BIFURCATION IN THE TRACTION PROBLEM FOR A TRANSVERSELY ISOTROPIC MATERIAL

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SUMMARY

The traction problem for a transversely isotropic incompressible elastic material is considered, and it is shown that when only pure homogeneous deformations are considered, the problem can be formulated as a two-dimensional $\mathbb{Z}_2$-equivariant bifurcation problem in which the bifurcation parameter is the dead-load. Using imperfect bifurcation theory conditions for bifurcation phenomena are given and the recognition problem is solved in the simplest cases considering a general non-linear form for the stored energy function. The last section treats transversely isotropic non-linear perturbations for a Mooney-Rivlin material and a neo-Hookean material and the corresponding bifurcations.

1. INTRODUCTION

The traction problem consists in the determination of the equilibrium configurations for an incompressible elastic material subject to a constant dead-load which acts normally on its boundary.

First results in this problem were obtained by Rivlin [1] who considered a non-linear isotropic elastic material described by a Mooney-Rivlin constitutive equation and pure homogeneous deformations only. Later, Ball and Schaeffer [2] analysed the same problem using the results of the imperfect bifurcation theory [3], [4]. The results from paper [2] improve those obtained in [1] as follows: homogeneous deformations and not only pure homogeneous ones were considered and the methods used make possible a greater generality in the stored energy form (applicable to the specific case of a Mooney-Rivlin material) and a qualitative analysis of the bifurcation of solutions in the case in which small imperfections (as a slightly compressibility) are present.

Other approaches to the traction problem were discussed in [5], [6], [7].

This paper is intended to perform an analysis similar to the one in [2] for a transversely isotropic material. Before formulating the problem we must observe some particular features of the isotropic case studied in [2]: The choice of the isotropic case leads to a D_3-equivariant bifurcation problem of the form: $g(x,\lambda) = 0$ with $g : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$. The equivariance condition results from the fact that the stored energy W depends on the deformation gradient through the principal invariants of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ (F is the deformation gradient with respect to a fixed reference configuration) and they are symmetric combinations of the eigenvalues of $\mathbf{C}$.

In order to extend the analysis to transversely isotropic materials some difficulties occur: if we want to maintain the generality from [2] by considering all homogeneous deformations (and not only pure homogeneous ones) we obtain a problem of the form $g(x,\lambda) = 0$ where g is defined on $\mathbf{R}^5 \times \mathbf{R}$. Without additional assumptions on the stored energy form g has, from the singularity theory point of view a

C^∞-codimension greater than 20 (see [8], Ch. IX Proposition 1.3). By an a priori restriction of the solution set in the class of the pure homogeneous deformations we can obtain a bifurcation problem which can be treated using the results from the singularity theory. We will use normal forms, defining conditions and universal unfoldings determined previously by Dangelmeyer and Armbruster in [9].

On the other hand in the isotropic case the results have been applied to a stored energy function of the form:

$$W(\lambda_1,\lambda_2,\lambda_3) = \phi(\lambda_1) + \phi(\lambda_2) + \phi(\lambda_3) \tag{1.1}$$

in order to study the specific cases of Mooney-Rivlin and neo-Hookean constitutive equations. In the transversely isotropic case we shall maintain the generality of the stored energy function without assuming its separable form (1.1). We show that the problem can be formulated as a $\mathbf{Z}_2$-equivariant bifurcation problem and in this way the $\mathbf{Z}_2$-universal unfolding of this problem can be viewed as describing the $\mathbf{Z}_2$ perturbations of an isotropic case; as might be for example a slightly extensibility to a direction. We underline here that the D_3-universal unfolding from [2] can show the bifurcation phenomena when a D_3-equivariant perturbation occurs.

2. EQUILIBRIUM EQUATIONS, NOTATIONS AND AUXILIARY RESULTS

Using the same notations as in [2] we denote the positions of the material points in the reference configuration by X and the deformation by $\mathbf{x}(\mathbf{X})$. $\mathbf{F} = \text{grad}\,\mathbf{x}(\mathbf{X})$ is the deformation gradient, $W(\mathbf{F})$ is the stored energy function and t is the surface dead-load which acts normally on the boundary in the reference configuration.

