INFINITESIMAL STUDY OF DIFFERENTIAL POLYNOMIAL FUNCTIONS

by

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INTRODUCTION

In a series of papers $[B_i]$ $(1 \leq i \leq 4)$ we initiated a program of study of differential polynomial functions (intuitively of non-linear differential operators) on projective varieties defined over a differential field. We shall freely reffer in what follows to $[B_1]$ with which the reader is assumed to be familiar.

The present paper is part of this program; its aim is to discuss the infinitesimal counterpart of the theory in $[B_i]$. More precisely E. Kolchin, P. Cassidy, J. Johnson ($[K_2]$, $[C]$, $[J]$) have defined Δ-tangent spaces of Δ-closed subsets of affine spaces and considered Δ-tangent maps between them (these concepts are an analogue in algebraic geometry of what in the C^∞ case are the linearisations of non-linear differential operators appearing in "global non-linear analysis" in the sense of R. Palais $[P]$). What we are doing in this paper is to study (and even "compute") the Δ-tangent maps of the most remarkable Δ-polynomial maps considered in $[B_1]$, $[B_4]$. To fix ideas note that if $f: X \longrightarrow Y$ is a Δ-polynomial map $[B_1]$ between two smooth algebraic $\mathcal{F}$- varieties ($\mathcal{F}$ a constrainedly closed ordinary Δ-field of characteristic zero with constant field $\mathcal{C}$, $[K_2]$) then the Δ-tangent map at x is a Δ-polynomial homomorphism

$T_x f: T_x X \longrightarrow T_{f(x)} Y$ (i.e. a linear differential operator); we shall review Δ-tangent maps in Section 1.

Our first result (cf. Section 2) is an infinitesimal analogue of our " Δ-algebraic analogue of Lang conjecture" from [B_4]. Recall that we proved in [B_4] the following result: let G be an irreductible algebraic $\mathcal{F}$-group, $\Sigma \subset G$ a Δ-closed subgroup of Δ-type zero and $X \subset G$ a closed subvariety possessing a dominant morphism $X \longrightarrow W$ into a variety W such that Alb(W) does not descend to $\mathcal{C}$ and W satisfies a certain "curvature condition" (e.g. W is smooth, projective, of general type); then $X \cap \Sigma$ is not Zariski dense in X. Recall from [B_4] that this result implied a geometric analogue of "Lang's conjecture" close to Raynaud's [R] saying that if A is an abelian $\mathbb{C}$-variety with $\mathbb{C}/\overline{\mathbb{Q}}$-trace zero, X is a smooth closed subvariety of A, not a translation of a non-zero abelian subvariety and $\Gamma \subset A$ is a finite rank subgroup then $X \cap \Gamma$ is not Zariski dense in X. But more important for us here, our result in [B_4] implied the following " Δ-geometric statement". Define for any smooth projective $\mathcal{F}$-variety X the Δ-character map $\psi_r: X \longrightarrow \mathcal{F}^{M_r}$ to be the composition $X \longrightarrow \mathrm{Alb}(X) \longrightarrow \mathcal{F}^{M_r}$ where the components of $\mathrm{Alb}(X) \longrightarrow \mathcal{F}^{M_r}$ are a basis of the space of Δ-polynomial characters of order $\leq r$ on Alb(X). Then our result in [B_4] implied that if X is a curve of genus $\geqslant 2$ not descending to $\mathcal{C}$ then the fibres of ψ_r are finite for $r \gg 0$. Our main result in Section 2 of the present paper implies for instance that if X is a curve as in the latter statement but assumed in addition non-hyperelliptic then the Δ-tangent map $T_x \psi_r$ is injective for all but finitely many points $x \in X$ provided $r \gg 0$. By the way this infinitesimal statement implies the "global statement" that the fibres of ψ_r are finite for $r \gg 0$ this providing (under the "non-hyperelliptic assumption") a purely algebraic proof of the latter (recall that the proof of the latter in [B$_4$] involved analytic arguments, especially a Big-Picard-type theorem!). Note however that the "infinitesimal main result" in the present paper, although reffers to the higher dimensional case, is far from implying the "global main result" from [B$_4$] (and conversely the "global" result does not imply the "infinitesimal" one!).

Next we want to "compute" some Δ-tangent maps. Let A be an abelian $\mathcal{F}$-variety of dimension g and Δ-rank g (cf. [B$_1$] (6.5)), pick a basis of the space of Δ-polynomial characters of order ≤ 2, let $\psi_2 : A \longrightarrow \mathcal{F}^g$ be the map whose components are the members of this basis and consider the tangent map $T_o\psi_2 : T_oA = \mathrm{Lie}A \longrightarrow T_o(\mathcal{F}^g) = \mathcal{F}^g$. As $T_o\psi_2$ is a Δ-polynomial homomorphism it can be viewed as a "linear differential operator" so it makes sense to consider its "symbol" $\sigma_2(T_o\psi_2)$ which is an $\mathcal{F}$-linear map from LieA to $\mathcal{F}^g$ (cf. (3.4) below). We will check that this map is invertible so we may consider the map $\psi_A = A \to \mathrm{Lie}A$ composition of $\psi_2 : A \to \mathcal{F}^g$ with $\sigma_2(T_o\psi_2)^{-1} : \mathcal{F}^g \to \mathrm{Lie}\ A$. Then ψ_A has an invariant meaning (it does not depend on choosing the basis of the space of Δ-polynomial characters of order ≤ 2 but only on A) and its Δ-tangent map $\tau_A := T_o\psi_A : \mathrm{Lie}\ A \to \mathrm{Lie}\ A$ has symbol the identity and is an important invariant of A. The map ψ_A might remind one of Kolchin's logarithmic derivative but it has nothing to do with it since the latter is never defined if $\mathrm{rank}_\Delta(A) \geq 1$! Our main result here "computes" τ_A in terms of the Gauss-Manin connection on $H^1_{DR}(A)$ as follows. Since $\mathcal{F}$ is constrainedly closed, $H^1_{DR}(A)$ has an $\mathcal{F}$-basis $e_1, \dots, e_2$ killed by δ (δ acting on $H^1_{DR}(A)$ via Gauss-Manin connection $\nabla^A : \mathrm{Der}_{\mathcal{C}}\mathcal{F} \longrightarrow \mathrm{End}_{\mathcal{C}}(H^1_{DR}(A))$); pick a basis $w_1, \dots, w_g$ of

$H^0(\Omega^1_{A/\mathcal{F}}) \subset H^1_{DR}(A)$ and express it as $w_i = \sum_{j=1}^{g} a_{ij}e_j + \sum_{j=1}^{g} b_{ij}e_{g+j}$ to obtain matrices $a = (a_{ij})$, $b = (b_{ij}) \in \mathrm{Mat}_{\mathcal{F}}(g,g)$. Permuting the e_i's we may assume $\det a \neq 0$ and put $z = a^{-1}b$ (this z can be viewed as a "Δ-period matrix" for A). Moreover $\mathrm{rank}_{\Delta}(A) = g$ implies $\det z' \neq 0$ (here $z', z'', \ldots$ stand for $\delta z, \delta^2 z, \ldots$). Put $\beta = \beta_A = (z''(z')^{-1})'/2 - (z''(z')^{-1})^2/4 \in \mathrm{Mat}_{\mathcal{F}}(g,g)$; we will check that the class of β modulo the adjoint action of $GL_{\mathcal{C}}(g)$ on $\mathrm{Mat}_{\mathcal{F}}(g,g)$ depends only on A (and not on the choice of the basis (e_i) and (w_j)). Our main result here is that Lie A has a basis such that upon identifying Lie A with $\mathrm{Mat}_{\mathcal{F}}(g,1)$ via this basis, τ_A is given by the formula $\tau_A(y) = y'' + \beta y$ for all $y \in \mathrm{Mat}_{\mathcal{F}}(g,1)$. Any basis of Lie A for which the above holds will be called a distinguished basis; all distinguished basis are conjugate under the action of $GL_{\mathcal{C}}(g)$. This will be proved in Section 5 after a digression on "Δ-Hodge structures" and "Δ-Torelli map" (cf. Sections 3,4). A consequence of Section 4 will be that the matrix z above (which reflects the "internal" Gauss-Manin connection $\nabla^A : \mathrm{Der}_{\mathcal{C}}\mathcal{F} \to \mathrm{End}_{\mathcal{C}}(H^1_{DR}(A))$) can be computed in terms of the "external" Gauss-Manin connection $\nabla^{X/Y} : \mathrm{Der}_{\mathcal{F}}\mathcal{O}_Y \to \underline{\mathrm{End}}_{\mathcal{F}}(H^1_{DR}(X/Y))$ (where Y is the moduli space of principally polarized abelian $\mathcal{F}$-varieties with level n structure and $X \to Y$ is the universal abelian scheme); here we will use a computation made in our monograph [B_5]. Finally note that if A is the elliptic curve defined by $Y^2 = X(X-1)(X-\lambda)$, $\lambda \in \mathcal{F}$, then $\beta_A = \frac{1}{4}(\lambda')\frac{\lambda^2-\lambda+1}{\lambda^2(\lambda-1)^2} + \frac{2\lambda'\lambda''' - 3(\lambda'')^2}{(\lambda')^2}$. So far we discussed the case of abelian varieties. Let's discuss now the case of curves of genus $g \geq 2$

(also cf. Section 5). Let X be such a curve assumed non-hyperelliptic of Δ-rank g, let A be its Jacobian and $\psi_X : X \longrightarrow \text{Lie } A$ the composition of $X \to A$ and $\psi_A : A \longrightarrow \text{Lie } A$. We will prove that if h: Lie A $\longrightarrow \mathcal{F}$ is a Δ-generic element of the dual $(\text{Lie } A)^o$ then upon viewing h as defining a hyperplane in $\mathbb{P}(\text{Lie } A)$ and hence a canonical divisor K on X and letting $\psi_h = h \circ \psi_X :$ $: X \longrightarrow \text{Lie } A \longrightarrow \mathcal{F}$ we have

$$a_\Delta(\text{Ker}(T_x \psi_h)) = \begin{cases} 2 & \text{if } x \in X \setminus \text{Supp } K \\ 1 & \text{if } x \in \text{Supp } K \end{cases}$$

Recall that $a_\Delta(\Sigma)$ denotes as usual the typical Δ-dimension of Σ (and we always understand when writing $a_\Delta(\Sigma)$ that Σ has Δ-type zero!). One should not be surprized by the fact that $x \mapsto a_\Delta(\text{Ker}(T_x \psi_h))$ is not upper semicontinuous on X as one should expect from "usual" algebraic geometry; this is a general phenomenon in Δ-algebraic geometry due to the systematic presence of "separants" (in the sense of Kolchin $[K_1]$, Chapter 1).

Putting together the last estimation and the finiteness result from Section 2 we get that there exist finite sets F_1 and F_2 in X such that

$$a_\Delta(\text{Ker}(T_x \psi_X)) = \begin{cases} 0 & \text{if } x \in X \setminus (F_1 \cup F_2) \\ 1 & \text{if } x \in F_1 \\ 2 & \text{if } x \in F_2 \end{cases}$$

As a corollary of our computations we are able to characterize the points in F_2; they are precisely the points $x \in X$ such that upon viewing X as canonically embedded in $\mathbb{P}^{g-1} = \mathbb{P}((\text{Lie } A)^o)$ and upon choosing coordinates in $\mathbb{P}^{g-1}$ corresponding to a distinguished basis of Lie A we have that $x \in \mathbb{P}^{g-1}_{\mathcal{C}}$ and any vector in the line of Lie A defined by x is an eigenvector of $\beta_A \in \text{Mat}_{\mathcal{F}}(g,g) =$

$= \mathrm{End}_{\mathcal{F}}$ (Lie A). In particular if β_A has distinct eigenvalues then card $F_2 \leq g$. And in any case card $F_2 \leq$ card $(X \cap \mathbb{P}^{g-1}_{\mathcal{C}})$.

In Section 6 we push the analogy with "global analysis" [P] further. Indeed the Δ-polynomial functions are the analogue in algebraic geometry of what in the C^{∞} case are the "lagrangians" [P]. So it is tempting to develop in the Δ-algebraic setting the formalism of the calculus of variations (Euler-Lagrange equations, geodesics,...). This is indeed possible, namely for any smooth $\mathcal{F}$-variety and any Δ-polynomial function $f: X \to \mathcal{F}$ we shall define a "Δ-polynomial section" $el(f): X \longrightarrow T^{*}X$ of the cotangent bundle. Its "zero locus" will be called Geo(f) (the geodesic locus of f); intuitively $x \in X$ belongs to Geo(f) iff x is a solution of the "Euler-Lagrange system" associated to the "lagrangian" f. Then one can hope that choosing "sufficiently general lagrangians f" of a certain type on a given X (usually f should be "quadratic in the top derivatives") Geo(f) will have Δ-type zero and the "expected" typical Δ-dimension. Morewer for "remarkable" pairs (X,f) one should expect that Geo(f) has a "remarkable" description. It is precisely what we shall do for X an abelian variety (or a curve) and $f = q \circ \psi_X$ where q is a quadratic form on Lie X (respectively on Lie A where A = Jacobian of X); for the precise results we send to Section 6.

An Appendix will be included in which we provide a "dictionary" between concepts of "Δ-algebraic geometry" à la Ritt-Kolchin [K_1], [K_2], [B_1] and concepts of "global non-linear analysis" à la Palais [P].

We close the Introduction by noting that in [B_1], [B_2], [B_3] we developed the theory over a universal Δ-field $\mathcal{U}$ with constant field $\mathcal{K}$; but everything which was said there holds if one replaces $\mathcal{U}$ and $\mathcal{K}$ by $\mathcal{F}$ and $\mathcal{C}$ where $\mathcal{F}$ is a constrainedly closed ordinary Δ-field of characteristic zero and $\mathcal{C}$ is its field of constants (as in $[B_4]$). We shall assume throughout the paper that $\mathcal{F}$ and $\mathcal{C}$ are as above.

