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by

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Traces of Runge domains on analytic subsets

by Mihnea COLTOIU

§o. Introduction

Let $A\subset\mathbb{C}^n$ be a closed analytic subset and $\widetilde{D}$ a Runge domain in $\mathbb{C}^n$. It is clear that $D=\widetilde{D}\cap A$ is a Runge open subset of A. In this paper we prove the converse of this statement (Corollary 1) : If D is a Runge open subset of a closed analytic subset $A\subset\mathbb{C}^n$ then there exists a Runge open subset $\widetilde{D}$ of $\mathbb{C}^n$ such that $\widetilde{D}\cap A=D$.

In fact we obtain a stronger result (Theorem 3) : If $D\subset A$ is a Runge open subset of A and $K\subset\mathbb{C}^n$ is a holomorphically convex compact subset such that $K\cap A\subset D$ then there exists a Runge open subset $\widetilde{D}\subset\mathbb{C}^n$ with $K\subset\widetilde{D}$ and $\widetilde{D}\cap A=D$.

As a consequence of this last statement we deduce the following extension property of plurisubharmonic functions (Proposition 2) :

Let $A\subset\mathbb{C}^n$ be a closed analytic subset, $K\subset\mathbb{C}^n$ a holomorphically convex compact subset and φ a plurisubharmonic function on A. If $\varphi<0$ on $K\cap A$ then φ can be extended to a plurisubharmonic function $\widetilde{\varphi}$ on $\mathbb{C}^n$ such that $\widetilde{\varphi}<0$ on K.

§1. Preliminaries

If $A\subset\mathbb{C}^n$ is a closed analytic subset and $D\subset A$ is a Stein open subset then D is said to be Runge in A if the restriction map $\mathcal{O}(A)\to\mathcal{O}(D)$ has dense image.

A compact subset $L\subset A$ is called holomorphically convex (with respect to A) if $L=\widehat{L}_A$ where $\widehat{L}_A=\{z\in A \mid |f(z)|\leq\sup_K|f| \quad \text{for any } f\in\mathcal{O}(A)\}$, or equivalently, L has a fundamental system of Runge neighbourhoods in A. It is easy to verify that a compact subset $L\subset A$ is holomorphically convex with respect to $\mathbb{C}^n$ iff it is holomorphically convex with respect to A.

In [1] it is proved the following :

Theorem 1. Let $A\subset\mathbb{C}^n$ be a closed analytic subset, $K\subset\mathbb{C}^n$ a holomorphically convex compact subset and U an open neighbourhood of $K\cup A$. Then there exists a C^∞ plurisubharmonic function ψ on $\mathbb{C}^n$ such that $\psi < o$ on $K\cup A$ and $\psi > o$ on $\complement U$. In particular, $K\cup A$ has a fundamental system of Runge neighbourhoods.

Let us recall also the following result due to Grauert and Docquier [2] :

Theorem 2. Let $D_o\subset D_\infty$ be Stein open subsets of $\mathbb{C}^n$. Assume that there exists a family $\{D_t\}_{t\in[o,\infty]}$ of Stein open subsets of $\mathbb{C}^n$ such that :

a) $D_{t_1}\subset D_{t_2}$ if $t_1<t_2$

b) for any $o<t_o\leq\infty$ $\bigcup\limits_{t<t_o} D_t=D_{t_o}$

c) for any $o\leq t_o<\infty$ $\overset{\circ}{\overline{\bigcap\limits_{t>t_o} D_t}}=D_{t_o}$

Then (D_o,D_∞) is a Runge pair.

§2. The main results

Proposition 1. Let $A\subset\mathbb{C}^n$ be a closed analytic subset, $K\subset\mathbb{C}^n$ a holomorphically convex compact subset and $L\subset A$ a holomorphically convex compact subset with $K\cap A\subset L$.

Then $K\cup L$ has a fundamental system of Runge neighbourhoods, hence it is holomorphically convex.

Proof

Let V be any open neighbourhood of $K\cup L$. Since $L\subset V$ is holomorphically convex there exists a C^∞ strongly plurisubharmonic function φ on $\mathbb{C}^n$ such that $L\subset\{\varphi<o\}\subset\subset V$ (in addition φ may be chosen to be an exhaustion function on $\mathbb{C}^n$ [4] but we need not this property).

If we set $K_1=K\cap\{\varphi\geq o\}$ then $K_1\cap A=\emptyset$ because $K\cap A\subset L$ and $\varphi<o$ on L. Let V_1 be an open neighbourhood of A such that $\overline{V}_1\cap K_1=\emptyset$.