The equilibrium equations are obtained from the Euler-Lagrange equations for the total free energy:

$$I(\mathbf{x}) = \int_\Omega [W(\mathbf{F}(\mathbf{X})) - t\,\mathrm{tr}\mathbf{F}(\mathbf{X})]d\mathbf{X}$$

In the incompressible case $\det \mathbf{F} = 1$, thus the equilibrium equations are obtained from the Euler-Lagrange equations for:

$$I(x) = \int_{\Omega} [W(F(X)) - t\,\mathrm{tr}F(X) - p(X)(\det \mathbf{F} - 1)]d\mathbf{X} \quad .$$

where p(X) is an arbitrary real-valued function. As we mentioned before we consider only incompressible homogeneous deformations, thus **F** is independent of **X**, and the Euler-Lagrange equations become:

$$\frac{\partial W}{\partial \mathbf{F}}(F) - t1 = pF^{-T}, \qquad \det \mathbf{F} = 1 \quad . \tag{2.1}$$

In [2] it is shown that if $t \neq 0$ any solution **F** of the equation (2.1) can be written as $\mathbf{F} = \mathbf{QDQ}^T$ with $\mathbf{Q} \in SO(3)$ (the proper orthogonal group) with **D** diagonal and **D** a purely homogeneous solution for (2.1). We prove the same result by using the principle of material objectivity (see [10], Ch. 19). For the rest of the paper we shall suppose that the stored energy is a C^{∞} function. The following two propositions show restrictions imposed on the energy form by principle of material obiectivity and material symmetry.

PROPOSITION 1. If $W(\mathbf{F}) = W(\mathbf{QF})$ for all $\mathbf{Q} \in SO(3)$ then every solution for (2.1) is of the form $\mathbf{F} = \mathbf{QDQ}^T$, where $\mathbf{Q} \in SO(3)$ and **D** is diagonal and is a solution for (2.1).

Proof. From $W(\mathbf{F}) = W(\mathbf{QF})$ we obtain for every $\mathbf{h} \in so(3)$ (the Lie algebra of SO(3)):

$$h_{ik}F_{kj}\frac{\partial W}{\partial F_{ij}} = 0 \tag{2.2}$$

and from (2.1) for $t \neq 0$, multiplying it by $h_{ik}F_{kj}$ and taking into account (2.2) we obtain:

$$F_{ik}h_{ki} = 0 \tag{2.3}$$

for every $\mathbf{h} \in so(3)$ so that F is symmetric, $\mathbf{F} = \mathbf{QDQ}^T$ with **D** diagonal and substituting this **F** in (2.1) we obtain the desired result.

By definition (see [10], Ch. 31) $\mathcal{G}_W \subset U(3)$ (the unimodular group) is called the symmetry group of the energy function of an elastic material if:

$$\mathcal{G}_W = \left\{ \mathbf{H} \in U(3) \,\middle|\, W(\mathbf{FH}) = W(\mathbf{F}),\ \forall\, \mathbf{F} \in GL(3) \right\}$$

We observe that $\mathcal{G}_W$ is a Lie group under our hypothesis on the stored energy function. For solid elastic materials $\mathcal{G}_W = \mathcal{G}_T$, where $\mathcal{G}_T$ stands for the symmetry group of the stress function (see [10], Ch. 31).

PROPOSITION 2. Any solution $\mathbf{F}$ of the equation (2.1) satisfies $F_{ik}h_{ki} = 0$ for every $\mathbf{h} \in \mathcal{L}(\mathcal{G}_W)$-the Lie algebra of the symmetry group $\mathcal{G}_W$.

The proof of this proposition involves the same steps as the proceeding one and we omit it. For solid elastic materials the result of proposition 2 is contained in that of proposition 1.

