ of tensor degree two, i.e. $y_i \in \mathsf{T}^2X$, for any i. Anick's beautiful result reads then:

THEOREM (see [A; theorems 3.2 and 3.1]). *The sequence $y_1, \ldots, y_n$ is strongly free in $\mathsf{T}X$ if the monomial sequence of its highest terms $\overline{y}_1, \ldots, \overline{y}_n$, is combinatorially free, i.e.*

($\star$) the monomials $\overline{y}_1, \ldots, \overline{y}_n$ are distinct, and

($\star\star$) the sets of indices $\{\mathrm{o}(\overline{y}_1), \ldots, \mathrm{o}(\overline{y}_n)\}$ and $\{\mathrm{e}(\overline{y}_1), \ldots, \mathrm{e}(\overline{y}_n)\}$ are disjoint.

As a first example, both simple and instructive, we shall again follow Anick and take $\{x'_1, \ldots, x'_r\} \cup \{x''_1, \ldots, x''_s\}$ as basis for X (concentrated in lower degree 2) – in this order – and consider the sequence of Lie elements $\{y_{ij} = [x'_i, x''_j] \in \mathsf{L}^2X \subset \mathsf{T}^2X\}$, $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Then any subsequence is strongly free, being combinatorially free (since $\overline{y}_{ij} = x'_i \otimes x''_j$, and the combinatorial conditions, ($\star$) and ($\star\star$), are obviously satisfied).

4.4. PROOF OF THEOREM B(ii). A general sufficient condition for the intrinsic spherical generation of a 1-connected algebra A^*, with Quillen model $\mathcal{L}_A = (\mathsf{L}^*W_*, \partial)$, may be found in [MP; 1.9. and 1.10]; it only requires the vanishing of $\mathrm{Hom}_{-1}(H^1_*\mathcal{L}_A, H^{\geq 2}_*\mathcal{L}_A)$, where $\mathrm{Hom}_{-1}(\ ,\ )$ denotes linear maps which are homogeneous of lower degree -1. Recall next that we know, by assumption, that $H^*\mathcal{L}_A$ is generated by $H^1\mathcal{L}_A$ (cf. Lemma 1.9(i)). On the other hand $H^1_*\mathcal{L}_A \cong \mathrm{Ker}(\partial|_{W_*}) \cong \# QA^{*+1}$. A simple degree argument (based on our hypotheses on the degrees of QA^*) shows that $\mathrm{Hom}_{-1}(H^1_*\mathcal{L}_A, H^{\geq 2}_*\mathcal{L}_A) = 0$ and finishes the proof. ∎

Here is a slightly more general version of B'(ii):

4.5. LEMMA. *Let A^* be any algebra (connected and of finite type), with associated grlie L^*_A. If $\mathcal{P}\mathrm{Ext}^{*,\geq 1}_A(\mathbb{Q}, \mathbb{Q}) = 0$ then we have an equality of formal power series*

$$\prod_{p=1}^{\infty} (1 - z^p)^{\dim L^p_A} = A^*(-z),$$

where $A^*(z)$ is the Hilbert series of A^*, $\sum_{n \geq 0} \dim A^n \cdot z^n$.

PROOF: As indicated in the proof of B'(i), the assumption $\mathcal{P}\mathrm{Ext}^{*,\geq 1}_A(\mathbb{Q}, \mathbb{Q}) = 0$ simply means that the bigraded model $\mathcal{B}_A$ coincides with the 1-bigraded model, $(\wedge Z^1_*, D)$. By a general formula of [HS; Proposition 3.10] we know that $A^*(-z) = \prod_{n=0}^{\infty}(1 - z^{n+1})^{\dim Z^1_n}$. On the other hand, Lemma 1.8 tells us that $Z^1_n \cong \# L^{n+1}_A$ (see (1.7)), which completes the proof. ∎

$i < n_i$, where n_j is the largest n_i for which $n_i < \infty$. For a given monomial $m = x_1 \otimes \cdots \otimes x_r$, define its weight to $w(\xi)$ with ... and the grading by $w(x_i) = n_i$. To simplify matters (having in mind our applications in 4.3(a) and 4.4(b)) we shall only consider sequences $y_1, \ldots, y_s$ of elements of degree two, i.e. $y_j \in T^2 X$ for any j. Anick's theorem [1] reads then:

THEOREM (see [A, theorems 2.2 and 3.1]). The sequence $y_1, \ldots, y_s$ is strongly free in TX if the associated monomial sequence with highest terms $\hat y_1, \ldots, \hat y_s$ is combinatorially free, i.e.:

(★) the monomials $\hat y_1, \ldots, \hat y_s$ are distinct, and

(★★) the sets of subwords $\{(a, b) \ldots\}$ and $\{(a, b) \ldots\}$ are disjoint.

As a first example, both simple and instructive, we shall again follow Anick and take $\{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_s\}$ as basis of X represented in lower degree 2 — in this order — and consider the sequence of the elements $z_{ij} = [x_i, y_j] \in T^2 X$, $i = 1, \ldots, r$ and $j = 1, \ldots, s$. This sequence is strongly free, being combinatorially free (since $\hat z_{ij} = x_i \otimes y_j$ and the combinatorial conditions (★) and (★★) are obviously satisfied).

4.4. PROOF OF THEOREM 1(ii). A general sufficient condition for the symmetric spherical realization of a 1-connected algebra H, with Quillen model $L_H = (\mathbb{L}(s^{-1}\bar H_*), \partial)$, may be found in [HP, 1.9] and [, 4.1]: it only requires the vanishing of $\mathrm{Hom}_{\mathbb{Q}}(H_* L_H, \bar H_*^{\geq 2} L_H)$, where $\mathrm{Hom}(-, -)$ denotes linear maps which are homomorphisms of lower degree [illegible]. Recall here that we know, by assumption, that $H_* L_H$ is generated by $H_0 L_H$ (cf. Lemma 1.9(i)). On the other hand $H_* L_H \cong$ [illegible]. A simple degree argument (based on our hypotheses on the dimension of H) shows that $\mathrm{Hom}_{\mathbb{Q}}(H_* L_H, \ldots) = 0$ and finishes the proof. ∎

Here is a slightly more general version of 3.4(ii).

4.5. LEMMA. Let H be any connected graded finite type algebra, with associated [illegible]. $H_*(\ldots) = 0$ implies the following formal power series

$$\prod_{j \geq 1} (1 + z^{2j+1})^{\dim \ldots} \ldots = (1 - H(z))^{-1},$$

where $H(z)$ is the Hilbert series of H: $\sum_{n \geq 0} \dim H^n z^n$.

PROOF. As indicated in the proof of 4.4, the assumption $\mathrm{Ext}(\mathbb{Q}, \mathbb{Q}) = 0$ simply means that the bigraded model $\mathcal{B}_*$ [illegible] the bigraded model $(\Lambda(\ldots), \partial)$. By a general formula of [HS, Proposition 1.2] we know that [illegible]. On the other hand, Lemma 1.9(i) tells us that [illegible] (see 1.4), which completes the proof. ∎

REMARKS. The above formula is in fact quite effective: one may uniquely express $\dim L_A^p$, for any p, with the aid of the Möbius function and of certain universal polynomials in the coefficients of $A^*(z)$, if $\dim A^* < \infty$, see [B].

4.6. PROPOSITION. *Given an arbitrary algebra A^* (connected and of finite type), the condition $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$ is equivalent to the fact that A^* is generated (as a graded algebra) by A^1, plus the equality of Hilbert series: $\mathrm{Tor}_*^{\mathcal{U}L_A}(\mathbb{Q},\mathbb{Q})(z) = A^*(z)$.*

PROOF: Recall the 1-modelling (b)dga map $f\colon (\wedge Z_*^1, D) \to (A^*, 0)$, where $(\wedge Z_*^1, D) \to (A^*, 0)$ is constructed out of L_A^* as explained in (1.7), see Lemma 1.8. The vanishing of $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q})$ is then equivalent to the fact that H^*f is an isomorphism. But we know that $H^*(\wedge Z_*^1, D) \cong H_0^*(\wedge Z_*^1, D) \oplus H_+^*(\wedge Z_*^1, D)$, where $H_0^* \cong \wedge^* Z_0^1/\mathrm{ideal}(DZ_1^1)$, and $H^*f(H_+^*) = 0$, by construction. Recalling from the proof of 1.8 that ($V_* \cong Z_*^1$!) $Z_0^1 \cong A^1$ and $DZ_1^1 \cong \mathrm{Ker}\,\mu$, H_0^*f being the canonical map, it follows, if H^*f is an isomorphism, that then A^* is generated by A^1, and we have an equality between the Hilbert series of A^* and of $H^*(\wedge Z^1, D)$. On the other hand we already have remarked (again in the proof of Lemma 1.8) that this last Hilbert series equals $H^*(L_A;\mathbb{Q})(z)$, hence also $\mathrm{Tor}_*^{\mathcal{U}L_A}(\mathbb{Q},\mathbb{Q})(z)$, which completes half of our proof, the other implication being immediate, with a dimension argument. ∎

4.7. EXAMPLE [KT]. Denote by P_n the n-th pure braid group and consider $A_n^* = H^*(P_n;\mathbb{Q})$. Then $\mathcal{P}\mathrm{Ext}_{A_n}^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$, for any n. This may be seen as follows: it is known that A_n^* is generated by A_n^1, for any n; the Hilbert series $A_n^*(z)$ equals $(1+z)(1+2z)\dots(1+nz)$ (hence A_n^* is <u>not</u> 2-skeletal for $n > 2$). The main result of [KT] also gives the equality $\mathrm{Tor}_*^{\mathcal{U}L_{A_n}}(\mathbb{Q},\mathbb{Q})(z) = A_n^*(z)$, for any n.

