THE COMMUTATOR METHOD FOR FORM RELATIVELY COMPACT PERTURBATIONS

by

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1. INTRODUCTION

In studying the spectral properties and the scattering theory for Schrödinger operators, a very powerful method has been initiated by Mourre [7] and developed in [2,4,8,10,14]. The main point is to construct a self - adjoint operator conjugated to the Hamiltonian [8] and to prove that it satisfies the "Mourre estimate" [7] and some regularity conditions [4,7]. Then one can use the Virial Theorem [4] in order to study the eigenvalues and one can construct locally smooth operators with respect to the Hamiltonian [13] in order to determine the absolute continuous spectrum. Moreover one can obtain a very precise form of the "Limiting Absorption Principle" [2]. This approach has been used with very good results for the two-body problem [4,5,7] and for the N-body problem [2,10,14]. In [2] the method is presented in a very general version, a powerful abstract result being obtained, which includes a large class of Schrödinger Hamiltonians (N-body, nonlocal perturbations, long-range interactions). An important condition for the result obtained in [2] is that the domain of the Hamiltonian should be invariant for the unitary group generated by its conjugate operator.

In principle, the method of the "Mourre estimate" does not need that the Hamiltonian should be decomposed in a "free part" and a "perturbation", but up to now it has been used only in this perturbative setting, considering the Laplacean as the free part and the generator of the dilation group as its conjugate [6]. Then, because of the domain condition underlined above, the case of form-relatively compact perturbations [15] is not covered by the general result in [2]. In fact the KLMN-theorem [12] does not give any information concerning the operator-domain of the form sum, and as one can easily see for the simple case: $H = -\frac{d^2}{dt^2} + V$ with $V \in L^1(\mathbb{R})$, the operator domain is no longer invariant for the dilation group.

In order to be able to deal with the form-relative compact case we shall consider a different conjugate operator for the Laplacean on $\mathbb{R}^n$. We shall denote by Q_j $(j = 1,\dots,n)$ the symmetric operator of multiplication by the variable x_j on $L^2(\mathbb{R}^n)$ by $\mathcal{F}: L^2(\mathbb{R}^n; dx) \to L^2(\mathbb{R}^n; dk)$ the Fourier transform and we shall use the notations:

$$(1.1)\quad \begin{cases} P_j = -i\dfrac{\partial}{\partial x_j} & (j = 1,\dots,n) \\ \langle v\rangle := \sqrt{1+|v|^2} & \text{for } v\in\mathbb{R}^n \\ f_j(x) := \dfrac{x_j}{\langle x\rangle} & \text{for } x\in\mathbb{R}^n,\ j = 1,\dots,n \\ A := \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\Big(Q_j f_j(P) + f_j(P)Q_j\Big) & \end{cases}$$

The above defined operator A is a conjugate for the Laplacean on $\mathbb{R}^n$ as one can see from the following lemma.

Lemma 1.1. *The following identity is verified on $\mathcal{S}(\mathbb{R}^n)$:*

$$i[-\Delta, A] = -2\,\Delta\langle P\rangle^{-1}.$$

Lemma 1.2. *The operator A defined by (1.1) is essentially self adjoint on $\mathcal{S}(\mathbb{R}^n)$.*

Proof: In fact A is symmetric on $\mathcal{S}(\mathbb{R}^n)$, and by performing a Fourier transform and taking into account that f is a bounded function with bounded derivatives of any order, it results that $\mathcal{S}(\mathbb{R}^n)$ is a dense, invariant domain of analytic vectors for A. The conclusion then follows from Corollary 2 of the theorem X.39 [12].

■

Lemma 1.3. *Let us denote $\tilde{A} = \mathcal{F}A\mathcal{F}^{-1}$. The unitary group generated by $\tilde{A}$ is given by the formula:*

$$\tilde{W}(t)f = |\mathfrak{J}(\Phi_t)|^{1/2}\, f\circ\Phi_t \qquad \text{for : } f\in L^2(\mathbb{R}^n; dk)$$

where Φ_t is the flow on $\mathbb{R}^n$ associated to the vector field:

$$X_j = -f_j(k), \qquad \text{for: } j = 1,\dots,n$$

and $\mathfrak{J}(\Phi_t)$ is its Jacobian.

Proof: If we consider the operator $\tilde{X} = i\sum f_j(k)\dfrac{\partial}{\partial k_j}$ and observe that the field X has globally Lipschitz components, it results that the associated flow is globally defined and satisfies

$$(1.2)\qquad \frac{d}{dt}\Big|_{t=0}\tilde{W}(t)f = -\frac{1}{2}(\mathrm{div}\,X)f - \tilde{X}f = -i\tilde{A}f.$$

Moreover one easily verifies that $\tilde{W}(t)$ as defined is in fact a one parameter unitary group.

■

Definition 1.4. *For any* $s \in \mathbb{R}$ *we consider the Hilbert space:*

$$\mathcal{H}^s := \{f \in \mathcal{S}'(\mathbb{R}^n) \mid \langle P\rangle^s f \in L^2(\mathbb{R}^n)\}$$

endowed with the norm: $\|f\|_s := \|\langle P\rangle^s f\|$.

Lemma 1.5. *For any* $s \in \mathbb{R}$, $\mathcal{H}^s$ *is left invariant by the unitary group* $\tilde{W}(t)$ *generated by* A.