In order to have an objective energy function we must have:

$$W(\mathbf{F}) = W(\mathbf{C}); \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}$$

and in order to have transverse isotropy in a direction (said $\mathbf{e}_3$) we must have (see [10], Ch. 31):

$$W(\mathbf{C}) = W(C_{11} + C_{22}, C_{13}^2 + C_{23}^2, C_{11}C_{22} - C_{12}^2, C_{33}, \det \mathbf{C})$$

In particular, on the set M of pure homogeneous deformations in the directions $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and which preserve the volume:

$$M = \left\{ \mathbf{F} \in GL(3) \,\middle|\, \mathbf{F}\mathbf{e}_i = \lambda_i \mathbf{e}_i,\ i = 1,2,3,\ \lambda_1\lambda_2\lambda_3 = 1 \right\}$$

$W(\mathbf{F})$ takes the form:

$$W(F) = W(\lambda_1^2 + \lambda_2^2, \lambda_1^2\lambda_2^2, \lambda_3^2) = W(\lambda_1^2, \lambda_2^2, \lambda_3^2) \tag{2.4}$$

Let us observe that interchanging λ_1 with λ_2 the value of $W(\mathbf{F})$ remains unchanged and it is this invariance property which permits us to formulate a $\mathbb{Z}_2$-equivariant bifurcation problem.

For $W(\mathbf{F})$ given by (2.4) the equations (2.1) become:

$$2\lambda_i^2 \frac{\partial W}{\partial \lambda_i^2} - t\lambda_i = p \quad , \quad i = 1,2,3. \tag{2.5}$$

From (2.5) results that $\mathbf{F} = \mathrm{diag}(1,1,1)$ is a solution for (2.1) for every $t \neq 0$ if and only if:

$$\frac{\partial w}{\partial \lambda_1^2}(1) = \frac{\partial w}{\partial \lambda_2^2}(1) = \frac{\partial w}{\partial \lambda_3^2}(1) \ . \tag{2.6}$$

The first of these two equalities hold from symmetry reasons of (2.4). We observe that M has 4 connected componenets M_+ and M_-^i , $i = 1,2,3$ and we shall study the problem (2.5) on $M_+ = \left\{ F \in M \mid Fe_i \cdot e_i > 0 \right\}$.

3. REDUCTION TO A $\mathbb{Z}_2$-EQUIVARIANT BIFURCATION PROBLEM; THE RECOGNITION PROBLEM

In order to show that (2.5) can be formulated as a $\mathbb{Z}_2$-equivariant bifurcation problem we will use the same method as in [2]. Denoting $w_i = \ln\lambda_i$ we have $w_1 + w_2 + w_3 = 0$ and considering the projection:

$$P : \mathbf{R}^3 \to V = \left\{(w_1,w_2,w_3) \in \mathbf{R}^3 \mid w_1 + w_2 + w_3 = 0\right\}$$

defined by:

$$P(x,y,z) = (x - \tfrac{1}{3}(x + y + z),\ y - \tfrac{1}{3}(x + y + z),\ z - \tfrac{1}{3}(x + y + z)) \tag{3.1}$$

we observe that the equations (2.5) are equivalent with:

$$Pf(w_i,t) = 0 \quad \text{where} \quad f_i = e^{w_i}\frac{\partial w}{\partial w_i} - te^{w_i}$$

Let us consider the permutations $1 = \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$ and $\sigma = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ and the action of the group $S_2 = (1, \sigma)$ on $\mathbf{R}^3$ given by:

$$\begin{aligned} 1 \cdot (x,y,z) &= (x,y,z) \\ \sigma \cdot (x,y,z) &= (y,x,z) \end{aligned} \tag{3.2}$$