1. Review of Δ-tangent maps (after Kolchin, Cassidy, Johnson)

In this section we transpose into the setting of $[B_1]$ some concepts of Δ-differential calculus due to Kolchin, Cassidy and Johnson (cf. $[K_2]$, $[C]$, $[J]$).

(1.1) Let V be a D-scheme. Then $\Omega_{V/\mathcal{F}}$ has a natural structure of D-module ($D = \mathcal{F}[\delta]$) such that the universal derivation $d: \mathcal{O}_V \to \Omega_{V/\mathcal{F}}$ is a D-module map. So $TV := \operatorname{Spec} S(\Omega_{V/\mathcal{F}})$ has an induced structure of D-scheme. For any D-scheme Z we have a functorial identification of $\operatorname{Hom}_{D\text{-sch}}(Z, TV)$ with the set of pairs (u, ∂) where $u \in \operatorname{Hom}_{D\text{-sch}}(Z,V)$ and $\partial: \mathcal{O}_V \to u_*\mathcal{O}_Z$ is an $\mathcal{F}$-derivation such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_V & \xrightarrow{\partial} & u_*\mathcal{O}_Z \\ \delta\downarrow & & \downarrow\delta \\ \mathcal{O}_V & \xrightarrow[\partial]{} & u_*\mathcal{O}_Z \end{array}$$

Such a derivation ∂ is called by Kolchin and Cassidy a Δ-$\mathcal{F}$-derivation of $\mathcal{O}_V$ into $u_*\mathcal{O}_Z$; call the set of such ∂'s $\operatorname{Der}_D(\mathcal{O}_V, u_*\mathcal{O}_Z)$.

(1.2) Let X be a smooth $\mathcal{F}$-variety. We claim that there is a natural D-scheme isomorphism $T(X^\infty) \simeq (TX)^\infty$. Indeed for any D-scheme Z the set $\operatorname{Hom}_D(Z, T(X^\infty))$ naturally identifies with the set of pairs (u, ∂) where $u \in \operatorname{Hom}_{D\text{-sch}}(Z, X^\infty) = \operatorname{Hom}_{\mathcal{F}\text{-sch}}(Z^!, X)$ and

$\partial \in \mathrm{Der}_D(\mathcal{O}_{X^\infty}, u_*\mathcal{O}_Z)$. Then u together with the composition

$$\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_{X^\infty} \xrightarrow{\pi_*\partial} \pi_* u_* \mathcal{O}_Z$$

(where $\pi: X^\infty \to X$ is the natural map arrising from adjunction) define an element in $\mathrm{Hom}_{\mathcal{F}\text{-sch}}(Z, TX) \simeq \mathrm{Hom}_{D\text{-sch}}(Z, (TX)^\infty)$. So we got a map $\mathrm{Hom}_{D\text{-sch}}(Z, T(X^\infty)) \to \mathrm{Hom}_{D\text{-sch}}(Z, (TX)^\infty)$. This map is injective because $\mathcal{O}_{X^\infty}$ is generated as a Δ-$\mathcal{F}$-algebra by $\mathcal{O}_X$. Once we dispose of injectivity, surjectivity can be checked locally in the Zariski topology; so we may assume $X = \mathrm{Spec}\, \mathcal{F}[y]/J$, $y = (y_1, \dots, y_N)$, $Z = \mathrm{Spec}\, R$, $X^\infty = \mathrm{Spec}\, \mathcal{F}\{y\}/[J]$, we may assume we are given an $\mathcal{F}$-derivation $\partial : \mathcal{F}[y]/J \to R$ and we must lift it to a Δ-$\mathcal{F}$-derivation $\tilde{\partial} : \mathcal{F}\{y\}/[J] \longrightarrow R$. This problem clearly reduces to the case $J = 0$ where it is trivial.

(1.3) Let $f: X \longrightarrow Y$ be a Δ-polynomial map of smooth $\mathcal{F}$-varieties; recall that by definition this is a map induced by a morphism of D-schemes $f^\infty: X^\infty \longrightarrow Y^\infty$. The latter induces a morphism of D-schemes $T(f^\infty) : (TX)^\infty = T(X^\infty) = \mathrm{Spec}\, S(\Omega_{X/\mathcal{F}}) \to \mathrm{Spec}\, S(f^{\infty *}\Omega_{Y^\infty/\mathcal{F}}) = T(Y^\infty) \times_{Y^\infty} X^\infty \longrightarrow (TY)^\infty$ hence a Δ-polynomial map $Tf : TX \longrightarrow TY$ called the Δ-tangent map of f. Above each $x \in X$ the induced Δ-polynomial map $T_xf = (Tf)_x : T_xX \longrightarrow T_{f(x)}Y$ is a group homomorphism; it corresponds to the morphism of D-group schemes $(T_xX)^\infty \simeq T_x(X^\infty) \longrightarrow T_{f(x)}(Y^\infty) \simeq (T_{f(x)}Y)^\infty$ induced by $T(f^\infty)$ (here for any $\mathcal{F}$-scheme V and any $\mathcal{F}$-point x of V we denote by T_xV the fibre of $TV \to V$ at x so $T_xV = \mathrm{Spec}\, S(m/m^2)$ where m is the maximal ideal of $\mathcal{O}_{V,x}$. We shall denote by T_x^*V the $\mathcal{F}$-linear space m/m^2). Note also that T_xX is in bijection with the space of Δ-$\mathcal{F}$-derivations from $\mathcal{O}_{X^\infty, x}$ to $\mathcal{F}$ while T_xf applied to such a Δ-$\mathcal{F}$-derivation $\partial : \mathcal{O}_{X^\infty, x} \to \mathcal{F}$ is nothing but the

Δ - $\mathcal{F}$ - derivation $\mathcal{O}_{Y^\infty, f(x)} \xrightarrow{f} \mathcal{O}_{X^\infty, x} \xrightarrow{\partial} \mathcal{F}$ viewed as an element of $T_{f(x)}Y$.

(1.4) Let's see how (1.3) looks "in coordinates". Assume $X = \operatorname{Spec} \mathcal{F}[y]/J$ and $Y = \mathbb{A}^1$. Then $TX = \operatorname{Spec} \mathcal{F}[y, dy]/J + (dJ)$ so $(TX)^\infty = \operatorname{Spec} \mathcal{F}\{y, dy\}/[J, dJ]$ and the map $(TX)^\infty \to (\mathbb{A}^1)^\infty$ corresponds to

$$df = \sum_{i,j} \frac{\partial F}{\partial(\delta^j y_i)} \delta^j dy_i \mod [J, dJ] \in \mathcal{O}((TX)^\infty)$$

where $f: X \to \mathbb{A}^1$ is defined by $F \in \mathcal{F}\{y\}$, $y = (y_1, \ldots, y_N)$.

(1.5) Let G and H be algebraic vector groups over $\mathcal{F}$ and $f: G \to H$ a Δ-polynomial homomorphism (we shall sometimes say that f is a linear differential operator). Identify G and H with Lie $G = T_0 G$ and Lie $H = T_0 H$ in the canonical way (if $G = \operatorname{Spec} S(V)$ for some $\mathcal{F}$-linear space V then $V \simeq X_a(G) (= \operatorname{Hom}(G, G_a))$ so $G \simeq V^0 \simeq L(G)$ where $L(G)$ is the space of left invariant $\mathcal{F}$-derivations on $\mathcal{O}_G$ which is in an obivous duality with $X_a(G)$ and identifies with $T_0 G$!). Then under this identification $T_0 f$: Lie $G \to$ Lie H coincides with f. Indeed fix isomorphisms $G \simeq \mathcal{F}^n$, $H \simeq \mathcal{F}^m$ and assume f has components $f_j(y) = \sum_{i,k} a_{jik} \delta^i y$. Then

$$(T_0 f)(y) = \sum_{i,k} \frac{\partial f_j}{\partial(\delta^i y_k)} \delta^i y_k = \sum_{i,k} a_{jik} \delta^i y_k$$

(1.6) Let Σ be a Δ-closed subset of a smooth $\mathcal{F}$-variety X and let $x \in \Sigma$. Then $T(\Sigma^\infty)$ is a closed D-subscheme of $T(X^\infty)$ hence the fibre $T_x(\Sigma^\infty)$ of $T(\Sigma^\infty)$ above (the point of Σ^∞ induced by x still denoted abusively by) $x \in \Sigma^\infty$ is a closed D-subscheme of $T_x(X^\infty) = (T_x X)^\infty$, hence (by [$B_1$] (3.9)) corresponds to a Δ-closed subgroup of $T_x X$ which we call the Δ-tangent space of Σ at x and which we denote by $T_x \Sigma$. So $T_x \Sigma$ is in bijection with the set of all Δ - $\mathcal{F}$ - derivations of $\mathcal{O}_{\Sigma^\infty,x}$ into $\mathcal{F}$. If x^n is the image of $x \in \Sigma^\infty$ via the map $\Sigma^\infty \to \Sigma^n$ then we have

$$T_x \Sigma \simeq Hom_{D\text{-mod}}(T_x^* \Sigma^\infty , \mathcal{F}) \subset Hom_{\mathcal{F}\text{-mod}}(T_x^* \Sigma^\infty , \mathcal{F}) =$$

$$= Hom_{\mathcal{F}\text{-mod}}(\varinjlim T_{x_n}^* \Sigma^n , \mathcal{F}) \doteq \varprojlim T_{x_n} \Sigma^n .$$

Assume in addition X above is an algebraic $\mathcal{F}$-group and Σ is a Δ-closed subgroup. Denote by Lie Σ the space $T_e\Sigma$ ($e \in G$ the identity). Then Σ has Δ-type zero iff Σ^∞ has finite dimension so in this case each of the spaces $T_e\Sigma^\infty$ and $T_e^*\Sigma^\infty$ is $\mathcal{F}$-isomorphic to the $\mathcal{F}$-dual of the other. This makes $T_e\Sigma^\infty$ a D-module. Moreover in this case the standard $\mathcal{F}$-Lie algebra isomorphism between Lie $\Sigma^\infty := T_e\Sigma^\infty$ and the space $L(\Sigma^\infty)$ of left invariant $\mathcal{F}$-derivations of $\mathcal{O}_{\Sigma^\infty}$ (the latter viewed with its obvious "adjoint" D-module structure) is a D-module isomorphism.

2. Infinitesimal Δ-algebraic Lang conjecture

(2.1.) Let X be a closed subvariety of a commutative irreducible algebraic $\mathcal{F}$-group G. Assume dim X = r. Then we dispose of the "Gauss map" $\gamma: X_{reg} \to$ Grass(r, Lie G) defined by $\gamma(x) = (T_eL_x)^{-1}(T_xX) \subset T_eG$ = Lie G where $L_x: G \to G$ is the translation with x. Let $\check{X}$ be the Zariski closure of $\gamma(X_{reg})$. With notations above our main result here is the following:

(2.2) THEOREM. Let $\chi: G \to \mathcal{F}^N$ be a Δ-polynomial homomorphism whose kernel has Δ-type zero and let $f: X \subset G \xrightarrow{\chi} \mathcal{F}^N$ be the composed map. Denote by C_f the set of all points $x \in X$ such that Ker(T_xf) is Zariski dense in T_xX. Assume Alb($\check{X}$) does not descent to $\mathcal{C}$. Then C_f is not Zariski dense in X.

We shall derive from (2.2) the following:

(2.3) COROLLARY. In notations of (2.1) let $\Sigma \subset G$ be a Δ-closed subgroup of Δ-type zero and assume Alb($\check{X}$) does not descend to $\mathcal{C}$. Then $X \cap \Sigma$ is not Zariski dense in X.

Recall from $[B_4]$ that the Δ-closure of any finite rank (subgroup) of a commutative algebraic $\mathcal{F}$-group has Δ-type zero, this making the connection between (2.3) and the original Lang conjecture $[La_1]$ (see $[B_4]$ for details). One would like to dispose of a statement like (2.2) in which the condition "Alb($\check{X}$) does not descend to $\mathcal{C}$" is replaced by "Alb(X) does not descend to $\mathcal{C}$" as in $[B_4]$. So it is of interest to emphasize situations when the map $X \dashrightarrow \check{X}$ is birational. This is the case when X is a non-hyperelliptic curve and $X \subset G$ is the embedding into its Jacobian (because then $X \longrightarrow \check{X}$ is nothing but the canonical embedding). More generally we have:

(2.4) LEMMA. Let S be a smooth projective curve over $\mathcal{F}$ of genus $g \geqslant 3$ embedded into its Jacobian A and let $X \subset A$ be the image of the symmetric product $S^{(d)}$ in A where $d < g/2$ is some positive integer. Assume S is not a covering of $\mathbb{P}^1$ with d + 1 sheets or less (by "Brill-Noether" this holds if S is generic). Then $S^{(d)} \longrightarrow X$ is an isomorphism and $X \longrightarrow \check{X}$ is birational.

Then (2.2), (2.3), (2.4) imply:

(2.5) COROLLARY. Under the hypothesis of (2.4) assume in addition S does not descend to $\mathcal{C}$. The following hold:

1) Let $\psi_r : X \longrightarrow \mathcal{F}^{M_r}$ be the Δ-character map of X and C_{ψ_r} be the locus of all $x \in X$ such that $\mathrm{Ker}(T_x \psi_r)$ is Zariski dense in X. Then for $r \gg 0$, C_{ψ_r} is not Zariski dense in X.

2) For any Δ-closed subgroup of Δ-type zero $\Sigma \subset A$ the set $X \cap \Sigma$ is not Zariski dense in X. In particular no fibre of ψ_r is Zariski dense in X if $r \gg 0$.

In the case of curves we get simply:

(2.6) COROLLARY. Let X be a smooth projective non-hyperelliptic curve over $\mathcal{F}$ which does not descend to $\mathcal{C}$. Then:

1) $T_x \psi_r$ is injective for all but finitely many $x \in X$ provided $r \gg 0$.