5. Link groups

We may now start the proof of Theorem C. Recall from the introduction that we have associated to an $m \times m$ symmetric matrix of rational numbers ℓ two objects: a 2-skeletal algebra A^* (with vector-valued 2-form $\mu_0\colon A^1 \wedge A^1 \to A^2$) and a finite unoriented graph Γ_0. If the entries of ℓ are integers we may do the same for any prime p. We just take $A_p^1 =$ the $\mathbb{Z}_p$-vector space with basis $\{e_1, \dots, e_m\}$, $A_p^2 = A_p^1 \wedge A_p^1$ modulo the same relations, and define $\mu_p\colon A_p^1 \wedge A_p^1 \to A_p^2$ by the same formula as before, taking of course the mod p residues of the linking coefficients l_{ij}. We also have an associated graph Γ_p, with the same vertices as Γ_0 (which is a subgraph of Γ_0 and in fact coincides with Γ_0 for all but a finite number of primes p); obviously the conectedness of Γ_p implies that of Γ_0. First we add to the conditions

REMARK. The above formula is rather quite circuitous: one may naturally express also P_n for any g with the aid of the Mumford function and of certain universal polynomials in the coefficients of $X^*(s)$, e.g. $\dim \Gamma(\ldots)$ — see [F].

4.7. PROPOSITION. [illegible] of a surface X of nonnegative Kodaira dimension [illegible] $\dim \Gamma(\Omega, \Omega^1_X)$ [illegible] to the fact that K^* is generated (as a graded algebra) by $\mathcal{K}^1_X$, plus the equality of Hilbert series $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{K}, \Omega^1_X) = K^*(1)$.

PROOF. [illegible] the [illegible] $\mathrm{Tor}_1(K^*_X, \Omega) = \mathrm{Ker}(\ldots)$, where $(K^*_X, \Omega) = (K^*\Omega)$ is [illegible] of $\mathcal{K}_X$ generated by Ω^1, see Lemma 1.3. The vanishing of $\mathrm{Tor}_1^{\mathcal{O}}(\Omega, \Omega)$ is then equivalent to the fact that K^*_X is an isomorphism. But we know that $K^*(X\mathbb{Z}, \Omega) \cong \bar{K}_X^{\ast} \cong \mathbb{Z}[\Omega] \cong K^*(X, \Omega)$, where $R_X \cong K^*\mathbb{Z}$ and $K^*(\Omega)$ [illegible] $= 0$. In one direction, it follows from the proof of 1.3 that $\mathcal{O}_n \cong K^*_n \mathbb{Z} \cong K^*$ and $\Omega(K^*) \cong \ldots$ [illegible]. If K^*_X is an isomorphism, then K^* is generated by K^1, and we have an equality between the Hilbert series of K^* and of $K^*(\mathbb{Z}\Omega)$. On the other hand we already have remarked (again in the proof of Lemma 1.3) that the Hilbert series equals $H(T \otimes \mathbb{Q}[\mathbb{Z}])$, hence also $\mathrm{Tor}_1^{\mathcal{O}}(\Omega, \Omega^1_X)$, which completes half of the proof, the other implication being immediate, with a diagram [illegible]. ∎

4.8. EXAMPLE [KL]. Denote by Σ_g the n-th pure braid group and consider $A_n = K^*(\Sigma_g, \mathbb{Q})$. Then $\mathrm{Tor}_1^{\mathcal{O}}(\mathbb{Q}, \mathbb{Q}) = 0$ for any n. This may be seen as follows: it is known that A_n is generated by A^1_n, for any n, the Hilbert series $A(t)$ equals $(1+t)(1+2t)\cdots(1+(n-1)t)$ (hence A_n is quite elliptic for $n \geq 2$). The main result of [K1] also gives the equality $P_{A_n}^{\mathcal{O}}(\mathbb{Q}, \mathbb{Q}) = A(t)$, for any n.

5. Lie groups

We may now start the proof of Theorem C. Recall from the Introduction that we have associated to each G in a certain [illegible] a two-dimensional $\mathbb{Z}$-skeletal, a fundamental graph (with vertex set and edge set [illegible]) $A(\Gamma) = \Lambda^*(\ldots)$ and a finite connected graph Γ_G. If the origin of a [illegible], we may do the same for any prime p: we just take A^1_p to be the $\mathbb{Z}_p$-vector space with basis $\{e_1, \ldots, e_n\}$, $A^2_p = A^1_p \wedge A^1_p$ (modulo the same relations) and define $\mu: A^1_p \times A^1_p \to A^2_p$ by the same formula as before, taking of course the mod p reduction of the usual coefficients. We will then have an associated graph $\Gamma_{G,p}$ with the same vertices as Γ_G, which is a subgraph of Γ_G, and which in fact coincides with Γ_G for all but a finite number of primes p. Accordingly, the critical cases of $\Gamma_{G,p}$ are the same as those of Γ_G. First we add to the conditions listed in the statement of C(i) one more convenient reformulation:

5.1. LEMMA. *Γ_p is connected if and only if μ_p is onto (p is a given prime, or zero).*

PROOF: Write $\Gamma_p = \Gamma_p^1 \cup \cdots \cup \Gamma_p^r$ (connected components). This gives a partition of the index set of ℓ, $I = I^1 \cup \cdots \cup I^r$, and, for each $k = 1, \ldots, r$, a submatrix of ℓ, namely $\ell^k = (l_{ij})_{(i,j) \in I^k \times I^k}$. Obviously the corresponding algebra $A_k^* := A_{\ell^k}^*$ (with associated two-form μ_p^k) is a subalgebra of A^*. It follows from these constructions that Im $\mu_p = \sum_{k=1}^r \text{Im } \mu_p^k$, hence $\dim \text{Im } \mu_p \le \sum_{k=1}^r \dim \text{Im } \mu_p^k \le \sum_{k=1}^r (\text{card}(I^k) - 1) = m - r$ (the second inequality is due to the easily seen fact that we always have $\dim A_p^2 = \text{card}(I) - 1$). Assuming that μ_p is onto, we infer that $r = 1$ and Γ_p must be connected. Conversely, the conectedness of Γ_p gives, for any $(i,j) \in I \times I$, the existence of a sequence of elements of I, $i_0, \ldots, i_s$ (with $i_0 = i$ and $i_s = j$), with the property that $\ell_{i_t i_{t+1}} \neq 0$, $t = 0, \ldots, s-1$, and consequently $e_{i_t} \wedge e_{i_{t+1}} \in \text{Im } \mu_p$. The defining relations for A_p^2 give the equality $e_i \wedge e_j = e_{i_0} \wedge e_{i_1} + \cdots + e_{i_{s-1}} \wedge e_{i_s}$, hence $e_i \wedge e_j \in \text{Im } \mu_p$ for any (i,j), therefore μ_p is onto and the proof is complete. ∎