Since $\varphi<o$ on $\overline{V}_1\cap K$, it follows that $\varphi<o$ on $\overline{V}_1\cap\overline{V}_2$ if $V_2\subset\subset V$ is a sufficiently small open neighbourhood of K. Let $U=V_1\cup V_2\supset K\cup A$ and choose a C^∞ plurisubharmonic function ψ on $\mathbb{C}^n$ such that $\psi<o$ on $K\cup A$ and $\psi>o$ on $\complement U$ (which exists by Theorem 1). We define the open set $D_o=\{x\in V_2|\psi(x)<o\}\cup\{x\in V_1|\ \psi(x)<o$ and $\varphi(x)<o\}$. Clearly $L\cup K\subset D_o\subset\subset V$. We shall prove that D_o is a Runge domain of $\mathbb{C}^n$. To see this we define $D_\infty=\{x\in\mathbb{C}^n|\ \psi(x)<o\}$ and D_t $o\leq t<\infty$ in the following way :

$D_t=\{x\in V_2|\ \psi(x)<o\}\cup\{x\in V_1|\ \psi(x)<o$ and $\varphi(x)<t\}$. We first verify that D_t is Stein for any $o\leq t<\infty$ (for $t=\infty$ it is clear that D_∞ is a Runge domain in $\mathbb{C}^n$). To show that D_t is Stein it suffices to prove that D_t is locally Stein.

Let $x_o\in\mathbb{C}^n$ be any point.

Case i) $x_o\notin\overline{V}_1$.

We choose an open neighbourhood V_{x_o} of x_o such that $V_{x_o}\cap V_1=\emptyset$. Hence $V_{x_o}\cap D_t=V_{x_o}\cap\{x\in V_2|\ \psi(x)<o\}$. We may assume that $x_o\in\overline{V}_2$. If $x_o\in\partial V_2$ then $\psi(x_o)>o$, so $\psi>o$ on V_{x_o} if V_{x_o} is sufficiently small, hence $V_{x_o}\cap D_t=\emptyset$. If $x_o\in V_2$ we choose $V_{x_o}\subset V_2$ hence $V_{x_o}\cap D_t=\{x\in V_{x_o}|\ \psi(x)<o\}$ which is Stein if V_{x_o} is chosen to be Stein.

Case ii) $x_o\in V_1$.

If $x_o\in\overline{V}_2$ then $\varphi(x_o)<o$. We choose a Stein neighbourhood V_{x_o} such that $\varphi<o$ on V_{x_o} (hence $\varphi<t$ on V_{x_o}) and $V_{x_o}\subset V_1$. Then $V_{x_o}\cap D_t=\{x\in V_{x_o}|\ \psi(x)<o\}$ which is Stein. If $x_o\notin\overline{V}_2$ we choose a Stein neighbourhood $V_{x_o}\subset V_1$, $V_{x_o}\cap V_2=\emptyset$. It follows that $V_{x_o}\cap D_t=\{x\in V_{x_o}|\ \psi(x)<o$ and $\varphi(x)<t\}$ which is Stein.

Case iii) $x_o\in\partial V_1$.

We may assume $x_o\in V_2$. We choose a Stein neighbourhood $V_{x_o}\subset V_2$ of x_o. Then $V_{x_o}\cap D_t=\{x\in V_{x_o}|\ \psi(x)<o\}$ which is Stein.

So we have proved that D_t is Stein for any $t\geq o$. It is also easy to verify that the family $\{D_t\}_{t\in[o,\infty]}$ verifies the conditions the conditions in the theorem of Grauert and Docquier (property c) is a direct consequence of the maximum principle for plurisubharmonic functions).

It follows that (D_0, D_∞) is a Runge pair, and since D_∞ is Runge in $\mathbb{C}^n$, we deduce that D_0 is Runge in $\mathbb{C}^n$.

The proof of Proposition 1 is complete.

Theorem 3. Let $A \subset \mathbb{C}^n$ be a closed analytic subset, $D \subset A$ a Runge open subset and $K \subset \mathbb{C}^n$ a holomorphically convex compact set such that $K \cap A \subset D$. Then there exists a Runge open subset $\tilde{D}$ in $\mathbb{C}^n$ with $\tilde{D} \cap A = D$ and $K \subset \tilde{D}$.

Proof

We first make the following remark : given a Stein manifold M there exists an exhaustion $M_1 \subset\subset M_2 \subset\subset \dots \subset\subset M_k \subset\subset \dots \subset\subset M$ of M with relatively compact Runge domains such that $\overline{M}_k$ is holomorphically convex for any $k \in \mathbb{N}$. This follows easily if we choose a C^∞ strongly plurisubharmonic exhaustion function $\varphi : M \to \mathbb{R}$ and we define $M_k = \{\varphi < \alpha_k\}$ where $\alpha_k \to \infty$ is such that M_k has smooth boundary, hence $\overline{M}_k = \{\varphi \leq \alpha_k\}$.

To prove now the theorem we choose an exhaustion $D_1 \subset\subset D_2 \subset\subset \dots \subset\subset D_k \subset\subset \dots \subset\subset D$ of D with relatively compact open subsets and we construct by induction a sequence of relatively compact Runge domains $V_i \subset\subset \mathbb{C}^n$ $i \geq 1$ such that $\overline{V}_i$ is holomorphically convex, $K \subset V_1 \subset\subset \dots \subset\subset V_i \subset\subset \dots$ and $D_i \subset \overline{V}_i \cap A \subset D$.