2) X meets any Δ-closed subgroup of Δ-type zero of its Jacobian in finitely many points. In particular ψ_r has finite fibres for $r \gg 0$ (indeed for $r \geqslant 2$ if X has Δ-rank g = genus of X).

Remark. Assertion 2) above was proved without the "non-hyperelliptic assumption" in $[B_4]$ using an analytic method (our proof here will be purely algebraic).

(2.7) Proof of (2.2). Put $\Sigma = \operatorname{Ker} \chi$. Then by $[K_2]$ p. 249 (or in this case by a direct verification using $[B_1]$, Section 3) we have $\operatorname{Lie} \Sigma = \operatorname{Ker}(T_e \chi)$ so for $x \in X_{reg}$ we have $\operatorname{Ker}(T_x f) = (T_x X) \cap (T_e L_x)(\operatorname{Lie} \Sigma)$.

So if $x \in C_f \cap X_{reg}$ then $\operatorname{Lie} \Sigma \cap \gamma(x)$ is not degenerate in $\gamma(x)$, i.e. not contained in a hyperplane. Let $V := \overset{r}{\wedge} \operatorname{Lie} G$ and let $\Gamma \subset V$ be the $\mathcal{C}$-linear span of the set of elements of V of the form $\sigma_1 \wedge \dots \wedge \sigma_r$ where $\sigma_i \in \operatorname{Lie} \Sigma$. Since Σ has Δ-type zero, by [C] it is a finite dimensional $\mathcal{C}$-linear subspace of Lie G hence Γ is a finite dimensional $\mathcal{C}$-linear subspace of V in particular Γ is a Δ-closed subgroup of V of Δ-type zero. Consequently Γ^∞ is an algebraic D-subgroup scheme of V^∞ (cf. $[B_1]$ (3.10), (3.12)). Since Γ is unipotent, by $[B_6]$ (pp. 98, 100) $\mathcal{O}(\Gamma^\infty)$ is a split D-module (i.e. it has an $\mathcal{F}$-basis killed by δ). Let $Z \subset V$ be the affine cone over $\check{X} \subset \operatorname{Grass}(r, \operatorname{Lie} G) \xrightarrow{p} \mathbb{P}(V^0)$ where p is the Plücker embedding and $Z^* = Z \setminus \{0\}$ where 0 is the vertex of Z. Consider the closed D-subscheme $W = (Z^\infty \cap \Gamma^\infty)_{red}$ of V^∞ and the open subscheme $W^* = ((Z^*)^\infty \cap \Gamma^\infty)_{red}$ of W.

The irreducible components W_i of W are D-subschemes of Γ^∞ hence their ideals in $\mathcal{O}(\Gamma^\infty)$ are D-submodules of $\mathcal{O}(\Gamma^\infty)$ so they are split D-modules too. Consequently $\mathcal{O}(W_i)$ are split D-modules, in particular the W_i's descend to $\mathcal{C}$.

Now if $x \in C_f \cap X_{reg}$ then one can choose a basis $\sigma_1, \dots, \sigma_r$ of $\gamma(x)$ contained in $\gamma(x) \cap \mathrm{Lie}\,\Sigma$ and then $\sigma_1 \wedge \dots \wedge \sigma_r \in$ $\in Z^* \cap \Gamma$ hence we are provided with an element in $\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, (Z^*)^\infty) \cap \mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, \Gamma^\infty) =$ $= \mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\,\mathcal{F}, W^*)$ which composed with the natural projection $W^* \longrightarrow \check{X}$ gives precisely the $\mathcal{F}$-point of $\check{X}$, image of x via $X_{reg} \longrightarrow \check{X}$. So the image of $W^* \longrightarrow \check{X}$ contains the image of $C_f \cap X_{reg} \subset$ $\subset X_{reg} \longrightarrow \check{X}$. Assume now C_f is Zariski dense in X and look for a contradiction. By what we just proved there is at least one component W_i of W such that $W_i \cap W^* \neq \emptyset$ and the rational map $W_i \dashrightarrow \check{X}$ is dominant. There exists a smooth projective model $\widetilde{W}_i$ of W_i which descends to $\mathcal{C}$. We dispose of a surjection $\mathrm{Alb}(\widetilde{W}_i) \longrightarrow \mathrm{Alb}(\check{X})$. Since $\mathrm{Alb}(\widetilde{W}_i)$ descends to $\mathcal{C}$, by the "rigidity theorem" in [La_2] p. 26, $\mathrm{Alb}(\check{X})$ must also descend to $\mathcal{C}$, contradiction. The theorem is proved.

(2.8) Proof of (2.3). Using [B_1] (5.1) one can construct a Δ-polynomial homomorphism $\chi : G \longrightarrow \mathcal{F}^N$ whose kernel has Δ-type zero and contains Σ. Of course we may assume $\Sigma = \mathrm{Ker}\,\chi$. Now assume $X \cap \Sigma$ is Zariski dense in X. Then the D-subscheme $Y = (X^\infty \cap \Sigma^\infty)_{reg}$ of X^∞ dominantes X. On the other hand Y being a subscheme of Σ^∞ which is of finite type over $\mathcal{F}$ is itself of finite type over $\mathcal{F}$. By "generic smoothness" for the map $\pi : Y \longrightarrow X$ we get that there exists a Zariski open set $Y_o \subset Y$ such that $T_y\pi : T_y Y \longrightarrow T_{\pi(y)} X$ is surjective for all $y \in Y_o$. Note that upon letting f denote the composition $X \subset G \xrightarrow{\lambda} \mathcal{F}^N$ we have $T_x(X \cap \Sigma) \subset \mathrm{Ker}(T_x f)$ for all $x \in X \cap \Sigma$. Since $\Omega := \mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\, \mathcal{F}, Y_o)$ is Zariski dense in $Y_o = \mathrm{Hom}_{\mathcal{F}\text{-sch}}(\mathrm{Spec}\, \mathcal{F}, Y_o)$ it follows that the image Ω' of Ω in X is Zariski dense in X. Now for $y \in \Omega$, $x = \pi(y) \in \in \Omega'$, $T_x(X \cap \Sigma)$ is by definition in bijection with $\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\, \mathcal{F}, T_y Y)$ via $T_y \pi$. Since $\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}\, \mathcal{F}, T_y Y)$ is Zariski dense in $T_y Y = \mathrm{Hom}_{\mathcal{F}\text{-sch}}(\mathrm{Spec}\, \mathcal{F}, T_y Y)$ we see that for $x \in \Omega'$, $T_x(X \cap \Sigma)$ is Zariski dense in $T_x X$. This implies that C_f is Zariski dense in X, this contradicting (2.2). The Corollary is proved.

(2.9) Proof of (2.4). The assertion that $\mu : S^{(d)} \longrightarrow X$ is an isomorphism is contained in [GH] p. 245. Let's prove that $X \longrightarrow \check{X}$ is birational. Let $D \in S^{(d)} = \mathrm{Div}^d(S)$, $D = P_1 + \dots + P_d$ where $P_1, \dots, P_d \in S$ are distinct points and put $x = \mu(D) \in X$; we have by our hypothesis $\dim |D| = 0$. One easily checks that $\gamma(x) \in \mathrm{Grass}(d, \mathrm{Lie}\, A)$ is nothing but the affine cone $C(\overline{D}) \subset \mathrm{Lie}\, A$ over the linear span $\overline{D} \subset \mathbb{P}(\mathrm{Lie}\, A)$ of $\varphi(P_1), \dots, \varphi(P_d) \in \mathbb{P}(\mathrm{Lie}\, A)$ where $\varphi : S \to \mathbb{P}^{g-1} = \mathbb{P}(\mathrm{Lie}\, A)$ is the canonical map of S (assumed a closed immersion). We claim that $\overline{D} \cap \varphi(S) = \varphi(D)$. Indeed if there exists $P \in S \setminus \{P_1, \dots, P_d\}$ such that $\varphi(P) \in \overline{D}$ then $\overline{D + P} = \overline{D}$. On the other hand by Riemann-Roch ([GH] p. 245) we have $\dim \overline{D + P} = d - \dim |D + P|$ and $\dim \overline{D} = d - 1 - \dim |D| = d - 1$ so $\dim |D + P| = 1$. But $|D + P|$ has degree $d + 1$ hence provides a covering of $\mathbb{P}^1$ with $d + 1$ sheets, contradiction. Now let D' be another reduced divisor of degree d and put $x' = \mu(D')$. If $\gamma(x) = \gamma(x')$ then $C(\overline{D}) = C(\overline{D'})$ hence $\overline{D} = \overline{D'}$ hence intersecting with $\varphi(S)$ we get $D = D'$ which proves birationality of $X \longrightarrow \check{X}$.

3. Δ - Hodge structures

We start a very elementary digression on a " Δ -linear" structure which will appear later in relation with the Δ -tangent maps of Δ -character maps. Recall that a Hodge structure of level 1 is a pair of $\mathbb{C}$ - linear spaces $W \subset V$ of dimensions g and $2g$ respectively together with a $\mathbb{Z}$ -submodule Λ of V of rank $2g$ and non-degenerate in V. Roughly speaking in a Δ -Hodge structure (over $\mathcal{F}$) the lattice Λ will be replaced by a $\mathcal{C}$-linear subspace of V. Here is the precise definition:

(3.1) By a Δ -Hodge structure (of level 1 and genus g) we understand a pair (V, W) consisting of a D-module V of dimension 2g over $\mathcal{F}$ and of an $\mathcal{F}$ -linear subspace $W \subset V$ of dimension g. Two Δ -Hodge structures (V, W) and (V', W') will be called isomorphic if there is a D-module isomorphism $\sigma : V \to V'$ such that $\sigma(W) = W'$. We let $\mathcal{H}_g$ be the set of isomorphism classes of Δ -Hodge structures (of level 1 and genus g); the ideal situation would be that in which $\mathcal{H}_g$ has a natural structure of Δ-closed subset in some $\mathcal{F}$-variety which would permit to examine the geometry of $\mathcal{H}_g$ as one does for "classifying spaces of Hodge structures" in algebraic geometry. Unfortunately there seems to be no such structure on $\mathcal{H}_g$. We will give instead several descriptions of $\mathcal{H}_g$ (or of parts of it) as "orbit sets" for several (not so obviously related) actions. Before starting this we give one more definition, A Δ - Hodge structure (V, W) has Δ -rank r (write $\mathrm{rank}_{\Delta}(V, W) = r$) if the $\mathcal{F}$ -linear map $W \hookrightarrow V \xrightarrow{\nabla_\delta} V \longrightarrow V/W$ has rank r (where $\nabla_\delta x = \delta x$, $x \in V$ is the multiplication by δ in V). Denote by $\mathcal{H}_g^{(r)}$ the subset of $\mathcal{H}_g$ corresponding to Δ-Hodge structures of Δ-rank r. A special role will be played by $\mathcal{H}_g^{(g)}$ which (for several reasons) should be viewed as a "big cell" in $\mathcal{H}_g$.

(3.2) The most obvious description of $\mathcal{H}_g$ as an orbit set is the following "double coset" representation:

$$\mathcal{H}_g = GL_{\mathcal{F}}(g) \backslash \operatorname{Mat}_{\mathcal{F}}(g, 2g)_o \,/\, GL_{\mathcal{C}}(2g)$$

(where $\operatorname{Mat}_{\mathcal{F}}(g, 2g)_o$ are the matrices in $\operatorname{Mat}_{\mathcal{F}}(g, 2g)$ of rank g) which arrises as follows. Since $\mathcal{F}$ is constrainedly closed, V has a basis $e_1, \dots, e_{2g}$ killed by δ. Pick a basis $w_1, \dots, w_g$ of W and write $w_i = \sum_{j=1}^{g} a_{ij}e_j + \sum_{j=1}^{g} b_{ij}e_{g+j}$, $a = (a_{ij})$, $b = (b_{ij})$; then (V, W) is represented by the double coset of $(a, b) \in \operatorname{Mat}_{\mathcal{F}}(g,2g)$. Note that $\operatorname{rank}_{\triangle}(V, W) = g$ iff $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0$.

(3.3) Each double coset in (3.2) contains a representative of the form $(1,z)$ where $1 \in GL_{\mathcal{F}}(g)$ is the identity matrix. The $\triangle$-rank is g iff $\det z' \neq 0$. Two matrices $(1, z_1)$ and $(1, z_2)$ belong to the same double coset iff there exist $c_{11}, c_{12}, c_{21}, c_{22} \in \operatorname{Mat}_{\mathcal{C}}(g,g)$ such that $\det\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \neq 0$, $\det(c_{11} + z_1c_{21}) \neq 0$ and

$z_2 = (c_{11} + z_1c_{21})^{-1}(c_{12} + z_1c_{22})$; if this happens write $z_1 \sim z_2$.

Then we get the representation:

$$\mathcal{H}_g \simeq \operatorname{Mat}_{\mathcal{F}}(g, g)/\sim$$

(which reminds of the picture: "Siegel upper half space modulo simplectic group"). If (V, W) is represented by $(1, z) \in \operatorname{Mat}_{\mathcal{F}}(g,2g)$ i.e. by $z (\mathrm{mod} \sim)$ we say that z is a $\triangle$-period matrix for (V, W). Of course if $\operatorname{Mat}_{\mathcal{F}}(g,g)_g$ is the set of all $z \in \operatorname{Mat}_{\mathcal{F}}(g,g)$ such that $\det z' \neq 0$ then $\mathcal{H}_g^{(g)} \simeq \operatorname{Mat}_{\mathcal{F}}(g,g)_g/\sim$.