Our plan of proving C(i) goes as follows: the first (and most serious) step will be the proof of the fact that the conectedness of Γ_0 implies that $\mathcal{P}\text{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$; here we shall rely upon the strongly freeness criterion for the vanishing of $\mathcal{P}\text{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})$ provided by B'(i), together with the result of Anick, relating combinatorial and strong freeness, that we have quoted in the preceding section, and finally upon a basic combinatorial argument (see Proposition 5.2 below). The vanishing of $\mathcal{P}\text{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})$ will then imply that A^* is 1-graded intrinsically formal, by A'(i). We shall next prove a result concerning the implications of the property of being 1-graded intrinsically formal, for a general class of 2-skeletal algebras A^* (in Proposition 5.3), and deduce as a corollary that in the case of a link algebra this implies that μ_0 is onto, hence Γ_0 is connected, by the above lemma. After thus completing the chain of equivalences stated in C(i), the remaining assertions may be proved as follows: the equality of Lie algebras with grading, $\text{gr}^* \pi_1 S \otimes \mathbb{Q} \cong L_A^*$, where $H^*(S;\mathbb{Q}) \cong A^*$, is provided again by A'(i); the equality $L_A^* = \mathbb{L}_{\mathbb{Q}}(x_1, \ldots, x_m)$ modulo the stated relations is just the explicit formulation of the construction defining L_A^*, $L_A^* = \mathbb{L}^*(X)/\text{ideal}(\partial Y)$ where $(\partial \colon Y \to X \wedge X) = \#(\mu_0 \colon A^1 \wedge A^1 \to A^2)$; here we only have to take $\{x_1, \ldots, x_m\}$ as the basis of X dual to the given basis $\{e_1, \ldots, e_m\}$ of A^1 (we may also notice that the m relations $\sum_{j \in I} l_{ij}[x_i, x_j] = 0$, for $i \in I$, are not independent, any of them being a consequence of the $m-1$ remaining ones). The Hilbert series $L_A^*(z)$ may be computed with the aid of B'(ii); since in our case $A^*(-z) = (1-(m-1)z)(1-z)$ (μ_0 is onto !), the last stated equality, namely $\text{gr}^* \pi_1 S \otimes \mathbb{Q}(z) \cong \text{gr}^*(\mathbb{F}_{m-1} \times \mathbb{F}_1) \otimes \mathbb{Q}(z)$ follows from the unique determination of the Hilbert series $L_A^*(z)$ by the exponential formula given in B'(ii) and the fact that $\text{gr}^*(\mathbb{F}_{m-1} \times \mathbb{F}_1) \otimes \mathbb{Q}(z)$

hard for the statement of C(i) not to be conveniently reformulated:

8.1. Lemma. $E_{\mathcal{J}}$ is connected if and only if $\mu_{\mathcal{J}}$ is onto (is a cross product of zero).

Proof: Write $\Gamma_{\mathcal{J}} = \Gamma_1 \cup \cdots \cup \Gamma_r$ (connected components). This gives a partition of the index set of Γ, $I = I_1 \cup \cdots \cup I_r$ and, for each $k = 1, 2, \ldots, r$, a subalgebra of $\mathcal{L}$, namely $\mathcal{L}^k$. Obviously the corresponding algebra $\mathcal{A}_{\mathcal{J}} = \mathcal{A}_{\mathcal{L}}$ (with associated two-form $\mu_{\mathcal{J}}$) is a subalgebra of $\mathcal{A}^1$. It follows from these constructions that $\operatorname{Im} \mu_{\mathcal{J}} = \sum_{k=1}^{r} \operatorname{Im} \mu_{\mathcal{J}}^k$, hence that $\dim \mu_{\mathcal{J}} \leq \sum_{k=1}^{r} (\operatorname{rank}(\Gamma_k) - 1) = m - r$ (the second inequality is due to the easily seen fact that we always have $\operatorname{rank} \mu_{\mathcal{J}}^k = \operatorname{rank}(\Gamma_k) - 1$). Assuming that $\mu_{\mathcal{J}}$ is onto, we infer that $r = 1$ and $\Gamma_{\mathcal{J}}$ must be connected. Conversely, the connectedness of $\Gamma_{\mathcal{J}}$ gives for any $(i,j) \in I \times I$, the existence of a sequence of elements of I, $i_0, \ldots, i_p$ (with $i_0 = i$ and $i_p = j$), with the property that $\ldots$ and consequently $e_i \wedge e_{i+1} \in \operatorname{Im} \mu_{\mathcal{J}}$. The defining relations for $\mathcal{A}^1$ give the equality $e_i \wedge e_j = e_{i_0} \wedge e_{i_1} + \cdots + e_{i_{p-1}} \wedge e_{i_p}$, hence $e_i \wedge e_j \in \operatorname{Im} \mu_{\mathcal{J}}$ for any (i,j), therefore $\mu_{\mathcal{J}}$ is onto and the proof is complete. ∎

Our plan of proving C(i) runs as follows. The first (and most serious) step will be the proof of the fact that the cotangent sheaf of $\Gamma_{\mathcal{J}}$ implies that $H^2(\ldots;\mathbb{Q}) = 0$; here we shall rely upon the cohomological criterion for the vanishing of $H^2(G;\mathbb{Q})$ provided by B(ii), together with the results of [illegible] concerning combinatorial arguments (see Proposition 8.2 below). The vanishing of $H^2(\ldots;\mathbb{Q})$ will then imply that $\mathcal{A}^1$ is 1-graded intrinsically formal, by [illegible]. We shall next prove a result concerning the implications of the property of being 1-formal, intrinsically formal, for a general class of 1-graded algebras $\mathcal{A}^1$ (in Proposition 8.3), and deduce as a corollary that in the case of a link diagram this implies that $\mu_{\mathcal{J}}$ is onto, hence $\Gamma_{\mathcal{J}}$ is connected, by the above lemma. After thus completing the chain of equivalences stated in C(i), the remaining assertions may be proved as follows: the equality of Lie algebras, which gives $\mathfrak{g} \otimes \mathbb{Q} \cong L_{\mathcal{J}}$, where $H^*(G;\mathbb{Q}) \cong \mathcal{A}^*$ is proved again by B(ii); the equality $L_{\mathcal{J}} = \mathbb{L}_{\mathbb{Q}}(x_1, \ldots, x_m)$ modulo the stated relations is just the explicit formulation of the construction recalling $\mathcal{L}_1^*$, $\mathcal{L}_2^* = \mathbb{L}(\mathcal{A}^1)$ (coming from the isomorphism $\mathcal{A}^1 \wedge \mathcal{A}^1 \to \mathcal{A}^2$, where we only have to take $\{x_1, \ldots, x_m\}$ as the basis of V dual to the given basis $\{e_1, \ldots, e_m\}$ of $\mathcal{A}^1$). We may also notice that the m relations $\sum_{j \neq i} [x_i, x_j] = 0$ for $i \leq m$ are not independent, any $m - 1$ of them being a consequence of the $m - 1$ remaining ones. The Hilbert series $\ldots$ may be computed with the aid of B(iii) once we know that $H^2(\ldots) = (m-1)$ (as in Corollary ...), the last stated equality, namely $\mathfrak{g}_r \otimes \mathbb{Q} \cong (\ldots)$, follows from the unique determination of the Hilbert series $E_{\mathcal{J}}(t)$ by the cohomology, as explained in B(ii), and the fact that $\mathfrak{g}_r / \Gamma_{r+1}\mathfrak{g} \otimes \mathbb{Q}$ [illegible] satisfies the same equality, which is an immediate consequence of a classical equality (see [Se]) $\prod_{n=1}^{\infty}(1-z^n)^{\mathrm{rk}\,\mathrm{gr}^n(\mathbb{F}_k)} = 1-kz$, valid for any free group $\mathbb{F}_k$.

Let us establish now our combinatorial setting: start with a finite unoriented graph Γ, with vertex set $\mathcal{V}$, $\mathrm{card}(\mathcal{V}) = m$, and arrow set $\mathcal{A}$, and fix a total ordering on $\mathcal{V}$. In all that follows one should bear in mind that we are just reformulating, in a more combinatorial form, Anick's test for combinatorial freeness explained in §4, where X has $\{x_v; v \in \mathcal{V}\}$ as basis, and the sequence we are considering consists of the $m-1$ elements $(y_v)_{v<\omega}$, where ω is the highest element of $\mathcal{V}$ in the given order, and $y_v = \sum_{u\in\mathcal{V}} l_{uv}[x_v, x_u]$, any v.

Given $u, v \in \mathcal{V}$, we shall say that u and v are neighbours if $\{u, v\} \in \mathcal{A}$, and we shall write this as $u \leftrightsquigarrow v$. Given the ordering on $\mathcal{V}$, we may also speak of the oriented arrows, $\overline{\mathcal{A}}$: any $a \in \mathcal{A}$ gives rise to an oriented arrow $\overline{a} \in \overline{\mathcal{A}}$; if $a = \{u, v\}$ then we write $\overline{a} = \overline{uv}$, where $u < v$. On $\mathcal{V}\times\mathcal{V}$ (and therefore on $\overline{\mathcal{A}}$) we shall consider the right lexicographic order ($(u, v) \leq (u', v')$ if either $v < v'$ or $v = v'$ and $u \leq u'$). Each oriented arrow $\overline{a} = \overline{uv}$ has an origin, $\mathrm{o}(\overline{a}) = u$, and an end, $\mathrm{e}(\overline{a}) = v$. To any vertex $u \in \mathcal{V}$ we shall associate an oriented arrow $\overline{a}_u$, defined as the highest oriented arrow having u either as an origin or as an end. Recalling how the graph was constructed from the matrix ℓ of linking coefficients, and also that $[x, x'] = x\otimes x' - x'\otimes x$ in T^2X, for any $x, x' \in X$, it follows that in our case of interest the highest terms are given by $\overline{y}_v = \overline{a}_v$, for each $v \in \mathcal{V}$ where we have identified an oriented arrow $\overline{a} = \overline{st}$ and the tensor $x_s \otimes x_t$.