Proof: We prove that $\tilde{W}(t)$ is bounded on any space $\mathcal{F}\mathcal{H}^s$:

$$(1.3) \qquad \|\tilde{W}(t)\|_{\mathcal{B}(\mathcal{F}\mathcal{H}^s)} = \|\langle P\rangle^s \tilde{W}(t) \langle P\rangle^{-s}\| \le \sup_{\mathbf{k}} \left(\frac{\langle \mathbf{k}\rangle}{\langle \Phi_t(\mathbf{k})\rangle}\right)^s \le C$$

using the fact that $|\Phi_t(\mathbf{k}) - \mathbf{k}| \le C\, exp(at)$ with some constants C and a. ■

Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$ the Banach space of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{K}^*$ a triad with $(\mathcal{K};\mathcal{H})$ and $(\mathcal{H};\mathcal{K}^*)$ Friedrich's couples [2], we shall constantly use the notations: $\|\cdot\|_+$ for the norm in $\mathcal{B}(\mathcal{K})$, $\|\cdot\|_-$ for that in $\mathcal{B}(\mathcal{K}^*)$, $\|\cdot\|_{+-}$ in $\mathcal{B}(\mathcal{K};\mathcal{K}^*)$, $\|\cdot\|_{0+}$ in $\mathcal{B}(\mathcal{H};\mathcal{K})$ $\|\cdot\|_{-0}$ in $\mathcal{B}(\mathcal{K}^*;\mathcal{H})$, and $\|\cdot\|_{-+}$ in $\mathcal{B}(\mathcal{K}^*;\mathcal{K})$, $\|\cdot\|_{+0}$ in $\mathcal{B}(\mathcal{K};\mathcal{H})$ and $\|\cdot\|_{0-}$ in $\mathcal{B}(\mathcal{H}:\mathcal{K}^*)$, .

In the next paragraph we shall prove an abstract result, close to the Theorem 1.3.1. in [2], but supposing only that the form domain of the Hamiltonian [17] is invariant for the unitary group generated by its conjugate. In the third paragraph we shall use this abstract result for the two-body Schrödinger operator with form-compact perturbations.

2. THE ABSTRACT RESULTS

Let us consider a Hilbert space $\mathcal{H}$, a self adjoint operator H and a unitary one parameter group $\{W(t)\}_{t\in\mathbb{R}}$ with a generator A. Let us identify $\mathcal{H}$ to its dual, by the Riesz isomorphism, and let us denote by $\mathcal{G}$ its domain endowed with the graph-norm of H. Let $\{\mathcal{G}^s\}_{s\in\mathbb{R}}$ be the scale of Hilbert spaces associated to H [2,12], so that $\mathcal{G} = \mathcal{G}^1$. We shall denote by $\mathcal{K}$ the form domain of H so that $\mathcal{K} = \mathcal{G}^{1/2} = \mathcal{D}(|H|^{1/2})$ and $\mathcal{K}^* = \mathcal{G}^{-1/2}$. On $\mathcal{K}^*$ we shall consider the dual norm given by:

$$(2.1) \qquad \|f\|_{\mathcal{K}^*} = \|(1+|H|)^{-1} f\| \qquad \text{for any } f \in \mathcal{H}.$$

We shall suppose that $\mathcal{K}$ is invariant for $W(t)$, for any $t\in\mathbb{R}$. By duality we can extend $W(t)$ to $\mathcal{K}^*$ which will also be left invariant. Evidently $H\in\mathcal{B}(\mathcal{K};\mathcal{K}^*)$. We denote by $E:\beta(\mathbb{R})\longrightarrow\mathcal{B}(\mathcal{H})$ the spectral measure of H. We shall also use the Hilbert space $\mathcal{E}\subseteq\mathcal{K}$ representing the domain of the generator of $W(t)|_{\mathcal{K}}$ endowed with the graph norm of A.

Theorem 2.1. *Let $\mathcal{H},\mathcal{K},H,W(t)$ and A be as above and suppose that $W(t)$ leaves $\mathcal{K}$ invariant for any $t\in\mathbb{R}$. Let us denote $B := i[H,A]$ defined a priori as a symmetric element of $\mathcal{B}(\mathcal{E};\mathcal{E}^*)$ and take $J\subseteq\mathbb{R}$ any bounded interval. Suppose that the following hypothesis are verified:*

α) $B\in\mathcal{B}(\mathcal{K};\mathcal{K}^*)$ and there is a strictly positive constant a and a compact operator on $\mathcal{H}$ denoted K such that:

$$E(J)BE(J) \geq aE(J) + K.$$

β) One can find a function: $[0,1]\ni\varepsilon\mapsto H(\varepsilon)\in\mathcal{B}(\mathcal{K};\mathcal{K}^*)$ such that:

β_1) The function $H(\varepsilon)$ is strongly continuous and $H(0) = H$.

β_2) The function: $(0,1)\ni\varepsilon\mapsto B_\varepsilon := i[H(\varepsilon),A]\in\mathcal{B}(\mathcal{K};\mathcal{K}^*)$ is strongly C^1 and there are two strictly positive constants C and δ such that:

$$\left\| \frac{d}{d\varepsilon} B_\varepsilon \right\|_{+-} \leq C\varepsilon^{-1+\delta}.$$

β_3) For any $\varepsilon\in(0,1]$ we have that $[B_\varepsilon, A]\in\mathcal{B}(\mathcal{K};\mathcal{K}^*)$ and there are some strictly positive constants C and δ such that:

$$\| [B_\varepsilon, A] \|_{+-} \leq C\varepsilon^{-1+\delta}.$$

Then J contains no singularly continuous spectrum of H and only a finite number of eigenvalues, each of finite multiplicity.

Remark: Theorem 2.1 is an analog of theorem 1.3.1. in [2] but we only impose the invariance of $\mathcal{K} = \mathcal{G}^{1/2}$ under $W(t)$, and we consider all the operators acting between the spaces $\mathcal{K} = \mathcal{G}^{1/2}$ and $\mathcal{K}^* = \mathcal{G}^{-1/2}$. In order to do that, in applications, we need the modified conjugate operator A for the Laplacean.

Proof: The proof of theorem 2.1 goes exactly through the same steps as that of theorem 1.3.1 in [2], with only a few modifications. In some sense our version of the theorem is more natural, involving only two levels of the scale of Hilbert spaces associated to H, namely $\mathcal{K}$ and $\mathcal{K}^*$.