(3.4) To get less obvious descriptions of $\mathcal{H}_g$ we make a preparation. Put $D_n = \sum_{i=0}^{n} \mathcal{F}\delta^i \subset D = \mathcal{F}[\delta]$. Note that D_n is an $\mathcal{F}$-submodule of D for both the right and the left $\mathcal{F}$-module structures of D. We define the n-symbol map $\sigma_n : D_n \longrightarrow \mathcal{F}$ by the formula $\sigma_n(\sum_{i=0}^{n} \lambda_i \delta^i) = \lambda_n$; it is $\mathcal{F}$-linear for both the right and left $\mathcal{F}$-module structures of D_n. Next let G, G' be two algebraic vector $\mathcal{F}$-groups and $f : G \longrightarrow G'$ a Δ-polynomial homomorphism. If $G = \operatorname{Spec} S(W)$, $G' = \operatorname{Spec} S(W')$ then giving f is equivalent to giving an $\mathcal{F}$-linear map (still denoted by) $f: W' \longrightarrow D \otimes_{\mathcal{F}} W$. (Indeed for any G as above the natural D-module map $D \otimes_{\mathcal{F}} X_a(G) \longrightarrow X_a(G^{\infty})$ is an isomorphism!). We say f has order $\leq n$ if $f(W') \subset D_n \otimes_{\mathcal{F}} W$; it is called of order n if it is of order $\leq n$ but not of order $\leq n-1$. If f has order $\leq n$, define the n-symbol of f to be the composition:

$$\sigma_n(f) : W' \xrightarrow{f} D_n \otimes_{\mathcal{F}} W \xrightarrow{\sigma_n \otimes 1} \mathcal{F} \otimes_{\mathcal{F}} W = W$$

identified with an algebraic group homomorphism (still denoted by) $\sigma_n(f) : G \longrightarrow G'$. In coordinates, if $G = \mathcal{F}^N = \operatorname{Mat}_{\mathcal{F}}(N, 1)$, $G' = \mathcal{F}^M = \operatorname{Mat}_{\mathcal{F}}(M, 1)$ and

$$f(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_o y \,, \quad y \in \operatorname{Mat}_{\mathcal{F}}(N, 1)$$

where $a_i \in \operatorname{Mat}_{\mathcal{F}}(M, N)$ then $\sigma_n(f)(y) = a_n y$. Note that if $f: G \longrightarrow G'$ and $g: G' \longrightarrow G''$ are Δ-polynomial homomorphisms of algebraic vector groups of order $\leq n$ and $\leq m$ respectively then $g \circ f$ has order $\leq n+m$ and $\sigma_{n+m}(g \circ f) = \sigma_m(g) \circ \sigma_n(f)$. Moreover if f has order 0 then $\sigma_o(f) = f$.

(3.5) By a Δ-Picard-Fuchs equation we will mean a pair (G, f) consisting of an algebraic vector $\mathcal{F}$-group G of dimension g and of a Δ-polynomial homomorphism $f: G \longrightarrow G$ of order 2 whose 2-symbol $\sigma_2(f) : G \longrightarrow G$ is the identity; (G, f) and (G', f') are called isomorphic if there exists an algebraic group isomorphism $\sigma : G \longrightarrow G'$ such that $\sigma \circ f' = f \circ \sigma$. We denote by $\mathcal{E}_g$ the set of isomorphism classes of Δ-Picard-Fuchs equations. Note that for any (G, f) as above, f is surjective!

(3.6) By a Δ-lattice we will mean a pair (W, Σ) where W is an $\mathcal{F}$-linear space of dimension g and $\Sigma \subset W$ is a $\mathcal{C}$-linear subspace of $\mathcal{C}$-dimension 2g. Once again we have an obvious notion of isomorphism and we denote by $\mathcal{L}_g$ the set of isomorphism classes of Δ-lattices.

(3.7) PROPOSITION. We have a natural bijection $\mathcal{H}_g^{(g)} \simeq \mathcal{E}_g$ and a natural injection $\mathcal{E}_g \to \mathcal{L}_g$.

Proof. Let's start by constructing a map $\mathcal{H}_g^{(g)} \to \mathcal{E}_g$. If (V, W) is a Δ-Hodge structure of Δ-rank g then the natural D-module map $D \otimes_{\mathcal{F}} W \to V$ induces an $\mathcal{F}$-linear isomorphism $\mu : D_1 \otimes_{\mathcal{F}} W \to V$ (everything with the left $\mathcal{F}$-module structure). Note that $\mu^{-1}(x) = 1 \otimes x$ and $\mu^{-1}(\delta x) = \delta \otimes x$ for all $x \in W$. Define the map $f: W \to D \otimes_{\mathcal{F}} W$ by the formula $f(x) = \delta^2 \otimes x - \mu^{-1}(\delta^2 x)$, $x \in W$; in fact $f(W) \subset D_2 \otimes W$. One checks easily that f is $\mathcal{F}$-linear (for the left module structures) so it defines a Δ-polynomial homomorphism (still denoted by) $f: G \to G$ (where $G = W^0 = \operatorname{Spec} S(W)$) of order 2 with $\sigma_2(f) = 1_G$. It will be useful later to "see" the map $\mathcal{H}_g^{(g)} \to \mathcal{E}_g$ in "coordinates". If $w_1, \dots, w_g$ is an $\mathcal{F}$-basis of W then $w_1, \dots, w_g$, $\delta w_1, \dots, \delta w_g$ form a basis of V so if $\delta^2 w_i = \sum \alpha_{ij} \delta w_j + \sum \beta_{ij} w_j$ in V ($\alpha_{ij}, \beta_{ij} \in \mathcal{F}$) then upon identifying G with $\operatorname{Mat}_{\mathcal{F}}(g, 1)$ using $w_1, \dots, w_g$ we have that

$$(3.7.1) \qquad f(y) = y'' - \alpha y' - \beta y \quad , \quad y \in \operatorname{Mat}_{\mathcal{F}}(g,1)$$

where $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij}) \in \operatorname{Mat}_{\mathcal{F}}(g,g)$.

Let's construct a map $\mathcal{E}_g \to \mathcal{H}_g^{(g)}$ (and leave to the reader the task of checking this is an inverse for the map defined above).

Assume we are given a Δ-Picard-Fuchs equation (G, f), put $G = \mathrm{Spec}\, S(W)$ and still denote by $f: W \longrightarrow D_2 \otimes_{\mathcal{F}} W$ the $\mathcal{F}$-liniar map defined by f. Then put $V = D_1 \otimes_{\mathcal{F}} W$ (viewed as a left $\mathcal{F}$-module in which W embeds via $x \longmapsto 1 \otimes x$) and define on V a structure of left D-module by the formulae

$$\delta(\lambda \otimes x) = (\delta\lambda) \otimes x + \lambda\delta \otimes x$$

$$\delta(\lambda\delta \otimes x) = (\delta\lambda)\delta \otimes x + \lambda(\delta^2 \otimes x - f(x))$$

for $\lambda \in \mathcal{F}$, $x \in W$. We get a Δ-Hodge structure (V, W) of Δ-rank g.

Finally let's define the map $\mathcal{E}_g \to \mathcal{L}_g$ by just sending a Δ-Picard-Fuchs equation (G, f) where $G \simeq W^0 \simeq \mathrm{Spec}\, S(W)$ into the Δ-lattice $(W^0, \mathrm{Ker}\, f)$ (one easily checks that $a_\Delta(\mathrm{Ker}\, f) = 2g!$). To check injectivity of $\mathcal{E}_g \longrightarrow \mathcal{L}_g$ let (G, f) and (G', f') be such that there exists an algebraic group isomorphism $\sigma: G \to G'$ taking $\mathrm{Ker}\, f$ into $\mathrm{Ker}\, f'$. Since f and f' are surjective, by [C] p. 910 there is a bijective Δ-polynomial homomorphism $\tau: G \to G'$ such that $\tau \circ f = f' \circ \tau$. Taking n - symbols and using (3.4) we immediately conclude that $\tau = \sigma$ and we are done.

(3.8) In what follows let's give an "orbit set description" for $\mathcal{E}_g$. We claim that

$$\mathcal{E}_g \simeq \mathrm{Mat}_{\mathcal{F}}(g,g) / \mathrm{Ad}\, GL_{\mathcal{C}}(g)$$

(by which we mean of course the set of orbits of the action of $GL_{\mathcal{C}}(g)$ on $\mathrm{Mat}_{\mathcal{F}}(g,g)$ via conjugation). In particular $\mathcal{E}_1 \simeq \mathcal{F}$.

To check our claim note that $\mathcal{E}_g$ is in bijection with the set of equivalence classes of maps $f: \mathrm{Mat}_{\mathcal{F}}(g,1) \to \mathrm{Mat}_{\mathcal{F}}(g,1)$ of the form $f(y) = y'' + \alpha y' + \beta y$, $y \in \mathrm{Mat}_{\mathcal{F}}(g,1)$, $\alpha, \beta \in \mathrm{Mat}_{\mathcal{F}}(g,g)$ where f and $\tilde{f}$ are equivalent iff there exists $u \in GL_{\mathcal{F}}(g)$ such that $\tilde{f} = u^{-1}fu$.

We have

$$u^{-1}fuy = u^{-1}(u''y + 2u'y' + uy'' + \alpha u'y + \alpha uy' + \beta uy)$$

so $\mathcal{E}_g \simeq \mathrm{Mat}_{\mathcal{F}}(g,g) \times \mathrm{Mat}_{\mathcal{F}}(g,g) / GL_{\mathcal{F}}(g)$ where $GL_{\mathcal{F}}(g)$ acts on the right on pairs of $g \times g$ matrices by the formula

$$(\alpha, \beta)u = (2u^{-1}u' + u^{-1}\alpha u,\ u^{-1}u'' + u^{-1}\alpha u' + u^{-1}\beta u)$$

We have a map

$$\mathrm{Mat}_{\mathcal{F}}(g,g) / \mathrm{Ad}\, GL_{\mathcal{C}}(g) \longrightarrow \mathrm{Mat}_{\mathcal{F}}(g,g) \times \mathrm{Mat}_{\mathcal{F}}(g,g) / GL_{\mathcal{F}}(g)$$

given by $\beta \longmapsto (0, \beta)$ which is clearly injective. We claim it is also surjective. Indeed given (α, β) mod $GL_{\mathcal{F}}(g)$, we may find (by Kolchin's surjectivity theorem for the logarithmic derivative $[K_1]$ p. 420) a matrix $u \in GL_{\mathcal{F}}(g)$ such that $u'u^{-1} = -\alpha/2$. Then

$$(3.8.1) \qquad (\alpha, \beta)u = (0,\ u^{-1}(\beta - \alpha^2/4 - \alpha'/2)u)$$

which proves our claim.

(3.9) In (3.3) we got two representations as "quotient sets" of $\mathcal{H}_g^{(g)}$ and $\mathcal{E}_g$ respectively. Since on the other hand there is a natural identification $\mathcal{H}_g^{(g)} \simeq \mathcal{E}_g$ (3.7) one would like to "see" what is the corresponding identification between the quotient sets:

$$\mathrm{Mat}_{\mathcal{F}}(g,g)_g/\sim \ \xrightarrow{\sim}\ \mathrm{Mat}_{\mathcal{F}}(g,g) / \mathrm{Ad}\, GL_{\mathcal{C}}(g)$$

We claim that it is given by

$$(3.9.1) \qquad z \longmapsto \beta = (z''(z')^{-1})'/2 - (z''(z')^{-1})^2/4$$

Indeed if z is viewed as a period matrix for some Δ-Hodge structure (V, W) we may choose a basis $w_1, \ldots, w_g$ of W and a basis $e_1, \ldots, e_{2g}$ of V with $\delta e_i = 0$ such that $w_i = e_i + \sum z_{ij} e_{g+j}$. Then $\delta w_i = \sum z'_{ij} e_{g+j}$, $\delta^2 w_i = \sum z''_{ij} e_{g+j}$ so

$\delta^2 w_i = \sum x_{ij} \delta w_j$ where $(x_{ij}) = z''(z')^{-1}$. By (3.7.1) (V, W) is represented in $\mathcal{E}_g$ by $f(y) = y'' - (z''(z')^{-1})y'$ hence by (3.8.1) the image of z in $\mathrm{Mat}_{\mathcal{F}}(g,g) / \mathrm{Ad}\, GL_{\mathcal{C}}(g)$ is $(1/4)u^{-1}(2(z''(z')^{-1})' - (z''(z')^{-1})^2)u$ where $u'u^{-1} = z''(z')^{-1}/2$. Our claim will be proved if we prove the following:

(3.10) LEMMA. Let $m \in \mathrm{Mat}_{\mathcal{F}}(g,g)$. Then there exists $u \in GL_{\mathcal{F}}(g)$ such that $u'u^{-1} = m$ and u commutes with m and m'.

Proof. Let H be the connected component of the centralizer $C(m)$ of m($C(m) = \{ h \in GL_{\mathcal{F}}(g) ; h m h^{-1} = m \}$) hence Lie H = $= \{ x \in \mathrm{Mat}_{\mathcal{F}}(g,g) ; xm = mx \}$. By Kolchin's surjectivity theorem the logarithmic derivative $\ell\delta : H \to \mathrm{Lie}\, H$, $\ell\delta(h) = h'h^{-1}$ is surjective so we may find $u \in H$ such that $u'u^{-1} = m$. Now $u^{-1}u' =$ $= u^{-1}mu = m = u'u^{-1}$ so $u'u = uu'$ so applying δ we get $u''u = uu''$. Since m' is a polynomial in u^{-1}, u', u'' we get that u commutes also with m' and we are done.