Then f is S_2-equivariant, which means:

$$f(\sigma(w),t) = \sigma f(w,t) \tag{3.3}$$

Through the isomorphism $H : V \to \mathbf{R}^2$ given by:

$$H(x,y,z) = (\frac{1}{\sqrt{6}}(x + y - 2z);\ \frac{1}{\sqrt{2}}(x - y)) = (\alpha\,;\beta) \tag{3.4}$$

the action (3.2) of S_2 on $\mathbf{R}^3$ induces a natural action of $\mathbf{Z}_2$ on $\mathbf{R}^2$ through:

$$\begin{aligned} 1\cdot(\alpha;\beta) &= HH^{-1}(\alpha;\beta) = (\alpha;\beta) \\ -1\cdot(\alpha;\beta) &= H\sigma H^{-1}(\alpha;\beta) = (\alpha;-\beta) \end{aligned} \tag{3.5}$$

and the map:

$$g(\alpha,\beta,t) = HPf(H^{-1}(\alpha;\beta),t) \tag{3.6}$$

becomes $\mathbf{Z}_2$-equivariant, which means:

$$\begin{aligned} g(\alpha,\beta,t) &= (g_1(\alpha,\beta,t);g_2(\alpha,\beta,t)) = \\ &= (g_1(\alpha,-\beta,t);-g_2(\alpha,-\beta,t)) \end{aligned} \tag{3.7}$$

We observe that $g(\alpha,\beta,t) = 0$ if and only if there exists $p\in\mathbf{R}$ such that (2.5) holds. We have proved:

PROPOSITION 3. The problem (2.5) with W of the form (2.4) can be formulated as a bifurcation problem of the form $g(\alpha,\beta,t) = 0$ with $g:\mathbf{R}^2\times\mathbf{R}\to\mathbf{R}^2$, $\mathbf{Z}_2$-equivariant. By the use of (3.6), (3.4) and (3.1) the problem (2.5) becomes:

$$\begin{aligned} g_1(\alpha,\beta,t) &= e^{w_1}\left(\frac{\partial W}{\partial w_1} - t\right) + e^{w_2}\left(\frac{\partial W}{\partial w_2} - t\right) - 2e^{w_3}\left(\frac{\partial W}{\partial w_2} - t\right) = 0 \\ g_2(\alpha,\beta,t) &= e^{w_1}\left(\frac{\partial W}{\partial w_1} - t\right) - e^{w_2}\left(\frac{\partial W}{\partial w_2} - t\right) = 0 \end{aligned} \tag{3.8}$$

where: $w_1 = \frac{\sqrt{6}}{6}\alpha + \frac{\sqrt{2}}{2}\beta$, $w_2 = \frac{\sqrt{6}}{6}\alpha - \frac{\sqrt{2}}{2}\beta$, $w_3 = -\frac{2\sqrt{6}}{6}\alpha$

In order to study the problem (3.8) we use results for the normal forms, defining conditions and universal unfoldings for $\mathbf{Z}_2$-equivariant bifurcation problems obtained by Golubitsky and Schaeffer [3] and Dangelmeyer and Armbruster [9].

It results from the works of Schwartz [11] and Poenaru [12] that in the ring of $\mathbf{Z}_2$-invariant C^∞ germs of real valued functions, a Hilbert basis is given (see notations from [13]) by $u = \alpha$ and $v = \beta^2$ and thus the general form of a bifurcation problem is:

$$g(\alpha, \beta, t) = (p(u,v,t), \beta q(u,v,t)) \tag{3.9}$$

where p and q are smooth germs at the bifurcation point. We use the notation [p,q] for a problem of the form (3.9). Subsequently we also use the following notation convention: an subscript following comma (e.g. $p_{,u}$; $W_{,1}$) means a partial derivative computed at $(0,0,t_o)$ (the bifurcation point) for p and q and in (1,1,1) for W respectively.