1] We shall first look at the eigenvalues contained in J. Observe that the Virial Theorem as stated in [2] (theorem 1.2.3.), can be applied taking $\mathcal{K}$ for $\mathcal{G}$ and $\mathcal{K}^*$ for $\mathcal{G}^*$, because any eigenfunction of H is in $\mathcal{K}$. Moreover in theorem 1.2.4. of [2], the condition that $\mathcal{G}$ is in fact the domain of H is not really needed, the necessary condition being that any eigenvector of H should be contained in $\mathcal{G}$, and that remains true also for the form domain of H. Thus theorem 1.2.4. in [2] gives us directly the desired conclusion concerning the eigenvalues of H in J.

2] Let us consider now the statement concerning the absence of singularly continuous spectrum. We shall prove that the operator $L=\langle A\rangle^{-1}$ is locally H-smooth on J by proving that the conclusion of proposition 1.3.6. in [2] remains true also in our hypothesis. Therefore the theorem XIII.23 in [13] gives us the desired conclusion on the singular continuous spectrum. Suppose $\lambda_0\in\mathring{J}$ is not an eigenvalue of H. From hypothesis β_2 we obtain by integration that $B=\lim_{\varepsilon\to 0} B_\varepsilon$ and:

(2.2) $$\|B-B_\varepsilon\|_{+-} \le \frac{C}{\delta}\varepsilon^\delta .$$

We can apply now point (b) of lemma 1.3.2. in [2], and we get a neighborhood $\hat{J}$ of λ_0 such that for any function $\phi\in C_0^\infty(\hat{J})$ we have:

(2.3) $$\phi(H)B_\varepsilon\phi(H) \ge \frac{a}{2}\phi(H)^2$$

for $\varepsilon\in(0,\varepsilon_1]$ with $\varepsilon_1\le 1$. We fix now J_0 a closed interval such that: $\lambda_0\in\mathring{J}_0\subseteq J_0\subseteq\hat{J}$, and a function $\phi\in C_0^\infty(J_0)$ such that $0\le\phi(\lambda)\le 1$ $\forall\lambda\in\mathbb{R}$ and $\phi(\lambda)=1$ on a neighborhood of J_0. We consider now the following operators as defined in [2]:

(2.4) $$\begin{cases} \Phi := \phi(H) \\ \Phi^\perp := 1-\Phi \\ M_\varepsilon := \frac{2\varepsilon}{a}\Phi B_\varepsilon\Phi \quad ,\text{for } \varepsilon\in(0,\varepsilon_1]. \end{cases}$$

We can now apply lemma 1.3.3. in [2] and obtain:

(2.5) $$\|[H-\lambda\pm i(M_\varepsilon+\mu)]f\| \ge \left(\frac{\varepsilon^2}{16}+\mu^2\right)^{1/2}$$

for: $f\in\mathcal{D}(H)\subseteq\mathcal{K}$, $\lambda\in J_0$, $\mu\ge 0$, $\varepsilon\in(0,\varepsilon_0]$, $\varepsilon_0\le\varepsilon_1$. Thus, the operator on the left side of (2.5) is an isomorphism from $\mathcal{G}=\mathcal{D}(H)$ to $\mathcal{H}$, for $\varepsilon\in[0,\varepsilon_0]$, $\mu\ge 0$, such that $\varepsilon+\mu\ge 0$ and $\lambda\in J_0$. By duality and interpolation we can see that it is also an isomorphism from $\mathcal{K}$ to $\mathcal{K}^*$, and thus it has an inverse in $\mathcal{B}(\mathcal{K}^*;\mathcal{K})$. We denote:

(2.6) $$G_\varepsilon(\lambda,\mu) := [H-\lambda+i(M_\varepsilon+\mu)]^{-1} \qquad \text{for } \lambda,\varepsilon,\mu \text{ as above.}$$

First observe that $L=L^*\in\mathcal{B}(\mathcal{H})$ and has dense range. We define:

(2.7) $$F_\varepsilon(\lambda,\mu) := LG_\varepsilon(\lambda,\mu)L$$

and we use the method described in [2] in order to control the ε-dependence of F_ε for $\varepsilon\to 0$. We have:

(2.8) $$\|F_\varepsilon\| \le \frac{C}{\varepsilon}.$$

Now we want to obtain a differential inequality for F_ε:

(2.9) $$\frac{d}{d\varepsilon}F_\varepsilon = -\frac{2i}{a}\left(\varepsilon L G_\varepsilon \Phi \frac{d}{d\varepsilon}B_\varepsilon \Phi G_\varepsilon L + L G_\varepsilon \Phi B_\varepsilon \Phi G_\varepsilon L\right)$$

so that by denoting ε-differentiation with a prime we get:

(2.10) $$\|F'_\varepsilon\| \le \frac{2}{a}\left[\varepsilon\|LG_\varepsilon\Phi B'_\varepsilon\Phi G_\varepsilon L\| + \|LG_\varepsilon\Phi B_\varepsilon\Phi G_\varepsilon L\|\right].$$

We have the following four estimations, a reformulation of lemma 1.3.4. in [2]:

(2.11) $$\|G_\varepsilon\|_{0+} + \|G^*_\varepsilon\|_{0+} \le \frac{C}{\varepsilon}$$

(2.12) $$\|\Phi^\perp G_\varepsilon\|_{0+} + \|\Phi^\perp G^*_\varepsilon\|_{0+} \le C$$

(2.13) $$\|\Phi G_\varepsilon L\|_{0+} + \|\Phi G^*_\varepsilon L\|_{0+} \le C\varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}$$

(2.14) $$\|G_\varepsilon L\|_{0+} + \|G^*_\varepsilon L\|_{0+} \le C\left(\|L\| + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}\right).$$