(3.11) We close this section with the remark (not to be used in the sequel) that the set $\mathrm{Mat}_{\mathcal{F}}(g,g) / \mathrm{Ad}\, GL_{\mathcal{C}}(g)$ identifies with $\mathrm{Mat}_{\mathcal{F}}(g,g) / \approx$ where if $x, y \in \mathrm{Mat}_{\mathcal{F}}(g,g)$ we write $x \approx y$ iff there exists $u \in GL_{\mathcal{F}}(g)$ such that $y = uxu^{-1}$ and $y' = ux'u^{-1}$, in other words iff the pairs $(x,x'), (y,y') \in \mathrm{Mat}_{\mathcal{F}}(g,g) \times \mathrm{Mat}_{\mathcal{F}}(g,g)$ are conjugate under the adjoint action of $GL_{\mathcal{F}}(g)$ on "pairs of matrices". The "only if" part of this is clear. To check the "if" part assume $y = uxu^{-1}$, $y' = ux'u^{-1}$ for some $u \in GL_{\mathcal{F}}(g)$. Then $y' = u'xu^{-1} +$ $+ ux'u^{-1} - uxu^{-1}u'u^{-1}$ hence $u'xu^{-1} = uxu^{-1}u'u^{-1}$ hence $xu^{-1}u' = u^{-1}u'x$

hence $u^{-1}u'$ belongs to the Lie algebra of the centralizer $C(x)$ of x. As in (3.10) we can find $b \in C(x)$ such that $b^{-1}b' = u^{-1}u'$ so $u = cb$ for some $c \in GL_{\mathcal{C}}(g)$ hence $y = uxu^{-1} = cbxb^{-1}c^{-1} = cxc^{-1}$ and we are done. The remark above may give a clue on finding a " Δ -algebraic structure" on a "big" subset of $Mat_{\mathcal{F}}(g,g)$ / Ad $GL_{\mathcal{C}}(g)$ by using the invariants of the adjoint action on pairs of matrices. *The coefficients of the characteristic polynomials of any (non-commutative) Δ-polynomial in $x \in Mat_{\mathcal{F}}(g,g)$ provide invariants of x modulo Ad $GL_{\mathcal{C}}(g)$.*

4. Δ-Torelli map

(4.1) Let $\mathcal{A}_g$ be the moduli $\mathcal{F}$-variety of g-dimensional principally polarized abelian $\mathcal{F}$-varieties. We define the Δ--Torelli map

$$h : \mathcal{A}_g \longrightarrow \mathcal{H}_g$$

by associating to each principally polarized abelian $\mathcal{F}$-variety A the Δ-Hodge structure $h_A = (H^1_{DR}(A), H^0(\Omega^1_{A/\mathcal{F}}))$ where $H^1_{DR}(A)$ is viewed as a D-module via the Gauss-Manin connection $\nabla^A : Der_{\mathcal{C}}\mathcal{F} \longrightarrow End_{\mathcal{C}}(H^1_{DR}(A))$, [Ka] . Note that $rank_{\Delta}(A) = r$ (in the sense of [B_1] (6.5)) iff $rank_{\Delta}(h_A) = r$ in the sense of Section 3 of this paper because by [Ka] the composition

$$H^0(\Omega^1_{A/\mathcal{F}}) \hookrightarrow H^1_{DR}(A) \xrightarrow{\nabla\delta} H^1_{DR}(A) \longrightarrow H^1_{DR}(A)/H^0(\Omega^1_{A/\mathcal{F}}) = H^1(\mathcal{O}_A)$$

coincides with the "cup product with the Kodaira-Spencer class" $\rho(\delta)$ (where $\rho : Der_{\mathcal{C}}\mathcal{F} \longrightarrow H^1(T_A)$ is the Kodaira-Spencer map). One easily checks also that h is constant on isogeny classes. On the other hand h clearly forgets polarisations and sends $(\mathcal{A}_g)_{\mathcal{C}}$ (= set of $\mathcal{C}$-points of $\mathcal{A}_g = (\mathcal{A}_g)_{\mathcal{F}}$) into a point; in any case h is far from being injective! The aim of this section will be to "compute" h as explicitely as possible. In the next section we shall relate it to Δ -tangent maps of Δ -character maps. For any abelian variety A of Δ-rank g we shall denote by $z_A \in \mathrm{Mat}_{\mathcal{F}}(g,g)_g$ and $\beta_A \in \mathrm{Mat}_{\mathcal{F}}(g,g)$ representatives of the classes corresponding to h_A via the identifications (3.3) and (3.8) respectively. One can choose z_A and β_A such that they are related by the formula (3.9.1). The most explicit result we are able to obtain is (as expected) in case g = 1. Here $\mathcal{A}_1$ is the "j-line" $\mathbb{A}^1 = \mathcal{F}$ while by (3.7), (3.8) we have $\mathcal{H}_1^{(1)} \simeq \mathcal{E}_1 \simeq \mathcal{F}$. Since the restriction of $h: \mathcal{A}_1 = \mathcal{F} \to \mathcal{H}_1$ to $(\mathcal{A}_1)_{\mathcal{C}} = \mathcal{C}$ is constant we may restrict our attention to the restriction of h to $\mathcal{F} \setminus \mathcal{C}$; since $h(\mathcal{F} \setminus \mathcal{C}) \subset \mathcal{H}_1^{(1)} \simeq \mathcal{F}$, h induces a map $h: \mathcal{F} \setminus \mathcal{C} \to \mathcal{F}$. Recall also that we dispose of the j - map $j: \mathcal{F} \setminus \{0,1\} \to \mathcal{F}$, $j(\lambda) = 2^8 \dfrac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$ and clearly $j(\mathcal{F} \setminus \mathcal{C}) = \mathcal{F} \setminus \mathcal{C}$ so after all we dispose of a map $\mathcal{F} \setminus \mathcal{C} \xrightarrow{j} \mathcal{F} \setminus \mathcal{C} \xrightarrow{h} \mathcal{F}$ given by $\lambda \longmapsto \beta_{A_\lambda}$ where A_λ is defined by $y^2 = x(x-1)(x-\lambda)$. Our first result is

(4.2) PROPOSITION. For any $\lambda \in \mathcal{F} \setminus \mathcal{C}$ we have

$$\beta_{A_\lambda} = h(j(\lambda)) = \frac{1}{4}\left[(\lambda')^2 \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda - 1)^2} + \frac{2\lambda'\lambda''' - 3(\lambda''}{(\lambda')^2}\right.$$

(4.3) COROLLARY. The fibres of the Δ -Torelli map $h: \mathcal{A}_1 \to \mathcal{H}_1$ are constructible in the Δ -topology and their Δ -closures have Δ -type zero and typical Δ -dimension ≤ 3. In particular the Δ -closure of any isogeny class in $\mathcal{A}_1$ has Δ -type zero and typical Δ-dimension ≤ 3.

(4.4) Proof of (4.2). Let $A = A_\lambda$ be given by $y^2 = x(x-1)(x-\lambda)$ By [Ka] the D-module $H^1_{DR}(A)$ is isomorphic to the space H/dL where L is the function field of A and $H \subset \Omega_{L/\mathcal{F}}$ is the space of differentials of the secont kind on A viewed as a D-module by letting δ act on L and $\Omega_{L/\mathcal{F}}$ by putting $\delta x = 0$ and $\delta(dx) = 0$ respectively. Then if $\omega = \frac{dx}{y}$ a computation similar to the classical one (cf. e.g. [Ka] p. 99) gives the following equality in $\Omega_{L/\mathcal{F}}$:

$$-\frac{1}{2}d\left(\frac{y}{(x-\lambda)^2}\right) = \frac{\omega}{4} + \left(\frac{2\lambda-1}{\lambda'} - \frac{\lambda\lambda''(\lambda-1)}{(\lambda')^3}\right)\delta\omega + \frac{\lambda(\lambda-1)}{(\lambda')^2}\delta^2\omega$$

So the image $\overline{\omega}$ of ω in $H^1_{DR}(A)$ satisfies the equality

$$\delta^2\overline{\omega} + \frac{(\lambda')^2(2\lambda-1)-\lambda\lambda''(\lambda-1)}{\lambda\lambda'(\lambda-1)}\delta\overline{\omega} + \frac{(\lambda')^2}{4\lambda(\lambda-1)}\overline{\omega} = 0$$

By (3.7.1) and (3.8.1) we get

$$h(j(\lambda)) = \frac{(\lambda')^2}{4\lambda(\lambda-1)} - \frac{1}{4}\left(\frac{(\lambda')^2(2\lambda-1)-\lambda\lambda''(\lambda-1)}{\lambda\lambda'(\lambda-1)}\right)^2 -$$

$$-\frac{1}{2}\left(\frac{(\lambda')^2(2\lambda-1)-\lambda\lambda''(\lambda-1)}{\lambda\lambda'(\lambda-1)}\right)'$$

A direct computation yealds then the formula from the statement of the Proposition.

(4.5) In what follows we consider the case $g \geq 2$ and seek to describe the composition of $h: \mathcal{A}_g \to \mathcal{H}_g$ with the projection $\mathcal{A}_g^{(n)} \to \mathcal{A}_g$ where $\mathcal{A}_g^{(n)}$ is the moduli space of principally polarized abelian varieties with level n structure $(n \geq 3)$. More generally given an abelian scheme $f: X \longrightarrow Y$ we seek to describe the composition $Y \to \mathcal{A}_g \to \mathcal{H}_g$. We make the assumption (which holds for the universal abelian scheme over $\mathcal{A}_g^{(n)}$) that $f: X \longrightarrow Y$ is deduced via base change from a morphism $f_o: X_o \longrightarrow Y_o$ of smooth

$\mathcal{C}$ -varieties. Next we assume (which is possible by taking a covering of $\mathcal{A}_g^{(n)}$) that Y_o is affine with trivial tangent bundle and that both $H^o(\Omega^1_{X_o/Y_o})$ and $H^1(\mathcal{O}_{X_o})$ are free $\mathcal{O}(Y_o)$-modules. Then we shall express the map $Y \to \mathcal{H}_g$ as a composition $Y \xrightarrow{\nabla} TY \xrightarrow{\Omega} \mathrm{Mat}_{\mathcal{F}}(2g,2g) \xrightarrow{\eta} \mathcal{H}_g$ where ∇ is the natural Δ -polynomial section of $TY \to Y$ defined in [B_1] (3.8) (so $\nabla y \in T_y Y$ for all $y \in Y$), Ω will be a morphism of $\mathcal{F}$ -varieties (which descends to $\mathcal{C}$!) constructed with the help of the Gauss--Manin connection of X_o/Y_o while η is map which does not depend on the geometry (i.e. of X/Y) being defined only in terms of " Δ -linear algebra".

Let's define Ω . One can pick a basis $\omega_1, \ldots, \omega_{2g}$ of $H^1_{DR}(X_o/Y_o)$ as an $\mathcal{O}(Y_o)$-module such that $\omega_1, \ldots, \omega_g$ is a basis of $H^o(\Omega^1_{X_o/Y_o})$, let $\nabla^{X_o/Y_o} : H^1_{DR}(X_o/Y_o) \longrightarrow$ $H^1_{DR}(X_o/Y_o) \otimes H^o(\Omega^1_{Y_o/\mathcal{C}})$ be the Gauss-Manin connection and write

$$\nabla^{X_o/Y_o} \omega_i = \sum_{j=1}^{2g} \omega_j \otimes \omega_{ij}, \quad 1 \leq i \leq 2g$$

where $\omega_{ij} \in H^o(\Omega^1_{Y_o/\mathcal{C}})$. The matrix of 1- forms $\Omega_o = (\omega_{ij})$ on Y_o identifies with a morphism of varieties $\Omega_o : TY_o \to \mathrm{Mat}_{\mathcal{C}}(2g, 2g)$ and let $\Omega = \Omega_o \otimes \mathcal{F} : TY \to \mathrm{Mat}_{\mathcal{F}}(2g, 2g)$ be the morphism deduced form Ω_o.

Let's define η . If $M \in \mathrm{Mat}_{\mathcal{F}}(2g, 2g)$ pick any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{\mathcal{F}}(2g) \text{ where } a, b, c, d \in \mathrm{Mat}_{\mathcal{F}}(g,g)$$

such that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = M \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and then the image of $(a, b) \in \mathrm{Mat}_{\mathcal{F}}(g, 2g)_o$ in $\mathrm{Mat}_{\mathcal{F}}(g,2g)_o$ / / $GL_{\mathcal{C}}(2g)$ is well defined (i.e. it depends only on M and not of the choice of a, b, c, d); we let $\eta(M)$ be the image of (a, b) in the double coset space $GL_{\mathcal{F}}(g) \backslash \mathrm{Mat}_{\mathcal{F}}(g, 2g)_o / GL_{\mathcal{C}}(2g) \simeq \mathcal{H}_g$ (cf. (3.2)). In notations above we have:

(4.6) THEOREM. The map $Y \longrightarrow \mathcal{A}_g \xrightarrow{h} \mathcal{H}_g$ coincides with the map $Y \xrightarrow{\nabla} TY \xrightarrow{\Omega} \mathrm{Mat}_{\mathcal{F}}(2g, 2g) \xrightarrow{\eta} \mathcal{H}_g$.

(4.7) To prove the above statement let's make some notations. Let

$$\nabla^{X/Y} : \mathrm{Der}_{\mathcal{F}} \mathcal{O}(Y) \longrightarrow \mathrm{End}_{\mathcal{F}}(H^1_{DR}(X/Y)), \ \theta \mapsto \nabla_\theta^{X/Y}$$

be the "external" Gauss-Manin connection and for each $y \in Y$, letting $X_y = f^{-1}(y)$ consider the "internal" Gauss-Manin connection

$$\nabla^{X_y} : \mathrm{Der}_{\mathcal{C}} \mathcal{F} \longrightarrow \mathrm{End}_{\mathcal{C}}(H^1_{DR}(X_y)), \quad p \longmapsto \nabla_p^{X_y}$$

On the other hand let δ^* and δ^{**} be the canonical liftings of δ to $Y = Y_o \otimes \mathcal{F}$ and $X = X_o \otimes \mathcal{F}$; then δ^{**} induces a $\mathcal{C}$-linear map $\nabla_{\delta^*} \in \mathrm{End}_{\mathcal{C}}(H^1_{DR}(X/Y))$ vanishing on $H^1_{DR}(X_o/Y_o)$ ($\subset H^1_{DR}(X_o/Y_o) \otimes \mathcal{F} = H^1_{DR}(X/Y)$).