We start the recognition problem with an observation: in order to have a bifurcation at a point $(0,0,t_o)$ we must have ([13], Ch. XIX, section 3):

$$p(0,0,t_o) = q(0,0,t_o) = p_{,u}(0,0,t_o) = 0 \tag{3.10}$$

A simple computation shows that:

$$p(0,0,t_o) = W_{,1} + W_{,2} - 2W_{,3} \tag{3.11}$$

$$q(0,0,t_o) = \frac{\sqrt{2}}{2}(W_{,1} + W_{,2} + W_{,11} + W_{,22} - 2W_{,12} - 2t_o) \tag{3.12}$$

$$\begin{aligned} p_{,u}(0,0,t_o) = \frac{\sqrt{6}}{6}(W_{,1} + W_{,2} + 4W_{,3} + W_{,11} + W_{,22} + \\ + 2W_{,12} - 4W_{,13} - 4W_{,23} + 4W_{,33} - 6t_o) \end{aligned} \tag{3.13}$$

The condition (3.11) is satisfied due to (2.6). It also represents the fact that the residual stress in the undeformed material is of the form **s1**. We note that in general in a transversely isotropic material this condition does not result only from symmetry considerations as in the case of an isotropic material.

From (3.12) and (3.13) we obtain a necessary condition in order to have a bifurcation:

$$W_{,1} + W_{,2} + W_{,11} + W_{,22} = 2(W_{,12} + W_{,3} + W_{,33} - W_{,13} - W_{,23}) \tag{3.14}$$

and the bifurcation can take place for:

$$t_o = W_{,3} + W_{,33} + W_{,12} - W_{,13} - W_{,23} \tag{3.15}$$

Moreover, we observe that the problem (3.8) is linear in t and that $p_{,t}(0,0,t_o) = 0$ so that using the results from [13] the simplest bifurcation which can occur is described by the normal form:

$$\begin{aligned} p &= \varepsilon_1 u^2 + \varepsilon_2 v + m(t - t_o)^2 \\ q &= \varepsilon_3 u + \varepsilon_4 (t - t_o) \end{aligned} \tag{3.16}$$

which represents a family of bifurcation problems, where m is a modal parameter (see [13], Ch. XIX, [3], section 3).

The defining conditions and the non-degenerancy conditions are:

$$\begin{aligned} \varepsilon_1 &= \mathrm{sgn} p_{,uu} \neq 0 \\ \varepsilon_2 &= \mathrm{sgn} p_{,v} \neq 0 \\ \varepsilon_3 &= \mathrm{sgn} q_{,u} \neq 0 \\ \varepsilon_4 &= \varepsilon_1 \mathrm{sgn}(q_{,t} p_{,uu} - q_{,u} p_{,ut}) \neq 0 \end{aligned} \tag{3.17}$$

$$m = \varepsilon_1 \frac{[p_{,uu} p_{,tt} - p_{,ut}^2] q_{,u}^2}{[q_{,t} p_{,uu} - q_{,u} p_{,ut}]^2}, \quad m \neq 0, \quad m \neq -\varepsilon_1$$

A straight forward computation leads to:

$$p_{,tt} = 0$$

$$\begin{aligned} p_{,uu} = \tfrac{1}{6}[&W_{,1} + W_{,2} - 2W_{,3} + 2W_{,11} + 2W_{,22} - 16W_{,33} + 4W_{,12} + \\ &+ 4W_{,13} + 4W_{,23} + W_{,111} + W_{,222} - 8W_{,333} + \\ &+ 3W_{,112} + 3W_{,122} - 6W_{,113} - 12W_{,123} + 12W_{,133} - \\ &- 6W_{,223} + 12W_{,233} + 6t_o] \end{aligned} \tag{3.18}$$

$$p_{,v} = \frac{1}{4}[W_{,1} + W_{,2} + 2W_{,11} + 2W_{,22} - 4W_{,12} + W_{,111} + W_{,222} - W_{,122} - W_{,112} - 2W_{,113} - 2W_{,223} + 4W_{,123} - 2t_o] \tag{3.19}$$

$$q_{,u} = \frac{\sqrt{3}}{6}[W_{,1} + W_{,2} + 2W_{,11} + 2W_{,22} - 2W_{,13} - 2W_{,23} + W_{,111} + W_{,222} - W_{,122} - W_{,112} - 2W_{,113} - 2W_{,223} + 4W_{,123} - 2t_o] \tag{3.20}$$