Given a fixed vertex $\omega \in \mathcal{V}$ we shall look for total orderings $(\mathcal{V}, \leq)$ with the following properties, which we shall call (P_ω)-orderings:

- $(P_\omega 0)$ ω is the maximal element of $\mathcal{V}$ with respect to $\leq$;
- $(P_\omega 1)$ the $m-1$ oriented arrows $\{\overline{a}_u; u < \omega\}$ are distinct;
- $(P_\omega 2)$ the sets $\{\mathrm{o}(\overline{a}_u); u < \omega\}$ and $\{\mathrm{e}(\overline{a}_u); u < \omega\}$ are disjoint;
- $(P_\omega 3)$ for any $u \in \mathcal{V}$ such that $u \leq \mathrm{o}(\overline{a}_\omega)$, there exists $v \in \mathcal{V}$ such that $v \leftrightsquigarrow u$ and $v > \mathrm{o}(\overline{a}_\omega)$.

Let us notice that $(P_\omega 1)$, respectively $(P_\omega 2)$, are equivalent to the properties $(\star)$, respectively $(\star\star)$, which represent the test for combinatorial freeness we are interested in. The proposition below will plainly help us to conclude that if a link algebra A^* has a connected associated graph Γ_0 then necessarily $\mathcal{P}\mathrm{Ext}^{*,1}_{A}(\mathbb{Q}, \mathbb{Q}) = 0$. This is our key combinatorial argument. The condition $(P_\omega 0)$ above just helps one to make a canonical choice of $m-1$ linearly independent relations for L^*_A, while $(P_\omega 3)$ is a technical property which enables one to argue by induction.

5.2. PROPOSITION. *If Γ is a connected graph then for any choice of vertex $\omega \in \mathcal{V}$, there* satisfies the same equality, which is an immediate consequence of the Gaussian equality (see [illegible] [illegible]) [illegible] $= 1$ [illegible] forming the [illegible] group [illegible].

Let us establish now our combinatorial setting. Start with a finite connected graph Γ, with vertex set V, $\mathrm{card}(V) = m$, and arrow set A, and fix a total ordering on V. In all that follows one should keep in mind the [illegible] in a more combinatorial form, which is a test for combinatorial [illegible] explained in §4, where X has $\{x_v \mid v \in V\}$ as basis, and the elements w_v are the following [illegible] consists of the $m-1$ elements $\{e_v\}_{v \neq c}$, where c is the highest element of V in the given ordering, $e_v = \sum$ [illegible], any v.

Given $v, w \in V$, we shall say that v and w are neighbours if $[v, w] \in A$, and we shall write this as $v \leftrightarrow w$. Given the ordering on V, we may also speak of the oriented arrows. Any $a \in A$ gives rise to an oriented arrow $\bar{a}$: if $a = [u, v]$ then we write $\bar{a} = \overline{uv}$, where $u < v$. On $V \times V$ (and therefore on A) we shall consider the right lexicographic order: $(u, v) \leq (u', v')$ if either $v < v'$, or $v = v'$ and $u \leq u'$. Each oriented arrow $\bar{a} = \overline{uv}$ has an origin, $o(\bar{a}) = u$, and an end, $e(\bar{a}) = v$. To any vertex $v \in V$ we shall associate an oriented arrow $\bar{a}_v$, defined as the highest oriented arrow having v either as an origin or as an end. Recalling how the graph was connected through the matrix [illegible] coefficients, and [illegible] $[v, v'] =$ [illegible] in Γ, for any v [illegible], it follows that in our case of interest the highest terms are given by [illegible] for each v [illegible] we have identified an oriented arrow $\bar{a}$ and the [illegible] $\bar{a}$.

Given a fixed vertex $c \in V$, we shall look for total orderings $(V, \leq)$ with the following properties, which we shall call (P) properties:

(P.1) c is the maximal element of V with respect to $\leq$;
(P.2) the $m-1$ oriented arrows $\{\bar{a}_v\}_{v \neq c}$ are distinct;
(P.3) there is [illegible];
(P.4) for any $v \in V$ such that $v \neq c$ there exists $w \in V$ such that $v \leftrightarrow w$ and [illegible].

Let us notice that (P.1), respectively (P.2), are equivalent to the properties (*), respectively (**), which appear in the text [illegible] we are interested in. The property [illegible] below will mainly help us to conclude that the [illegible] A' has a connected associated graph Γ, [illegible] $= 0$ [illegible] argument. The condition (P.3) [illegible] a canonical choice of $m-1$ linearly independent [illegible], while (P.4) is a technical property, which enables one to argue by induction.

6.2. Proposition. If Γ is a connected graph then for any choice of vertex $c \in V$, there *exists a (P_ω)-ordering on $\mathcal{V}$.*

PROOF: We shall use induction on $m = \mathrm{card}(\mathcal{V})$. If $m = 2$, there is only one possibility, forced by $(P_\omega 0)$, and this choice evidently satisfies all our requirements. Let us describe the induction step. Since Γ is connected, the given vertex ω has neighbours. Let us pick such a neighbour of ω, say v, and also fix it. Define now a graph Γ' by just deleting the vertex v from Γ, and write the connected components of Γ', $\Gamma' = \Gamma'_1 \cup \cdots \cup \Gamma'_r$ (and similarly for the vertices: $\mathcal{V}' = \mathcal{V}'_1 \cup \cdots \cup \mathcal{V}'_r$). Remember that $\omega \in \mathcal{V}'$ and order the components such that $\omega \in \mathcal{V}'_r$. Due to the connectedness of Γ again, we may pick, in each component Γ'_i, a neighbour of v, say $\omega'_i \in \mathcal{V}'_i$ (we shall take $\omega'_r = \omega$). By the induction hypothesis, there exists a $(P_{\omega'_i})$-ordering on each Γ'_i, $i = 1, \ldots, r$. We shall now extend these orderings to a total order on Γ, as follows: denote by $\mathrm{o}'_i \in \mathcal{V}'_i$, for each i, the vertex $\mathrm{o}(\overline{a}_{\omega'_i})$, and split $\mathcal{V}'_i$ as $\mathcal{V}'_i = \mathcal{OV}'_i \cup \mathcal{EV}'_i$, where $\mathcal{OV}'_i = \{u \in \mathcal{V}'_i;\ u \leq \mathrm{o}'_i\}$ and $\mathcal{EV}'_i = \{u \in \mathcal{V}'_i;\ u > \mathrm{o}'_i\}$ (if Γ'_i happens to have only one vertex, ω'_i, set $\mathcal{OV}'_i = \emptyset$ and $\mathcal{EV}'_i = \mathcal{V}'_i$). The order on $\mathcal{V}$ is obtained by arranging first the vertices of $\mathcal{OV}'_1$ (if any), in the chosen $(P_{\omega'_1})$-order, then those of $\mathcal{OV}'_2, \ldots$, up to those of $\mathcal{OV}'_r$; put next the vertex v, and continue with the vertices of $\mathcal{EV}'_1$, arranged in the chosen $(P_{\omega'_1})$-order, then those belonging to $\mathcal{EV}'_2, \ldots$, ending with $\mathcal{EV}'_r$. Property $(P_\omega 0)$ is obviously satisfied.

In order to check $(P_\omega 3)$, notice first that, in general, $\mathrm{o}(\overline{a}_\omega)$ (where ω is the highest vertex of a graph) equals the highest neighbour of ω. In our case, one such a neighbour is v, by construction; any other neighbour must belong to $\mathcal{V}'_r$ (which is the connected component of $\Gamma' = \Gamma \setminus \{v\}$ which contains ω!), more precisely to $\mathcal{OV}'_r$, being therefore smaller than v. Hence $\mathrm{o}(\overline{a}_\omega) = v$. The property $(P_\omega 3)$ is immediately checked for $u = v$, by considering the vertex ω itself. If $u \in \mathcal{V}$ and $u < v$, then u must lie in some $\mathcal{OV}'_i$, hence, by the inductively established property $(P_{\omega'_i}3)$ there exists $v_i \in \mathcal{V}'_i$ such that $v_i \leftrightsquigarrow u$ and $v_i > \mathrm{o}'_i$; this means that $v_i \in \mathcal{EV}'_i$, hence, by construction, $v_i > v = \mathrm{o}(\overline{a}_\omega)$, as desired.