We define V_1 as follows : we choose a holomorphically convex compact subset $L_1 \subset D$ with $D_1 \subset L_1$, $K \cap A \subset L_1$. From Proposition 1 there exists a Runge domain $U_1 \subset\subset \mathbb{C}^n$ such that $K \cup L_1 \subset U_1$ and $U_1 \cap A \subset D$. From the remark at the beginning of the proof we can choose a Runge domain $V_1 \subset\subset U_1$, $K \cup L_1 \subset V_1$ and $\overline{V}_1$ is holomorphically convex. Clearly $D_1 \subset \overline{V}_1 \cap A \subset D$.

Now assume that we have constructed $V_1, \dots, V_k$ with the required properties and we define V_{k+1} as follows : we choose a holomorphically convex compact subset $L_{k+1} \subset D$ such that $D_{k+1} \subset L_{k+1}$ and $\overline{V}_k \cap A \subset$

L_{k+1}. By Proposition 1 we get a Runge domain $U_{k+1} \subset\subset \mathbb{C}^n$ with $\overline{V}_k \cup L_{k+1} \subset U_{k+1}$ and $U_{k+1} \cap A \subset D$. Exhausting U_{k+1} we obtain a Runge domain $V_{k+1} \subset\subset U_{k+1}$, $\overline{V}_{k+1}$ holomorphically convex and $\overline{V}_k \cup L_{k+1} \subset V_{k+1}$. Clearly $D_{k+1} \subset \overline{V}_{k+1} \cap A \subset D$.

Hence we have proved the existence of $\{V_i\}$ with the stated properties. If we set $\widetilde{D} = \bigcup_{i \geq 1} V_i$ then obviously $\widetilde{D}$ is Runge in $\mathbb{C}^n$, $K \subset \widetilde{D}$ and $\widetilde{D} \cap A = D$.

The proof of Theorem 3 is thus complete.

Corollary 1. Let $A \subset \mathbb{C}^n$ be a closed analytic subset and $D \subset A$ a Runge open subset of A. Then there exists a Runge open set $\widetilde{D} \subset \mathbb{C}^n$ with $\widetilde{D} \cap A = D$.

We deduce now from Theorem 3 a result concerning the extension of plurisubharmonic functions with some growth conditions.

Proposition 2. Let $A \subset \mathbb{C}^n$ be a closed analytic subset, $K \subset \mathbb{C}^n$ a holomorphically convex compact subset and φ a plurisubharmonic function on A. If $\varphi < 0$ on $K \cap A$ then φ can be extended to a plurisubharmonic function $\widetilde{\varphi}$ on $\mathbb{C}^n$ such that $\widetilde{\varphi} < 0$ on K.

Proof

We begin with the following remark : If B is a Stein space and $D \subset B$ is an open subset such that $(D \cap B_i, B_i)$ is a Runge pair for any irreducible component B_i of B then (D, B) is a Runge pair. This follows easily from the invariance of the Runge property at normalization [5] .

Now we consider $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$ and the closed analytic subset $B \subset \mathbb{C}^{n+1}$ defined by $B = A \times \mathbb{C} \cup \mathbb{C}^n \times \{0\}$. In B we define the following open subset $D = \{(z,w) \in A \times \mathbb{C} \mid |w| < e^{-\varphi(z)}\} \cup \mathbb{C}^n \times \{0\}$. The function $\log|w| + \varphi(z)$ is plurisubharmonic on $A \times \mathbb{C}$, hence $\{(z,w) \in A \times \mathbb{C} \mid \log|w| + \varphi(z) < 0\}$ is Runge in $A \times \mathbb{C}$. From the remark at the beginning of the proof we deduce that D is Runge in B.

We consider also the holomorphically convex compact $Q = K \times \{|w| \leq 1\} \subset \mathbb{C}^{n+1}$

The intersection $Q\cap B$ can be written $Q\cap B=(A\cap K)\times\{|w|\leq 1\}\cup K\times\{o\}$ and it follows $(A\cap K)\times\{|w|\leq 1\}\subset\{(z,w)\in A\times\mathbb{C}\mid |w|<e^{-\varphi(z)}\}$ because $\varphi<o$ on $A\cap K$, hence $Q\cap B\subset D$.

By Theorem 3 there exists a Runge domain Ω in $\mathbb{C}^{n+1}$ such that $\Omega\cap B=D$ and $Q\subset\Omega$. For any point $(z,w)\in\Omega$ we consider the distance $\delta_\Omega(z,w)$ at the boundary $\partial\Omega$ along the w-direction. The function $-\log\delta_\Omega$ is an upper semi-continuous plurisubharmonic function on Ω because Ω is Stein [6] and we define $\tilde{\varphi}=-\log\delta_\Omega|_{\mathbb{C}^n}$. From the definitions it follows that $\tilde{\varphi}|_A=\varphi$ and since $\Omega\supset Q=K\times\{|w|\leq 1\}$ we get $\delta_\Omega(z)>1$ if $z\in K$, so $\tilde{\varphi}(z)=-\log\delta_\Omega(z)<o$ when $z\in K$.

The proof of Proposition 2 is complete.

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