Recall that we proved in [B_1] a formula relating the "external" and "internal" Kodaira-Spencer maps. Using an analogue reasoning one can prove the formula (4.8) below relating the "external" and "internal" Gauss-Manin connections (for details see our monograph $[B_4]$, Chapter 5). Before writing down the formula we need more notations. For any $\mathcal{O}(Y)$-module E and any $\varphi \in E$ denote by $\varphi(y)$ the image of φ in $E/m_y E$ where $y \in Y$ and m_y is the maximal ideal of $\mathcal{O}_{Y,y}$. For instance if $\varphi \in H^1_{DR}(X/Y)$ then $\varphi(y) \in H^1_{DR}(X_y)$ while if $\varphi \in H^0(\Omega^1_{Y/\mathcal{F}})$ then $\varphi(y) \in T^*_y Y$ and if $\varphi \in H^0(T_Y)$ then $\varphi(y) \in T_y Y$.

We have:

(4.8) LEMMA $[B_4]$. Let $y \in Y$, $\omega \in H^1_{DR}(X/Y)$ and $\theta \in H^0(T_Y)$ such that $\theta(y) = \nabla y$. Then

$$\nabla_\delta^{X_y}(\omega(y)) = (\nabla_{\delta^*}\omega + \nabla_\theta^{X/Y}\omega)(y)$$

(4.9) Proof of (4.6). Put $\omega = {}^t(u, v)$ where $u = {}^t(\omega_1, \ldots, \omega_g)$ $v = {}^t(\omega_{g+1}, \ldots, \omega_{2g})$. By definition of the D-module structure on $H^1_{DR}(X_y)$ we have $\delta\sigma = \nabla_\delta^{X_y}\sigma$ for all $\sigma \in H^1_{DR}(X_y)$. Assume $\nabla^{X/Y}\omega = \Omega\omega$ as in (4.5) where we view ω_i as elements of $H^1_{DR}(X/Y)$ and Ω as a matrix of 1-forms on Y. Then by (4.8) we have:

$$\begin{pmatrix} \delta(u(y)) \\ \delta(v(y)) \end{pmatrix} = \langle \Omega(y), \nabla y \rangle \begin{pmatrix} u(y) \\ v(y) \end{pmatrix}$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{\mathcal{F}}(2g)$ is such that

$$\begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix} = \langle \Omega(y), \nabla y \rangle \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and if $\tilde{u}(y)$ and $\tilde{v}(y)$ are defined by

$$\begin{pmatrix} u(y) \\ v(y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{u}(y) \\ \tilde{v}(y) \end{pmatrix}$$

then applying δ to the above equality we get that $\delta(\tilde{u}(y)) = \delta(\tilde{v}(y)) = 0$. So $\begin{pmatrix} \tilde{u}(y) \\ \tilde{v}(y) \end{pmatrix}$ is a basis of $H^1_{DR}(X_y)$ killed by δ and $u(y) = a\,\tilde{u}(y) + b\,\tilde{v}(y)$ is a basis of $H^0(\Omega_{X_y/\mathcal{F}})$. By (3.2), (a, b) represents the image of X_y in $\mathcal{H}_g$ via the Δ-Torelli map which proves the Theorem.

(4.10) Remark. Although (4.6) gives an "explicit" way of computing the Δ-Torelli map it seems rather hard to evaluate the Δ-type and typical Δ-dimension of the fibres of the Δ-Torelli map and hence to get dimensional upper bounds for the Δ-closures of isogeny classes as in case $g = 1$.
In any case we have:

(4.11) COROLLARY. The fibres of the Δ-Torelli map $h: \mathcal{A}_g \to \mathcal{H}_g$ are constructible for the Δ-topology of $\mathcal{A}_g$.

Proof. One easily checks that the fibres of η are constructible in the Δ-topology of $\mathrm{Mat}_{\mathcal{F}}(2g, 2g)$ and use (4.6) and "Δ-Chevalley constructibility" [B_7].

5. Δ-tangent maps of Δ-character maps

(5.1) Start with Δ-character maps of abelian varieties $\psi_r : A \to \mathcal{F}^{M_r}$.

By [B_1] there are two cases which are better understood namely when $\mathrm{rank}_\Delta(A) = 0$ (i.e. A descends to $\mathcal{C}$) and $\mathrm{rank}_\Delta(A) = g$

(where $g = \dim A$ as usual). We shall be concerned here with the Δ -tangent map $T_0 \psi_r$ in the second case (which is more interesting). By [B_1] (6.1) if $\operatorname{rank}_\Delta(A) = g$ then $\psi_2 : A \to \mathcal{F}^g$ is surjective (with kernel $A^\#$ of Δ-type zero and typical Δ-dimension $2g$, equal to the Δ-closure of the torsion subgroup of A) and any ψ_r factors through ψ_2. So it is sufficient to look at $T_0 \psi_2$. Note that the map ψ_2 is not an "invariant of A" since it depends upon choosing a basis in the space of Δ-polynomial characters of order ≤ 2. But if one is able to prove that the 2--symbol $\sigma = \sigma_2(T_0 \psi_2)$: Lie $A \to \mathcal{F}^g$ is invertible then the composed map $\psi_A : A \xrightarrow{\psi_2} \mathcal{F}^g \xrightarrow{\sigma^{-1}}$ Lie A is an "invariant of A" and if $\tau_A := T_0 \psi_A$: Lie $A \to$ Lie A then by (3.4) $\sigma_2(\tau_A) = 1_{\text{Lie}}$ so (Lie A, τ_A) is a Δ-Picard-Fuchs equation (3.5). (see also the Introduction). Our main result is:

(5.2) THEOREM. Let A be a principally polarized abelian $\mathcal{F}$ -variety of dimension g and Δ -rank g and let $h_A \in \mathcal{H}_g^{(g)}$ be its image under the Δ -Torelli map. Then:

1) $\sigma_2(T_0 \psi_2)$ is invertible (so by (5.1) we may define ψ_A and $\tau_A = T_0 \psi_A$)

2) The class of the Δ-Picard Fuchs equation (Lie A, τ_A) in $\mathcal{E}_g$ coincides with the image of h_A in $\mathcal{E}_g$ via the isomorphism $\mathcal{H}_g^{(g)} \xrightarrow{\sim} \mathcal{E}_g$ in (3.7).

3) The class of the Δ -lattice (Lie A, Lie $A^\#$) in $\mathcal{L}_g$ coincides with the image of h_A in $\mathcal{L}_g$ via the embedding $\mathcal{H}_g^{(g)} \to \mathcal{L}_g$ in (3.7).

Remark. Assertion 2) implies that if $z_A \in \operatorname{Mat}_{\mathcal{F}}(g,g)$ is a Δ -period matrix for A (cf. (4.1)) then Lie A has a basis (called in what follows a distinguished basis) such that upon identifying Lie A with $\mathrm{Mat}_{\mathcal{F}}(g,1)$ via this basis we have $\tau_A(y) = y'' + \beta_A y$ for all $y \in \mathrm{Mat}_{\mathcal{F}}(g,1)$ where $\beta_A \in \mathrm{Mat}_{\mathcal{F}}(g,g)$ is given by

$\beta_A = (z''_A(z'_A)^{-1})'/2 - (z''_A(z'_A)^{-1})^2/4.$

By (3.8) all distinguished basis of Lie A are $GL_{\mathcal{C}}(g)$ -conjugate.

Proof of the Theorem. Assertion 3) follows directly from 2). So we concentrate ourselves on 1) and 2). Recall from $[B_1]$ the "standard picture" giving ψ_2. Let $\dashrightarrow A^n \longrightarrow A^{n-1} \longrightarrow \dots \longrightarrow A^o$ be the infinite prolongation sequence associated to A (cf. $[B_1]$ (3.2)), let $L^n = \mathrm{Spec}\ \mathcal{O}(A^n)$ and $C^n = \mathrm{Ker}(A^n \to L^n)$. By the proof of $[B_1]$ (6.1) L^n are algebraic vector groups, $C^o = A^o = A$, $C^1 = A^1$ and $C^n \longrightarrow C^{n-1}$ are isomorphisms for $n \geqslant 2$. So $C^{\infty} := \varprojlim C^n$ is a D-group scheme whose underlying group scheme is isomorphic to A^1 and $C^{\infty} \subset A^{\infty}$ is the D-scheme closed immersion corresponding to the inclusion $A^{\#} \subset A$ so after all $C^{\infty} = (A^{\#})^{\infty}$, $[B_1]$ (3.9). Moreover the components of ψ_2 viewed as elements of $\mathcal{O}(A^2)$ form a basis of $X_a(L^2) = X_a(A^2)$ (see the proof of $[B_1]$ (6.1)).

Now it follows from $[B_5]$, Chapter III, (2.14) that the Δ-Hodge structure $h_A = (H^1_{DR}(A), H^o(\Omega^1_{A/\mathcal{F}}))$ is isomorphic to the Δ-Hodge structure $(T^{\#}_o C^{\infty}, T^{\#}_o C^o)$ where $T^{\#}_o C^{\infty}$ has the structure of D-module induced from the D-module structure of the maximal ideal of $\mathcal{O}_{C^{\infty},o}$ (see (1.6)); to be able to apply $[B_5]$, loc. cit. we have to use the D-module isomorphism $T_o C^{\infty} \simeq L(C^{\infty})$ (cf. (1.6)) and the fact implicitely noted in $[B_1]$ (6.6) that A^1 is the universal extension E(A) of A by a vector group (Instead of $[B_5]$ one could probably invoke the characteristic zero version of the main result from [MM] plus the "duality theorem" from [BBM]). Fix an $\mathcal{F}$-basis $y = (y_1,\dots,y_g)$ of $T^{\#}_o A$ and pick any basis $z = (z_1,\dots,z_g)$ of $X_a(L^2)$. Then the map $\psi_2 : A \longrightarrow S := \mathcal{F}^g$ corresponds to a D-map $A^{\infty} \to S^{\infty} = \mathrm{Spec}\ \mathcal{F}\{z_1,\dots,z_y\}$ so the map

$T_o\Psi_2 : (T_oA)^\infty = \operatorname{Spec} \mathcal{F}\{y_1,\dots,y_g\} \to (T_oS)^\infty =$ $= \operatorname{Spec} \mathcal{F}\{z_1,\dots,z_g\}$ is given precisely by the inclusion $\mathcal{F}\{z_1,\dots,z_g\} \to \mathcal{F}\{y_1,\dots,y_g\}$ taking each z_i into the corresponding element of $X_a(L^2) = T_o^*L^2 \hookrightarrow T_o^*A^2 \hookrightarrow T_o^*(A^\infty) =$ $= \sum_{i=1}^{g} Dy_i$. So to compute $T_o\Psi_2$ we must choose a basis $z_1,\dots,z_g$ and express its elements as linear Δ-polynomials in $y_1,\dots,y_g$. We have a commutative diagram

$$\begin{tikzcd}
C^\infty \arrow[r] \arrow[d, "\wr"'] & A^\infty \arrow[d] & \\
C^2 \arrow[r] \arrow[d, "\wr"'] & A^2 \arrow[r] \arrow[d] & L^2 \\
C^1 \arrow[r] \arrow[d] & A^1 \arrow[d] & \\
C^o \arrow[r] & A^o &
\end{tikzcd}$$

inducing a commutative diagram

$$\begin{tikzcd}
T^*_oC^\infty & T^*_oA^\infty \arrow[l, "\sigma_\infty"'] & & \\
T_o^*C^2 \arrow[u] & T^*_oA^2 \arrow[l, "\sigma_2"'] \arrow[u] \arrow[r, hookleftarrow] & T_o^*L^2 = X_a(L^2) & \\
T_o^*C^1 \arrow[u] & T_o^*A^1 \arrow[l, "\sigma_1"'] \arrow[u] & & \\
T_o^*C^o \arrow[u] & T_o^*A^o \arrow[l, "\sigma_o"'] \arrow[u] & &
\end{tikzcd}$$

where the vertical arrows are thought of as inclusions in particular $T_o^*C^1 = T_o^*C^\infty$. Clearly $z_i = \delta^2 y_i - \sigma_1^{-1}\delta^2\sigma_o y_i \in$ $\in T_o^*A^2$ belong to $T_o^*L^2 = X_a(L^2)$ and form a basis of it. Write

$$(*) \quad \delta^2(\sigma_o y_i) = \sum_j \alpha_{ij}\,\delta(\sigma_o y_j) + \sum_j \beta_{ij}(\sigma_o y_j),$$

with $\alpha_{ij}, \beta_{ij} \in \mathcal{F}$

in the D-module $T_0^*C^\infty$. Since $\delta^k(\sigma_0 y_i) = \sigma_\infty(\delta^k y_i)$ we get

$$(**) \quad z_i = \delta^2 y_i - \sum \alpha_{ij}\,\delta y_j - \sum \beta_{ij}\, y_j$$

So $T_0\psi_2$ is given by $y \mapsto y'' - \alpha y' - \beta y$ where $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij})$, in particular $\sigma_2(T_0\psi_2)$ is invertible, this proving assertion 1). On the other hand by (3.7.1) the Δ-Picard-Fuchs equation associated to the Δ-Hodge structure $(T_0^*C^\infty, T_0^*C^0)$ is also given by $y \mapsto y'' - \alpha y' - \beta y$ this proving assertion 2). Our Theorem is proved.