Passing to the key properties, namely $(P_\omega 1)$ and $(P_\omega 2)$, let us begin by making some preliminary observations on the construction of the highest arrows $\overline{a}_u$, where u is a vertex of an arbitrary (connected) graph Γ, with highest vertex ω. It turns out that we must consider the following partition of its vertex set, $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$, where:

$$\mathcal{M} = \{u \in \mathcal{V};\ v \leftrightsquigarrow u \Longrightarrow v < u\}, \text{ and}$$
$$\mathcal{N} = \{u \in \mathcal{V};\ \text{there exists } v \in \mathcal{V} \text{ such that } v \leftrightsquigarrow u \text{ and } v > u\}.$$

If $u \in \mathcal{M}$, then $\overline{a}_u = \overline{vu}$, for some $v \in \mathcal{V}$ (by construction), and $u = \mathrm{e}(\overline{a}_u)$. If $u \in \mathcal{N}$, then $\overline{a}_u = \overline{uv}$, for some $v \in \mathcal{V}$, again by construction, hence $u = \mathrm{o}(\overline{a}_u)$. For example, denoting by o the origin of $\overline{a}_\omega$, we have: $\mathrm{o} \in \mathcal{N}$, $\omega \in \mathcal{M}$ and $\overline{a}_\mathrm{o} = \overline{a}_\omega = \overline{\mathrm{o}\omega}$. Let us also notice the exists a (P_r) ordering on V.

PROOF. We shall use induction on $m = \max|V_i|$. If $m = 2$, there is only one possibility, forced by $(P_r 1)$, and this clearly satisfies all our requirements. Let us describe the induction step. Since V is connected, there is a vertex [illegible] Let us pick such a [illegible] and also [illegible]. Define now a graph F by first deleting the vertex v from [illegible] connected component [illegible] and similarly for the vertices [illegible]. Remember that [illegible] consider the component such that [illegible]. Due to the connectedness of F again, we may pick in each component F_i a neighbour of [illegible]. We shall denote [illegible] by the induction hypothesis, there exists a (P_r) ordering on [illegible]. We shall now extend these orderings to a total order on V as follows: denote by [illegible] the [illegible] vertex [illegible] and split V; set [illegible] where [illegible] happens to have only one vertex, [illegible] set [illegible]. The order on P is obtained by arranging first the vertices of [illegible] in the chosen (P_r) order, then those of [illegible] up to those of [illegible], put next the vertex v, and continue with the vertices of [illegible] arranged in the chosen (P_r) order, then those belonging to [illegible]. Property $(P_r 3)$ is obviously satisfied.

In order to check $(P_r 4)$ we notice [illegible] (where u is the highest vertex of [illegible]) equals the highest neighbour of u [illegible], one such a neighbour is v, by construction, any other neighbour must belong to P_i (which is the connected component of $F - \{v\}$), which contains all [illegible] vertices smaller than u. Hence [illegible]. The property $(P_r 3)$ [illegible] immediately the case $u = v$, by considering the vertex u itself. If $u \neq v$ and u is a [illegible] in some CP_i, hence, by the inductively established property $(P_r 3)$ there exists w such that [illegible] and $v_i > u$; this means that [illegible] hence, by construction [illegible] as desired.

Passing to the last properties, namely $(P_r 1)$ and $(P_r 2)$, let us begin by making some preliminary observations on the connected [illegible] of the highest vertex $\hat{u}_i$, where u is a vertex of an arbitrary [illegible] subgraph F [illegible] vertex u. It turns out that we must consider the following partition of its vertex set, $V = M \cup N$, where

$$M = \{u \in V : [illegible]\} \text{ and}$$

$$N = \{u \in V : \text{there exists } x \text{ such that } [illegible] \text{ and } x > u\}.$$

If $u \in M$ then [illegible] for some $v \in V$ (by construction), and u [illegible]. If $u \in N$, then [illegible] but [illegible] by construction, [illegible]. For example, denoting by o the origin of [illegible], we have [illegible] $\in M$ and [illegible]. Let us also notice the equality $\{o(\overline{a}_u);\ u < \omega\} = \mathcal{N}$. Indeed, we clearly have $o(\overline{a}) \in \mathcal{N}$, for any $\overline{a} \in \overline{\mathcal{A}}$; conversely, if $u \in \mathcal{N}$ then $u = o(\overline{a}_u)$, as already noticed, while $o(\overline{a}_\omega) = o(\overline{a}_o)$. We also have an inclusion $\{e(\overline{a}_u);\ u < \omega\} \supset \mathcal{M}$, due to the fact that $e(\overline{a}_u) = u$ for any $u \in \mathcal{M}$, as remarked before, plus the observation that $e(\overline{a}_\omega) = e(\overline{a}_o)$. We may thus equivalently reformulate $(P_\omega 2)$ as

$(P_\omega 4)$ $\{e(\overline{a}_u);\ u \in \mathcal{V},\ u \neq o\} \subset \mathcal{M}$, or
$(P_\omega 5)$ $\{e(\overline{a}_u);\ u \in \mathcal{V}\} \subset \mathcal{M}$.

In our case $(P_\omega 2)$ will be checked by proving that $e(\overline{a}_{u_i}) \in \mathcal{M}$ for any $u_i \in \mathcal{V}'_i$, $i = 1, \ldots, r$. We must first understand the difference between $\overline{a}_{u_i}$ and $\overline{a}^i_{u_i}$, where the second oriented arrow is computed in the subgraph Γ'_i. If $u_i \in \mathcal{OV}'_i$ then, by the inductively established property $(P_{\omega'_i}3)$, there exists $v_i \in \mathcal{V}'_i$ such that $v_i \leftrightsquigarrow u_i$, and moreover $v_i \in \mathcal{EV}'_i$, hence $v_i > v$, by construction; the only possible difference between $\overline{a}_{u_i}$ and $\overline{a}^i_{u_i}$ might be caused, in general, by the fact that $u_i \leftrightsquigarrow v$; in our case that would imply the existence of an additional arrow (in Γ) $\overline{u_i v}$ which might contribute to the computation of $\overline{a}_{u_i}$ – but we know that $\overline{u_i v_i} > \overline{u_i v}$, hence $\overline{a}^i_{u_i} = \overline{a}_{u_i}$. If $u_i \in \mathcal{EV}'_i \cap \mathcal{N}'_i$, a similar argument shows that $\overline{a}^i_{u_i} = \overline{a}_{u_i}$ in this case too. If $u_i \in \mathcal{EV}'_i \cap \mathcal{M}'_i$ but $u_i \not\leftrightsquigarrow v$, then plainly again $\overline{a}^i_{u_i} = \overline{a}_{u_i}$. Finally if $u_i \in \mathcal{EV}'_i \cap \mathcal{M}'_i = \mathcal{M}'_i$ (use $(P_{\omega'_i}3)$!), $u_i \leftrightsquigarrow v$ but there exists $v_i \in \mathcal{EV}'_i$ such that $v_i \leftrightsquigarrow u_i$, the same argument shows again that $\overline{a}^i_{u_i} = \overline{a}_{u_i}$. The only exceptions appear when $u_i \in \mathcal{M}'_i$, $u_i \leftrightsquigarrow v$ and all the other neighbours of u_i lie in $\mathcal{OV}'_i$ (in short when $u_i \in \mathcal{B}'_i$); in this case the arrows containing u_i in Γ are $\{\overline{v_i u_i}\}$, where $\{v_i\}$ are the neighbours of u_i in Γ'_i, plus $\overline{v u_i}$; clearly $\overline{a}_{u_i} = \overline{v u_i}$, and it is different from $\overline{a}^i_{u_i}$. On the other hand, since $\mathcal{M}'_i \subset \mathcal{EV}'_i$ and $v < v_i$ for any $v_i \in \mathcal{EV}'_i$, we know that $\mathcal{M}'_i \subset \mathcal{M}$, for any i. Finally, if $u_i \notin \mathcal{B}'_i$ then $e(\overline{a}_{u_i}) = e(\overline{a}^i_{u_i}) \in \mathcal{M}'_i \subset \mathcal{M}$, by $(P_{\omega'_i}5)$ and the induction; if $u_i \in \mathcal{B}'_i$ then $e(\overline{a}_{u_i}) = e(\overline{v u_i}) = u_i \in \mathcal{M}'_i \subset \mathcal{M}$ (where $u_i \in \mathcal{M}'_i$ follows directly from $u_i \in \mathcal{B}'_i$).