$$p_{,ut} = -\sqrt{6}$$

$$q_{,t} = -\sqrt{2}$$

We obtain by taking into account (3.15) at the bifurcation point:

$$p_{,uu} = \frac{1}{6}[W_{,1} + W_{,2} + 4W_{,3} + 2W_{,11} + 2W_{,22} - 10W_{,33} + 10W_{,12} - 2W_{,13} - 2W_{,23} + W_{,111} + W_{,222} - 8W_{,333} + 3W_{,112} - 6W_{,113} + 3W_{,122} - 12W_{,123} + 12W_{,133} - 6W_{,223} + 12W_{,233}] \tag{3.21}$$

$$p_{,v} = \frac{1}{4}[W_{,1} + W_{,2} - 2W_{,3} + 2W_{,11} + 2W_{,22} - 6W_{,12} - 2W_{,33} + 2W_{,13} + 2W_{,23} + W_{,111} + W_{,222} - W_{,112} - W_{,122} - 2W_{,113} - 2W_{,223} + 4W_{,123}] \tag{3.22}$$

$$q_{,u} = \frac{\sqrt{3}}{6}[W_{,1} + W_{,2} - 2W_{,3} + 2W_{,11} + 2W_{,22} - 2W_{,12} - 2W_{,33} + W_{,111} + W_{,222} - W_{,112} - W_{,122} - 2W_{,113} - 2W_{,223} + 4W_{,123}] \tag{3.23}$$

The normal form (3.16) has $\mathbb{Z}_2$-codimension C^∞ 3 and $\mathbb{Z}_2$-topological codimension 2. The $\mathbb{Z}_2$-universal unfolding for this normal form is given by (see [13]):

$$G(u,v,t,\mu,a,b) = [\varepsilon_1 u^2 + \varepsilon_2 v + \mu(t - t_o)^2 + b; \quad \varepsilon_3 u + \varepsilon_4 (t - t_o) + a] \tag{3.24}$$

where a and b are small and μ is close to m.

Let us observe that if in (3.10)

$$p(0,0,t_o) = q(0,0,t_o) = 0, \quad p_{,u}(0,0,t_o) \neq 0 ,$$

then the first equation in (3.8) is locally solved by the implicit function theorem and we obtain near the bifurcation point: $u = u(v,t)$ and substituting this in the second equation of (3.8) we obtain a bifurcation problem of the form: $\tilde{g}_2(\beta,t) = 0$ with $\tilde{g}_2 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ where $\tilde{g}_2(\beta,t) = g_2(u(\beta,t),v,t)$ such that $\tilde{g}_2$ is even in β. In the same way as above, all the derivatives of $\tilde{g}_2$ can be computed as depending on the derivatives of the stored energy W. We are led in the same way as above, to a 1-dimensional bifurcation problem without any symmetry this time if in (3.10) $q(0,0,t_o) \neq 0$ and $p_{,u}(0,0,t_o) = 0$; (in this case $g_{2,\beta} \neq 0$ and is this equation which can be locally solved with respect to β). For these two special cases the normal forms, the universal unfoldings and the defining conditions were computed by Golubitsky and Langford [14] for the $\mathbf{Z}_2$-invariant case and by Golubitsky, Schaeffer and Keyfitz [15] for the case when there is no symmetry.

For the basic two-dimensional bifurcation problem the normal form (3.16) was studied in [16], [17], [18], in the context of an interaction between a Hopf bifurcation and a steady state bifurcation. Moreover, for any $\mathbf{Z}_2$-equivariant perturbation p the bifurcation problem $g + \varepsilon p$ with small ε is $\mathbf{Z}_2$-equivalent (as refered to in the imperfect bifurcation theory) with one of the problems described by the universal unfolding (3.24). The bifurcation diagrams for the normal form (3.16) are indicated in [3] and [13].