(5.3) Next we pass to computing Δ-tangent maps of Δ-character maps of curves. So let X be a smooth projective non-hyperelliptic curve over $\mathcal{F}$ of genus g and Δ-rank g and denote by ψ_X the composition $X \xrightarrow{\mu} A \xrightarrow{\psi_A} \text{Lie } A$ where A is the Jacobian of X. For any functional $0 \neq h \in (\text{Lie } A)^0$ denote by ψ_h the composition $X \xrightarrow{\psi_X} \text{Lie } A \xrightarrow{h} \mathcal{F}$ (cf. the Introduction); clearly the space of the maps ψ_h (as h varies) coincides with the image of the restriction map $Ch_\Delta(A) \longrightarrow \mathcal{O}^\delta(X)$. Now fix a distinguished basis in Lie A (cf. the remark after (5.2)) and identify as there Lie A $\simeq$ $\text{Mat}_{\mathcal{F}}(g,1)$ via this basis so that $\tau_A y = y'' + \beta y$, $\beta = \beta_A \in \text{Mat}_{\mathcal{F}}(g,g)$, $y \in \text{Mat}_{\mathcal{F}}(g,1)$. Then $(\text{Lie } A)^0$ identifies with $\text{Mat}_{\mathcal{F}}(1,g)$ so we may speak about the derivatives h, h', h'', ... $\in \text{Mat}_{\mathcal{F}}(1,g)$. Moreover view X as canonically embedded into $\mathbb{P}^{g-1} = \mathbb{P}((\text{Lie } A)^0)$ and note that the distinguished basis of Lie A provides distinguished coordinates on $\mathbb{P}^{g-1}$. In particular it makes sense to speak about the set $\mathbb{P}^{g-1}_{\mathcal{C}}$ of $\mathcal{C}$-points of $\mathbb{P}^{g-1}$; these are the points which can be represented with respect to the distinguished coordinates as $(c_1:\ldots:c_g)$ with $c_i \in \mathcal{F}$. Now define some canonical divisors on X as follows. First let $K \in \mathrm{Div}(X)$ be the canonical divisor pull-back of h, next we let $K' \in \mathrm{Div}(X)$ be the pull-back of h' if $h' \neq 0$ and write Supp $K' = X$ if $h' = 0$ and finally let $\widetilde{K} \in \mathrm{Div}(X)$ be the pull back of $h'' + h\beta$ if $h'' + h\beta \neq 0$ and write Supp $\widetilde{K} = X$ if $h'' + h\beta = 0$. It will be also convenient to consider a certain union of linear subspaces of $\mathbb{P}^{g-1}$ as follows: let $V_1, \dots, V_s \subset \mathrm{Lie}\, A$ be the eigenspaces of β and let $L_1, \dots, L_s$ be the linear subspaces of $\mathbb{P}^{g-1}$ defined by $V_1, \dots, V_s$. Then the union $\cup L_i$ will play a role below; of course $\cup L_i \neq \mathbb{P}^{g-1}$ iff β is not a scalar matrix while $\dim(\cup L_i) = 0$ if β has distinct eigenvalues. With the notations above our result is:

(5.4) THEOREM. The following hold:

1) $a_\Delta(\mathrm{Ker}(T_x \psi_h)) = 2$ iff $x \notin$ Supp K

2) $a_\Delta(\mathrm{Ker}(T_x \psi_h)) = 1$ iff $x \in$ Supp K and $x \notin$ Supp K'

3) $a_\Delta(\mathrm{Ker}(T_x \psi_h)) = 0$ (equivalently $T_x \psi_h$ is injective) iff $x \in$ Supp K $\cap$ Supp K' and $x \notin$ Supp $\widetilde{K}$.

4) $T_x \psi_h = 0$ iff $x \in$ Supp K $\cap$ Supp K' $\cap$ Supp $\widetilde{K}$

5) If h is Δ-generic in $(\mathrm{Lie}\, A)^0$ then Supp K $\cap$ Supp K' = $= \emptyset$ so only cases 1) and 2) above may occur.

6) $a_\Delta(\mathrm{Ker}(T_x \psi_X)) \leq 2$ for all $x \in X$

7) $a_\Delta(\mathrm{Ker}(T_x \psi_X)) \geq 1$ for only finitely many $x \in X$

8) $a_\Delta(\mathrm{Ker}(T_x \psi_X)) = 2$ iff $x \in X \cap \mathbb{P}^{g-1}_{\mathscr{C}} \cap (\cup L_i)$

Remarks. 1) Since $T_x \psi_h : T_x X \simeq \mathcal{F} \longrightarrow \mathcal{F}$ is a linear differential operator it follows that $a_\Delta(\mathrm{Ker}(T_x \psi_h))$ coincides with the order of $T_x \psi_h$ (with the convention that the "zero operator" has order, say, $-\infty$). Of course this interpretation fails for $a_\Delta(\mathrm{Ker}(T_x \psi_X))$!

2) A heuristic principle in algebraic geometry says that "any dimensional invariant depending algebraically on a parameter is uper semicontinuous". This principle is violated here (as noted also in the Introduction). But there is nothing misterious here for one can give very down-to-earth examples when this principle is violated in Δ-algebraic geometry. For instance let $f: \mathcal{F} \to \mathcal{F}$ be the Δ-polynomial map $f(y) = y + yy'$. Then for $y \in \mathcal{F}$, $(T_y f)(t) = (1 + y')t + yt'$ hence

$$a_{\Delta}(\mathrm{Ker}(T_y f)) = \begin{cases} 0 & \text{if} \quad y = 0 \\ 1 & \text{if} \quad y \neq 0 \end{cases}$$

Proof of (5.4). Let $x_0 \in X$ and let $s \in \mathcal{O}_{X,\, x_0}$ be a local parameter. Then the tangent map $T\mu : TX \to TA = A \times \mathrm{Lie}\, A$ has the form $(x, t\frac{\partial}{\partial s}) \longmapsto (\mu(x), t\, v(x))$ for x in a neighbourbrood of x_0, $t \in \mathcal{F}$ and $v(x) \in \mathrm{Lie}\, A = \mathrm{Mat}_{\mathcal{F}}(g,1)$; note that the image $[v(x)] \in \mathbb{P}^{g-1}$ of $v(x)$ is precisely the image of x under the canonical map. Consequently for x around x_0 and writing v instead of $v(x)$ we have the following formulae for $T_x \psi_X : T_x X \to \mathrm{Mat}_{\mathcal{F}}(g,1)$ and $T_x \psi_h : T_x X \to \mathcal{F}$:

$$(T_x \psi_X)\left(t \frac{\partial}{\partial s}\right) = (tv)'' + t\beta v = t''v + 2t'v' + t(v'' + \beta v)$$

$$(T_x \psi_h)\left(t \frac{\partial}{\partial s}\right) = hvt'' + 2hv't' + h(v'' + \beta v)t$$

Now $T_x \psi_h$ has order 2 iff $hv \neq 0$ which proves 1). $T_x \psi_h$ has order 1 iff $hv = 0$ and $hv' \neq 0$; since $h'v + hv' = 0$ this is equivalent to $hv = 0$ and $h'v \neq 0$ which proves 2). Similarily one proves 3) and 4). To prove 5) start with the Δ-polynomial map $\varphi : \mathrm{Mat}_{\mathcal{F}}(1,g) \to \mathrm{Mat}_{\mathcal{F}}(1,g) \times \mathrm{Mat}_{\mathcal{F}}(1,g)$, $\varphi(h) = (h, h')$ which clearly has a Zariski dense image and let $U \subset \mathrm{Mat}_{\mathcal{F}}(1,g) \times$

$\times$ $Mat_{\mathcal{F}}(1,g)$ be the Zariski open set of all pairs (h_1, h_2) such that $Supp\ D_1 \cap Supp\ D_2 = \emptyset$ where $D_i \in Div(X)$ is the pull-back of h_i on X. Then $\varphi^{-1}(U)$ is a Δ-open subset of $Mat_{\mathcal{F}}(1,g)$ such that for any $h \in \varphi^{-1}(U)$, $Supp\ K \cap Supp\ K' = \emptyset$. Assertion 6) is clear from the formula of $T_x \Psi_X$. Assertion 7) follows from (2.6). Let's check the "only if" part of assertion 8) (the "if" part follows similarily). We must have $v' \in \mathcal{F} v$ because otherwise there exists $h \in Mat_{\mathcal{F}}(1,g)$ with $hv = 0$ and $hv' \neq 0$ hence $a_\Delta(Ker(T_x \Psi_X)) \leq a_\Delta(Ker(T_x \Psi_h)) = 1$, contradiction. So $v' = \lambda v$ ($\lambda \in \mathcal{F}$); taking $w = \rho v$ where $\rho \in \mathcal{F}$, $\rho'/\rho = \lambda$ we get $w' = 0$ so $x \in \mathbb{P}^{g-1}_{\mathcal{C}}$. Finally we get $v'' = (\lambda' + \lambda^2)v$; if v was not an eigenvector of β we would get that $v'' + \beta v \notin \mathcal{F} v$ hence one could choose $h \in Mat_{\mathcal{F}}(1,g)$ such that $hv = hv' = 0$ and $h(v'' + \beta v) \neq 0$ hence once again $a_\Delta(Ker(T_x \Psi_X)) \leq a_\Delta(Ker(T_x \Psi_h)) = 0$, contradiction.

6. Calculus of variations

(6.1) We define the "adjoint map" ad: $D \to \mathcal{F}$ where D means as usual the ring $\mathcal{F}[\delta] = \sum_{i \geq 0} \mathcal{F} \delta^i$; this map will replace "integration by parts" from "usual" calculus of variations. First, some notations: if $p_1, p_2 \in D$ we let $p_1 \circ p_2 \in D$ be their product. If $p_1 \in \mathcal{F}$ then we simply write $p_1 p_2$ instead of $p_1 \circ p_2$. But if $p_2 \in \mathcal{F}$ then we keep the notation $p_1 p_2$ to denote $p_1(p_2) \in \mathcal{F}$ ($p_1(p_2)$ = result of applying the operator p_1 to the scalar p_2); therefore we have $\delta \circ \lambda = \lambda \delta + \delta \lambda$ for all $\lambda \in \mathcal{F}$. Define ad: $D \to \mathcal{F}$ by $ad(\sum_{i \geq 0} \lambda_i \delta^i) = \sum (-1)^i \delta^i \lambda_i$.

(6.2) LEMMA. The map ad: $D \to \mathcal{F}$ is $\mathcal{F}$-linear where D is viewed with its right $\mathcal{F}$-module structure.

Proof. First note that if $p \in D$ then $ad(p \circ \delta) = -\delta(ad(p))$. Then prove that $ad(p \circ \lambda)) = \lambda(ad(p)$ by induction on ord(p) = minimum n such that $p \in D_n$.

(6.3) Let G be an algebraic vector $\mathcal{F}$-group. To any Δ-polynomial character $f: G \to \mathcal{F}$ we will associate an algebraic group character $ad(f): G \to \mathcal{F}$ as follows. Assume $G = \mathrm{Spec}\ S(W)$, $W = X_a(G)$. Then f corresponds to an element of $D \otimes_{\mathcal{F}} W$ and we let $ad(f) \in W$ be the image of f under the map

$$D \otimes_{\mathcal{F}} W \xrightarrow{ad \otimes 1} \mathcal{F} \otimes_{\mathcal{F}} W = W$$

which makes sense by (6.2). In coordinates, if $G = \mathcal{F}^g$ and $f: \mathcal{F}^g \to \mathcal{F}$ is given by $f(y_1, \ldots, y_g) = \sum_{i,j} a_{ij} \delta^i y_j$ then $ad(f): \mathcal{F}^g \to \mathcal{F}$ is given as expected by $ad(f)(y_1, \ldots, y_g) = \sum_{i,j} (-1)^j (\delta^j a_{ij}) y_i$.

(6.4) Let X be a smooth $\mathcal{F}$-variety and $\pi: G \to X$ be a vector bundle on X (viewed as a group scheme $G = \mathrm{Spec}\ S(W)$, W locally free on X). By a Δ-polynomial section of G/X (or of W) we mean a Δ-polynomial map $s: X \to G$ such that $\pi \circ s = 1_X$. To give a Δ-polynomial section of G/X is equivalent to giving a map $s: X \to G$ with $\pi \circ s = 1_X$ such that for each $x_0 \in X$ there exists a neighbourhood U of x_0 in X and a frame $w_1, \ldots, w_g \in H^0(U, W)$ such that $s(x) = \sum f_i(x) w_i$ for $x \in U$ where $f_i \in \mathcal{O}^{\Delta}(U)$. So the space of Δ-polynomial sections of G/X equals $H^0(X, \check{W}^{\Delta})$ where $\check{W}^{\Delta} := \check{W} \otimes_{\mathcal{O}_X} \mathcal{O}^{\Delta}_X$. Note also that to give a Δ-polynomial section of G/X is equivalent to giving a Δ-polynomial map $\check{G} \to \mathcal{F}$

where $\check{G} := \operatorname{Spec} S(\check{W})$ such that for each $x \in X$ the induced map $\check{G}_x \to \mathcal{F}$ is an algebraic group character (where $\check{G}_x$ = preimage of x in $\check{G}$).

(6.5) Let $G \longrightarrow X$ be as in (6.4). By a relative Δ-polynomial character of G/X we mean a Δ-polynomial map $f: G \longrightarrow \mathcal{F}$ such that for each $x \in X$ the induced map $f_x: G_x \to \mathcal{F}$ is a Δ-polynomial homomorphism. To any relative Δ-polynomial character $f: G \to \mathcal{F}$ we can associate a Δ-polynomial map $ad(f): G \to \mathcal{F}$ by the formula $ad(f_x) = ad(f_x)$ for $x \in X$ (one checks that $ad(f)$ if Δ-polynomial by a local computation: we may assume $G = X \times \mathcal{F}^g$ and using the fact that $\mathcal{O}^{\Delta}(X \times \mathcal{F}^g) = \mathcal{O}^{\Delta}(X) \otimes \mathcal{O}^{\Delta}(\mathcal{F}^g)$ [B3] section 1, f has the form $f(x, y_1, \dots, y_g) = \sum f_{ij}(x)\, \delta^j y_i$ with $f_{ij} \in \mathcal{O}^{\Delta}(X)$ so by (6.3) $ad(f)(x, y_1, \dots, y_g) = \sum (-1)^j (\delta^j f_{ij}(x)) y_i)$). Since each $ad(f)_x : G_x \to \mathcal{F}$ is an algebraic group character by (6.4) $ad(f)$ corresponds to a Δ-polynomial section of $\check{G}/X$ i.e. to an element in $H^o(X, W^{\Delta})$ which we still denote by $ad(f)$.