As far as $(P_\omega 1)$ is concerned, it may be equivalently reformulated as

$(P_\omega 6)$ $\mathrm{card}\{\overline{a}_u;\ u \neq o\} = \mathrm{card}(\mathcal{V}) - 1$

First of all, the above discussion implies that $e(\overline{a}_{u_i}) = e(\overline{a}^i_{u_i}) \in \mathcal{V}'_i$ for any $u_i \in \mathcal{V}'_i$, $i = 1, \ldots, r$. With this remark $(P_\omega 6)$ will follow from the equalities $\mathrm{card}\{\overline{a}_{u_i};\ u_i \in \mathcal{V}'_i\} = \mathrm{card}(\mathcal{V}'_i)$, to be proven for any i. The above arrow set splits as $\{\overline{a}_{u_i};\ u_i \in \mathcal{B}'_i\} \cup \{\overline{a}_{u_i};\ u_i \in \mathcal{V}'_i \setminus \mathcal{B}'_i\}$. The first set equals $\{\overline{v u_i};\ u_i \in \mathcal{B}'_i\}$ and the second equals $\{\overline{a}^i_{u_i};\ u_i \in \mathcal{V}'_i \setminus \mathcal{B}'_i\}$. The two above sets are clearly disjoint, the first contains $\mathrm{card}(\mathcal{B}'_i)$ elements and the second contains $\mathrm{card}(\mathcal{V}'_i)$-$\mathrm{card}(\mathcal{B}'_i)$ elements (due to the inductive hypothesis $(P_{\omega'_i}1)$ and the remark that $\omega'_i \in \mathcal{B}'_i$). Our inductive proof is thus completed. ∎

5.3. PROPOSITION. *Let A^* be a 2-skeletal algebra, with associated 2-form $\mu\colon A^1 \wedge A^1 \to A^2$ and Lie algebra with grading L_A^*. If A^* is 1-graded intrinsically formal and μ is not onto, then $L_A^{\geq 3} = 0$.*

COROLLARY. *A 1-graded intrinsically formal link algebra A^* has a connected associated graph Γ_0.*

PROOF OF THE COROLLARY: We must show that μ is onto (Lemma 5.1). If not, the above proposition tells us that $L_A^3 = 0$, i.e. the linear map $f\colon X \otimes Y \to \mathbb{L}^3 X$ given by $f(x \otimes y) = [x, \partial y]$ is onto. Counting dimensions, this gives the inequalities $(m^3 - m)/3 \leq m \cdot \dim(\partial Y) = m \cdot \dim(\operatorname{Im} \mu) < m(m-1)$, hence $m = 1$ and $A^2 = 0$, a contradiction. ∎

PROOF OF THE PROPOSITION: Assuming that μ is not onto, one has a decomposition $Y = Y' \oplus C$, where $\partial|_{Y'}$ is monic, $\partial C = 0$ and $C \neq 0$, say C has $\{z_1, \ldots, z_c\}$ as basis. We must see that $L_A^3 = 0$. Pick then an arbitrary element $p \in \mathbb{L}^3 X$; we have to prove that $p \in I =$ the ideal generated by $\partial Y = \partial Y'$. Suppose on the contrary that $p \notin I$ and consider then the (larger) perturbed ideal $I_p =$ ideal generated by ∂Y and p and the natural grlie surjection f, $f\colon L_A^* = \mathbb{L}^* X/I \to \mathbb{L}^* X/I_p =: L_p^*$. Perform then on L_p^* the construction described in (1.7) to obtain a 1-minimal (b)dga $(\wedge V_*, d)$, with the property that $\mathrm{gr}^*(\wedge V, d) = L_p^*$ (see also Proposition 1.4). If we are able to exhibit a dga $\mathcal{M}$ with the property that $H^*\mathcal{M} \cong A^*$ and having $(\wedge V, d)$ as 1-minimal model, then the property of A^* of being 1-graded intrinsically formal may be eventually exploited, giving a grlie isomorphism between L_p^* and L_A^*; by a dimension argument f must be an isomorphism and consequently $p \in I$, a contradiction.

Construct $\mathcal{M}$ by starting with $(\wedge V, d) \otimes (\wedge(z_2, \ldots, z_c), 0)$, where $\deg z_i = 2$ and the second differential is trivial. Consider then $H^{>2}((\wedge V, d) \otimes (\wedge(z_2, \ldots, z_c), 0)$ and add new generators, $U_1^* = \oplus_{k \geq 2} U_1^k$, so as to kill $H^{\geq 3}$; look next at $H^{>2}(\wedge V \otimes \wedge(z) \otimes \wedge U_1)$ and kill it by adding new generators U_2^*, iterate and obtain $\mathcal{M}$ as the inductive limit of this process. By construction $H^*\mathcal{M}$ will be a 2-skeletal algebra. It is equally easy to see that when killing $H^{\geq 3}\mathcal{M}$ as above one does not change $H^{\leq 2}\mathcal{M}$, hence $(\wedge V, d)$ is indeed the 1-minimal model of $\mathcal{M}$ and $H^*\mathcal{M} \cong H^{\leq 2}((\wedge V, d) \otimes (\wedge(z_2, \ldots, z_c), 0) \cong \mathbb{Q} \cdot 1 \oplus H^1(\wedge V, d) \oplus H^2(\wedge V, d) \oplus \mathrm{span}_{\mathbb{Q}}\{z_2, \ldots, z_c\}$. Recalling the construction (1.7), Lemma 1.6(iv) tells that $H^1(\wedge V_*, d) \cong H_0^1(\wedge V_*, d) \cong V_0 \cong \# L_p^1 \cong \# X \cong A^1$. Likewise $H^2(\wedge V_*, d) \cong H_0^2(\wedge V_*, d) \oplus H_+^2(\wedge V_*, d)$, where $H_0^2(\wedge V_*, d) \cong V_0 \wedge V_0 / dV_1$ and the multiplication $H^1(\wedge V, d) \wedge H^1(\wedge V, d) \twoheadrightarrow H_0^2(\wedge V, d)$ is by construction the dual of the inclusion $K \rightarrowtail L_p^1 \wedge L_p^1$, where $K = \operatorname{Ker}(L_p^1 \wedge L_P^1 \xrightarrow{[\ ,\ ]} L_p^2)$. Because f is an isomorphism in degrees 1 and 2, we may safely replace $L_p^{\leq 2}$ by $L_A^{\leq 2}$ and thus identify $\#(K \rightarrowtail L_p^1 \wedge L_p^1)$ with $\#(\partial\colon Y' \rightarrowtail X \wedge X) \cong \mu\colon A^1 \wedge A^1 \twoheadrightarrow \operatorname{Im} \mu$. In order to show that $H^*\mathcal{M} \cong A^*$ and thus finish our proof, we only have to see that $\dim H_+^2(\wedge V_*, d) = 1$, or equivalently that $\dim H^2(\wedge V, d) = 1 + \dim Y'$.

5.3. PROPOSITION. Let $\mathcal{C}$ be a $\mathbb{Z}_2$-graded algebra, with associated 2-form $\mu: A^1 \wedge A^1 \to A^2$ and Lie algebra [illegible] $L_\bullet^{\mathcal{C}}$. If $\mathcal{C}$ is 1-minimally formal and μ is not onto, then $L_3^{\mathcal{C}} \neq 0$.

COROLLARY. A 1-minimally formal link algebra $\mathcal{C}$ has a [illegible] associated graded group $L_\bullet$.

PROOF OF THE COROLLARY. We must show that μ is onto (Lemma 1.1). Indeed, the above proposition tells us that $L_3^{\mathcal{C}} = 0$, i.e. the linear map $f: \Lambda^2 Y \to Y$ [illegible] given by [illegible] is onto. Counting dimensions, this gives the inequality $(n^3 - n)/3 \le \dim(\ker \mu) = n(n-1)/2$, hence $n = 1$ and $L_2 = 0$, a contradiction. ∎

PROOF OF THE PROPOSITION. Assuming that μ is not onto, one has a decomposition $A^2 = \operatorname{Im}\mu \oplus C$, where $\operatorname{Im}\mu$ is minimal, $\partial C = 0$ and $C \neq 0$, say C has $\{c_1, \dots, c_s\}$ as basis. We must see that $L_3^{\mathcal{C}} = 0$. Pick then an arbitrary element $p \in L_3^{\mathcal{C}}$; we have to prove that $p \in I$ = the ideal generated by [illegible]. Suppose on the contrary that $p \notin I$ and consider then the (larger) perturbed ideal J generated by ∂I and p and the natural projection $f: L_\bullet = L(V^*)/I \to L(V^*)/J = L_\bullet'$. From then on $L_\bullet'$ the construction described in [HS] is able to produce [illegible] $(\mathcal{M}, d)$, with the property that $\mathcal{H}^*(\mathcal{M}, d) = L_\bullet'$ (see also Proposition 1.1). If we are able to exhibit a dga $\mathcal{A}$ with the property that $H^*\mathcal{M} \cong \mathcal{A}^*$ and having $(\mathcal{M}, d)$ as 1-minimal model, then the assumption of $\mathcal{A}^*$ of being 1-minimally formal [illegible] giving a graded isomorphism between $L_\bullet$ and $L_\bullet'$, by a dimension argument f must be an isomorphism and consequently $p \in I$, a contradiction.