In proving lemma 1.3.4. in [2] one shows that: $\|G_\varepsilon\|\le C/\varepsilon$ and $\|\Phi G_\varepsilon L\|^2 \le C\varepsilon^{-1}\|F_\varepsilon\|$, and thus all we have to do is to show that one can get this estimations in $\mathcal{B}(\mathcal{H};\mathcal{K})$ instead of $\mathcal{B}(\mathcal{H};\mathcal{G})$. But as the operator $\sqrt{|H|}(i+H)^{-1}$ is in $\mathcal{B}(\mathcal{H})$ and the norm on $\mathcal{K}$ is the graph norm of $\sqrt{|H|}$ we see that:

(2.15) $$\begin{cases} \sqrt{|H|}\,G_\varepsilon = \sqrt{|H|}\,(i+H)^{-1}\left[1 + [\lambda - i(M_\varepsilon+\mu-1)G_\varepsilon\right] \\ \|\sqrt{|H|}\,\Phi G_\varepsilon L\| \le \sup\,\{\sqrt{|x|}\mid x\in \operatorname{supp}\phi\}\,\|\Phi G_\varepsilon L\| \end{cases}$$

Using the inequalities 2.11 - 2.14, by a similar procedure to that described in [2], we can estimate the two terms on the right side of 2.10. For the first one we easily get:

(2.16) $$\varepsilon\|LG_\varepsilon\Phi B'_\varepsilon\Phi G_\varepsilon L\| \le C\|F_\varepsilon\|\varepsilon^{-1+\delta}.$$

For the second term we obtain:

(2.17) $$LG_\varepsilon\Phi B_\varepsilon\Phi G_\varepsilon L = LG_\varepsilon B_\varepsilon G_\varepsilon L - LG_\varepsilon\Phi^\perp B_\varepsilon\Phi^\perp G_\varepsilon L - LG_\varepsilon\Phi^\perp B_\varepsilon\Phi G_\varepsilon L - \\ - LG_\varepsilon\Phi B_\varepsilon\Phi^\perp G_\varepsilon L$$

(2.18) $$\|LG_\varepsilon\Phi^\perp B_\varepsilon\Phi^\perp G_\varepsilon L\| \le \|L\|^2\|G_\varepsilon\Phi^\perp\|_{-0}\|B_\varepsilon\|_{+-}\|\Phi^\perp G_\varepsilon\|_{0+} \le C\|L\|^2$$

(2.19) $$\|LG_\varepsilon\Phi B_\varepsilon\Phi^\perp G_\varepsilon L\| \le \|LG_\varepsilon\Phi\|_{-0}\|B_\varepsilon\|_{+-}\|\Phi^\perp G_\varepsilon\|_{0+}\|L\| \le \\ \le C\|L\|\,\varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}$$

(2.20) $LG_\varepsilon B_\varepsilon G_\varepsilon L = i(LAG_\varepsilon L - LG_\varepsilon AL) + LG_\varepsilon(B_\varepsilon - B)G_\varepsilon L + LG_\varepsilon[M_\varepsilon, A]G_\varepsilon L$

(2.21) $\|LAG_\varepsilon L\| \le \|LA\|_{+0}\|G_\varepsilon L\|_{0+} \le C\|LA\|_{+0}\left(\|L\| + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}\right)$

(2.22) $\|LG_\varepsilon(B_\varepsilon - B)G_\varepsilon L)\| \le \|LG_\varepsilon\|_{-0}\|B_\varepsilon - B\|_{+-}\|G_\varepsilon L\|_{0+} \le$

$$\le C\varepsilon^\delta\left(\|L\| + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}\right)$$

(2.23) $[M_\varepsilon, A] = \varepsilon\left\{[\Phi, A]B_\varepsilon\Phi + \Phi B_\varepsilon[\Phi, A] + \Phi[B_\varepsilon, A]\Phi\right\}$

(2.24) $\|LG_\varepsilon\Phi[B_\varepsilon, A]\Phi G_\varepsilon L\| \le \|LG_\varepsilon\Phi\|_{-0}\|[B_\varepsilon, A]\|_{+-}\|\Phi G_\varepsilon L\|_{0+} \le C\varepsilon^{-2+\delta}\|F_\varepsilon\|$

We only have to study the terms containing $[\Phi, A]$. We use a procedure similar to that in [2]. We remind that for $\phi \in C_0^\infty(\mathbb{R})$ and T a self adjoint operator, the following representation formula is true (Section 4.2 in [1]):

(2.25) $\phi(T) = 1/(2\pi)^{1/2} \int_{\mathbb{R}} \hat{\phi}(t)\, e^{iTt} dt$

with $\hat{\phi}$ them Fourier transform of ϕ. Using 2.25 one immediately proves that $[\Phi, A] \in \mathcal{B}(\mathcal{E}; \mathcal{E}^*)$. By a method similar to that in the proof of lemma 1.2.8. in [2] we shall see that in fact $[\Phi, A] \in \mathcal{B}(\mathcal{K}^*; \mathcal{K})$. In fact one can observe that the only hypothesis needed for proving points (a) and (b) of Lemma 1.2.8. in [2] are that $(\mathcal{E}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{H})$ must be Friedrich's couples (1.1.4. in [2]), with $\mathcal{G}$ invariant for $W(t)$, things that remain true if we replace $\mathcal{G}$ by $\mathcal{K}$ Thus we obtain that:

(2.26) $\begin{cases} [e^{iHt}, A] \in \mathcal{B}(\mathcal{K}; \mathcal{K}^*) \\ \|[e^{iHt}, A]\|_{+-} \le C|t|\ \|B\|_{+-} \end{cases}$

so that using 2.25 we obtain: $[\Phi, A] \in \mathcal{B}(\mathcal{K}; \mathcal{K}^*)$. Now we remark that because $(i+H)$ is an isomorphism from $\mathcal{K}$ to $\mathcal{K}^*$ we have that $[(i+H)^{-1}, A] \in \mathcal{B}(\mathcal{K}^*; \mathcal{K})$. Finally we get:

(2.27) $\|F'_\varepsilon\| \le C\left[\left(\|L\| + \|LA\|_{0+}\right)\left(\varepsilon^{-1/2}\|F_\varepsilon\|^{1/2} + \|L\|\right) + \varepsilon^{-1+\delta}\|F_\varepsilon\|\right].$

From this inequality, by integration, we improve the estimation 2.8 and by repeating the procedure for a finite number of times we obtain:

(2.28) $\sup\{\ \|F_\varepsilon(\lambda, \mu)\|\ \mid 0<\varepsilon<\varepsilon_0,\ \lambda \in J_0,\ \mu \ge 0\ \} < \infty$

so that L turns out to be H-smooth on J_0. ■

Besides this result concerning the spectral properties of H, we can extend the result of theorem 1.4.4. in [2] in order to obtain a Limiting Absorption Principle valid for our situation. We shall use the same class of "weight operators" as in [2]:

Definition 2.2. *For* $\nu\in[0,1/2)$, *we denote by* $\mathbb{L}_\nu(\mathcal{G};\mathcal{H};A)$ *the class of self adjoint, injective operators in* $\mathcal{B}(\mathcal{H})$ *for which a family of operators* $\{L_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, $(\varepsilon_0\leq 1)$ *in* $\mathcal{B}(\mathcal{H})$ *can be found, that satisfies the following properties:*

1. *For* $\varepsilon\to 0$, L_ε *is weakly convergent to* L *in* $\mathcal{B}(\mathcal{H})$.
2. $L_\varepsilon A\in\mathcal{B}(\mathcal{G};\mathcal{H})$ *for any* $\varepsilon\in(0,\varepsilon_0]$.
3. *The function:* $(0,\varepsilon_0)\ni\varepsilon\mapsto L_\varepsilon\in\mathcal{B}(\mathcal{G};\mathcal{H})$ *is weakly* C^1.
4. *There is a positive constant* $C<\infty$ *such that:*

$$\|L_\varepsilon A\|_{+0} + \|L'_\varepsilon\|_{+0} \leq C\varepsilon^{-\nu}.$$

Theorem 2.3. *(Limiting Absorption Principle)*
Suppose H *satisfies the hypothesis of theorem 2.1. (for a given conjugate operator* A*) and that* $L\in\mathbb{L}_\nu(\mathcal{G};\mathcal{H};A)$ *for a given* $\nu\in[0,1/2)$. *Let us denote by* $\mathcal{H}_L$ *the domain of* L *endowed with the graph norm and let* $\mathcal{H}^*_L$ *be its dual. If* $\lambda\in J$ *is not an eigenvalue of* H *we have:*

1. $(H-\lambda\pm i\mu)^{-1}\in\mathcal{B}(\mathcal{H}_L,\mathcal{H}^*_L)$ *for any* $\mu>0$ *and the following limit exists:*

$$\lim_{\mu\to 0}(H-\lambda\pm i\mu)^{-1} := (H-\lambda\pm i0)^{-1}\in\mathcal{B}(\mathcal{H}_L,\mathcal{H}^*_L)$$

uniformly in λ *on compact subintervals in* $J\setminus\sigma_p(H)$.

2. *The mappings:* $J\setminus\sigma_p(H)\ni\lambda\mapsto(H-\lambda\pm i0)^{-1}\in\mathcal{B}(\mathcal{H}_L,\mathcal{H}^*_L)$ *are norm continuous and even norm Hölder-continuous of order* θ, *with:*

$$\theta = \min\left\{\frac{\delta}{1+\delta},\frac{1-2\nu}{3-2\nu}\right\}.$$

Proof: The proof goes through exactly the same lines as that of theorems 1.4.3. and 1.4.4. in [2] taking into account the arguments given in the proof of theorem 2.1 and replacing L by L_ε ∎

3. Application to the Schroedinger two-body problem

As announced in the Introduction, our aim is to extend the general abstract formulation of the Mourre method, in order to be able to deal with the two-body Schroedinger operators with a potential having strong local singularities and also some "long-range" part. In order to see that the abstract results of the theorems 2.1 and 2.3 cover the situation described above we prove theorem 3.6., a result similar to Lemma 2.9.4. in [2], giving a general procedure to verify condition (β) in Theorem 2.1.

We shall work in $\mathbb{R}^3$ and we begin with some technical results that we shall repeatedly use in the following. Suppose $\xi\in\mathbb{R}^3\setminus\{0\}$, we denote $\underline{\xi}=|\xi|^{-1}\xi$ and $P_{\underline{\xi}}:=\underline{\xi}\cdot P=|\xi|^{-1}\sum_j \xi_j P_j$. We denote by BC^∞ the space of bounded C^∞ functions with bounded derivatives.

Lemma 3.1. *Suppose* $g\in BC^\infty$ *and* T *is a self adjoint operator such that:* $[T,P_j]\in\mathcal{B}(\mathcal{K},\mathcal{K}^*)$ *for* $j=1,2,3$. *Then, in* $\mathcal{B}(\mathcal{K},\mathcal{K}^*)$ *we have the formula:*

$$[T,g(P)] = -i(2\pi)^{-3/2}\int_{\mathbb{R}^3} d\xi\ \hat{g}(\xi)\int_0^{|\xi|} exp\{i\tau P_{\underline{\xi}}\}[T,P_{\underline{\xi}}]exp\,iP_{\underline{\xi}}\{(|\xi|-\tau)\}d\tau$$

where $\hat{g}$ *is the Fourier transform of* g, *(thus* $\hat{g}$ *is a measure of rapid decrease).*

Proof: It is an evident reformulation of the result in Section 4.2 of [1] and point (a) of Lemma 1.2.8. in [2], by observing that $\xi\cdot P=|\xi|P_{\underline{\xi}}$. ■

Lemma 3.2. *For* $s,\alpha,\beta\in\mathbb{R}$, *there is some finite positive constant* C *such that:*

$$\|\langle Q\rangle^{\alpha}\langle P\rangle^{\beta}(Q\cdot P)\langle P\rangle^{-\beta-1}\langle Q\rangle^{-\alpha-1}\|_{\mathcal{B}(\mathcal{G}^s)} \le C.$$