4. NON-LINEAR TRANSVERSELY ISOTROPIC PERTURBATIONS FOR THE MOONEY-RIVLIN AND NEO-HOOKEAN CONSTITUTIVE EQUATIONS

The aim of the analysis presented in [2] was the application of the general results of imperfect bifurcation theory to the case of a Mooney-Rivlin material and a neo-Hookean material which have the property that the stored energy function has a separable form (1.1). In what follows we apply the results from the preceeding section to a separable stored energy function but more general than (1.1)

$$W(w_1, w_2, w_3) = A(w_1) + A(w_2) + B(w_3) \tag{4.1}$$

The analysis given in section 3 permits us to study this problem even in the case in which A = B but in this special case it is more convenient to use the entire symmetry of the problem (which in this case is D_3 as was stated in [2]). But this approach leads to the D_3 universal unfolding of the problem, and this describes its behaviour under D_3-equivariant perturbations.

If we want to study the transversely isotropic perturbations we have to find out the $\mathbf{Z}_2$ universal unfolding and it is this observation that we use to study transversely isotropic perturbations for the Mooney-Rivlin and neo-Hookean materials.

We must have by taking into account the results of the last section

$$A' + A'' = t_o, \quad A' = B', \quad A'' = B'', \tag{4.2}$$

in order to have a two-dimensional bifurcation phenomena (the prime means the derivative of the function computed at 0). For a stored energy of the form:

$$W(\lambda_1, \lambda_2, \lambda_3) = \mu(\lambda_1^2 + \lambda_2^2) + \vartheta(\lambda_1^{-2} + \lambda_2^{-2}) + \mu_1 \lambda_3^2 + \vartheta_1 \lambda_3^{-2}$$

(which reduced to the Mooney-Rivlin material when $\mu = \mu_1$ and $\vartheta = \vartheta_1$) the two dimensional bifurcation does not occur if we are not in the isotropic case.

From (3.17), taking into account (4.1) and (4.2) we have:

$$\begin{aligned}
&\varepsilon_1 = \mathrm{sgn}(A' - A'' - A''' + \tfrac{4}{3}(A''' - B''')) \neq 0 \\
&\varepsilon_2 = \mathrm{sgn}(A''' + A'') \neq 0 \\
&\varepsilon_3 = \mathrm{sgn}(A' + A'' + A''') \neq 0 \qquad\qquad (4.3) \\
&\varepsilon_4 = \varepsilon_1 \mathrm{sgn}(3A'' + A''' + 2B''') \neq 0 \\
&m = \varepsilon_1 \frac{-9(A'+A''+A''')^2}{4(3A''+A'''+2B''')^2}, \quad m \neq 0, -\varepsilon_1
\end{aligned}$$

and considering B as a perturbation of A, $B = A - 3T$ we obtain for T: $T'(0) = T''(0) = 0$ and from (4.3) we have:

$$\begin{aligned} \varepsilon_1 &= \operatorname{sgn}(A' - A'' - A''' + 4T''') \\ \varepsilon_4 &= \varepsilon_1 \operatorname{sgn}(A'' + A''' - 2T''') \end{aligned} \qquad (4.4)$$

and we observe that only A', $A'' + A'''$ and T''' have a significant influence on the normal form and also on the bifurcation diagram.

If for A we consider the form from the Mooney-Rivlin material: $A(w) = \varphi(l(w))$ with $l(w) = e^w$ and $\varphi(l) = \mu l^2 + \vartheta l^{-2}$ we obtain for t_o at the bifurcation point $t_o = 4(\mu + \vartheta)$ (and this value is the same as the one obtained in [2] for the isotropic problem) and:

$$\begin{aligned} \varepsilon_1 &= \operatorname{sgn}(T''' - 8\vartheta) \neq 0 \\ \varepsilon_2 &= \operatorname{sgn}(\mu - 9\vartheta) \neq 0 \\ \varepsilon_3 &= \operatorname{sgn}(\mu - 5\vartheta) \neq 0 \\ \varepsilon_4 &= \varepsilon_1 \operatorname{sgn}(\mu - 9\vartheta - T''') \neq 0 \end{aligned} \qquad (4.5)$$

$$m = -\varepsilon_1 \frac{9(\mu - 5\ \ )^2}{4(\mu - 9\ \ -T''')^2}, \quad m \neq 0, -\varepsilon_1 .$$