Construct $\mathcal{M}$ by starting with $(\Lambda(x_1, \dots, x_n), 0)$, where $\deg x_i = 1$ and the second differential is trivial. Consider then $H^2(\Lambda(x_1, \dots, x_n), 0)$ and add new generators [illegible] to kill [illegible] and kill it by adding new generators [illegible], and take $\mathcal{M}$ as the inductive limit of this process. By construction $H^*\mathcal{M}$ will be a [illegible] algebra. It is equally easy to see that, when killing $H^2\mathcal{M}$ as above, one does not change [illegible], hence [illegible] is indeed the 1-minimal model of $\mathcal{A}$ and [illegible]. Recalling the construction of [illegible] ([HS]) tells that [illegible], where [illegible] is by construction the kernel of the morphism [illegible]. Because f is an isomorphism [illegible] we may safely replace [illegible] and thus identify [illegible]. In order to show that [illegible] and thus finish the proof, we only have to see that $\dim H_3(\mathcal{M}, d) = 1$, or equivalently that $\dim L_3 = 1 + \dim$ [illegible].

This will be accomplished by remarking that in general one has $H^k(\wedge V, d) = H^k(L_p; \mathbb{Q}) = \# H_k(L_p; \mathbb{Q})$, for any k, and the same reasoning as in 1.8 indicates that $\dim H_2(L_p; \mathbb{Q}) = \dim I_p/[\mathbb{L}X, I_p]$, and our assumption $p \notin I$ helps to conclude that this last dimension equals $\dim Y' + 1$, as claimed.

5.4. REMARKS. Theorem C(i) considerably enlarges the class of 1-graded intrinsically formal link algebras, whose first examples appeared in [KS], namely the series $H^*(S^1 \times (\vee_{m-1} S^1); \mathbb{Q})$ corresponding to m-component links consisting of $m-1$ unlinked circles stringed with simple linking on another circle. Let us look at a simple example, by taking $m = 4$ and $l_{ij} = 0$ except $l_{12} = l_{21} = l_{23} = l_{32} = l_{34} = l_{43} = 1$ (a chain of four simply linked circles). This may be handled by our methods (one even knows that Γ_p is connected, for any p, see C(ii)), while the corresponding link algebra A^* is not isomorphic to $H^*(S^1 \times (\vee_3 S^1); \mathbb{Q})$. It is not difficult to see this, by observing that $\mathrm{cg}(H^*(S^1 \times (\vee_3 S^1); \mathbb{Q})) = 3$, while $\mathrm{cg}(A^*) = 2$.

The newt two lemmas will finally give the proof of C(ii).

5.5. LEMMA. *If the graphs Γ_p are connected, for any prime p, then $gr^*\pi \cong \mathbb{L}^*_{\mathbb{Z}}(x_1, \ldots, x_m)$ modulo the ideal generated by $r_i = \sum_j l_{ij}[x_i, x_j]$, $i = 1, \ldots, m$, and it is torsion free as a graded abelian group.*

5.6. LEMMA. *The graphs Γ_p are connected, for any prime p, if and only if $gr^2\pi$ is a free abelian group of rank $(m-1)(m-2)/2$.*

PROOF OF LEMMA 5.5: Denote by $L^*_{\mathbb{Z}}$ the Lie algebra with grading constructed out of the matrix ℓ of linking coefficients, as in our statement. It is well-known that $L^*_{\mathbb{Z}}$ is torsion free if and only if $L_{\mathbb{Z}}(p, z) = L_{\mathbb{Z}}(0, z)$, for any prime p, where $L_{\mathbb{Z}}(p, z)$ stands for the Hilbert series of the graded $\mathbb{Z}_p$-vector space $L^*_{\mathbb{Z}} \otimes \mathbb{Z}_p$ and likewise $L_{\mathbb{Z}}(0, z)$ denotes the Hilbert series of the graded $\mathbb{Q}$-vector space $L^*_{\mathbb{Z}} \otimes \mathbb{Q} \cong L^*_A$ (and of course $L^*_{\mathbb{Z}} \otimes \mathbb{Z}_p = L^*_{\mathbb{Z}_p}(x_1, \ldots, x_m)$ modulo the relations r_i taken mod p). For $p =$ prime and for $p = 0$, we may double the upper degrees and convert them into lower degrees, as we have already done before (see 4.2), replacing the above power series equality by $\overline{L}_{\mathbb{Z}}(p, z) = \overline{L}_{\mathbb{Z}}(0, z)$. Thanks to the (version for connected glie) PBW theorem (see [Se; III.4, Theorem 3]) we may check instead the equality $\mathcal{U}\overline{L}_{\mathbb{Z}}(p, z) = \mathcal{U}\overline{L}_{\mathbb{Z}}(0, z)$. Since $\mathcal{U}(\overline{L}_{\mathbb{Z}} \otimes \mathbb{Q}) = \mathbb{T}_{\mathbb{Q}}(x_1, \ldots, x_m)$ modulo the relations r_i (where $\deg x_i = 2!$) – and similarly for $\mathcal{U}(\overline{L}_{\mathbb{Z}} \otimes \mathbb{Z}_p)$ – we may again appeal to Anick's result [A; Theorem 2.6], which is valid for <u>any</u> field coefficients and says that the Hilbert series of a connected graded algebra, together with the number and the degrees of a strongly free set of its elements, uniquely determine the Hilbert series of the quotient algebra. It only remains to notice that our proof of the fact that the connectedness of Γ_0 implies the combinatorial freeness of $\{r_1, \ldots, r_{m-1}\}$ given in Proposition 5.2 applies verbatim to Γ_p (which is known to be connected, by assumption) and

This will be accomplished by remarking that in general one has $H^*(\Lambda X, A) = E_2^{*}$ [illegible] for arrangements with the same structure as in 1.5 implies that $\dim H^*(\ldots, \mathbb{Q})$ [illegible] and [illegible], from our assumption [illegible], to conclude that the top dimension equals $\dim V^*$ [illegible], as claimed.

5.4. Example. There is a [illegible] of the [illegible] infinitely formal Lie algebras whose first example appeared in [illegible]; the series [illegible] corresponding to the complement of a consisting of n [illegible] of circles wrapped with another, [illegible] for example, by taking [illegible] except [illegible], the chain of four simply linked circles). This may be handled by our methods (formal) for [illegible], while the corresponding Lie algebras [illegible]. It is not difficult to see this, by observing that [illegible], while [illegible] $= 2$.

The next two lemmas will finally give the proof of [illegible].

5.5. Lemma. If the graded Lie [illegible] are connected, for any prime p, then [illegible] $\mathbb{L}(\ldots)$ modulo the [illegible] generated by [illegible], and it is torsion free as a graded abelian group.

5.6. Lemma. The groups [illegible] are connected, for any prime p, if and only if [illegible] is a free abelian group of rank [illegible].

Proof of Lemma 5.5. Regarding [illegible] the Lie algebra with graded construction of the [illegible] of finite [illegible], as noted [illegible]. It is well-known that [illegible] is torsion free if and only if [illegible], where [illegible] stands for the Hilbert series of the graded [illegible] vector space [illegible], and therefore [illegible] denotes the Hilbert series of the graded [illegible]-vector space [illegible]. [illegible] modulo the relations is taken modulo [illegible]. For [illegible], we may denote the upper degree and convert them into lower degrees, as we have already done before (see [illegible]), replacing the above power series equality by [illegible]. Thanks to [illegible] becomes [illegible] connected [illegible] the identities (see [illegible]). [illegible] we may check instead the equality [illegible]. Since [illegible] inside [illegible] (where [illegible]) and similarly for [illegible] — equality [illegible] to [illegible], which is valid for any field coefficients (and [illegible] for Hilbert series of connected graded algebras, together with the number and the degrees of a minimal [illegible] set of its elements, uniquely determine the Hilbert series of the [illegible] algebra). It only remains to [illegible] proof of the fact that the connectedness of [illegible] the [illegible] of [illegible] given in Proposition [illegible] [illegible], which is known [illegible] connected [illegible] and to the sequence $\{r_1 \bmod p, \ldots, r_{m-1} \bmod p\}$, for any prime p. It follows that $\{r_1, \ldots, r_{m-1}\}$ is strongly free, viewed either with $\mathbb{Q}$ or with $\mathbb{Z}_p$ coefficients, for any p, hence the $\mathcal{U}$-Hilbert series are equal and $L^*_{\mathbb{Z}}$ is torsion free. It is now easy to prove that $\mathrm{gr}^*\pi \cong L^*_{\mathbb{Z}}$, as grlie, by making use of an old result of Chen [C3], who established the existence of a grlie map $f\colon L^*_{\mathbb{Z}} \to \mathrm{gr}^*\pi$, which is isomorphic in degree one, hence epimorphic. Knowing that $f \otimes \mathbb{Q}$ is epimorphic and also that $\mathrm{gr}^*\pi \otimes \mathbb{Q}(z) \cong L^*_{\mathbb{Z}} \otimes \mathbb{Q}(z)$ (cf. C(i)) we infer that $f \otimes \mathbb{Q}$ is also monic, hence, due to the torsion freeness of $L^*_{\mathbb{Z}}$, f itself must be monic, giving thus the asserted isomorphism. ∎