Proof: It follows immediately from the results in [11]. ■

Lemma 3.3. *For* $\alpha,t\in\mathbb{R}$, *there is a finite positive constant* C *such that:*

$$\|\langle Q\rangle^{\alpha} exp\{i\tau P_{\underline{\xi}}\}\langle Q\rangle^{-\alpha}\|_{\mathcal{B}(\mathcal{G}^t)} \le C|\tau|^{|\alpha|}.$$

Proof: We remind that: $\|u\|_{\mathcal{G}^t} = \|\langle P\rangle^t u\|$. Thus:

$$\|\langle Q\rangle^{\alpha}exp\{i\tau P_{\underline{\xi}}\}\langle Q\rangle^{-\alpha}\|_{\mathcal{B}(\mathcal{G}^t)} = \|\langle P\rangle^t\langle Q\rangle^{\alpha}exp\{i\tau P_{\underline{\xi}}\}\langle Q\rangle^{-\alpha}\langle P\rangle^{-t}\| \le$$

$$\le C\|\langle Q\rangle^{\alpha}exp\{i\tau P_{\underline{\xi}}\}\langle Q\rangle^{-\alpha}\| = C\,sup\left\{\frac{1+|x|^2}{1+|x-\tau\underline{\xi}|^2}\ \Big|\ x\in\mathbb{R}^3\right\} \le C|\tau|^{|\alpha|}$$

We have intertwined $\langle Q\rangle^{\alpha}\langle P\rangle^{t}$ by using the results in [11]. ■

Lemma 3.4. *For* $\theta\in C_0^\infty(\mathbb{R}^3)$ *satisfying:* $0\le\theta(x)\le1$, $\theta(x)=0$ *for* $|x|\ge2$ *and* $\theta(x)=1$ *for* $|x|\le1$, *let us define* $\theta(\varepsilon Q)$ *the bounded operator of multiplication with* $\theta_\varepsilon(x)=\theta(\varepsilon x)$ *on* $L^2(\mathbb{R}^3)$. *Then* $\frac{d}{d\varepsilon}\theta(\varepsilon Q)$ *and* $[\theta(\varepsilon Q),P_j]$, *for* $j=1,2,3$, *are operators of multiplication with functions of class* $C_0^\infty(\mathbb{R}^3)$ *with support in* $\{x\in\mathbb{R}^3 \mid 1\le|x|\le2\}$. *More* *specifically we have:*

$$(3.1)\qquad \frac{d}{d\varepsilon}\theta(\varepsilon Q) = \sum_j Q_j(\partial_j\theta)(\varepsilon Q) = \varepsilon^{-1}\,\hat{\theta}(\varepsilon Q)$$

$$(3.2)\qquad [\theta(\varepsilon Q),P_j] = i\varepsilon(\partial_j\theta)(\varepsilon Q)$$

where $\hat{\theta}(x) := \sum_j (x_j\partial_j\theta)(x)$.

We shall use the following result proved in [2] (Lemma 2.9.2)

$$(3.3)\qquad \|\psi(\varepsilon Q)\langle Q\rangle^{\delta}\|_{\mathcal{B}(\mathcal{G}^s)} \le C\varepsilon^{-\delta}$$

for $\psi\in C_0^{\infty}(\mathbb{R}^3\setminus\{0\})$, $\delta, s\in\mathbb{R}$, $\varepsilon\in(0,1]$.

Lemma 3.5. *Let us consider* $\chi\in C_0^{\infty}(\mathbb{R}^3)$ *such that* χ=*const. for* $|x|\le 1$ *and* $s,\alpha,\beta\in\mathbb{R}$. *Then there is a finite, positive constant C such that for* $\varepsilon\in(0,1]$;

$$\|\langle Q\rangle^{\alpha}[\chi(\varepsilon Q),A]\langle Q\rangle^{\beta}\|_{\mathcal{B}(\mathcal{G}^s)} \le C\varepsilon^{-(\alpha+\beta)}.$$

Proof:

$$[\chi(\varepsilon Q),A] = (1/2)\sum_j\left\{Q_j[\chi(\varepsilon Q),P_j\langle P\rangle^{-1}] + [\chi(\varepsilon Q),P_j\langle P\rangle^{-1}]Q_j\right\}$$

as operators in $\mathcal{B}(\mathcal{G}^s)$. We use Lemma 3.1 with $g=f_j$ and observe that due to Lemma 3.4 we have that:

$$[\chi(\varepsilon Q),P_{\xi}] = i\sum_j \xi_j\,\varepsilon(\partial_j\chi)(\varepsilon Q)$$

If we define now: $\psi(x) := \langle x\rangle\sum_j \xi_j(\partial_j\chi)(x)$ and use (3.3), Lemma 3.2 and the fact that $\hat{f}_j$ has rapid decrease we obtain the desired inequality. ■

Theorem 3.6. *Suppose V is a symmetric operator in* $\mathcal{B}(\mathcal{K},\mathcal{K}^*)$ *such that there are two positive constants* δ *and C for which:*

i) $\|\langle Q\rangle^{\delta}V\|_{+-} \le C$

ii) $\|\langle Q\rangle^{\delta}[V,A]\|_{+-} \le C$

Let $\theta\in C_0^{\infty}(\mathbb{R}^3)$ *be such that:* $0\le\theta(x)\le 1$, $\theta(x)=1$ *for* $|x|\le 1$ *and* $\theta(x)=0$ *for* $|x|\ge 2$ *and let us denote:* $V(\varepsilon) := \theta(\varepsilon Q)V\theta(\varepsilon Q)$. *Then:*

a) The function: $[0,1]\ni\varepsilon\mapsto V(\varepsilon)\in\mathcal{B}(\mathcal{K},\mathcal{K}^*)$ *is strongly continuous and:*

$$V = \lim_{\varepsilon\to 0} V(\varepsilon).$$

b) The function: $(0,1)\ni\varepsilon\mapsto[V(\varepsilon),A]\in\mathcal{B}(\mathcal{K},\mathcal{K}^*)$ *is strongly* C^1 *and:*

$$\left\|\frac{d}{d\varepsilon}[V(\varepsilon),A]\right\|_{+-} \le C\varepsilon^{-1+\delta}$$

c) $\|[[V(\varepsilon),A],A]\|_{+-} \le C\varepsilon^{-1+\delta}$.