As an example, let us consider $\theta \neq 0$, and:

$$T(\lambda_3) = -\theta\lambda_3^2 - 3\theta\lambda_3^{-2} + \theta\lambda_3^{-4} \qquad (4.6)$$

We have: $T'(1) = T''(1) = 0$, $T'''(1) = -48\theta$ and $T'(\lambda_3) = -2\theta(\lambda_3 - 3\lambda_3^{-3} + 2\lambda_3^{-5})$. For a transversely isotropic material for which the stored energy function has the form (4.1) with $B = A - 3T$ the stress in the e_3 direction will be greater (or smaller respectively) than that in a direction in the (e_1,e_2)-plane if the sign of T' is smaller (or greater respectively) than 0. In the example it can be easily checked that $\operatorname{sgn}T' = -\operatorname{sgn}\theta$ for $\lambda_3 > 0$; thus for $\theta > 0$ the e_3-direction is more rigid that any other one from the (e_1,e_2)-plane while for $\theta < 0$ it is less rigid that any other direction from the (e_1,e_2)-plane. From (4.6) we obtain for $B = A - 3T$:

$$B(\lambda_3) = (\mu + 3\theta)\lambda_3^2 + (\vartheta + 9\theta)\lambda_3^{-2} - 3\theta\lambda_3^{-4}.$$

In order to obtain a monotonic dependence in the relation stress-strain we can put for example :

$$\theta < 0, \quad \mu + 3\theta > 0, \quad \vartheta + 9\theta > 0.$$

Before studing a perturbation for the neo-Hookean material let us observe that if the nondegenerancy conditions (4.5) are not satisfied we can have more degenerate singularities (the normal forms are classified up to $\mathbf{Z}_2$ topological codimension ≤ 5 in [9]).

Ball and Schaeffer have shown in [2] that for a neo-Hookean material only the simplest singularity can occur and this happens if $t_0 = 4(\mu + \vartheta)$ (in [2] the coefficient is 2 but the energy has a 1/2 coefficient compared to that from (4.1)). By taking into account the results from the Mooney-Rivlin case we obtain the neo-Hookean one ($\vartheta = 0$) for transversely isotropic perturbations. Because $\mu > 0$ we have:

$$\begin{aligned} &\varepsilon_2 = \varepsilon_3 = \operatorname{sgn}\mu = 1, && \varepsilon_1 = \operatorname{sgn}T''', \\ &\varepsilon_4 = \varepsilon_1 \operatorname{sgn}(\mu - T'''), && m = -\varepsilon_1 \frac{9\mu^2}{4(\mu - T''')^2} \end{aligned} \tag{4.7}$$

We observe that if $T''' = 5\mu$, $T''' = \mu$, $T''' = 0$, $T''' = -\mu/2$ then more degenerate singularities can appear. Moreover, from (4.7) we see that for the pair $(\varepsilon_1, \varepsilon_4)$ only the cases (1,1), (1,-1) and (-1,1) are possible for $0 < T''' < \mu$, $\mu < T'''$ and $T''' < 0$ respectively. In this case the normal forms are:

$$(x^2 + y^2 - m(t - t_0)^2, \; y(x + t - t_0)) \tag{4.8}$$

$$(x^2 + y^2 - m(t - t_0)^2, \; y(x - t + t_0)) \tag{4.9}$$

$$(-x^2 + y^2 + m(t - t_0)^2, \; y(x + t - t_0)) \tag{4.10}$$

For perturbations of these problems see also [3] . All bifurcation diagrams coresponding to normal forms (3.16) and it's perturbations are shown in [13].

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