(6.6) Let X be a smooth $\mathcal{F}$-variety and $f: X \to \mathcal{F}$ a polynomial map, i.e. $f \in \mathcal{O}^{\Delta}(X)$. Then the Δ-tangent map $Tf: TX \longrightarrow T\mathcal{F} = \mathcal{F} \times \mathcal{F}$ composed with the second projection $p_2: \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ yelds a relative Δ-polynomial character $p_2 o Tf: TX \longrightarrow \mathcal{F}$ hence we may consider the associated Δ-polynomial section $ad(p_2 o Tf)$ of the cotangent bundle $T^*X \to X$; write $el(f) = ad(p_2 o Tf)$ and call it the Euler-Lagrange section (it lies in $H^o(X, \Omega^{\Delta})$). Its zero locus in X (i.e. the inverse image via $el(f): X \longrightarrow T^*X$ of the zero section of $T^*X \to X$) will be called $Geo(f)$ and is Δ-closed in X.

In coordinates, if $X \subset \mathbb{A}^N$, $X = \operatorname{Spec} \mathcal{F}[y_1, \dots, y_N]/J$ and $f \in \mathcal{O}^{\Delta}(X) = \mathcal{F}\{y_1, \dots, y_N\}/[J]$, $f = F \bmod [J]$, $F \in \mathcal{F}\{y_1, \dots, y_N\}$ then $el(f): TX \to \mathcal{F}$ is induced by $el(F)$:

$: T\Delta^N = \Delta^N \times \Delta^N \to \mathcal{F}$, $el(F)(y_1,\dots,y_N; dy_1,\dots,dy_N) =$

$$= \sum_{i,j} (-1)^j \delta^j \left(\frac{\partial F}{\partial(\delta^j y_i)}\right) dy_i$$

which is precisely the expression which vanishes in the classical Euler-Lagrange equations corresponding to the "lagrangian" F (see [P]). In what follows we examine Geo(f) in two particular situations which we were very fond of in this paper namely when f is a "quadratic form in the Δ-polynomial characters of order 2" on an abelian variety of maximum Δ-rank respectively on a curve of maximum Δ-rank. We need a preparation.

(6.7) Let G be an algebraic vector $\mathcal{F}$-group, viewed as on $\mathcal{F}$-linear space and let $q: G \to \mathcal{F}$ be a non-degenerate quadratic form. Then for any Δ-Picard-Fuchs equation (G, f) we can define "the adjoint" Δ-Picard-Fuchs equation (G, f^*) with respect to q as follows. Pick any basis of G which is orthonormal with respect to q, identify (using this basis) G with $Mat_{\mathcal{F}}(g,1)$, write

$$f(y) = y'' + \alpha y' + \beta y$$

where $\alpha, \beta \in Mat_{\mathcal{F}}(g,g)$ and define

$$f^*(y) = y'' - ({}^t\alpha\, y)' + {}^t\beta\, y$$

where α^t , ${}^t\beta$ are the transposed of α, β . One easily checks that the definition of f^* does not depend upon choosing our orthonormal basis.

(6.8) THEOREM. Let A be a principally polarized abelian $\mathcal{F}$-variety of dimension g and Δ-rank g, let $q: \text{Lie } A \to \mathcal{F}$ be a non-degenerate quadratic form, let $f := q \circ \psi_A : A \to \text{Lie } A \to \mathcal{F}$

and let (Lie A, τ_A^*) be the adjoint of (Lie A, τ_A) with respect to q (where ψ_A and τ_A are as in (5.1)). Then Geo(f) = Ker($\tau_A^* \circ \psi_A$) hence Geo(f) is an irreducible Δ-closed subgroup of A of Δ-type zero and typical Δ-dimension 4g with Lie algebra Lie(Geo(f)) = = Ker($\tau_A^* \circ \tau_A$).

Proof. Under the identifications TA = A $\times$ Lie A, TLieA = = Lie A $\times$ Lie A, $T\mathcal{F} = \mathcal{F} \times \mathcal{F}$ the map $T\psi_A$ identifies with $\psi_A \times \tau_A$ while the map $p_2 \circ Tq$ identifies with the bilinear map b: Lie A $\times$ Lie A $\longrightarrow \mathcal{F}$ where $2b(y,y) = q(y)$, $y \in$ Lie A. So the map $p_2 \circ Tf: TA \longrightarrow \mathcal{F}$ in (6.6) is given by $(x,y) \mapsto b(\psi_A(x), \tau_A(y))$. Hence the map el(f): $TA \to \mathcal{F}$ is given by $(x,y) \longmapsto$ $\mathrm{ad}(b(\psi_A(x), \tau_A(y)))$. Now choose an orthonormal basis of Lie A with respect to q and identify as usual Lie A with $\mathrm{Mat}_{\mathcal{F}}(g,1)$ so $b(y,z) = {}^t yz$ for $y,z \in \mathrm{Mat}_{\mathcal{F}}(g,1)$. Then if $\tau_A(y) = y'' + \alpha y' + \beta y$

$$\mathrm{el}(f)(x,y) = \mathrm{ad}({}^t\psi_A(x)(y'' + \alpha y' + \beta y)) = (({}^t\psi_A(x))'' - ({}^t\psi_A(x))\alpha)' + {}^t\psi_A(x)\beta)y = {}^ty((\psi_A(x))'' - ({}^t\alpha\,\psi_A(x))' + {}^t\beta\,\psi_A(x))$$

So Geo(f) = $\psi_A^{-1}(\Sigma)$ where $\Sigma = \mathrm{Ker}\,\tau_A^*$ and we are done.

(6.9) THEOREM. Let X be a smooth projective non-hyperelliptic curve over $\mathcal{F}$ of genus g and Δ-rank g embedded into its Jacobian A and also viewed as embedded in $\mathbb{P}^{g-1} = \mathbb{P}((\mathrm{Lie}\ A)^{\circ})$, let $q : \mathrm{Lie}\,A \longrightarrow \mathcal{F}$ be a non-degenerate quadratic form and let $f : X \longrightarrow \mathcal{F}$ be the composed map $X \xrightarrow{\psi_X} \mathrm{Lie}\,A \xrightarrow{q} \mathcal{F}$. Then Geo(f) has Δ-type zero and for any $x \in X$ not belonging to the quadric $Q \subset \mathbb{P}^{g-1}$ defined by q, $T_x\mathrm{Geo}(f)$ has typical Δ-dimension ≤ 4.

Proof. Let's borrow the notations from the proof of (6.8) and exactly as in the proof of (5.4) express the map $TX \xrightarrow{T\mu} TA = A \times \mathrm{Lie}\ A$

as $(x, t\frac{\partial}{\partial s}) \mapsto (\mu(x), tv(x))$, $t \in \mathcal{F}$, $v(x) \in \mathrm{Mat}_{\mathcal{F}}(g,1)$. Then we get:

$$(T_xf)(t\frac{\partial}{\partial s}) = {}^t\psi_X(x)((tv(x))'' + \alpha(tv(x))' + \beta\, tv(x))$$

For a fixed x write $v = v(x)$, $\psi = \psi_X(x)$ and $\varphi = {}^t\psi \in \mathrm{Mat}_{\mathcal{F}}(1,g)$. Then

$$\begin{aligned} \mathrm{ad}(T_xf)(t\frac{\partial}{\partial s}) &= \mathrm{ad}(\varphi(t''v + 2t'v' + tv'' + \alpha t'v + \alpha tv' + \beta tv)) \\ &= t(\varphi'' - (\varphi\alpha)' + \varphi\beta)v \\ &= t({}^tv(x))(\tau_A^*(\psi_X(x))) \end{aligned}$$

Let X_1 be the Zariski open set where $\frac{\partial}{\partial s}$ is a basis of T_X. Then $\mathrm{Geo}(f) \cap X_1 = E^{-1}(o)$ where $E : X_1 \to \mathcal{F}$ is the Δ-polynomial map $E(x) = {}^tv(x)(\tau_A^*\psi_X(x))$. Let's compute the Δ-tangent map $T_xE : T_xX \to \mathcal{F}$; it will be sufficient to check that T_xE has order ≤ 4 and $x \notin Q$ iff T_xE has order 4 (Because $T_x\mathrm{Geo}(f) \subset$ $\subset \mathrm{Ker}(T_xE)$).

We have:

$$(T_xE)(t) = {}^tv(x)(T_x(\tau^*\psi)(t)) + ({}^t(T_xv)(t))(\tau^*\psi(x))$$

where we put $\tau^* = \tau^*_A$, $\psi = \psi_X$, $\tau = \tau_A$. Since $T\psi = T\psi_A \circ T\mu = \tau \circ T\mu$ we get by (1.5):

$$(T_xE)(t) = {}^tv(x)((\tau^* \circ \tau)(tv(x))) + (\text{a term of order 0 in } t)$$

Since $\tau^* \circ \tau$: Lie A $\to$ Lie A has order 4 and $\sigma_4(\tau^* \circ \tau) =$ $= 1_{\text{Lie } A}$ we get that $\sigma_4(T_xE) = {}^tv(x)v(x)$ and the Theorem is proved.

APPENDIX. A HEURISTIC DICTIONARY

We already noted several times that there is an analogy between "Ritt-Kolchin Theory" and "global non-linear analysis" as presented in Palais'book [P]. We would like to present here a short "dictionary" between the two theories. This "dictionary" motivated (and on the other hand suggested) some of our results and might be significant for further developments.

Ritt-Kolchin [K_1] [K_2] [B_1]		Palais [P]
- affine line $\mathbb{A}^1$ (identified with $\mathcal{F}$, where $\mathcal{F}$ is an ordinary Δ-field)	.....	unit circle $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ ("identified" with the algebra $C^\infty(S^1)$)
- derivation $\delta : \mathcal{F} \to \mathcal{F}$	.....	derivation operator $d/d\theta : C^\infty(S^1) \to C^\infty(S^1)$
- $\mathcal{F}$-varieties X (identified with their sets $X_{\mathcal{F}}$ of $\mathcal{F}$-points)	.....	C^∞ fibre bundles E over S^1 ("identified" with their C^∞ manifolds of sections $C^\infty(E)$)
- Δ-polynomial maps of $\mathcal{F}$-varieties $X_1 \longrightarrow X_2$	.....	non-linear differential operators $C^\infty(E_1) \to C^\infty(E_2)$
- Δ-polynomial functions $X \longrightarrow \mathcal{F}$	.....	lagrangians $C^\infty(E) \to C^\infty(S^1)$
- algebraic vector $\mathcal{F}$-groups (identified with $\mathcal{F}^N$)	.....	vector bundles η over S^1 (identified with their manifolds of sections $C^\infty(\eta)$)

– Δ-polynomial homomorphism between algebraic vector groups	linear differential operators $C^{\infty}(\eta_1) \to C^{\infty}(\eta_2)$
– symbols of the above	symbols of the above
– Δ-tangent maps of Δ-polynomial maps	linearizations of non-linear differential operators
– $\mathcal{F}$-varieties X which descend to $\mathcal{C}$	trivial bundles $M \times S^1$ "identified" with the loop spaces $\mathrm{Map}(S^1, M)$
– subset $X_{\mathcal{C}}$ of X for X descending to $\mathcal{C}$	embedding of M into $\mathrm{Map}(S^1, M)$ as "constant loops".

The list can be continued but should also be taken with a grain of salt since these analogies cannot be "perfect" and don't go too far (as this is the case with the analogies between "algebraic geometry" and "finite dimensional differential topology").

REFERENCES

[BBM] P. Berthelot, L. Breen, W. Messing, Théorie de Dieudonné cristalline II, Lecture Notes in Math. 930, Springer 1982

[B_1] A. Buium, Geometry of differential polynomial functions, I: algebraic groups, Preprint INCREST 55/1989

[B_2] A. Buium, Geometry of differential polynomial functions II: algebraic curves, Preprint INCREST 1/1990

[B_3] A. Buium, Some results on differential polynomial functions, to appear.

[B_4] A. Buium, Differential algebraic groups and a conjecture of Serge Lang, to appear.

[B_5] A. Buium, Algebraic D-groups, to appear

[B_6] A. Buium, Splitting differential algebraic groups, J. Algebra, 130, 1(1990), 97-105

[B_7] A. Buium, Ritt schemes and torsion theory, Pacific J. Math. 98(1982), 281-293

[C] P. Cassidy, Differential algebraic groups, Amer. J. Math., 94(1972), 891-954.

[J] J. Johnson, Prolongations of integral domains, J. Algebra 94-1(1985), 173-211.

[Ka] N. Katz, Algebraic solutions of differential equations, Invent. Math., 18, 1-2(1972), 1-119

[K_1] E. Kolchin, Differential Algebra and Algebraic Groups, Academic Press, New York 1973.

[K_2] E. Kolchin, Differential Algebraic Groups, Academic Press, New York 1975.

[La_1] S. Lang, Division points on curves, Anali di Matematica pura ed applicata, IV, LXX (1965), 229-234

[La_2] S. Lang, Abelian Varieties, Springer Verlag 1983

[MM] B. Mazur, W. Messing, Universal Extensions and One Dimensional Crystalline Cohomology, Lecture Notes in Math. 370, Springer 1974.

[P] R. Palais, Foundations of Global non-linear Analysis, W.A. Benjamin, New York 1968.

[R] M. Raynaud, Around the Mordell conjecture for function fields and a conjecture of Serge Lang, in: Algebraic Geometry, Lecture Notes in Math. 1016, Springer, 1-20.

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