Proof: The proof follows that of Lemma 2.9.4. in [2], but some technical points are more complicated because the commutator $[V(\varepsilon),A]$ is no longer a multiplication operator as in [2] and we have to make a systematic use of Lemma 3.1. Point (*a*) is identical to that in Lemma 2.9.4. in [2].

For point (*b*) we use Lemma 3.4 to obtain:

$$\frac{d}{d\varepsilon}[V(\varepsilon),A] = \varepsilon^{-1}\Big\{[\hat{\theta}(\varepsilon Q),A]V\theta(\varepsilon Q) + \hat{\theta}(\varepsilon Q)V[\theta(\varepsilon Q),A] + \hat{\theta}(\varepsilon Q)[V,A]\theta(\varepsilon Q) + [\theta(\varepsilon Q),A]V\hat{\theta}(\varepsilon Q) + \theta(\varepsilon Q)V[\hat{\theta}(\varepsilon Q),A] + \theta(\varepsilon Q)[V,A]\hat{\theta}(\varepsilon Q)\Big\} \tag{3.4}$$

Observe that the last three terms are the adjoints of the first three and hence will have the same bounds. Using Hypothesis (*i*) and (*ii*) we have: $\|\langle Q\rangle^{\delta}V\chi(\varepsilon Q)\|_{+-}\le C$ and $\|\langle Q\rangle^{\delta}[V,A]\chi(\varepsilon Q)\|_{+-}\le C$ for any $\chi\in C_0^{\infty}(\mathbb{R}^3)$. Therefore we can estimate the second and third terms by using 3.3 because $[\theta(\varepsilon Q),A]\in\mathcal{B}(\mathcal{G}^s)$ for any $s\in\mathbb{R}$. For the first term we use Lemma 3.5 with $\alpha=0$ and $\beta=-\delta$.

c) A direct computation gives:

$$[[V(\varepsilon),A],A] = 2[\theta(\varepsilon Q),A]V[\theta(\varepsilon Q),A] + 2[\theta(\varepsilon Q),A][V,A)\theta(\varepsilon Q) + [[\theta(\varepsilon Q),A],A]V\theta(\varepsilon Q) + \theta(\varepsilon Q)[[V,A],A]\theta(\varepsilon Q) + 2\theta(\varepsilon Q)[V,A][\theta(\varepsilon Q),A] + \theta(\varepsilon Q)V[[\theta(\varepsilon Q),A],A] \tag{3.5.}$$

We observe that the last two terms are the adjoints of the second and third ones, so that we only have to estimate the first four terms. Let us denote $S=[\theta(\varepsilon Q),A]$ and remark that it can be controlled by Lemma 3.5. We also denote $T=[V,A]$ and observe that it is not an operator of multiplication, but it is controlled by hypothesis (ii). We thus have to consider the operators : SVS, $ST\theta(\varepsilon Q)$, $[S,A]V\theta(\varepsilon Q)$ and $\theta(\varepsilon Q)[T,A]\theta(\varepsilon Q)=\theta(\varepsilon Q)TA\theta(\varepsilon Q)-\theta(\varepsilon Q)AT\theta(\varepsilon Q)$

$$\|SVS\|_{+-} \le \|S\langle Q\rangle^{-\delta}\|_{-}\,\|\langle Q\rangle^{\delta}V\|_{+-}\,\|S\|_{+} \le C\varepsilon^{\delta} \tag{3.6}$$

$$\|ST\theta(\varepsilon Q)\|_{+-} \le \|S\langle Q\rangle^{-\delta}\|_{-}\,\|\langle Q\rangle^{\delta}T\|_{+-}\,\|\theta(\varepsilon Q)\|_{+} \le C\varepsilon^{\delta} \tag{3.7}$$

$$AS V\theta(\varepsilon Q) = \left[\frac{1}{2}\left(Q\cdot\frac{P}{\langle P\rangle} + \frac{P}{\langle P\rangle}\cdot Q\right)\langle Q\rangle^{-1}\right]\left(\langle Q\rangle^{\delta}S\langle Q\rangle^{-\delta}\right)\left(\langle Q\rangle^{\delta}V\theta\right) \tag{3.8}$$

$$SAV\theta(\varepsilon Q) = \left(S\langle Q\rangle^{1-\delta}\right)\left[\frac{1}{2}\langle Q\rangle^{-1+\delta}\left(Q\cdot\frac{P}{\langle P\rangle} + \frac{P}{\langle P\rangle}\cdot Q\right)\langle Q\rangle^{-\delta}\right]\langle Q\rangle^{\delta}V\theta \tag{3.9}$$

and by using Lemma 3.3 and 3.5 we get:

$$\|[S,A]V\theta(\varepsilon Q)\|_{+-} < C\varepsilon^{-1+d} \tag{3.10}$$

For the last operator it is enough to control $\theta(\varepsilon Q)AT\theta(\varepsilon Q)$, the first term being its adjoint

$$(3.11)\qquad \theta(\varepsilon Q)AT\theta(\varepsilon Q) = \left(\theta(\varepsilon Q)<Q>^{1-\delta}\right)\left(<Q>^{-1+\delta}A<Q>^{-\delta}\right)\left(<Q>^{\delta}T\right)\theta(\varepsilon Q)$$

$$(3.12)\qquad \theta(\varepsilon Q)<Q>^{1-\delta} = \theta(Q)<Q>^{1-\delta} - \int_{\varepsilon}^{1}\tau<Q>\hat{\theta}(\tau Q)<Q>^{-\delta}\tau^{-2}d\tau.$$

$$(3.13)\qquad \|\theta(\varepsilon Q)[T,A]\theta(\varepsilon Q)\|_{+-} \le C(1+\varepsilon^{-1+\delta})$$