PROOF OF LEMMA 5.6: At the beginning we shall prove that $\mathrm{gr}^2\pi$ is free if and only if $\dim \mathrm{Im}\, \mu_p = \dim \mathrm{Im}\, \mu_0$, for any prime p. The torsion freeness of the finitely generated abelian group $\mathrm{gr}^2\pi$ is equivalent, as before, to the fact that $\dim_{\mathbb{Z}_p} \mathrm{gr}^2\pi \otimes \mathbb{Z}_p = \dim_{\mathbb{Q}} \mathrm{gr}^2\pi \otimes \mathbb{Q}$, for any p. By Chen's work [C3], $\mathrm{gr}^2\pi$ is isomorphic to the quotient group of the second exterior power of the free abelian group with $\mathbb{Z}$-basis $\{x_1, \ldots, x_m\}$ by the relations $r_1, \ldots, r_m$ (and evidently $\mathrm{gr}^2\pi \otimes \mathbb{K}$ may be similarly presented, $\mathbb{K} = \mathbb{Q}$ or $\mathbb{Z}_p$). By elementary duality we infer that $\mathrm{gr}^2\pi \otimes \mathbb{Z}_p \cong \#(\mathrm{Ker}\, \mu_p)$ and $\mathrm{gr}^2\pi \otimes \mathbb{Q} \cong \#(\mathrm{Ker}\, \mu_0)$, whence our assertion.

Assuming that Γ_p is connected, for any prime p, we deduce from Lemma 5.1 that $\dim \mathrm{Im}\, \mu_p = \dim \mathrm{Im}\, \mu_0 = m-1$, for any p, $\mathrm{gr}^2\pi$ being therefore free; its stated rank is immediately obtained from $\mathrm{gr}^2\pi \otimes \mathbb{Q} \cong \#(\mathrm{Ker}\, \mu_0)$. Conversely assume $\mathrm{gr}^2\pi$ free, therefore $\dim \mathrm{Im}\, \mu_p = \dim \mathrm{Im}\, \mu_0$, for any p, by the previous discussion. If moreover $\mathrm{rk}\, \mathrm{gr}^2\pi = (m-1)(m-2)/2$, we deduce that this common value must be equal to $m-1$, hence μ_p must be onto and Γ_p must be connected, for any p, again by 5.1. ∎

5.7. REMARKS. An alternative way to prove that if Γ_p is connected, for any p, then $\mathrm{gr}^*\pi \cong L^*_{\mathbb{Z}} =$ torsion free, as stated in C(ii), would be to start with Hain's geometric presentation for π [H] and then try to apply Labute's independence criterion [La1]. This raises the following (difficult) task: given a sequence $r_1, \ldots, r_n$, of homogeneous elements in a free grlie $\mathbb{L} = \mathbb{L}^*_{\mathbb{Z}}(x_1, \ldots, x_m)$, decide, for a given prime p, whether the following condition is satisfied or not:

$$(*_p)\ \mathrm{r}(p)/[\mathrm{r}(p), \mathrm{r}(p)] \text{ is a free } \mathcal{U}(\mathbb{L}(p)/\mathrm{r}(p))\text{-module with basis given by the images of } r_1, \ldots, r_n$$

(where r is the ideal generated by $r_1, \ldots, r_n$ in $\mathbb{L}$, $\mathrm{r}/[\mathrm{r}, \mathrm{r}]$ is an $\mathbb{L}/\mathrm{r}$-module in a natural way, via the adjoint action, and $(\ )(p)$ stands for taking mod p residues).

In our situation, $n = m-1$ and $r_i = \sum_j l_{ij}[x_i, x_j]$, and everything would follow if the conditions $(*_p)$ were satisfied for any p. For a fixed p, $(*_p)$ holds as soon as $r_1(p), \ldots, r_n(p)$ is strongly free in $\mathbb{T}_{\mathbb{Z}_p}(x_1, \ldots, x_m)$ [HL; Théorème 3.3]. Here we meet the really delicate point: the argument of 5.2, establishing that if Γ_p is connected then $\overline{r}_1(p), \ldots, \overline{r}_{m-1}(p)$ is by [illegible] (by mod p, [illegible] mod p), for any prime p. It follows that [illegible] is strongly [illegible] with [illegible] $\mathbb{Z}_p$-coefficients, for all p, hence their Hilbert series are equal and [illegible] is torsion free. It is now easy to prove that [illegible] $\cong C_q$, by making use of an idea of [illegible] (cf. [illegible]), who established the existence of a [illegible] map $f: C_q \to$ [illegible], which is isomorphic in degree one. In particular, knowing that $f \otimes \mathbb{Q}$ is epimorphic and also that [illegible] $\cong C_q \otimes \mathbb{Q}$ (cf. [illegible]) we infer that $f \otimes \mathbb{Q}$ is also mono; hence, due to the torsion freeness of C_q, f itself must be mono, giving thus the asserted isomorphism. ∎

PROOF OF LEMMA 5.6. At the beginning we shall prove that [illegible] is free if and only if $\dim H_1 \varphi_p = \dim H_1 \varphi_0$ for any prime p. The torsion [illegible] of the finitely generated abelian group [illegible] is equivalent, as before, to the fact that [illegible] $\otimes \mathbb{F}_p = \dim$ [illegible] for any p. By [illegible] [illegible] is isomorphic to the quotient group of the second exterior power of the free abelian group with [illegible] by the relations [illegible] (and, evidently, is [illegible] finitely presented, [illegible] $\otimes \mathbb{Z}_p$). By elementary duality we infer that [illegible] and [illegible], whence our assertion.

Assuming that Γ_q is connected, for any prime p, we derive from Lemma 6.1 that $\dim H_1 \varphi_p = \dim H_1 \varphi_0 = n_q - 1$. It now follows [illegible] from the stated fact is immediately obtained from [illegible]. Conversely, assume [illegible] free, therefore $\dim H_1 \varphi_p = \dim H_1 \varphi_0$ for any p. By the preceding discussion, if [illegible] $= (n_q - 1)(n_q - 2)/2$, we deduce that this common value must be equal to $n_q - 1$, hence [illegible] must be zero and Γ_q must be connected, [illegible] Lemma 6.1. ∎

6.7. REMARKS. An alternative way to prove that if Γ_q is connected for any q then [illegible] $\cong L_q$ is torsion free, as stated in [illegible], would be to start with Hall's [illegible] presentation for [illegible] and then try to apply [illegible] introduced in [illegible]. This requires the following difficult task: given [illegible] of homogeneous elements in a free graded Lie algebra $L = L_{\mathbb{Z}}(x_1, \ldots, x_n)$, decide, for a given prime p, whether the following condition is satisfied or not:

(C_p) [illegible] $\cong$ [illegible] module, with basis given by images of [illegible],

where r is the ideal generated by [illegible] in $L \otimes \mathbb{F}_p$, [illegible] and L [illegible] in a natural way on the adjoint action, and [illegible] stands for [illegible] basis.

In our situation, it is [illegible] and [illegible], and everything would follow if the condition (C_p) were satisfied for any p. For a fixed p, (C_p) holds as soon as [illegible] is strongly free in [illegible] [illegible], Theorem 5.3]. Here we meet the really delicate point: the argument of [illegible], establishing that if Γ_q is connected then [illegible] is combinatorially free. Conversely, if $(*_p)$ holds in our case then plainly $r_1(p), \ldots, r_{m-1}(p)$ must be linearly independent, hence μ_p is onto and Γ_p must be connected (Lemma 5.1). This connectedness criterion answers (partially) a question of Labute [La2; Problem 5]: if $r_i = \sum c_i^{jk}[x_j, x_k]$, find a necessary and sufficient condition for $(*_p)$ to hold, in terms of the coefficients c_i^{jk}.

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