■

Let us prove now that the case of a potential with strong local singularities and long-range can be covered by theorem 2.1. From now on we consider $\mathcal{H}=L^2(\mathbb{R}^3)$, $H_0=-\Delta$ and $\mathcal{K}=\mathcal{H}^1(\mathbb{R}^3)$, (so that $\mathcal{K}^*=\mathcal{H}^{-1}(\mathbb{R}^3)$). We shall consider a potential function $V=V_s+V_L$ such that:

(3.14) V is compact in $\mathcal{B}(\mathcal{H}^1,\mathcal{H}^{-1})$,

where V_s is the "short range" part satisfying:

(3.15) $\|<Q>^{1+\delta}V_s\|_{+-} \le C$, for some $\delta > 0$

and V_L is the "long range" part satisfying:

(3.16) $\|<Q>^{\delta}V_L\|_{+-} \le C$

(3.17) $\|<Q>^{1+\delta}(\partial_j V_L)\|_{+-} \le C$, for $j = 1,2,3$.

In order to apply theorem 2.1. we observe that conditions 3.15 and 3.16 imply that $\|<Q>^{\delta}V\|_{+-} \le C$ for some $\delta>0$. Thus $V\in\mathcal{B}(\mathcal{H}^1,\mathcal{H}^{-1})$ so that it is form-relatively bounded with respect to the Laplacean and it is even form-relatively compact (3.14). Thus we can apply the KLMN theorem and define $H=-\Delta+V$ starting from the form sum. H so defined is semibounded from below and has $\mathbb{R}_+$ as essential spectrum. We shall consider for the Laplacean the conjugate operator A discussed in the introduction and using Lemma.1.5 $\mathcal{K}$ is invariant for the unitary group generated by A.

Lemma 3.7. *If V satisfies condition 3.15 then we have:*

$$\|<Q>^{\delta}[V,A]\|_{+-} \le C$$

Proof: In fact we see that:

$$(3.18)\qquad <Q>^{\delta}[V,A] = \left(<Q>^{\delta}V<Q>\right)\left(<Q>^{-1}A\right) - \left(<Q>^{\delta}A<Q>^{-1-\delta}\right)\left(<Q>^{\delta}V\right)$$

and using Lemma 3.3 one gets the stated result. ■

Lemma 3.8. *If V satisfies condition 3.17 then we have:*

$$\|<Q>^{\delta}[V,A]\|_{+-} \le C$$

Proof: From lemma 3.1 we get:

(3.19) $$\langle Q\rangle^{\delta}[V,A] = -\,i(2\pi)^{-3/2}\sum_j\int_{\mathbb{R}^3} d\xi\,\hat{f}_j\int_0^{|\xi|}\langle Q\rangle^{\delta}\times$$
$$\times\Big(Q_j\,exp\{i\tau P_\xi\}[V,P_\xi]exp\{i(|\xi|-\tau)P_\xi\} -$$
$$-\,exp\{i\tau P_\xi\}[V,A]exp\{i(|\xi|-\tau)P_\xi\}Q_j\Big)\,d\tau$$

(3.20) $$[V,P_\xi] = i\sum|\xi|^{-1}\xi_1\partial_1 V$$

so that using Lemma 3.2 we get the conclusion. ∎

We study first condition (α) of theorem 2.1. We have:

(3.21) $$B := i[H,A] = -2\Delta\langle P\rangle^{-1} + i[V,A]$$

and due to the Lemma above, it will belong to $\mathcal{B}(\mathcal{H}^1,\mathcal{H}^{-1})$. But:

(3.22) $$\Delta\langle P\rangle^{-1} - H\langle\sqrt{|H|}\rangle^{-1} = (\Delta-H)\langle P\rangle^{-1} + H\Big(\langle P\rangle^{-1}-\langle\sqrt{|H|}\rangle^{-1}\Big).$$

Due to conditions 3.15 and 3.16 we have: $V\langle P\rangle^{-1}\in\mathcal{B}(\mathcal{H},\mathcal{H}^{-1})$ and is even compact [14]. Thus the operator:

$$\Big(\langle P\rangle^{-1} - \langle\sqrt{|H|}\rangle^{-1}\Big)\langle P\rangle^{-1}$$

will also be compact in $\mathcal{B}(\mathcal{H},\mathcal{H}^{-1})$, due to the fact that the function $h(t)=\langle\sqrt{t}\rangle^{-1}$ is a continuous function vanishing at infinity [4]. Lemma 3.7 and 3.8 imply that $E(J)[V,A]E(J)$ is compact in $\mathcal{B}(\mathcal{H})$. Thus condition (α) is satisfied with:

$$a < \inf\left\{\frac{t}{(1+|t|)^{1/2}}\ \Big|\ t\in J\right\}$$

Now using theorem 3.6 and lemma 3.7 and 3.8 we see that condition (β) of theorem 2.1 is also verified.

Remarks: *i*) If V satisfies conditions 3.14 - 3.17 and H is defined as above one can apply theorem 2.3 with an operator L defined as in [2] (the proof of proposition 1.4.5), and obtain the Limiting Absorption Principle.

ii) Condition 3.15 covers the situation discussed by M.Schechter in [15].

iii) Let us observe that if V is of Rollnik class [17] and satisfies: $|V(\mathbf{x})| \le C|\mathbf{x}|^{-\delta}$, $|\partial_j V(\mathbf{x})| \le C|\mathbf{x}|^{-1-\delta}$ for $|\mathbf{x}| \ge R$, and j=1,2,3, then it satisfies conditions 3.15 - 3.17.

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